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A Note on a Paper of S.G. Kim

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Abstract: We answer a question posed by S.G. Kim in [3] and show that some of the results of his paper are immediate consequences of known results.

Key words: Bohnenblust-Hille inequality.

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The recent paper [3] deals with extreme multilinear forms and polynomials and the constants of the Bohnenblust-Hille inequalities. In this note we answer a question posed in [3] and show that two theorems stated in [3] are immediate corollaries of well known results of this field.

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. The multilinear Bohnenblust-Hille inequality asserts that, given a positive integer m, there is an optimal constant $C(m:\mathbb{K}) \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left| U(e_{i_1},\ldots,e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C(m:\mathbb{K}) \, \|U\| ,$$

for all bounded m-linear forms $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$. The case of complex scalars was first investigated in [1] and the case of real scalars seems to have been just explored more recently. It is well known that the exponent $2m/(m+1)$ is sharp, so one of the main goals of the research in this field is to investigate the constants involved. The following result was proved in [4]:

Theorem 1. ([4, Corollary 5.4], 2018) *Let $m \geq 2$ be a positive integer. If the optimal constant $C(m:\mathbb{R})$ is attained in a certain $T : c_0 \times \cdots \times c_0 \to \mathbb{R}$, then the quantity of non zero monomials of T is bigger than $4^{m-1} - 1$.*

As an immediate corollary we conclude that if $N_1, \ldots, N_m \geq 1$ are positive integers such that

$$\prod_{j=1}^{m} N_j \leq 4^{m-1} - 1 ,$$

then

$$\sup\left(\sum_{i_1,\ldots,i_m=1}^{N_1,\ldots,N_m}\left|T\left(e_{i_1},\ldots,e_{i_m}\right)\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} < C\left(m:\mathbb{R}\right),$$

where the sup runs over all norm one m-linear forms $T:\ell_\infty^{N_1}\times\cdots\times\ell_\infty^{N_m}\to\mathbb{R}$. In particular,

$$\sup\left(\sum_{i,j,k=1}^{2}\left|T\left(e_i,e_j,e_k\right)\right|^{\frac{6}{4}}\right)^{\frac{4}{6}} < C\left(3:\mathbb{R}\right),$$

where the sup runs over all norm one m-linear forms $T:\ell_\infty^2\times\ell_\infty^2\times\ell_\infty^2\to\mathbb{R}$, and this is the content of [3, Theorem 4.9].

The polynomial Bohnenblust-Hille inequality for real scalars asserts that, given a positive integer m, there is an optimal constant $C_p\left(m:\mathbb{R}\right)\geq 1$ such that

$$\left(\sum_{|\alpha|=m}|a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C_p\left(m:\mathbb{R}\right)\|Q\|,$$

for all $N\geq 1$ and for all m-homogeneous polynomials $Q:\ell_\infty^N\left(\mathbb{R}\right)\to\mathbb{R}$ given by

$$Q(z)=\sum_{|\alpha|=m}a_\alpha z^\alpha.$$

To the best of our knowledge, the case of real scalars became unexplored until the publication of the paper [2] in 2015, where it is proved that the constants $C_p\left(m:\mathbb{R}\right)$ cannot be chosen with a sub-exponential growth. More precisely,

Theorem 2. ([2, Theorem 2.2], 2015)

$$C_p\left(m:\mathbb{R}\right) > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^m > (1.177)^m,$$

for all positive integers $m\geq 2$.

The above result is, obviously, by far, rather precise than [3, Theorem 4.5], which states that

$$C_p\left(m:\mathbb{R}\right)\geq 2^{\frac{m+1}{2m}},$$

for all positive integers $m \geq 2$. The only case that deserves a little bit more of attention is the case $m = 2$, since

$$\left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^2 < 2^{\frac{3}{4}},$$

but in the case $m = 2$ a quick look at the proof of [2, Proof of Theorem 2.2] shows that

$$C_p\,(2:\mathbb{R}) \geq \frac{3^{\frac{3}{4}}}{\frac{5}{4}} \approx 1.8236 > 2^{\frac{3}{4}},$$

and this answers in the negative the Question (2) posed by the author in [3, Question (2)].

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