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ON THE MAXIMAL OPERATOR OF A GENERAL ORNSTEIN–UHLENBECK SEMIGROUP

VALENTINA CASARINO, PAOLO CIATTI, AND PETER SJÖGREN

ABSTRACT. If Q is a real, symmetric and positive definite $n \times n$ matrix, and B a real $n \times n$ matrix whose eigenvalues have negative real parts, we consider the Ornstein–Uhlenbeck semigroup on \mathbb{R}^n with covariance Q and drift matrix B . Our main result is that the associated maximal operator is of weak type $(1, 1)$ with respect to the invariant measure. The proof has a geometric gist and hinges on the “forbidden zones method” previously introduced by the third author. For large values of the time parameter, we also prove a refinement of this result, in the spirit of a conjecture due to Talagrand.

1. INTRODUCTION

Let Q be a real, symmetric and positive definite $n \times n$ matrix, and B a real $n \times n$ matrix whose eigenvalues have negative real parts; here $n \geq 1$. We first introduce the covariance matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad t \in (0, +\infty]. \quad (1.1)$$

Observe that both Q_t and Q_∞ are well defined, symmetric and positive definite. Then we define the family of normalized Gaussian measures in \mathbb{R}^n

$$d\gamma_t(x) = (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_t^{-1}x, x \rangle} dx, \quad t \in (0, +\infty].$$

On the space $\mathcal{C}_b(\mathbb{R}^n)$ of bounded continuous functions, we consider the Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$, explicitly given by Kolmogorov’s formula

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y) d\gamma_t(y), \quad x \in \mathbb{R}^n. \quad (1.2)$$

The Gaussian measure γ_∞ is the unique invariant measure of the semigroup \mathcal{H}_t . We are interested in the maximal operator defined as

$$\mathcal{H}_* f(x) = \sup_{t>0} |\mathcal{H}_t f(x)|.$$

Under the above assumptions for B and Q , our main result will be the following.

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Theorem 1.1. *The Ornstein–Uhlenbeck maximal operator \mathcal{H}_* is of weak type $(1, 1)$ with respect to the invariant measure γ_∞ , with an operator quasinorm that depends only on the dimension and the matrices Q and B .*

In other words, the inequality

$$\gamma_\infty\{x \in \mathbb{R}^n : \mathcal{H}_*f(x) > \alpha\} \leq \frac{C}{\alpha} \|f\|_{L^1(\gamma_\infty)}, \quad \alpha > 0, \quad (1.3)$$

holds for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$.

The history of \mathcal{H}_* is quite long and started with the first attempts to prove that \mathcal{H}_* maps the L^p space into L^p . When $(\mathcal{H}_t)_{t>0}$ is symmetric, i.e., when each operator \mathcal{H}_t is self-adjoint on $L^2(\gamma_\infty)$, then \mathcal{H}_* is bounded on $L^p(\gamma_\infty)$ for $1 < p \leq \infty$, as a consequence of the general Littlewood–Paley–Stein theory for symmetric semigroups of contractions on L^p spaces [17, Ch. III].

It is easy to see that the maximal operator is unbounded on $L^1(\gamma_\infty)$. This led, about fifty years ago, to the study of the weak type $(1, 1)$ of \mathcal{H}_* . The first positive result is due to B. Muckenhoupt [14], who proved an estimate like (1.3) in the one-dimensional case with $Q = I$ and $B = -I$. The analogous question in the higher-dimensional case was an open problem until 1983, when the third author [16] proved the weak type $(1, 1)$ in any finite dimension. Other proofs are due to Menárguez, Pérez and Soria [12] (see also [11, 15]) and to García-Cuerva, Mauceri, Meda, Sjögren and Torrea [8]. Moreover, a different proof of the weak type $(1, 1)$ of \mathcal{H}_* , based on a covering lemma halfway between covering results by Besicovitch and Wiener, was given by Aimar, Forzani and Scotto [1].

In [4] the present authors recently considered a normal Ornstein–Uhlenbeck semigroup in \mathbb{R}^n , that is, we assumed that \mathcal{H}_t is for each $t > 0$ a normal operator on $L^2(\gamma_\infty)$. Under this extra assumption, we proved that the associated maximal operator is of weak type $(1, 1)$ with respect to the invariant measure γ_∞ . This extends some earlier work in the non-symmetric framework by Mauceri and Noselli [10], who proved some ten years ago that, if $Q = I$ and $B = \lambda(R - I)$ for some positive λ and a real skew-symmetric matrix R generating a periodic group, then the maximal operator \mathcal{H}_* is of weak type $(1, 1)$.

In this paper we go beyond the hypothesis of normality, which underlies the results in [4] and [10]. In Theorem 1.1 we prove the estimate (1.3) under only the aforementioned spectral assumptions on B and Q . The proof has a geometric core and strongly relies on the *ad hoc* technique developed by the third author in [16].

Since the maximal operator \mathcal{H}_* is trivially bounded from L^∞ to L^∞ , we obtain by interpolation the following corollary.

Corollary 1.2. *The Ornstein–Uhlenbeck maximal operator \mathcal{H}_* is bounded on $L^p(\gamma_\infty)$ for all $p > 1$.*

This result improves Theorem 4.2 in [10], where the L^p boundedness of \mathcal{H}_* is proved for all $p > 1$ in the normal framework and under the additional assumption that the infinitesimal generator of $(\mathcal{H}_t)_{t>0}$ is a sectorial operator of angle less than $\pi/2$.

A question related to the Ornstein–Uhlenbeck semigroup and the weak type $(1, 1)$ inequality was recently addressed by Ball, Barthe, Bednorz, Oleszkiewicz and Wolff [2]. Inspired by a conjecture formulated by Talagrand in a slightly different context [18], they conjectured the following, in the standard case $Q = I$ and $B = -I$: For each fixed $t > 0$, there exists a function $\psi_t = \psi_t(\alpha)$, satisfying

$$\lim_{\alpha \rightarrow +\infty} \psi_t(\alpha) = 0$$

and

$$\gamma_\infty \{x \in \mathbb{R}^n : |\mathcal{H}_t f(x)| > \alpha\} \leq C \frac{\psi_t(\alpha)}{\alpha} \quad (1.4)$$

for all large $\alpha > 0$ and all $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$. In [2] this conjecture is proved with $\psi_t(\alpha) = C(t)/\sqrt{\log \alpha}$ in dimension 1 and with $\psi_t(\alpha) = C(n, t) \log \log \alpha / \sqrt{\log \alpha}$ as $n > 1$; in the latter case the constant tends to ∞ with the dimension. Then Eldan and Lee [6] improved the result in [2] for $n > 1$, proving (1.4) with $\psi_t(\alpha) = C(t) (\log \log \alpha)^4 / \sqrt{\log \alpha}$, where the constant $C(t)$ is independent of the dimension. Finally Lehec [9], revisiting the argument in [6], proved the conjecture in any dimension with $\psi_t(\alpha) = C(t)/\sqrt{\log \alpha}$, which turns out to be sharp. All the results in [2, 6, 9] are established for $Q = I$ and $B = -I$.

In analogy with these results, we prove in Proposition 6.1 that the maximal operator with t large, associated to a general Ornstein–Uhlenbeck semigroup, satisfies

$$\gamma_\infty \left\{ x \in \mathbb{R}^n : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \leq C \frac{\psi(\alpha)}{\alpha} \quad (1.5)$$

for $\alpha > 0$ large and for all normalized functions $f \in L^1(\gamma_\infty)$. Here $\psi(\alpha) = 1/\sqrt{\log \alpha}$ and $C = C(n, Q, B)$, and this estimate is shown to be sharp. It cannot be extended to \mathcal{H}_* , since the maximal operator corresponding to small values of t only satisfies an inequality with $\psi(\alpha) = 1$.

In this paper we focus our attention on the Ornstein–Uhlenbeck maximal function in \mathbb{R}^n . In view of possible applications to stochastic analysis and to SPDE's, it would be very interesting to investigate the case of the infinite-dimensional Ornstein–Uhlenbeck maximal operator as well (see [5, 19, 3] for an introduction to the infinite-dimensional setting). The Riesz transforms associated to a general Ornstein–Uhlenbeck semigroup in \mathbb{R}^n will be considered in a forthcoming paper.

The scheme of the paper is as follows. In Section 2 we introduce the Mehler kernel $K_t(x, u)$, that is, the integral kernel of \mathcal{H}_t . Some estimates for the norm and the determinant of Q_t and related matrices are provided in Section 3. As a consequence, we obtain precise bounds for the Mehler kernel. In Section 4 we consider the relevant geometric features of the problem; in particular, we introduce in Subsection 4.1 a system of polar-like coordinates. We also express Lebesgue measure in terms of these coordinates. Sections 5, 6, 7 and 8 are devoted to the proof of Theorem 1.1. First, Section 5 introduces some preliminary simplifications of the proof; in particular, we reduce most of the problem to an ellipsoidal annulus. In Section 6 we consider the supremum in the definition of the maximal operator taken only over $t > 1$ and prove the sharpened version (1.5) of (1.3). Section 7 is devoted to the case of small t under

an additional local condition. Finally, in Section 8 we treat the remaining case and conclude the proof of Theorem 1.1, by proving the estimate (1.3) for small t under a global assumption.

In the following, we use the “variable constant convention”, according to which the symbols $c > 0$ and $C < \infty$ will denote constants which are not necessarily equal at different occurrences. They all depend only on the dimension and on Q and B . For any two nonnegative quantities a and b we write $a \lesssim b$ instead of $a \leq Cb$ and $a \gtrsim b$ instead of $a \geq cb$. The symbol $a \simeq b$ means that both $a \lesssim b$ and $a \gtrsim b$ hold.

By \mathbb{N} we mean the set of all nonnegative integers. If A is an $n \times n$ matrix, we write $\|A\|$ for its operator norm on \mathbb{R}^n with the Euclidean norm $|\cdot|$.

2. THE MEHLER KERNEL

For $t > 0$, the difference

$$Q_\infty - Q_t = \int_t^\infty e^{sB} Q e^{sB^*} ds \quad (2.1)$$

is a symmetric and strictly positive definite matrix. So is the matrix

$$Q_t^{-1} - Q_\infty^{-1} = Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}, \quad (2.2)$$

and we can define

$$D_t = (Q_t^{-1} - Q_\infty^{-1})^{-1} Q_t^{-1} e^{tB}. \quad (2.3)$$

Then formula (1.2), the definition of the Gaussian measure and some elementary computations yield

$$\begin{aligned} \mathcal{H}_t f(x) &= (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \int f(e^{tB}x - y) \exp \left[-\frac{1}{2} \langle Q_t^{-1} y, y \rangle \right] dy \\ &= (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \int f(u) \exp \left[-\frac{1}{2} \langle Q_t^{-1} (e^{tB}x - u), e^{tB}x - u \rangle \right] du \\ &= \left(\frac{\det Q_\infty}{\det Q_t} \right)^{\frac{1}{2}} \exp \left[-\frac{1}{2} \langle Q_t^{-1} e^{tB}x, e^{tB}x \rangle \right] \\ &\quad \times \exp \left[-\frac{1}{2} \langle Q_t^{-1} e^{tB}x, (Q_\infty^{-1} - Q_t^{-1})^{-1} Q_t^{-1} e^{tB}x \rangle \right] \\ &\quad \times \int f(u) \exp \left[\frac{1}{2} \langle (Q_\infty^{-1} - Q_t^{-1})(u - D_t x), u - D_t x \rangle \right] d\gamma_\infty(u), \end{aligned}$$

that is,

$$\begin{aligned} \mathcal{H}_t^{Q,B} f(x) &= \left(\frac{\det Q_\infty}{\det Q_t} \right)^{1/2} \exp \left[\frac{1}{2} \langle Q_t^{-1} e^{tB}x, D_t x - e^{tB}x \rangle \right] \\ &\quad \times \int f(u) \exp \left[\frac{1}{2} \langle (Q_\infty^{-1} - Q_t^{-1})(u - D_t x), u - D_t x \rangle \right] d\gamma_\infty(u), \end{aligned} \quad (2.4)$$

where we repeatedly used the fact that $Q_\infty^{-1} - Q_t^{-1}$ is symmetric. We now express the matrix D_t in various ways.

Lemma 2.1. *For all $x \in \mathbb{R}^n$ and $t > 0$ we have*

- (i) $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$;
- (ii) $D_t = e^{tB} + Q_t e^{-tB^*} Q_\infty^{-1}$.

Proof. (i) Formulae (2.1) and (1.1) imply

$$Q_\infty - Q_t = e^{tB} Q_\infty e^{tB^*} \quad (2.5)$$

(see also [13, formula (2.1)]). From (2.3) and (2.2) it follows that

$$D_t = Q_\infty (Q_\infty - Q_t)^{-1} e^{tB},$$

and combining this with (2.5) we arrive at (i).

(ii) Multiplying (2.5) by $e^{-tB^*} Q_\infty^{-1}$ from the right, we obtain

$$Q_\infty e^{-tB^*} Q_\infty^{-1} - Q_t e^{-tB^*} Q_\infty^{-1} = e^{tB},$$

and (ii) now follows from (i). □

By means of (i) in this lemma, we can define D_t for all $t \in \mathbb{R}$, and they will form a one-parameter group of matrices.

Now (ii) in Lemma 2.1 yields

$$\langle Q_t^{-1} e^{tB} x, D_t x - e^{tB} x \rangle = \langle Q_t^{-1} e^{tB} x, Q_t e^{-tB^*} Q_\infty^{-1} x \rangle = \langle Q_\infty^{-1} x, x \rangle.$$

Thus (2.4) may be rewritten as

$$\mathcal{H}_t f(x) = \int K_t(x, u) f(u) d\gamma_\infty(u),$$

where K_t denotes the Mehler kernel, given by

$$\begin{aligned} K_t(x, u) &= \left(\frac{\det Q_\infty}{\det Q_t} \right)^{1/2} \exp(R(x)) \\ &\quad \times \exp \left[-\frac{1}{2} \langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x), u - D_t x \rangle \right] \end{aligned} \quad (2.6)$$

for $x, u \in \mathbb{R}^n$. Here we introduced the quadratic form

$$R(x) = \frac{1}{2} \langle Q_\infty^{-1} x, x \rangle, \quad x \in \mathbb{R}^n.$$

3. SOME AUXILIARY RESULTS

In this section we collect some preliminary bounds, which will be essential ingredients in the proof of the weak type (1, 1) for the maximal operator \mathcal{H}_* .

Lemma 3.1. *For $s > 0$ the matrices D_s and $D_{-s} = D_s^{-1}$ satisfy*

$$\|D_s\| \lesssim e^{Cs} \quad \text{and} \quad \|D_{-s}\| \lesssim e^{-Cs}. \quad (3.1)$$

Proof. First we prove estimates for $\|e^{sB^*}\|$ and $\|e^{-sB^*}\|$. They can be obtained by means of a Jordan decomposition of sB^* , that is, writing sB^* as the sum of a complex diagonal matrix and a triangular, nilpotent matrix, and these two terms will commute. Another possibility is to use standard theory of strongly continuous semigroups, see [7, Theorem 3.14 and Theorem 5.5 in Chapter 1]. Both arguments rely on the fact that the eigenvalues of B have negative real parts. The result will be

$$\|e^{-sB^*}\| \lesssim e^{Cs} \quad \text{and} \quad \|e^{sB^*}\| \lesssim e^{-cs}, \quad s > 0. \quad (3.2)$$

Finally, (3.2) implies (3.1) for $D_s = Q_\infty e^{-sB^*} Q_\infty^{-1}$ and $D_{-s} = Q_\infty e^{sB^*} Q_\infty^{-1}$. \square

In the following lemma, we collect estimates of some basic quantities related to the matrices Q_t .

Lemma 3.2. *For all $t > 0$ we have*

- (i) $\det Q_t \simeq (\min(1, t))^n$;
- (ii) $\|Q_t^{-1}\| \simeq (\min(1, t))^{-1}$;
- (iii) $\|Q_\infty - Q_t\| \lesssim e^{-ct}$;
- (iv) $\|Q_t^{-1} - Q_\infty^{-1}\| \lesssim t^{-1} e^{-ct}$;
- (v) $\|(Q_t^{-1} - Q_\infty^{-1})^{-1/2}\| \lesssim t^{1/2} e^{Ct}$.

Proof. (i) and (ii) Using (3.2), we see that for each $t > 0$ and for all $v \in \mathbb{R}^n$

$$\begin{aligned} \langle Q_t v, v \rangle &= \left\langle \int_0^t e^{sB} Q e^{sB^*} v ds, v \right\rangle = \int_0^t \langle Q^{1/2} e^{sB^*} v, Q^{1/2} e^{sB^*} v \rangle ds \\ &= \int_0^t |Q^{1/2} e^{sB^*} v|^2 ds \simeq \int_0^t |e^{sB^*} v|^2 ds \\ &\lesssim \int_0^t e^{-cs} ds |v|^2 \simeq \min(1, t) |v|^2. \end{aligned}$$

Since $\|(e^{sB^*})^{-1}\| = \|e^{-sB^*}\| \lesssim e^{Cs}$, there is also a lower estimate

$$\int_0^t |e^{sB^*} v|^2 ds \gtrsim \int_0^t e^{-Cs} ds |v|^2 \simeq \min(1, t) |v|^2.$$

Thus any eigenvalue of Q_t has order of magnitude $\min(1, t)$, and (i) and (ii) follow.

(iii) From the definition of Q_t and (3.2), we get

$$\|Q_\infty - Q_t\| = \left\| \int_t^\infty e^{sB} Q e^{sB^*} ds \right\| \lesssim e^{-ct}.$$

(iv) Using now (ii) and (iii), we have

$$\begin{aligned} \|Q_t^{-1} - Q_\infty^{-1}\| &= \|Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}\| \lesssim \|Q_t^{-1}\| \|Q_\infty - Q_t\| \\ &\lesssim (\min(1, t))^{-1} e^{-ct} \lesssim t^{-1} e^{-ct}. \end{aligned}$$

(v) Since $\|A^{1/2}\| = \|A\|^{1/2}$ for any symmetric positive definite matrix A , we consider $(Q_t^{-1} - Q_\infty^{-1})^{-1}$, which can be rewritten as

$$(Q_t^{-1} - Q_\infty^{-1})^{-1} = (Q_\infty^{-1}(Q_\infty - Q_t)Q_t^{-1})^{-1} = Q_t(Q_\infty - Q_t)^{-1}Q_\infty. \quad (3.3)$$

It follows from (2.5) that $(Q_\infty - Q_t)^{-1} = e^{-tB^*} Q_\infty^{-1} e^{-tB}$, so that

$$\|(Q_\infty - Q_t)^{-1}\| \lesssim e^{Ct},$$

as a consequence of (3.2). Inserting this and the simple estimate $\|Q_t\| \lesssim t$ in (3.3), we obtain $\|(Q_t^{-1} - Q_\infty^{-1})^{-1}\| \lesssim te^{Ct}$, and (v) follows. \square

Proposition 3.3. *For $t \geq 1$ and $w \in \mathbb{R}^n$, we have*

$$\langle (Q_t^{-1} - Q_\infty^{-1})D_t w, D_t w \rangle \simeq |w|^2.$$

Proof. By (2.3) and Lemma 2.1 (i) we have

$$\begin{aligned} \langle (Q_t^{-1} - Q_\infty^{-1})D_t w, D_t w \rangle &= \langle Q_t^{-1} e^{tB} w, Q_\infty e^{-tB^*} Q_\infty^{-1} w \rangle \\ &= \langle Q_\infty Q_t^{-1} e^{tB} w, e^{-tB^*} Q_\infty^{-1} w \rangle. \end{aligned}$$

Since $Q_\infty Q_t^{-1} = I + (Q_\infty - Q_t)Q_t^{-1}$, this leads to

$$\begin{aligned} \langle (Q_t^{-1} - Q_\infty^{-1})D_t w, D_t w \rangle &= \langle e^{tB} w, e^{-tB^*} Q_\infty^{-1} w \rangle + \langle (Q_\infty - Q_t)Q_t^{-1} e^{tB} w, e^{-tB^*} Q_\infty^{-1} w \rangle \\ &= \langle Q_\infty^{-1} w, w \rangle + \langle e^{-tB} (Q_\infty - Q_t)Q_t^{-1} e^{tB} w, Q_\infty^{-1} w \rangle. \end{aligned}$$

Using (2.1) and the definition of Q_∞ , we observe that the last term here can be written as

$$\begin{aligned} &\left\langle \int_t^\infty e^{(s-t)B} Q e^{(s-t)B^*} ds e^{tB^*} Q_t^{-1} e^{tB} w, Q_\infty^{-1} w \right\rangle \\ &= \langle Q_\infty e^{tB^*} Q_t^{-1} e^{tB} w, Q_\infty^{-1} w \rangle \\ &= \langle e^{tB^*} Q_t^{-1} e^{tB} w, w \rangle \\ &= |Q_t^{-1/2} e^{tB} w|^2. \end{aligned}$$

Since $|Q_t^{-1/2} e^{tB} w|^2 \lesssim e^{-ct}|w|^2$ for $t \geq 1$, the claim of the proposition follows if t is large enough. In the opposite case $1 < t < C$, we apply Lemma 3.2 (v) to conclude that

$$\langle (Q_t^{-1} - Q_\infty^{-1})D_t w, D_t w \rangle \gtrsim e^{-Ct} |D_t w|^2 \sim |w|^2.$$

The converse inequality is clear, and the claim follows again. \square

We can now write the estimates for the kernel K_t which we will use later. If $t > 1$, we combine (2.6) with Proposition 3.3 and write $u - D_t x = D_t(D_{-t} u - x)$. Because of Lemma 3.2 (i), the result will be

$$\begin{aligned} \exp(R(x)) \exp(-C |D_{-t} u - x|^2) \\ \lesssim K_t(x, u) \lesssim \exp(R(x)) \exp(-c |D_{-t} u - x|^2), \quad t > 1. \end{aligned} \quad (3.4)$$

For $t \leq 1$ we use Lemma 3.2 (v) to see that

$$\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x), u - D_t x \rangle = |(Q_t^{-1} - Q_\infty^{-1})^{1/2}(u - D_t x)|^2 \gtrsim t^{-1} |u - D_t x|^2,$$

Then (2.6) and Lemma 3.2 (i) imply

$$K_t(x, u) \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp\left(-c \frac{|u - D_t x|^2}{t}\right), \quad t \leq 1. \quad (3.5)$$

4. GEOMETRIC ASPECTS OF THE PROBLEM

4.1. A system of adapted polar coordinates. We first need a technical lemma.

Lemma 4.1. *For all x in \mathbb{R}^n and $s \in \mathbb{R}$, we have*

$$\langle B^* Q_\infty^{-1} x, x \rangle = -\frac{1}{2} |Q^{1/2} Q_\infty^{-1} x|^2; \quad (4.1)$$

$$\frac{\partial}{\partial s} D_s x = -Q_\infty B^* Q_\infty^{-1} D_s x = -Q_\infty e^{-sB^*} B^* Q_\infty^{-1} x; \quad (4.2)$$

$$\frac{\partial}{\partial s} R(D_s x) = \frac{1}{2} |Q^{1/2} Q_\infty^{-1} D_s x|^2 \simeq |D_s x|^2. \quad (4.3)$$

Proof. To prove (4.1), we use the definition of Q_∞ to write for any $z \in \mathbb{R}^n$

$$\begin{aligned} \langle B^* z, Q_\infty z \rangle &= \int_0^\infty \langle B^* z, e^{sB} Q e^{sB^*} z \rangle ds \\ &= \int_0^\infty \langle e^{sB^*} B^* z, Q e^{sB^*} z \rangle ds \\ &= \frac{1}{2} \int_0^\infty \frac{d}{ds} \langle e^{sB^*} z, Q e^{sB^*} z \rangle ds \\ &= -\frac{1}{2} |Q^{1/2} z|^2. \end{aligned}$$

Setting $z = Q_\infty^{-1} x$, we get (4.1).

Further, (4.2) easily follows if we observe that

$$\frac{\partial}{\partial s} D_s x = \frac{\partial}{\partial s} (Q_\infty e^{-sB^*} Q_\infty^{-1} x) = -Q_\infty B^* Q_\infty^{-1} Q_\infty e^{-sB^*} Q_\infty^{-1} x = -Q_\infty B^* Q_\infty^{-1} D_s x.$$

Finally, we get by means of (4.2) and (4.1)

$$\begin{aligned} \frac{\partial}{\partial s} R(D_s x) &= \frac{1}{2} \frac{\partial}{\partial s} \langle Q_\infty^{-1/2} D_s x, Q_\infty^{-1/2} D_s x \rangle \\ &= -\langle Q_\infty^{-1/2} Q_\infty B^* Q_\infty^{-1} D_s x, Q_\infty^{-1/2} D_s x \rangle \\ &= \frac{1}{2} |Q^{1/2} Q_\infty^{-1} D_s x|^2, \end{aligned}$$

and (4.3) is verified. \square

Fix now $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n : R(x) = \beta\}.$$

As a consequence of (4.3), the map $s \mapsto R(D_s z)$ is strictly increasing for each $0 \neq z \in \mathbb{R}^n$. Hence any $x \in \mathbb{R}^n$, $x \neq 0$, can be written uniquely as

$$x = D_s \tilde{x}, \quad (4.4)$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We consider s and \tilde{x} as the polar coordinates of x . Our estimates in what follows will be uniform in β .

Next, we write Lebesgue measure in terms of these polar coordinates. A normal vector to the surface E_β at the point $\tilde{x} \in E_\beta$ is $\mathbf{N}(\tilde{x}) = Q_\infty^{-1}\tilde{x}$, and the tangent hyperplane at \tilde{x} is $\mathbf{N}(\tilde{x})^\perp$. For $s > 0$ the tangent hyperplane of the surface $D_s E_\beta = \{D_s \tilde{x} : \tilde{x} \in E_\beta\}$ at the point $D_s \tilde{x}$ is $D_s(\mathbf{N}(\tilde{x})^\perp)$, and a normal to $D_s E_\beta$ at the same point is $w = (D_s^{-1})^*(\mathbf{N}(\tilde{x})) = D_{-s}^* Q_\infty^{-1} \tilde{x} = Q_\infty^{-1} e^{sB} \tilde{x}$.

The scalar product of w and the tangent of the curve $s \mapsto D_s \tilde{x}$ at the point $D_s \tilde{x}$ is, because of (4.2) and (4.1),

$$\begin{aligned} & \left\langle \frac{\partial}{\partial s} D_s \tilde{x}, w \right\rangle \\ &= -\langle Q_\infty e^{-sB^*} B^* Q_\infty^{-1} \tilde{x}, Q_\infty^{-1} e^{sB} \tilde{x} \rangle = -\langle B^* Q_\infty^{-1} \tilde{x}, \tilde{x} \rangle = \frac{1}{2} |Q^{1/2} Q_\infty^{-1} \tilde{x}|^2 > 0. \end{aligned} \quad (4.5)$$

Thus the curve $s \mapsto D_s \tilde{x}$ is transversal to each surface $D_s E_\beta$. Let dS_s denote the area measure of $D_s E_\beta$. Then Lebesgue measure is given in terms of our polar coordinates by

$$dx = H(s, \tilde{x}) dS_s(D_s \tilde{x}) ds, \quad (4.6)$$

where

$$H(s, \tilde{x}) = \left\langle \frac{\partial}{\partial s} D_s \tilde{x}, \frac{w}{|w|} \right\rangle = \frac{|Q^{1/2} Q_\infty^{-1} \tilde{x}|^2}{2 |Q_\infty^{-1} e^{sB} \tilde{x}|}.$$

To see how dS_s varies with s , we take a continuous function $\varphi = \varphi(\tilde{x})$ on E_β and extend it to $\mathbb{R}^n \setminus \{0\}$ by writing $\varphi(D_s \tilde{x}) = \varphi(\tilde{x})$. For any $t > 0$ and small $\varepsilon > 0$, we define the shell

$$\Omega_{t,\varepsilon} = \{D_s \tilde{x} : t < s < t + \varepsilon, \tilde{x} \in E_\beta\}.$$

Then $\Omega_{t,\varepsilon}$ is the image under D_t of $\Omega_{0,\varepsilon}$, and the Jacobian of this map is $\det D_t = e^{-t \operatorname{tr} B}$. Thus

$$\int_{\Omega_{t,\varepsilon}} \varphi(x) dx = e^{-t \operatorname{tr} B} \int_{\Omega_{0,\varepsilon}} \varphi(D_t x) dx,$$

which we can rewrite as

$$\begin{aligned} & \int_{t < s < t + \varepsilon} \int_{\tilde{x} \in E_\beta} \varphi(\tilde{x}) H(s, \tilde{x}) dS_s(D_s \tilde{x}) ds \\ &= e^{-t \operatorname{tr} B} \int_{0 < s < \varepsilon} \int_{\tilde{x} \in E_\beta} \varphi(\tilde{x}) H(s, \tilde{x}) dS_s(D_s \tilde{x}) ds. \end{aligned}$$

Now we divide by ε and let $\varepsilon \rightarrow 0$, getting

$$\int_{E_\beta} \varphi(\tilde{x}) H(t, \tilde{x}) dS_t(D_t \tilde{x}) = e^{-t \operatorname{tr} B} \int_{E_\beta} \varphi(\tilde{x}) H(0, \tilde{x}) dS_0(\tilde{x}).$$

Since this holds for any φ , it follows that

$$dS_t(D_t \tilde{x}) = e^{-t \operatorname{tr} B} \frac{H(0, \tilde{x})}{H(t, \tilde{x})} dS_0(\tilde{x}).$$

Together with (4.6), this implies the following result.

Proposition 4.2. *The Lebesgue measure in \mathbb{R}^n is given in terms of polar coordinates (t, \tilde{x}) by*

$$dx = e^{-t \operatorname{tr} B} \frac{|Q^{1/2} Q_\infty^{-1} \tilde{x}|^2}{2 |Q_\infty^{-1} \tilde{x}|} dS_0(\tilde{x}) dt.$$

We also need estimates of the distance between two points in terms of the polar coordinates.

Lemma 4.3. *Fix $\beta > 0$. Let $x^{(0)}, x^{(1)} \in \mathbb{R}^n \setminus \{0\}$ and assume $R(x^{(0)}) > \beta/2$. Write*

$$x^{(0)} = D_{s^{(0)}}(\tilde{x}^{(0)}) \quad \text{and} \quad x^{(1)} = D_{s^{(1)}}(\tilde{x}^{(1)})$$

with $s^{(0)}, s^{(1)} \in \mathbb{R}$ and $\tilde{x}^{(0)}, \tilde{x}^{(1)} \in E_\beta$.

(i) *Then*

$$|x^{(0)} - x^{(1)}| \gtrsim c |\tilde{x}^{(0)} - \tilde{x}^{(1)}|. \quad (4.7)$$

(ii) *If also $s^{(1)} \geq 0$, then*

$$|x^{(0)} - x^{(1)}| \gtrsim c \sqrt{\beta} |s^{(0)} - s^{(1)}|. \quad (4.8)$$

Proof. Let $\Gamma : [0, 1] \rightarrow \mathbb{R}^n \setminus \{0\}$ be a differentiable curve with $\Gamma(0) = x^{(0)}$ and $\Gamma(1) = x^{(1)}$. It suffices to bound the length of any such curve from below by the right-hand sides of (4.7) and (4.8).

For each $\tau \in [0, 1]$, we write

$$\Gamma(\tau) = D_{s(\tau)} \tilde{x}(\tau),$$

with $\tilde{x}(\tau) \in E_\beta$ and $\tilde{x}(i) = \tilde{x}^{(i)}$, $s(i) = s^{(i)}$ for $i = 0, 1$. Thus

$$\Gamma'(\tau) = -s'(\tau) \frac{\partial}{\partial s} D_s \Big|_{s=s(\tau)} \tilde{x}(\tau) + D_{s(\tau)} \tilde{x}'(\tau).$$

The group property of D_s implies that

$$\frac{\partial}{\partial s} D_s \Big|_{s=s(\tau)} = D_{s(\tau)} \frac{\partial}{\partial s} D_s \Big|_{s=0},$$

and so

$$\Gamma'(\tau) = D_{s(\tau)} v,$$

with

$$v = -s'(\tau) \frac{\partial}{\partial s} D_s \Big|_{s=0} \tilde{x}(\tau) + \tilde{x}'(\tau).$$

The vector $\tilde{x}'(\tau)$ is tangent to E_β and so orthogonal to $\mathbf{N}(\tilde{x})$. Then (4.5) (with $s = 0$) and the triangle inequality on the unit sphere imply that the angle between $\frac{\partial}{\partial s} D_s \Big|_{s=0} \tilde{x}(\tau)$ and $\tilde{x}'(\tau)$ is larger than some positive constant. It follows that

$$|v|^2 \gtrsim |s'(\tau)|^2 \left| \frac{\partial}{\partial s} D_s \Big|_{s=0} \tilde{x}(\tau) \right|^2 + |\tilde{x}'(\tau)|^2 \gtrsim |s'(\tau)|^2 \beta + |\tilde{x}'(\tau)|^2, \quad (4.10)$$

where we also used the fact that, by (4.2),

$$\left| \frac{\partial}{\partial s} D_s \Big|_{s=0} \tilde{x}(\tau) \right| \simeq |\tilde{x}(\tau)| \simeq \sqrt{\beta}.$$

Since

$$|v| = |D_{-s(\tau)}\Gamma'(\tau)| \leq \|D_{-s(\tau)}\| |\Gamma'(\tau)| \lesssim e^{-C \min(s(\tau), 0)} |\Gamma'(\tau)|$$

because of Lemma 3.1, we obtain from (4.10)

$$|\Gamma'(\tau)| \gtrsim e^{C \min(s(\tau), 0)} (\sqrt{\beta} |s'(\tau)| + |\tilde{x}'(\tau)|). \quad (4.11)$$

Next, we derive a lower bound for $s(0)$; assume first that $s(0) < 0$. The assumption $R(x^{(0)}) > \beta/2$ implies, together with Lemma 3.1,

$$\beta/2 \leq R(D_{s(0)} \tilde{x}^{(0)}) \lesssim |D_{s(0)} \tilde{x}^{(0)}|^2 \lesssim e^{c s(0)} |\tilde{x}^{(0)}|^2 \simeq e^{c s(0)} \beta.$$

It follows that

$$s(0) > -\tilde{s},$$

for some \tilde{s} with $0 < \tilde{s} < C$, and this obviously holds also without the assumption $s(0) < 0$.

Assume now that $s(\tau) > -2\tilde{s}$ for all $\tau \in [0, 1]$. Then (4.11) implies

$$|\Gamma'(\tau)| \gtrsim \sqrt{\beta} |s'(\tau)|$$

and

$$|\Gamma'(\tau)| \gtrsim |\tilde{x}'(\tau)|.$$

Integrating these estimates with respect to τ in $[0, 1]$, we immediately see that the length of Γ is bounded below by the right-hand sides of (4.7) and (4.8).

If instead $s(\tau) \leq -2\tilde{s}$ for some $\tau \in [0, 1]$, we can proceed as in the proof of Lemma 4.2 in [4]. More precisely, since the image $s([0, 1])$ contains the interval $[-2\tilde{s}, \max(s(0), s(1))]$, we can find a closed subinterval I of $[0, 1]$ whose image $s(I)$ is exactly the interval $[-2\tilde{s}, \max(s(0), s(1))]$. Thus we may control the length of Γ , in the light of (4.11), by

$$\int_0^1 |\Gamma'(\tau)| d\tau \geq \int_I |\Gamma'(\tau)| d\tau \gtrsim \sqrt{\beta} \int_I |s'(\tau)| d\tau \geq \sqrt{\beta} (\max(s(0), s(1)) + 2\tilde{s}).$$

Here

$$\sqrt{\beta} (\max(s(0), s(1)) + 2\tilde{s}) \gtrsim \sqrt{\beta} \gtrsim \text{diam } E_\beta \geq |\tilde{x}^{(0)} - \tilde{x}^{(1)}|,$$

and (4.7) follows. Under the additional hypotheses of (b), we have

$$\max(s(0), s(1)) + 2\tilde{s} \geq |s(0) - s(1)|,$$

which implies (4.8). \square

4.2. The Gaussian measure of a tube. We fix a large $\beta > 0$. Define for $x^{(1)} \in E_\beta$ and $a > 0$ the set

$$\Omega = \{x \in E_\beta : |x - x^{(1)}| < a\}.$$

This is a spherical cap of the ellipsoid E_β , centered at $x^{(1)}$. Observe that $|x| \simeq \sqrt{\beta}$ for $x \in \Omega$, and that the area of Ω is $|\Omega| \simeq \min(a^{n-1}, \beta^{(n-1)/2})$. Then consider the tube

$$Z = \{D_s \tilde{x} : s \geq 0, \tilde{x} \in \Omega\}. \quad (4.12)$$

Lemma 4.4. *There exists a constant C such that $\beta > C$ implies that the Gaussian measure of the tube Z fulfills*

$$\gamma_\infty(Z) \lesssim \frac{a^{n-1}}{\sqrt{\beta}} e^{-\beta}.$$

Proof. Proposition 4.2 yields, since $H(0, \tilde{x}) \simeq |\tilde{x}| \simeq \sqrt{\beta}$,

$$\gamma_\infty(Z) \simeq \int_0^\infty e^{-s \operatorname{tr} B} e^{-R(D_s \tilde{x})} \int_\Omega H(0, \tilde{x}) dS(\tilde{x}) ds \lesssim \sqrt{\beta} a^{n-1} \int_0^\infty e^{-s \operatorname{tr} B} e^{-R(D_s \tilde{x})} ds.$$

By (4.3) we have

$$R(D_s \tilde{x}) - R(\tilde{x}) \simeq \int_0^s |D_{s'} \tilde{x}|^2 ds' \gtrsim s |\tilde{x}|^2 \simeq s\beta,$$

which implies

$$\gamma_\infty(Z) \lesssim \sqrt{\beta} a^{n-1} e^{-\beta} \int_0^\infty e^{-s \operatorname{tr} B} e^{-cs\beta} ds.$$

Assuming β large enough, one has $c\beta > -2 \operatorname{tr} B$, and then the last integral is finite and no larger than C/β . The lemma follows. \square

5. SOME SIMPLIFICATIONS

In this section, we introduce some preliminary simplifications and reductions in the proof of (1.3), i.e., of Theorem 1.1.

(1) We may assume that f is nonnegative and normalized in the sense that

$$\|f\|_{L^1(\gamma_\infty)} = 1,$$

since this involves no loss of generality.

(2) We may assume that our fixed α is large, $\alpha > C$, since otherwise (1.3) is trivial.

(3) In many cases, we may restrict x in (1.3) to the ellipsoidal annulus

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n : \frac{1}{2} \log \alpha \leq R(x) \leq 2 \log \alpha \right\}.$$

To begin with, we can always forget the unbounded component of the complement of \mathcal{E} , since

$$\gamma_\infty\{x \in \mathbb{R}^n : R(x) > 2 \log \alpha\} \quad (5.1)$$

$$\lesssim \int_{R(x) > 2 \log \alpha} \exp(-R(x)) dx \lesssim (2 \log \alpha)^{(n-2)/2} \exp(-2 \log \alpha) \lesssim \frac{1}{\alpha}.$$

(4) When $t > 1$, we may forget also the inner region where $R(x) < \frac{1}{2} \log \alpha$. Indeed, from (3.4) we get, if $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ with $R(x) < \frac{1}{2} \log \alpha$,

$$K_t(x, u) \lesssim e^{R(x)} < \sqrt{\alpha} \leq \alpha,$$

since α is large. In other words, for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$

$$R(x) < \frac{1}{2} \log \alpha \quad \Rightarrow \quad K_t(x, u) \lesssim \alpha, \quad (5.2)$$

for all $t > 1$.

Replacing α by $C\alpha$ for some C , we see from (5.1) and (5.2) that we can assume $x \in \mathcal{E}$ in the proof of (1.3), when the supremum of the maximal operator is taken only over $t > 1$.

Before introducing the last simplification, we need to define a global region

$$G = \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^n : |x - u| > \frac{1}{1 + |x|} \right\}$$

and a local region

$$L = \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^n : |x - u| \leq \frac{1}{1 + |x|} \right\}.$$

(5) When $t \leq 1$ and $(x, u) \in G$, we shall see that (5.2) is still valid, and it is again enough to consider $x \in \mathcal{E}$.

To prove this, we need a lemma which will also be useful later.

Lemma 5.1. *If $(x, u) \in G$ and $0 < t \leq 1$, then*

$$\frac{1}{(1 + |x|)^2} \lesssim t^2 |x|^2 + |u - D_t x|^2.$$

Proof. From the definition of G we have

$$\begin{aligned} \frac{1}{1 + |x|} &\leq |x - u| \\ &\lesssim |x - D_t x| + |D_t x - u| \\ &= |Q_\infty(Q_\infty^{-1}x - e^{-tB^*}Q_\infty^{-1}x)| + |u - D_t x| \\ &\lesssim |(I - e^{-tB^*})Q_\infty^{-1}x| + |u - D_t x| \\ &\lesssim t|x| + |u - D_t x|. \end{aligned}$$

The lemma follows. □

To verify now (5.2) in the global region with $t \leq 1$, we recall from (3.5) that

$$K_t(x, u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c \frac{|u - D_t x|^2}{t}\right).$$

It follows from Lemma 5.1 that

$$t^2 \gtrsim \frac{1}{(1+|x|)^4} \quad \text{or} \quad \frac{|u - D_t x|^2}{t} \gtrsim \frac{1}{(1+|x|)^2 t}.$$

The first inequality here implies that

$$K_t(x, u) \lesssim e^{R(x)} (1+|x|)^n \lesssim e^{2R(x)},$$

and (5.2) follows. If the second inequality holds, we have

$$K_t(x, u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-\frac{c}{(1+|x|)^2 t}\right) \lesssim e^{R(x)} (1+|x|)^n,$$

and we get the same estimate. Thus (5.2) is verified.

Finally, let

$$\mathcal{H}_*^G f(x) = \sup_{0 < t \leq 1} \left| \int K_t(x, u) \chi_G(x, u) f(u) d\gamma_\infty(u) \right|,$$

and

$$\mathcal{H}_*^L f(x) = \sup_{0 < t \leq 1} \left| \int K_t(x, u) \chi_L(x, u) f(u) d\gamma_\infty(u) \right|.$$

6. THE CASE OF LARGE t

In this section, we consider the supremum in the definition of the maximal operator taken only over $t > 1$, and we prove (1.5).

Proposition 6.1. *For all functions $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$,*

$$\gamma_\infty \left\{ x : \sup_{t > 1} |\mathcal{H}_t f(x)| > \alpha \right\} \lesssim \frac{1}{\alpha \sqrt{\log \alpha}}, \quad \alpha > 2. \quad (6.1)$$

In particular, the maximal operator

$$\sup_{t > 1} |\mathcal{H}_t f(x)|$$

is of weak type $(1, 1)$ with respect to the invariant measure γ_∞ .

Proof. We can assume that $f \geq 0$. Looking at the arguments in Section 5, items (3) and (4), we see that it suffices to consider points $x \in \mathcal{E}$. For both x and u we use the coordinates introduced in (4.4) with $\beta = \log \alpha$, that is,

$$x = D_s \tilde{x}, \quad u = D_{s'} \tilde{u},$$

where $\tilde{x}, \tilde{u} \in E_{\log \alpha}$ and $s, s' \in \mathbb{R}$.

From (3.4) we have

$$K_t(x, u) \lesssim \exp(R(x)) \exp(-c |D_{-t} u - x|^2)$$

for $t > 1$ and $x, u \in \mathbb{R}^n$. Since $x \in \mathcal{E}$ and $D_{-t} u = D_{-t} D_{s'} \tilde{u} = D_{s'-t} \tilde{u}$, we can apply Lemma 4.3 (i), getting

$$|D_{-t} u - x| \gtrsim |\tilde{x} - \tilde{u}|,$$

so that

$$\int K_t(x, u) f(u) d\gamma_\infty(u) \lesssim \exp(R(D_s \tilde{x})) \int \exp(-c|\tilde{x} - \tilde{u}|^2) f(u) d\gamma_\infty(u).$$

In view of (4.3), the right-hand side here is strictly increasing in s , and therefore the inequality

$$\exp(R(D_s \tilde{x})) \int \exp(-c|\tilde{x} - \tilde{u}|^2) f(u) d\gamma_\infty(u) > \alpha \quad (6.2)$$

holds if and only if $s > s_\alpha(\tilde{x})$ for some function $\tilde{x} \mapsto s_\alpha(\tilde{x})$, with equality for $s = s_\alpha(\tilde{x})$. Since $\alpha > 2$ and $\|f\|_{L^1(\gamma_\infty)} = 1$, it follows that $s_\alpha(\tilde{x}) > 0$.

For some C , the set of points $x \in \mathcal{E}$ where the supremum in (6.1) is larger than $C\alpha$ is contained in the set $\mathcal{A}(\alpha)$ of points $D_s \tilde{x} \in \mathcal{E}$ fulfilling (6.2). We use Proposition 4.2 to estimate the γ_∞ measure of this set. Observe that $H(0, \tilde{x}) \simeq |\tilde{x}| \simeq \sqrt{\log \alpha}$ and that $D_s \tilde{x} \in \mathcal{E}$ implies $s \lesssim 1$, so that also $e^{-s \operatorname{tr} B} \lesssim 1$. We get

$$\begin{aligned} \gamma_\infty(\mathcal{A}(\alpha) \cap \mathcal{E}) &= \int_{\mathcal{A}(\alpha) \cap \mathcal{E}} e^{-R(x)} dx \\ &\lesssim \sqrt{\log \alpha} \int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^C e^{-R(D_s \tilde{x})} dS(\tilde{x}) ds \\ &\lesssim \sqrt{\log \alpha} \int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{+\infty} \exp(-R(D_{s_\alpha(\tilde{x})} \tilde{x}) - c \log \alpha (s - s_\alpha(\tilde{x}))) ds dS(\tilde{x}), \end{aligned}$$

where the last inequality follows from (4.3), since $|D_s \tilde{x}|^2 \gtrsim |\tilde{x}|^2 \simeq \log \alpha$. Integrating in s , we obtain

$$\gamma_\infty(\mathcal{A}(\alpha) \cap \mathcal{E}) \lesssim \frac{1}{\sqrt{\log \alpha}} \int_{E_{\log \alpha}} \exp(-R(D_{s_\alpha(\tilde{x})} \tilde{x})) dS(\tilde{x}).$$

Now combine this estimate with the case of equality in (6.2) and change the order of integration, to get

$$\begin{aligned} \gamma_\infty(\mathcal{A}(\alpha) \cap \mathcal{E}) &\lesssim \frac{1}{\alpha \sqrt{\log \alpha}} \int \int_{E_{\log \alpha}} \exp(-c|\tilde{x} - \tilde{u}|^2) dS(\tilde{x}) f(u) d\gamma_\infty(u) \\ &\lesssim \frac{1}{\alpha \sqrt{\log \alpha}} \int f(u) d\gamma_\infty(u), \end{aligned}$$

which proves Proposition 6.1. \square

Finally, in analogy with [9], we show that the factor $1/\sqrt{\log \alpha}$ in (6.1) is sharp.

Proposition 6.2. *For any $t > 1$ and any large α , there exists a function f , normalized in $L^1(\gamma_\infty)$ and such that*

$$\gamma_\infty \{x : |\mathcal{H}_t f(x)| > \alpha\} \simeq \frac{1}{\alpha \sqrt{\log \alpha}}.$$

Proof. Take a point z with $R(z) = \log \alpha$, and let f be (an approximation of) a Dirac measure at the point $u = D_t z$. Then, as a consequence of (3.4), $K_t(x, u) \simeq$

$\exp(R(x))$ in the ball $B(D_{-t}u, 1) = B(z, 1)$. We then have $\mathcal{H}_t f(x) = K_t(x, u) \gtrsim \alpha$ in the set $\mathcal{B} = \{x \in B(z, 1) : R(x) > R(z)\}$, whose measure is

$$\gamma_\infty(\mathcal{B}) \simeq e^{-R(z)} \frac{1}{\sqrt{R(z)}} = \frac{1}{\alpha \sqrt{\log \alpha}}.$$

□

7. THE LOCAL CASE FOR SMALL t

Proposition 7.1. *If $(x, u) \in L$ and $0 < t \leq 1$, then*

$$|K_t(x, u)| \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp\left(-c \frac{|u-x|^2}{t}\right).$$

Proof. In view of (3.5), it is enough to show that

$$\frac{|u - D_t x|^2}{t} \geq \frac{|u - x|^2}{t} - C. \quad (7.1)$$

We write

$$\begin{aligned} |u - D_t x|^2 &= |u - x + x - D_t x|^2 = |u - x|^2 + 2\langle u - x, x - D_t x \rangle + |x - D_t x|^2 \\ &\geq |u - x|^2 - 2|u - x| |x - D_t x|. \end{aligned}$$

But

$$|u - x| |x - D_t x| = |u - x| |Q_\infty(I - e^{-tB^*})Q_\infty^{-1}x| \lesssim |u - x| t |x| \leq t$$

since $(x, u) \in L$, and (7.1) follows. □

Proposition 7.2. *The maximal operator \mathcal{H}_*^L is of weak type $(1, 1)$ with respect to the invariant measure γ_∞ .*

Proof. The proof is standard, since Proposition 7.1 implies

$$\mathcal{H}_*^L f(x) \lesssim \sup_{0 < t \leq 1} \frac{\exp(R(x))}{t^{n/2}} \int \exp\left(-c \frac{|x-u|^2}{t}\right) \chi_L(x, u) f(u) d\gamma_\infty(u).$$

The supremum here defines an operator of weak type $(1, 1)$ with respect to the Lebesgue measure in \mathbb{R}^n . From this the proposition follows, cf. [8, Section 3]. □

8. THE GLOBAL CASE FOR SMALL t

In this section, we conclude the proof of Theorem 1.1.

Proposition 8.1. *The maximal operator \mathcal{H}_*^G is of weak type $(1, 1)$ with respect to the invariant measure γ_∞ .*

Proof. For $m \in \mathbb{N}$ and $0 < t \leq 1$, we introduce regions \mathcal{S}_t^m . If $m > 0$, we let

$$\mathcal{S}_t^m = \left\{ (x, u) \in G : 2^{m-1}\sqrt{t} < |u - D_t x| \leq 2^m\sqrt{t} \right\}.$$

If $m = 0$, we replace the condition $2^{m-1}\sqrt{t} < |u - D_t x| \leq 2^m\sqrt{t}$ by $|u - D_t x| \leq \sqrt{t}$. Note that for any fixed $t \in (0, 1]$ these sets form a partition of G .

In the set \mathcal{S}_t^m we have, because of (3.5),

$$K_t(x, u) \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp(-c2^{2m}).$$

Then setting

$$\mathcal{K}_t^m(x, u) = \frac{\exp(R(x))}{t^{n/2}} \chi_{\mathcal{S}_t^m}(x, u), \quad (8.1)$$

one has, for all $(x, u) \in G$ and $0 < t < 1$,

$$K_t(x, u) \lesssim \sum_{m=0}^{\infty} \exp(-c2^{2m}) \mathcal{K}_t^m(x, u).$$

Hence, it suffices to prove that for $m = 0, 1, \dots$ and $f \geq 0$ normalized in $L^1(\gamma_\infty)$

$$\gamma_\infty \left\{ x \in \mathcal{E} : \sup_{0 < t \leq 1} \int \mathcal{K}_t^m(x, u) f(u) d\gamma_\infty(u) > \alpha \right\} \lesssim \frac{2^{Cm}}{\alpha}, \quad (8.2)$$

for large α , since this will allow summing in m in the space $L^{1,\infty}$.

Fix $m \in \mathbb{N}$. Then $(x, u) \in \mathcal{S}_t^m$, $t \in (0, 1]$ implies $|u - D_t x| \leq 2^m\sqrt{t}$. Now Lemma 5.1 leads to

$$1 \lesssim (1 + |x|)^4 t^2 + (1 + |x|)^2 2^{2m} t \leq ((1 + |x|)^2 2^{2m} t)^2 + (1 + |x|)^2 2^{2m} t.$$

Consequently,

$$(1 + |x|)^2 2^{2m} t \gtrsim 1 \quad (8.3)$$

as soon as there exists a point u with $\mathcal{K}_t^m(x, u) \neq 0$, and then $t \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\alpha, m) > 0$. Hence the supremum in (8.2) can as well be taken over $\varepsilon \leq t \leq 1$, and this supremum is a continuous function of $x \in \mathcal{E}$.

To prove (8.2), the idea, which goes back to [16], is to construct a finite sequence of pairwise disjoint balls $(\mathcal{B}^{(\ell)})_{\ell=1}^{\ell_0}$ in \mathbb{R}^n and a finite sequence of sets $(\mathcal{Z}^{(\ell)})_{\ell=1}^{\ell_0}$ in \mathbb{R}^n , called forbidden zones. These zones will together cover the level set in (8.2). We will show that

$$\left\{ x \in \mathcal{E} : \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^m(x, u) f(u) d\gamma_\infty(u) \geq \alpha \right\} \subset \bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}, \quad (8.4)$$

and that for each ℓ

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u). \quad (8.5)$$

Since the $\mathcal{B}^{(\ell)}$ will be pairwise disjoint, we could then conclude

$$\gamma_\infty\left(\bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}\right) \lesssim \frac{2^{Cm}}{\alpha} \sum_{\ell=1}^{\ell_0} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha}.$$

This would imply (8.2) and so complete the proof of Proposition 8.1.

The sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ will be introduced by means of a sequence of points $x^{(\ell)}$, $\ell = 1, \dots, \ell_0$, which we define by recursion. To find the first point $x^{(1)}$, consider the minimum of the quadratic form $R(x)$ in the compact set

$$\mathcal{A}_1(\alpha) = \left\{ x \in \mathcal{E} : \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^m(x, u) f(u) d\gamma_\infty \geq \alpha \right\}.$$

Should this set be empty, (8.2) is immediate. By continuity, this minimum is attained at some point $x^{(1)}$ of the set.

We now describe the recursion to construct $x^{(\ell)}$ for $\ell \geq 2$. Like $x^{(1)}$, these points will satisfy

$$\sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^m(x^{(\ell)}, u) f(u) d\gamma_\infty \geq \alpha.$$

Once an $x^{(\ell)}$, $\ell \geq 1$, is defined, we can thus by continuity choose $t_\ell \in [\varepsilon, 1]$ such that

$$\int \mathcal{K}_{t_\ell}^m(x^{(\ell)}, u) f(u) d\gamma_\infty \geq \alpha. \quad (8.6)$$

Using this t_ℓ , we associate with $x^{(\ell)}$ the tube

$$\mathcal{Z}^{(\ell)} = \{ D_s \eta \in \mathbb{R}^n : s \geq 0, R(\eta) = R(x^{(\ell)}), |\eta - x^{(\ell)}| < A 2^{3m} \sqrt{t_\ell} \},$$

Here the constant $A > 0$ is to be determined, depending only on n , Q and B .

All the $x^{(\ell)}$ will be minimizing points. To avoid having them too close to one another, we will not allow $x^{(\ell)}$ to be in any $\mathcal{Z}^{(\ell')}$ with $\ell' < \ell$. More precisely, assuming $x^{(1)}, \dots, x^{(\ell)}$ already defined, we will choose $x^{(\ell+1)}$ as a minimizing point of $R(x)$ in the set

$$\mathcal{A}_{\ell+1}(\alpha) = \left\{ x \in \mathcal{E} \setminus \bigcup_{\ell'=1}^{\ell} \mathcal{Z}^{(\ell')} : \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^m(x, u) f(u) d\gamma_\infty(u) \geq \alpha \right\}, \quad (8.7)$$

provided this set is nonempty. But if $\mathcal{A}_{\ell+1}(\alpha)$ is empty, the process stops with $\ell_0 = \ell$ and (8.4) follows. We will soon see that this actually occurs for some ℓ .

Now assume that $\mathcal{A}_{\ell+1}(\alpha) \neq \emptyset$. In order to assure that a minimizing point exists, we must verify that $\mathcal{A}_{\ell+1}(\alpha)$ is closed and thus compact, although the $\mathcal{Z}^{(\ell')}$ are not open. To do so, observe that for $1 \leq \ell' \leq \ell$, the minimizing property of $x^{(\ell')}$ means that there is no point in $\mathcal{A}_{\ell'}(\alpha)$ with $R(x) < R(x^{(\ell')})$. Thus we have the inclusions

$$\mathcal{A}_{\ell+1}(\alpha) \subset \mathcal{A}_{\ell'}(\alpha) \subset \left\{ x : R(x) \geq R(x^{(\ell')}) \right\}, \quad 1 \leq \ell' \leq \ell.$$

It follows that

$$\mathcal{A}_{\ell+1}(\alpha) = \mathcal{A}_{\ell+1}(\alpha) \cap \bigcap_{1 \leq \ell' \leq \ell} \left\{ x : R(x) \geq R(x^{(\ell')}) \right\} =$$

$$\bigcap_{\ell'=1}^{\ell} \left\{ x \in \mathcal{E} \setminus \mathcal{Z}^{(\ell')} : R(x) \geq R(x^{(\ell')}), \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^m(x, u) f(u) d\gamma_\infty(u) \geq \alpha \right\}.$$

The sets $\{x \in \mathcal{E} \setminus \mathcal{Z}^{(\ell')} : R(x) \geq R(x^{(\ell')})\}$ are closed in view of the choice of $\mathcal{Z}^{(\ell')}$. This makes $\mathcal{A}_{\ell+1}(\alpha)$ compact, and a minimizing point $x^{(\ell+1)}$ can be chosen. Thus the recursion is well defined.

We observe that (8.3) applies to t_ℓ and $x^{(\ell)}$, so that

$$|x^{(\ell)}|^2 2^{2m} t_\ell \gtrsim 1. \quad (8.8)$$

Further, we define balls

$$\mathcal{B}^{(\ell)} = \{u \in \mathbb{R}^n : |u - D_{t_\ell} x^{(\ell)}| \leq 2^m \sqrt{t_\ell}\}.$$

Because of (8.1) and the definitions of \mathcal{K}_t^m and \mathcal{S}_t^m , the inequality (8.6) implies

$$\alpha \leq \frac{\exp(R(x^{(\ell)}))}{t_\ell^{n/2}} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u). \quad (8.9)$$

We now verify that the sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ have the required properties. The proof follows the lines of the proof of Lemma 6.2 in [4], with only slight modifications.

Lemma 8.2. *The collection of balls $\mathcal{B}^{(\ell)}$ is pairwise disjoint.*

Proof. Two balls $\mathcal{B}^{(\ell)}$ and $\mathcal{B}^{(\ell')}$ with $\ell < \ell'$ will be disjoint if

$$|D_{t_\ell} x^{(\ell)} - D_{t_{\ell'}} x^{(\ell')}| > 2^m (\sqrt{t_\ell} + \sqrt{t_{\ell'}}). \quad (8.10)$$

By means of the coordinates from Subsection 4.1 with $\beta = R(x^{(\ell)})$, we write

$$x^{(\ell')} = D_s \tilde{x}^{(\ell')}$$

for some $\tilde{x}^{(\ell')}$ with $R(\tilde{x}^{(\ell')}) = R(x^{(\ell)})$ and some $s \in \mathbb{R}$. Note that $s \geq 0$, because $R(x^{(\ell')}) \geq R(x^{(\ell)})$. Since $x^{(\ell')}$ does not belong to the forbidden zone $\mathcal{Z}^{(\ell)}$, we must have

$$|\tilde{x}^{(\ell')} - x^{(\ell)}| \geq A 2^{3m} \sqrt{t_\ell}. \quad (8.11)$$

We first assume that $t_{\ell'} \geq M 2^{4m} t_\ell$, for some $M \geq 2$ to be chosen. Lemma 4.3 (ii) implies

$$|D_{t_\ell} x^{(\ell)} - D_{t_{\ell'}} x^{(\ell')}| = |D_{t_\ell} x^{(\ell)} - D_{t_{\ell'}+s} \tilde{x}^{(\ell')}| \gtrsim |x^{(\ell)}| (t_{\ell'} + s - t_\ell) \gtrsim |x^{(\ell)}| t_{\ell'}.$$

Using our assumption and then (8.8), we get

$$|x^{(\ell)}| t_{\ell'} \gtrsim |x^{(\ell)}| \sqrt{M} 2^{2m} \sqrt{t_\ell} \sqrt{t_{\ell'}} \gtrsim \sqrt{M} 2^m \sqrt{t_{\ell'}} \simeq \sqrt{M} 2^m (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Fixing M suitably large, we obtain (8.10) from the last two formulae.

It remains to consider the case when $t_{\ell'} < M 2^{4m} t_\ell$. Then

$$\sqrt{t_\ell} > \frac{2^{-2m-1}}{\sqrt{M}} (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Applying this to (8.11), we obtain (8.10) by choosing A so that A/\sqrt{M} is large enough. \square

We next verify that the sequence $(x^{(\ell)})$ is finite. For $\ell < \ell'$, we have (8.11), as in the preceding proof. Then Lemma 4.3 (i) implies

$$|x^{(\ell')} - x^{(\ell)}| \gtrsim A 2^{3m} \sqrt{t_\ell}.$$

Since $t_\ell \geq \varepsilon$, we see that the distance $|x^{(\ell')} - x^{(\ell)}|$ is bounded below by a positive constant. But all the $x^{(\ell)}$ are contained in the bounded set \mathcal{E} , so they are finite in number. Thus the set considered in (8.7) must be empty for some ℓ , and the recursion stops. This implies (8.4).

We finally prove (8.5). Observe that the forbidden zone $\mathcal{Z}^{(\ell)}$ is a tube as defined in (4.12), with $a = A 2^{3m} \sqrt{t_\ell}$ and $\beta = R(x^{(\ell)})$. This value of β is large since $x^{(\ell)} \in \mathcal{E}$, and thus we can apply Lemma 4.4 to obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{(A 2^{3m} \sqrt{t_\ell})^{n-1}}{\sqrt{R(x^{(\ell)})}} \exp(-R(x^{(\ell)})).$$

We bound the exponential here by means of (8.9) and observe that $R(x^{(\ell)}) \sim |x^{(\ell)}|^2$, getting

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{1}{\alpha |x^{(\ell)}| \sqrt{t_\ell}} (A 2^{3m})^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u).$$

As a consequence of (8.8), we obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^m}{\alpha} (A 2^{3m})^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u),$$

proving (8.5). This concludes the proof of Proposition 8.1. \square

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UNIVERSITÀ DEGLI STUDI DI PADOVA, STRADELLA SAN NICOLA 3, I-36100 VICENZA, ITALY
E-mail address: valentina.casarino@unipd.it

UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA MARZOLO 9, I-35100 PADOVA, ITALY
E-mail address: paolo.ciatti@unipd.it

MATHEMATICAL SCIENCES, UNIVERSITY OF GOTHENBURG AND MATHEMATICAL SCIENCES,
CHALMERS UNIVERSITY OF TECHNOLOGY, SE - 412 96 GÖTEBORG, SWEDEN
E-mail address: peters@chalmers.se