A New Method for the Analytical Determination of the Complex Relative Permeance Function in Slotted Electrical Machines

M. Bortolozzi ♠, N.Barbini ♣, M. Olivo ♥, A. Tessarolo ♦

University of Trieste, Via A. Valerio 10, Trieste (Italy). E-mail: ♠mauro.bortolozzi@gmail.com, ♣nicola.barbini@phd.units.it, ♥matteo.olivo@nidec-asi.it, ♦atessarolo@units.it

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Abstract

The complex relative permeance function is a suitable tool to predict the air gap magnetic field in slotted stator electrical machines. In the literature, the complex permeance function is usually identified by means of numerical techniques based on complex techniques such as conformal mapping or subdomain method. In this paper, an alternative approach is proposed. This method is based on solving the magnetostatic Laplace equation over a portion of the slotted air gap by imposing appropriate boundary conditions in the slot opening region. Such boundary condition tries to represent the theoretical trend of the magnetic field divergence near the corner-shaped ferromagnetic regions. Furthermore, the Carter theory for the slot opening consideration is used. A fully analytical formulation for the complex relative permeance function is obtained, and its accuracy is assessed using finite element analysis.

1 Introduction

In electrical machine theory is quite known that the flux density in the air gap is not a perfect sinusoid but it is affected by an inevitable grade of distortion. This distortion is caused by multiple phenomena, one of these is the slotting effect [1]. This parasitic effect causes in the machine additional losses and torque pulsations. In permanent magnet machines the slotting effect is also responsible for the cogging torque ([2][3]). The knowledge of the air gap magnetic field distribution is essential in order to estimate these parasitic phenomena.

The magnetic field analytical evaluation in the air-gap is a hot topic in literature and the use of complex permeance function is a quite common approach in order to fulfil this task [4][5][6]. In this paper, an innovative method for complex permeance function determination is proposed, without use of numerical other tools such as subdomain technique or conformal mapping theory ([3][7][8][9]). In fact, the standard method for the permeance function determination is using conformal mapping, this leads to an accurate result but with many non-linear calculations. This analytical solution is made possible by directly solving the Laplace equation in the air gap region and imposing a well-defined function for the tangential component of the flux density in the slot opening region. Furthermore, the approximated Carter coefficient theory is used to determine the final expression of the complex permeance function [10][11]. This work follows the same approach already adopted in [12] for the linear machine case. In the end, the analytical formula obtained is assessed using finite element analysis showing a satisfactory good accordance. The study is performed in regard to a circular shaped electrical machine and all the electromagnetic equations are expressed in polar coordinates. End effects are disregarded, hence the vector potential is everywhere parallel to the rotational axis, so that its axial component is always considered as a scalar quantity.

2 Complex permeance function theory

The aim of this work is to derive an analytical expression for the complex relative permeance function components. As the first step, let us consider two different models one called "slotted" and the other "slotless" (Fig.1). These two models are both equipped with a smooth rotor and energized with only two current points placed as shown in the figure and carrying the current respectively $+I$ and $-I$. All the iron parts are assumed as infinitely permeable ($\mu_r \to \infty$).

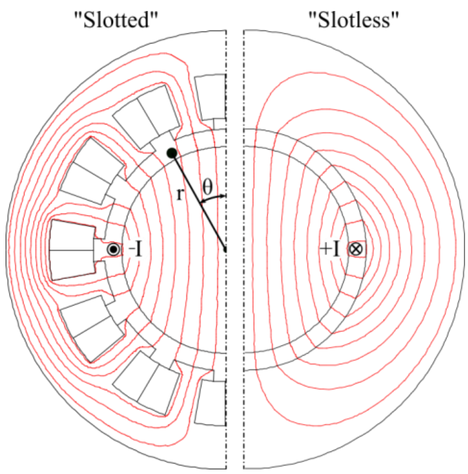


Figure 1: Portions of "slotted" and "slotless" models with reference polar coordinate system energized with two current points carrying the current +I and -I respectively

Referring to these models, the relative complex permeance function theory ([12]) allows to express the magnetic field in a slotted air-gap using a complex notation as follows:

$$\bar{B}(r,\theta) = B_r(r,\theta) + jB_\theta(r,\theta) = \overline{B_k}(r,\theta) \cdot \bar{\lambda}^*(r,\theta) \quad (1)$$

Where, $\bar{\lambda}^*$ is the complex conjugate of the relative complex permeance function and $\overline{B_k}(r,\theta)$ is the slotless magnetic field represented with a complex notation. The expressions of both quantities are reported below:

$$\overline{B_k}(r,\theta) = B_{kr}(r,\theta) + jB_{k\theta}(r,\theta) \quad (2)$$

$$\bar{\lambda}(r,\theta) = \lambda_a(r,\theta) + j\lambda_b(r,\theta) \quad (3)$$

After using this complex definition, it is possible to evaluate the components of the air gap flux density separately:

$$B_r(r,\theta) = B_{kr}(r,\theta)\lambda_a(r,\theta) + B_{k\theta}(r,\theta)\lambda_b(r,\theta) \quad (4)$$

$$B_\theta(r,\theta) = B_{k\theta}(r,\theta)\lambda_a(r,\theta) - B_{kr}(r,\theta)\lambda_b(r,\theta) \quad (5)$$

Using expressions (4) and (5) under some simplifying hypothesis is possible to obtain a fully analytical definition for both complex relative permeance function components.

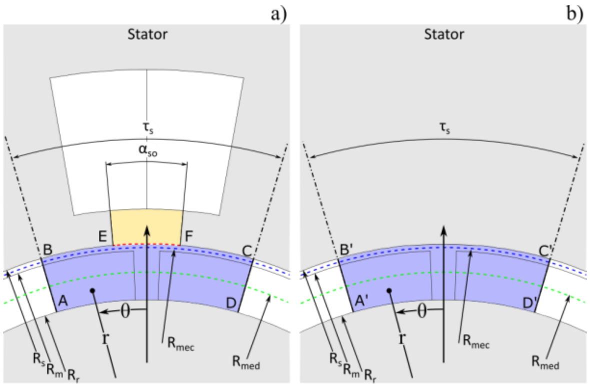


Figure 2: Slot opening region in slotted (a) and slotless (b) model

Considering the slotless model and focusing the attention only in the region reported in Figure 2b, it is legit to consider that in the blue highlighted zone, the flux density has only radial component and it is possible to express the slotless field components as follows:

$$B_{kr}(r,\theta) = B_{k0}\frac{R_{med}}{r} \quad (6)$$

$$B_{k\theta}(r,\theta) = 0 \quad (7)$$

Where B_{k0} is the radial flux density mean value along the circumference with radius R_{med} (Figure 2b).

Considering the conditions reported at (6) and (7), (4) and (5) can be rewritten as:

$$B_r(r,\theta) = B_{kr}(r,\theta)\lambda_a(r,\theta) \quad (8)$$

$$B_\theta(r,\theta) = -B_{kr}(r,\theta)\lambda_b(r,\theta) \quad (9)$$

That implies:

$$\lambda_a(r,\theta) = \frac{B_r(r,\theta)}{B_{kr}(r,\theta)} \quad (10)$$

$$\lambda_b(r,\theta) = -\frac{B_\theta(r,\theta)}{B_{kr}(r,\theta)} \quad (11)$$

After the complex permeance function components definition it is necessary to express the magnetic field distribution in the slotted ($\bar{B}(r,\theta)$) and slotless ($\overline{B_k}(r,\theta)$) models.

3 Slotted air-gap magnetic field evaluation

3.1 Problem definition

As already shown in the previous section it is possible to express the complex permeance function components as a function of the magnetic field of both models (slotless and slotted). The aim of this section is to evaluate the flux density distribution in the airgap region (ABCD in Figure 2a) of the slotted model using a direct solution of the Laplace equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \mathrm{A}(r,\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \mathrm{A}(r,\theta)}{\partial \theta^2} = 0 \quad (12)$$

Where A is the vector potential of the magnetic field in this area. For this kind of partial differential equations, it is possible to derive a class of solutions that can respect (12). In this particular case, using also the Fourier expansion theory, the solving function that fits in (12) can be expressed as:

$$\mathrm{A}(r,\theta) = A_0\theta + \sum_{n=1}^{\infty}((A_n^+ \cdot r^{Zn} + A_n^- \cdot r^{-Zn})\sin(Zn\theta)) \quad (13)$$

Where Z is the number of slots, n is the harmonic order and A_0, A_n^+, A_n^- are numerical constants that can be evaluated imposing correct boundary conditions. Considering the vector potential theory, the flux density components are immediately defined as:

$$B_r(r,\theta) = \frac{1}{r}\frac{\partial \mathrm{A}(r,\theta)}{\partial \theta}\ ;\ B_\theta(r,\theta) = -\frac{\partial \mathrm{A}(r,\theta)}{\partial r} \quad (14)$$

3.2 Boundary conditions

Referring to Figure 2a, to evaluate the constants in (13), it is necessary to impose the appropriate boundary conditions at the borders of the region identified with the letters ABCD. Regarding AB and CD segments it is legit to consider the flux density have only radial component along that direction, so:

$$B_\theta^{\mathrm{AB}} = 0\ ;\ B_\theta^{\mathrm{CD}} = 0 \quad (15)$$

The condition that must be imposed in the circular arc AD is again the nullity of the tangential component of the magnetic field along the curve, due to the ideal permeability of the rotor iron:

$$B_\theta^{\mathrm{AD}} = 0 \quad (16)$$

The last condition is referred to the circular arc BC. In this curve, the flux density tangential component B_θ must be equal to a prescribed function $f(x)$ (the same approach is adopted in [12] for a linear case). This boundary condition function is represented in Figure 3. Looking at the figure, it is clear that under the teeth the magnetic field tangential component is zero, due to the infinite iron permeability of the stator core. Under the slot opening the flux density B_θ can be represented with an analytical function *f(x)*. Thus, the tangential component of the flux density along the BC segment can be expressed as follows:

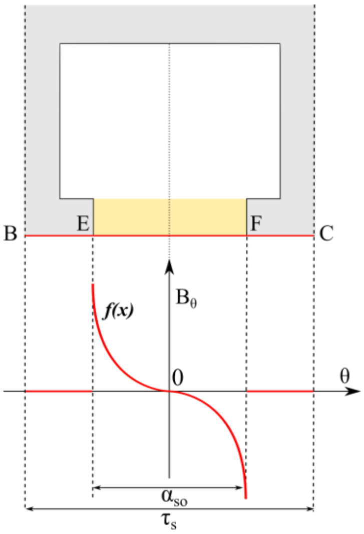


Figure 3: function that represents the magnetic field tangential component along the circular arc segment BC.

$$B_\theta^{\mathrm{BC}}(\theta) = \begin{cases} c \cdot \left[\dfrac{1}{\sqrt[3]{\frac{\alpha_{so}}{2}+\theta}} - \dfrac{1}{\sqrt[3]{\frac{\alpha_{so}}{2}-\theta}} \right] & \forall \theta \in \left]-\frac{\alpha_{so}}{2}; \frac{\alpha_{so}}{2}\right[\\ 0 & \forall \theta \in \left[-\frac{\tau_s}{2}; -\frac{\alpha_{so}}{2}\right] \cup \left[\frac{\alpha_{so}}{2}; \frac{\tau_s}{2}\right] \end{cases} \quad (17)$$

where the constant c is an unknown parameter to be suitably decided. The function reported in equation (17) can be represented using Fourier expansion as follows:

$$B_\theta^{\mathrm{BC}}(\theta) = c \sum_n b_n \cdot \sin(Zn\theta) \quad (18)$$

With:

$$b_n = \frac{2Z}{\pi} \int_0^{\frac{\alpha_{so}}{2}} \left[\frac{1}{\sqrt[3]{\frac{\alpha_{so}}{2}+\theta}} - \frac{1}{\sqrt[3]{\frac{\alpha_{so}}{2}-\theta}} \right] \sin(Zn\,\theta)\, d\theta \quad (19)$$

3.3 Laplace equation solution

Imposing all the boundary conditions defined in the previous section, it is possible to obtain the final solution for the magnetic field in the slotted air-gap. Based on (14), the solution (13) corresponds to the following flux density components in the studied domain:

$$B_r(r,\theta) = \frac{B_0 R_{med}}{r} + c \sum_n \mathrm{K}(n,r)\cos(Z\theta n)\,\mathrm{C}_r(n,r) \quad (20)$$

$$B_\theta(r,\theta) = c \sum_n \mathrm{K}(n,r)\sin(Z\theta n)\,\mathrm{C}_\theta(n,r) \quad (21)$$

With:

$$\mathrm{K}(n,r) = \frac{2Z\, b_n R_s}{\pi\, r} \quad (22)$$

$$\mathrm{C}_r(n,r) = \frac{\left(\frac{R_s}{R_r}\right)^{Zn} \cdot \left(\frac{r}{R_r}\right)^{Zn} + \left(\frac{R_s}{r}\right)^{Zn}}{1 - \left(\frac{R_s}{R_r}\right)^{2Zn}} \quad (23)$$

$$\mathrm{C}_\theta(n,r) = \frac{\left(\frac{R_s}{R_r}\right)^{Zn} \cdot \left(\frac{r}{R_r}\right)^{Zn} - \left(\frac{R_s}{r}\right)^{Zn}}{1 - \left(\frac{R_s}{R_r}\right)^{2Zn}} \quad (24)$$

Where B_0 is the radial flux density mean value along the circumference with radius R_{med} (Figure 2).

Carter's coefficient

In the common practice of the electrical machine analysis and design, the slotting effect is considered using Carter's coefficient. This quantity is defined as the ratio between the maximum and the average value of the radial flux density along the medium radius circumference in a slot pitch (Fig. 4):

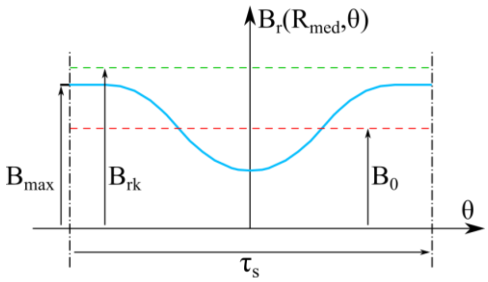


Figure 4: Radial flux density distribution along the R_{med} radius circumference under a slot pitch.

$$k_c = \frac{B_{max}}{B_0} \quad (25)$$

It can be proven that, with a good approximation and under some hypothesis, the carter coefficient can be expressed as follows [10][11]:

$$k_c = \frac{\frac{\tau_s}{\alpha_{so}}}{\frac{\tau_s}{\alpha_{so}} - \frac{\frac{\alpha_{so} R_s}{g}}{5 + \frac{\alpha_{so} R_s}{g}}} \quad (26)$$

Where τ_s is the slot pitch, α_{so} is the slot opening angle and g is the air gap (Figure 2a).

3.4 Determination of c using Carter coefficient

After the Laplace equation solution founded previously it is clear that the next task is to find an expression for c. At this purpose, it has been used the same approach used in [12] using the geometrical definition of the carter coefficient. Starting from its theoretical definition (25); it is possible to evaluate the maximum value of the radial flux density along the R_{med} radius circumference using (20).

In fact:

$$B_{max} = B_0 + c\sum_n \mathrm{K}(n, R_{med})\cos(\pi n)\, \mathrm{C}_r(n, R_{med}) \quad (27)$$

Substituting (27) into (25) and comparing the result with (26) it is possible to obtain a final expression for the coefficient c:

$$c = B_0 \frac{k_c - 1}{\sum_n \mathrm{K}(n, R_{med})\cos(\pi n)\, \mathrm{C}_r(n, R_{med})} \quad (28)$$

It is very important to pay attention at the fact that the coefficient found in (28) is dependent from B_0 that is an unknown quantity.

4 Slotless air-gap field evaluation

As already said in section 2, the flux density distribution in the considered region of the slotless model can be assumed as purely radial and dependent only by the inverse of the radius. This fact allows to write expressions (6) and (7).

A further consideration should be made regarding the relationship between $B_{kr}(r)$ and the radial flux density distribution in the slotted model. At this purpose, looking at Figure 5, it is clear that, applying the ampere circuital law along the closed paths Γ and Γ' we can write the following expressions:

$$\oint_\Gamma \overline{H_k}(r)\cdot \mathrm{dl} = I \text{ and } \oint_{\Gamma'} \overline{H}(r,\theta)\cdot \mathrm{dl} = \mathrm{I} \quad (29)$$

considering that the iron is infinitely permeable, the integrals in (29) can be rewritten as:

$$2\int_{\mathrm{A}}^{\mathrm{B}} \overline{H_k}(r)\cdot \mathrm{dr} = \mathrm{I} \quad \text{and} \quad 2\int_{\mathrm{A'}}^{\mathrm{B'}} \overline{H}(r,\theta)\cdot \mathrm{dr} = \mathrm{I} \quad (30)$$

The flux density distribution along the AB segment is completely radial due to the hypothesis of the slotless model. The magnetic field distribution along the A'B' segment is again purely radial this time due to symmetry reasons. After these considerations, it is possible to express the formulas in (30) as follows:

$$2\int_{\mathrm{R_r}}^{\mathrm{R_s}} \frac{B_{kr}(r)}{\mu_0}\mathrm{dr} = \frac{2}{\mu_0}B_{kr}(r)\cdot r\ln\left(\frac{R_s}{R_r}\right) = \mathrm{I} \quad (31)$$

$$2\int_{\mathrm{R_r}}^{\mathrm{R_s}} H_r(r,\frac{\tau_s}{2})\mathrm{dr} = 2\int_{\mathrm{R_r}}^{\mathrm{R_s}} \frac{B_r\left(r,\frac{\tau_s}{2}\right)}{\mu_0}\cdot \mathrm{dr} = \mathrm{I} \quad (32)$$

Now, comparing (31) and (32) we obtain:

$$B_{kr}(r) = \frac{1}{\mathrm{r}\cdot\ln\left(\frac{R_s}{R_r}\right)}\cdot \int_{\mathrm{R_r}}^{\mathrm{R_r}} B_r\left(r,\frac{\tau_s}{2}\right)\,\mathrm{dr} \quad (33)$$

Looking at the expressions (33) it is clear that the radial flux density in the slotless model is fully determined as a function of the radial flux density of the slotted model.

5 Analytical expression for the relative complex permeance function components

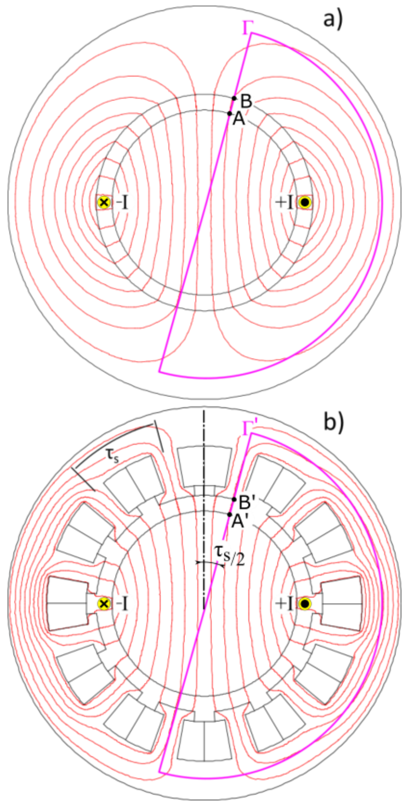


Figure 5: Slotless (a) and slotted (b) models for the ampere's circuital law application

The solution of the magnetic field in the area ABCD in the slotted model and the relationship between $B_{kr}(r)$ with the slotted field makes it possible to determine an analytical expression for λ_a and λ_b.

In order to obtain an analytical expression for the complex permeance function components it is necessary to substitute (33), (20) (21) and (28) in (10) and (11). After some algebraic manipulations, the result is an analytical expression for the permeance function components:

$$\lambda_a(r,\theta) = \frac{\mathrm{H} + \sum_n b_n\, \mathrm{C}_r(n,r)\cos(Z n\theta)}{\mathrm{H} + \frac{1}{Z\ln(R_r/R_s)}\sum_n \frac{b_n\cos(\pi n)}{n}} \quad (34)$$

$$\lambda_b(r,\theta) = -\frac{\sum_n b_n\, \mathrm{C}_\theta(n,r)\sin(Z n\theta)}{\mathrm{H} + \frac{1}{Z\ln(R_r/R_s)}\sum_n \frac{b_n\cos(\pi n)}{n}} \quad (35)$$

where:

$$\mathrm{H} = \frac{\sum_n b_n\, \mathrm{C}_r(n, R_{med})\cos(\pi n)}{k_c - 1} \quad (36)$$

6 Validation through finite element analysis

The proposed method for the relative complex permeance function determination has been assessed using the finite element analysis. The main geometric and material data characterizing the machine used for the validation are reported in the following table:

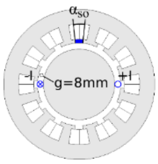
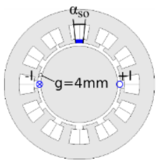
	Model A	Model B
		
Air gap width g	8 mm	4 mm
Slot pitch τ_s	30 deg	30 deg
Slot opening angles α_{so}	5, 10, 20 deg	5, 10, 20 deg
Stator inner radius R_s	55 mm	55 mm
Iron magnetic permeability	10000	10000
Current value	5000 A	5000 A

Table 1: Main data characterizing the machines used for the validations

The validation is made between the relative permeance function components along the R_{med} circumference obtained with the proposed method and with the finite element simulations. For the sake of clarity, the comparison between these quantities is represented in Figure 7 and Figure 6 restricting the angular coordinate only at one slot pitch.

Looking at Figure 7 and Figure 6, it can be noticed that the proposed method is in a good accordance with the FEA. The gap between the results obtained using the analytical approach and the results obtained with the FEA is very small when the air gap is small; this fact is in accordance with the proposed method assumptions. These assumptions mainly derive from the simplifying hypothesis that allows us to express the carter's coefficient as (26).

7 Conclusions

Surface permanent machines are very important for several applications, especially where high values of efficiency need to be achieved. This kind of machines can be equipped with slotted or slotless stator. The slotted architecture it is known to suffer various parasitic phenomena caused by the slotting effect. The complex relative permeance function theory is a useful tool to estimate the air gap magnetic field distortion.

In this work an innovative approach for the evaluation of the complex relative permeance function components is proposed. The output formulas are entirely analytical except the presence of the numerical integration for the b_n coefficient calculation and the presence of the infinite summations. Results have been compared with finite elements showing a satisfactory accordance.

Future work will be dedicated to the c parameter correct identification using also alternative approaches.

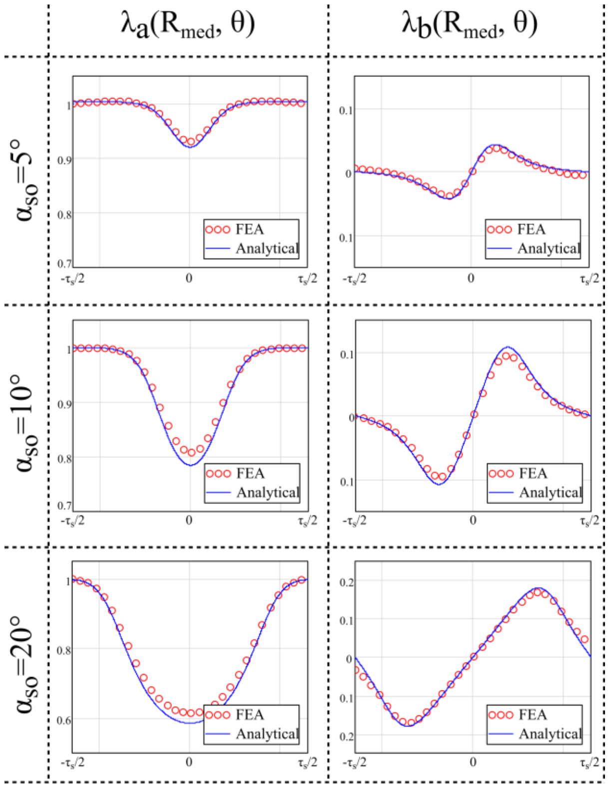


Figure 7: Complex relative permeance function components for various slot opening angles α_{so} and $g = 8$ mm

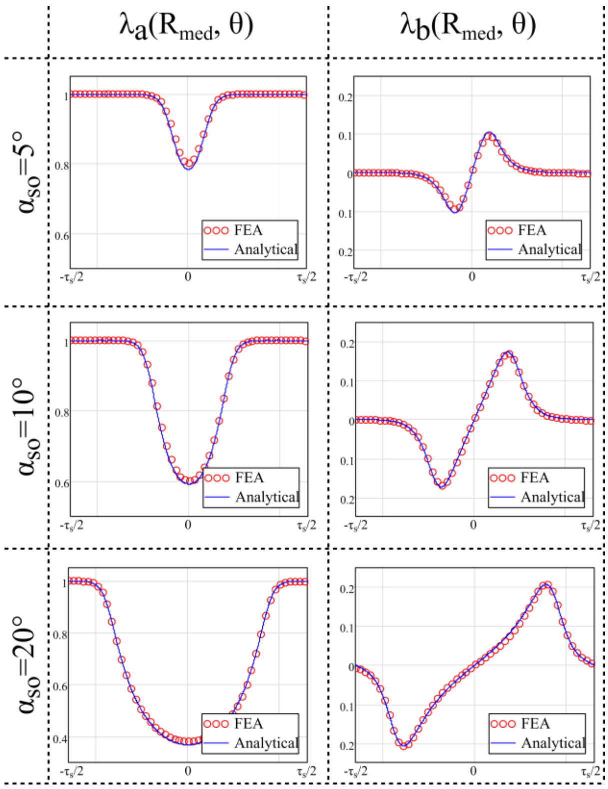


Figure 6: Complex relative permeance function components for various slot opening angles α_{so} and $g = 4$ mm

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