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A method based on 3D stiffness matrices in Cartesian coordinates for computation of 2.5D elastodynamic Green's functions of layered half-spaces

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Abstract

This article elaborates on an extension to the classical stiffness matrix method to obtain the Green's functions for two-and-a-half dimensional (2.5D) elastodynamic problems in homogeneous and horizontally layered half-spaces. Exact expressions for the three-dimensional (3D) stiffness matrix method for isotropic layered media in Cartesian coordinates are used to determine the stiffness matrices for a system of horizontal layers underlain by an elastic half–space. In the absence of interfaces, virtual interfaces are considered at the positions of external loads. The analytic continuation is used to find the displacements at any receiver point placed within a layer. The responses of a horizontally layered half-space subjected to a unit harmonic load obtained using the present method are compared with those calculated using a well-established methodology, achieving good agreement.

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1. Introduction

Elastic wave propagation is a very significant topic in fields such as seismology, soil dynamics, noise and vibration, and soil-structure interaction. Specifically, elastodynamic fundamental solutions for homogeneous and layered elastic solid media are of great interest in ground modeling. Calculation of the dynamic response of a homogeneous or layered soil in response to some specific dynamic sources calls for methods that depend on how the loads are distributed in space. If the loads are two-dimensional (2D) or they satisfy plane–strain conditions, the problem can be addressed by various well–known 2D (plane–strain) formulations. Otherwise, when the 2D load is allowed to change harmonically in the third dimension, the resulting dynamic problem ceases to be plane–strain or axisymmetric, and is referred to instead as the 2.5D problem. 2.5D elastodynamic fundamental solutions, also known as 2.5D elastodynamic Green's functions, for these types of problems correspond to the system's response to spatially sinusoidal harmonic line load.

Regarding homogeneous elastic media, Tadeu and Kausel analytically derived the 2.5D elastodynamic fundamental solution for a homogeneous full-space [1]. In addition, semi-analytical solutions have been obtained for the case of a homogeneous half-space [2] and that of a free solid layer [3]. These analytical and semi-analytical expressions have been employed by numerous authors in various problems such as railway-induced vibration [4], acoustics [5] and soil-structure interaction [6].

Several researchers have focused on methodologies to determine the 2.5D elastodynamic fundamental solutions for elastic, horizontally layered media. The thin-layer method (TLM) [7, 8], direct stiffness matrix method [9] and the method of potentials [10] are the approaches most commonly used to obtain Green's functions for layered media. In some research fields, authors prefer to use the direct stiffness matrix method to calculate the Green's functions rather than other methods; for instance, those dealing with railway-induced ground-borne vibration [11–13]. Kausel and Roësset first introduced the stiffness matrix approach in the analysis of elastodynamic wave propagation [9]. More computationally efficient versions of their method were presented later [14]. The direct stiffness matrix method for two-dimensional (2D) problems in Cartesian coordinates and for 3D ones in cylindrical coordinates were proposed by Kausel [15]. An extension of the stiffness matrix method to elastodynamic systems composed of cylindrical and of spherical layers is also included in Ref. [15].

The aim of this paper is to present a method to calculate 2.5D Green's functions for both homogeneous and layered half-spaces in the wavenumber-frequency domain. Derived Green's functions are based on explicit expressions of the 3D stiffness matrices for layered media in Cartesian coordinates rather than in cylindrical coordinates. The rest of the paper is organized as follows: firstly, it addresses explicit expressions of the 3D stiffness matrices for a horizontal layer and a lower half-space; secondly, it explains how to use the stiffness matrices to compute 2.5D Green's functions; and finally, 2.5D Green's functions for a layered half-space, calculated using the proposed method, are compared against the established formulations based on the stiffness matrix method in cylindrical coordinates.

2. Stiffness matrices of a layer and a lower half-space in Cartesian coordinates

Consider a free, horizontal, homogeneous, isotropic and elastic layer with arbitrary thickness h subjected to arbitrarily distributed tractions $\mathbf{P}_u$ and $\mathbf{P}_l$ on the upper and lower interfaces, respectively. The displacements and tractions at the upper and lower interfaces can be related through the stiffness matrix $\hat{\mathbf{K}}$ as:

$$\begin{Bmatrix} \hat{\mathbf{P}}_u \\ \hat{\mathbf{P}}_l \end{Bmatrix} = \begin{bmatrix} \hat{\mathbf{K}}_{11} & \hat{\mathbf{K}}_{12} \\ \hat{\mathbf{K}}_{21} & \hat{\mathbf{K}}_{22} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{U}}_u \\ \hat{\mathbf{U}}_l \end{Bmatrix}, \tag{1}$$

where capital letters with hat (e.g. $\hat{\mathbf{K}}$) denote variables cast in the frequency–wavenumber domain, i.e. (k_x, k_y, ω), with ω being the frequency in rad/s and k_x, k_y the two horizontal wavenumbers, in m^{-1}.

Kausel presented the elements of matrix $\hat{\mathbf{K}}$ for the plane strain case [15]. The same process can be followed for the case of 3D wave propagation, resulting in the following closed-form expressions for the elements of matrix $\hat{\mathbf{K}}$:

$$\hat{\mathbf{K}}_{11} = \frac{\mu}{D} \frac{k_\beta^2}{k^2} \mathbf{A}_1 + \mu \mathbf{A}_2, \tag{2}$$

$$\hat{\mathbf{K}}_{12} = \frac{\mu}{D} \frac{k_\beta^2}{k^2} \mathbf{A}_3 + \mu \mathbf{A}_4, \qquad \hat{\mathbf{K}}_{21} = \hat{\mathbf{K}}_{12}^{\mathrm{T}}, \tag{3}$$

$$\hat{\mathbf{K}}_{22} = \hat{\mathbf{K}}_{11} \circ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \tag{4}$$

where matrices A_1, A_2, A_3 and A_4 are defined in the ensuing in Eqs. 10, 13, 14 and 18, respectively. Also, in the above equations, μ is the shear modulus (Lamé's second parameter), the superscript T denotes the transpose operator, $\circ$ is the Hadamard product and

$$k = \sqrt{k_x^2 + k_y^2}, \qquad k_\alpha = \frac{\omega}{\alpha}, \qquad k_\beta = \frac{\omega}{\beta}, \tag{5}$$

where α and β represent the phase velocities of the P- and S-waves, respectively. Coefficient D is calculated from:

$$D = 2\mathrm{e}^{(\upsilon_\beta - \upsilon_\alpha)h} - 2C_\alpha C_\beta + \left(\frac{k^2}{\upsilon_\alpha \upsilon_\beta} + \frac{\upsilon_\alpha \upsilon_\beta}{k^2}\right) B_\alpha B_\beta, \tag{6}$$

$$\upsilon_\alpha = \sqrt{k^2 - {k_\alpha}^2}, \qquad \upsilon_\beta = \sqrt{k^2 - {k_\beta}^2}, \tag{7}$$

$$B_\alpha = \frac{1 - \mathrm{e}^{-2\upsilon_\alpha h}}{2}, \qquad B_\beta = \frac{1 - \mathrm{e}^{-2\upsilon_\beta h}}{2}, \tag{8}$$

$$C_\alpha = \frac{1 + \mathrm{e}^{-2\upsilon_\alpha h}}{2}, \qquad C_\beta = \frac{1 + \mathrm{e}^{-2\upsilon_\beta h}}{2}. \tag{9}$$

The matrices A_1, A_2, A_3 and A_4 are obtained from:

$$\mathbf{A}_1 = \begin{bmatrix} \frac{k_x^2}{\upsilon_\beta} D_1 & \frac{k_x k_y}{\upsilon_\beta} D_1 & k_x D_3 \\ \frac{k_x k_y}{\upsilon_\beta} D_1 & \frac{k_y^2}{\upsilon_\beta} D_1 & k_y D_3 \\ k_x D_3 & k_y D_3 & \frac{k^2}{\upsilon_\alpha} D_2 \end{bmatrix}, \tag{10}$$

$$D_1 = B_\beta C_\alpha - \frac{\upsilon_\alpha \upsilon_\beta}{k^2} B_\alpha C_\beta, \qquad D_2 = B_\alpha C_\beta - \frac{\upsilon_\alpha \upsilon_\beta}{k^2} B_\beta C_\alpha, \tag{11}$$

$$D_3 = \mathrm{e}^{-(\upsilon_\alpha + \upsilon_\beta)h} - C_\alpha C_\beta + \left(\frac{k^2}{\upsilon_\alpha \upsilon_\beta}\right) B_\alpha B_\beta, \tag{12}$$

$$\mathbf{A}_2 = \begin{bmatrix} \upsilon_\beta \dfrac{k_y^2}{k^2} \dfrac{C_\beta}{B_\beta} & -\upsilon_\beta \dfrac{k_x k_y}{k^2} \dfrac{C_\beta}{B_\beta} & -2k_x \\ -\upsilon_\beta \dfrac{k_x k_y}{k^2} \dfrac{C_\beta}{B_\beta} & \upsilon_\beta \dfrac{k_x^2}{k^2} \dfrac{C_\beta}{B_\beta} & -2k_y \\ -2k_x & -2k_y & 0 \end{bmatrix}, \tag{13}$$

$$\mathbf{A}_3 = \begin{bmatrix} -\dfrac{k_x^2}{\upsilon_\beta} D_4 & -\dfrac{k_x k_y}{\upsilon_\beta} D_4 & k_x D_6 \\ -\dfrac{k_x k_y}{\upsilon_\beta} D_4 & -\dfrac{k_y^2}{\upsilon_\beta} D_4 & k_y D_6 \\ -k_x D_6 & -k_y D_6 & -\dfrac{k^2}{\upsilon_\alpha} D_5 \end{bmatrix}, \tag{14}$$

$$D_4 = B_\beta \mathrm{e}^{-\upsilon_\alpha h} - \frac{\upsilon_\alpha \upsilon_\beta}{k^2} B_\alpha \mathrm{e}^{-\upsilon_\beta h}, \quad D_5 = B_\alpha \mathrm{e}^{-\upsilon_\beta h} - \frac{\upsilon_\alpha \upsilon_\beta}{k^2} B_\beta \mathrm{e}^{-\upsilon_\alpha h}, \tag{15}$$

$$D_5 = B_\alpha \mathrm{e}^{-\upsilon_\beta h} - \frac{\upsilon_\alpha \upsilon_\beta}{k^2} B_\beta \mathrm{e}^{-\upsilon_\alpha h}, \tag{16}$$

$$D_6 = C_\beta \mathrm{e}^{-\upsilon_\alpha h} - C_\alpha \mathrm{e}^{-\upsilon_\beta h}, \tag{17}$$

and

$$\mathbf{A}_4 = \begin{bmatrix} -\upsilon_\beta \dfrac{k_y^2}{k^2}\dfrac{\mathrm{e}^{-\upsilon_\beta h}}{B_\beta} & \upsilon_\beta \dfrac{k_x k_y}{k^2}\dfrac{\mathrm{e}^{-\upsilon_\beta h}}{B_\beta} & 0 \\ \upsilon_\beta \dfrac{k_x k_y}{k^2}\dfrac{\mathrm{e}^{-\upsilon_\beta h}}{B_\beta} & -\upsilon_\beta \dfrac{k_x^2}{k^2}\dfrac{\mathrm{e}^{-\upsilon_\beta h}}{B_\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

Now consider a homogeneous, isotropic and elastic lower half-space subjected to an arbitrarily distributed surface traction. The displacements and traction at the upper interface can be related by means of the stiffness matrix for that half–space. The elements of the 3D stiffness matrix for the lower half-space, $\hat{\mathbf{K}}_h$, can be found as

$$\hat{\mathbf{K}}_h = \frac{\mu}{k^2 - \upsilon_\alpha \upsilon_\beta} \begin{bmatrix} k_y^2 \upsilon_{\beta\alpha} + k_\beta^2 \upsilon_\alpha & -k_x k_y \upsilon_{\beta\alpha} & k_x D_H \\ -k_x k_y \upsilon_{\beta\alpha} & k_x^2 \upsilon_{\beta\alpha} + k_\beta^2 \upsilon_\alpha & k_y D_H \\ k_x D_H & k_y D_H & k_\beta^2 \upsilon_\beta \end{bmatrix}, \tag{19}$$

where

$$\upsilon_{\beta\alpha} = \upsilon_\beta - \upsilon_\alpha, \quad D_H = 2\upsilon_\alpha \upsilon_\beta - {\upsilon_\beta}^2 - k^2. \tag{20}$$

3. 2.5D Green's functions for homogeneous and layered half-spaces

Consider an arbitrarily layered half-space with an arbitrary distribution of receiver points and sources located within the half-space. Based on the methodology proposed in this paper, the following steps must be taken to obtain the 2.5D Green's functions that relate all the desired receiver points and sources:

1. Add a virtual interface for every source that has not been placed on any of the physical layer interfaces.

2. Follow the same approach used in the finite element method to assemble the global stiffness matrix for the layered media while considering physical and virtual interfaces.

3. Use the concept of analytic continuation, as described by Kausel [15], to obtain the displacements at receiver points within a layer, i.e. those not located at physical or virtual interfaces, in terms of the layers' interface displacements. Then, by means of the inverse of the global stiffness matrix, $\hat{\mathbf{H}}(k_x, k_y, \omega)$ which defines the displacements of all the receiver points as a function of the tractions at the sources' location can be constructed.

4. Compute the 2.5D Green's functions of the system $\bar{\mathbf{H}}$, in the (k_x, y, ω) domain, accounting for all the selected receiver points and sources by applying an inverse Fourier transform (FT) in the y direction on $\hat{\mathbf{H}}(k_x, k_y, \omega)$.

4. Results and discussion

We applied the proposed method to determine 2.5D Green's functions for a layered half-space subjected to a buried harmonic load. The results were compared with those obtained using an alternative methodology [16], which formulates the stiffness matrices in cylindrical coordinates [9].

Fig. 1 shows an example of a layered half-space. A virtual interface, represented by the dashed line, was added where the load was applied. A receiver point was located within the first physical layer of the layered half-space.

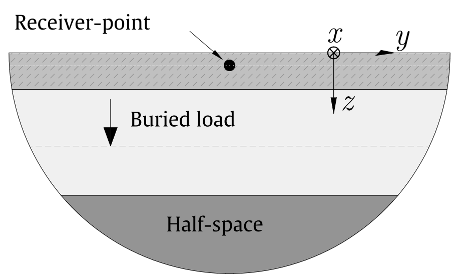

Fig. 1: A three-layered half-space with a virtual interface (dashed line) placed at the position of the buried load.

The receiver point and the source were placed at $(y_r, z_r) = (6, 8)$ m and $(y_f, z_f) = (2, 23)$ m, respectively. We calculated the 2.5D Green's functions for this particular case. A sampling vector with 2^9 points and increments of 0.01 rad/m was selected for k_x. To transform the Green's functions from (k_x, k_y, ω) domain to the 2.5D domain (k_x, y, ω), the inverse fast Fourier transform (FFT) is carried out over a set of 2^{10} values of k_y with increments of 0.04 rad/m. Noteworthy, if the response is needed only at a small subset of k_y, a direct evaluation of the FT is more effective. The mechanical parameters of the soil used in the calculations are given in Table 1. Hysteretic damping ratios D_p and D_s corresponding to P- and S-waves, respectively, were used to account for viscoelasticity; an outcome of the damping is preventing large oscillations in the integrands of the FFT, i.e. to minimize integration errors.

Table 1: Mechanical parameters used to model the layered half-space

Soil parameters	1st layer	2nd layer	3rd layer
E (MPa)	366	390	420
ρ (kg m^{-3})	2000	2200	2500
ν (-)	0.3	0.25	0.2
D_p (-)	0.03	0.03	0.03
D_s (-)	0.03	0.03	0.03

Fig. 2 features a comparison between the results obtained using the method presented herein and those calculated following the method proposed in [16]. The values presented in the figure correspond to a frequency of 30 Hz. As can be noted, this is good agreement between these two sets of results.

From a computational point of view, the two methods follow different schemes. In our method, the computational steps were: i) inversion of the global matrix based on 6×6 elementary matrices; ii) an inverse FT along k_y; and iii) a one-dimensional (1D) interpolation. On the other hand, the method based on stiffness matrices in cylindrical coordinates was developed according to the following steps: i) inversion of the global matrices associated with P-SV and SH waves, separately, to obtain the Green's functions in the (k_r, ω) domain; ii) computation of the Green's functions in the (r, θ, ω) domain by means of an inverse Hankel transform and an inverse Fourier series expansion in the radial and circumferential directions, respectively; iii) a 2D interpolation process to find the Green's functions in the (x, y, ω) domain; and iv) a numerical FT along x to obtain the 2.5D Green's functions in the (k_x, y, ω) domain.

Fig. 3 shows an example of a sampling grid for computing the 2.5D Green's function with the proposed method. The y axis could be linear or logarithmic, depending on the sampling used for the FT. Interpolation is only needed along the y direction, since the required positions may not coincide with the sampling associated with the FT.

Fig. 4 is an example of a sampling grid for the method based on stiffness matrices in cylindrical coordinates. Black solid points represent the sampling associated with the direct output of the method. Grey circles correspond to the required sampling points. This example considers the same sampling along x for any required y value. The data obtained directly from this method must be translated from a cylindrical to a Cartesian sampling grid using 2D interpolation. This process will generally induce larger numerical errors than the proposed method, particularly for large values of y, due to the geometrical relationship between cylindrical and Cartesian coordinate systems.

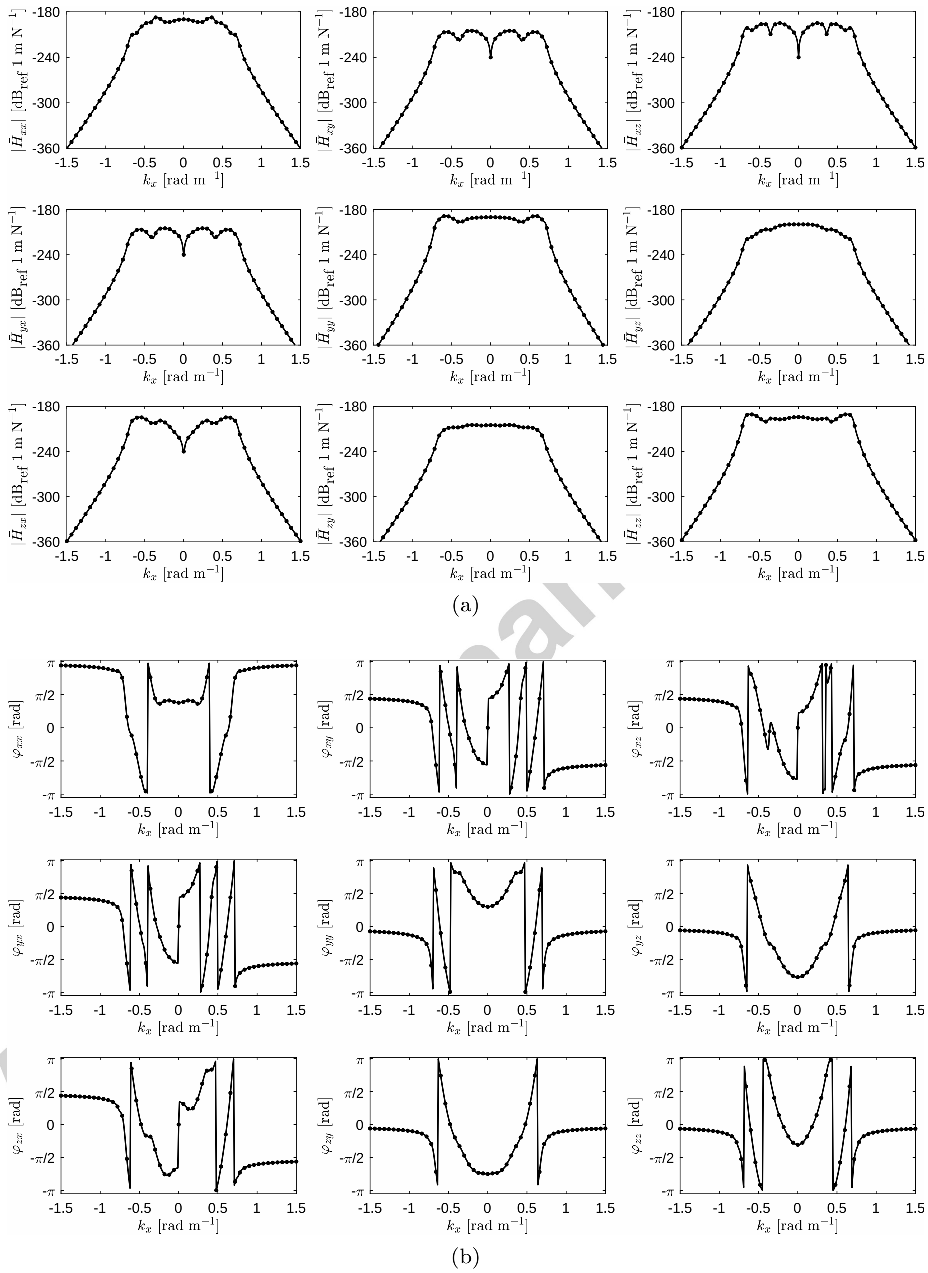

Fig. 2: Amplitude (a) and phase value (b) of the 2.5D Green's functions at 30 Hz. Solid and dotted lines are used to represent the results obtained using the stiffness matrix method in cylindrical coordinates [16] and the present method, respectively.

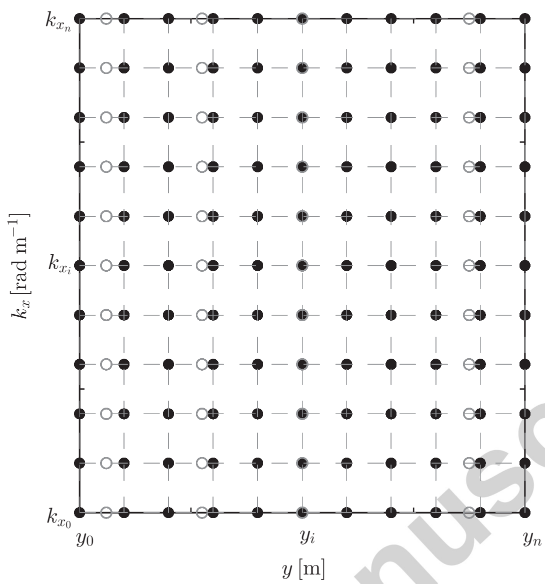

Fig. 3: Sampling grid obtained directly using the proposed method (black solid points). The required points are denoted by grey circles.

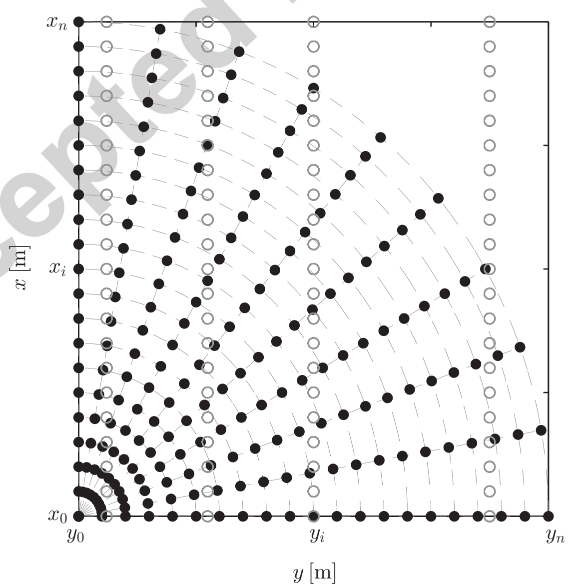

Fig. 4: Sampling grid obtained directly from the method based on the stiffness matrices in cylindrical coordinates (black solid points). The required points are denoted by grey circles.

5. Conclusions

We have put forward an extension of the stiffness matrix method that allows calculating the Green's functions for a homogeneous and layered medium when it is subjected to 2.5D loads. The method is based on explicit expressions of the 3D stiffness matrices in Cartesian coordinates defined in the (k_x, k_y, ω) domain. The 2.5D Green's functions can be derived by applying an inverse Fourier transform in the y direction on the inverse of the global stiffness matrix, which is calculated using the 3D stiffness matrices in the (k_x, k_y, ω) domain.

The results obtained through the present method are in good agreement with those obtained using the method based on stiffness matrices in cylindrical coordinates. However, the present method can be used to improve the accuracy of the required interpolations, especially for large values of y. As explained above, the present method requires fewer numerical steps to obtain the 2.5D Green's functions of the system. In general, this more streamlined process should result in more accurate results.

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