A new general family of mixed graphs

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Abstract

A new general family of mixed graphs is presented, which generalizes both the pancake graphs and the cycle prefix digraphs. The obtained graphs are vertex transitive and, for some values of the parameters, they constitute the best infinite families with asymptotically optimal (or quasi-optimal) diameter for their number of vertices.

Mathematics Subject Classifications: 05C35.

Keywords: Mixed graph, pancake graph, cycle prefix digraph, degree/diameter problem, Moore bound, diameter, mean distance.

1 Preliminaries

For the description of mixed graphs, their applications, the used notation, and the theoretical background, see, for example, Nguyen and Miller [12], and Nguyen, Miller, and Gimbert [13].

A mixed graph can be seen as a type of digraph containing some edges (or two opposite arcs). A *mixed graph* G with vertex set V may contain (undirected) *edges* as well as directed edges (also known as *arcs*). From this point of view, a *graph* has all its edges undirected, whereas a *directed graph* or *digraph* has all its edges directed. In fact, we can identify the mixed graph G with its associated digraph G^* obtained by replacing all the edges by digons (that is, two opposite arcs or a directed 2-cycle). The *undirected degree* of a

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$k\diagdown d$	1	2	3	4	5
1	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$
2	$z+\mathbf{2}$	$z+\mathbf{5}$	$z+\mathbf{10}$	$z+\mathbf{17}$	$z+\mathbf{26}$
3	$2z+\mathbf{2}$	$4z+\mathbf{7}$	$6z+\mathbf{22}$	$8z+\mathbf{53}$	$10z+\mathbf{106}$
4	$z^2+z+\mathbf{2}$	$z^2+9z+\mathbf{9}$	$\boldsymbol{z}^2+22z+\mathbf{46}$	$z^2+41z+\mathbf{161}$	$z^2+66z+\mathbf{426}$
5	$2z^2+2z+\mathbf{2}$	$5z^2+16z+\mathbf{11}$	$8z^2+66z+\mathbf{94}$	$11z^2+176z+\mathbf{485}$	$14z^2+370z+\mathbf{1706}$

Table 1: The Moore bound in (1) when $r=d-z$ and $0\le z\le d$.

vertex v, denoted by $d(v)$ is the number of edges incident to v. The *out-degree* of vertex v, denoted by $d^+(v)$ is the number of arcs emanating from v. Similarly, the *in-degree* of v, denoted by $d^-(v)$, is the number of arcs going to v. If $d^+(v)=d^-(v)=z$ and $d(v)=r$, for all $v\in V$, then G is said to be *totally regular* of degree (r,z), with $r+z=d$ (or, simply, (r,z)*-regular*).

The *degree/diameter problem* for mixed graphs can be viewed as a generalization of the corresponding problem for undirected and directed graphs: Given three natural numbers r,z and k, find the largest possible number of vertices $n(r,z,k)$ in a mixed graph with maximum undirected degree r, maximum directed out-degree z and diameter k.

All versions of this problem have received special attention for their application in modeling dense networks (for example, telecommunication, multiprocessor, or local area networks). In many real-world networks there is a mixture of both unidirectional and bidirectional connections. Most of the main results can be found in the comprehensive survey of Miller and Širáň [11]. Two recent contributions on mixed graphs are due to the author, Fiol, and López, which study the sequence mixed graphs [5] and the bipartite mixed graphs [6].

Buset, Amiri, Erskine, Miller, and Pérez-Rosés [2] proved that the maximum number of vertices for a mixed graph of diameter k with maximum undirected degree r and maximum out-degree z is

$$M(r,z,k)=A\frac{u_1^{k+1}-1}{u_1-1}+B\frac{u_2^{k+1}-1}{u_2-1}, \tag{1}$$

where

$$v=(z+r)^2+2(z-r)+1, \tag{2}$$

$$u_1=\frac{z+r-1-\sqrt{v}}{2},\qquad u_2=\frac{z+r-1+\sqrt{v}}{2}, \tag{3}$$

$$A=\frac{\sqrt{v}-(z+r+1)}{2\sqrt{v}},\qquad B=\frac{\sqrt{v}+(z+r+1)}{2\sqrt{v}}.$$

This bound applies when G is totally regular with degrees (r,z). Moreover, if we

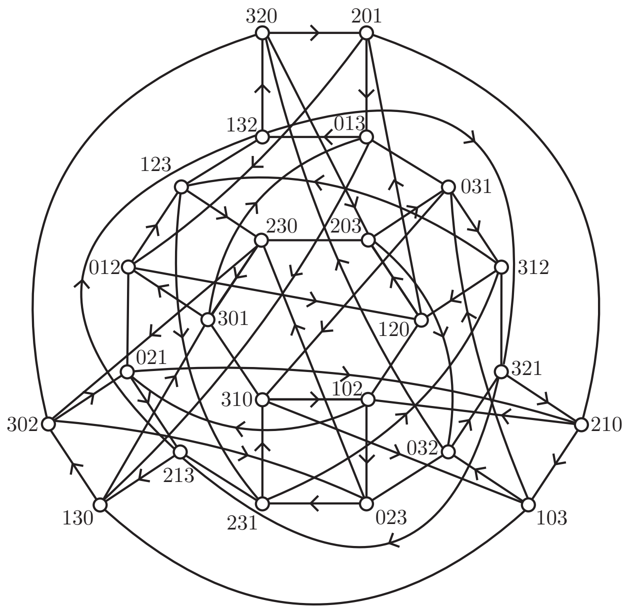

Figure 1: The cycle prefix digraph $\Gamma_3(3)$ (the lines without direction represent two arcs in opposite directions).

bound the total degree $d = r + z$, the largest number is always obtained when $r = 0$ and $z = d$. That is, when the mixed graph is, in fact, a digraph with no edges. As in the case of graphs and digraphs, the Moore bound (1) for mixed graphs is asymptotically equivalent to d^k, when $k \to \infty$. In Table 1 we show the values of (1) when $r = d - z$, with $0 \le z \le d$, for different values of d and the diameter k. In particular, when $z = 0$, the bound corresponds to the Moore bound for graphs (numbers in boldface).

In this paper, we present a new family of mixed graphs, which generalizes both the pancake graphs (when $z = 0$) and the Faber-Moore or cycle-prefix digraphs (when $r = 1$). Both are two well-known families of graphs and digraphs, respectively, which have been studied in the context of interconnection networks, and certain Gray codes. In some cases of our proposed mixed graphs, we are able to compute their exact diameters, showing that they are asymptotically optimal for their number of vertices.

1.1 The cycle prefix digraphs

For any given integers Δ and D, with $\Delta \ge D$, the *cycle prefix digraph* $\Gamma_\Delta(D)$, was proposed as a Cayley coset digraph by Faber and Moore in [8]. The digraph $\Gamma_\Delta(D)$ may be defined on an alphabet in the following way (see Comellas and Fiol [4], and Faber, Moore, and Chen [8]). The vertices are labeled with different words of length D, $x_1x_2\dots x_D$, such that they form a D-permutation of an alphabet with $\Delta+1$ symbols, say $[0, \Delta] := \{0, 1, \dots, \Delta\}$.

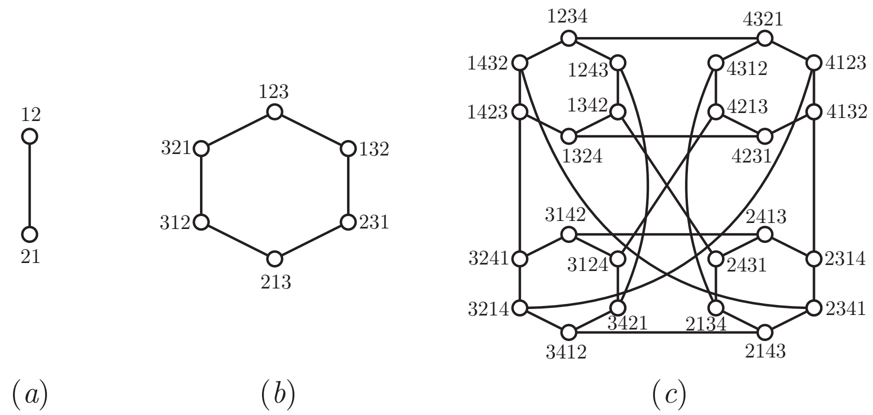

Figure 2: Pancakes graphs: (a) $P(2)$, (b) $P(3)$, and (c) $P(4)$.

The adjacencies are given by

$$x_1x_2\ldots x_D \;\rightarrow\; \begin{cases} x_2x_3x_4\ldots x_Dx_{D+1}, & x_{D+1}\neq x_1,x_2,\ldots,x_D,\\ x_2x_3x_4\ldots x_Dx_1, & \\ x_1x_3x_4\ldots x_Dx_2, & \\ x_1x_2x_4\ldots x_Dx_3, & \\ \qquad\vdots & \\ x_1x_2x_3\ldots x_Dx_{D-1}. & \end{cases} \tag{4}$$

Then, $\Gamma_\Delta(D)$ is a Δ-regular and vertex-symmetric digraph, with diameter $k=D$ and $N=\frac{(\Delta+1)!}{(\Delta+1-D)!}$ vertices. For instance, the digraph $\Gamma_3(3)$ is shown in Figure 1.

1.2 The pancake graphs

The *n-dimensional pancake graph*, proposed by Dweighter [7] (see also Akers and Krishnamuthy [1]), and denoted by $P(n)$, is a graph with the vertex set $V(P(n))=\big\{x_1x_2\ldots x_n|x_i\in\{1,2,\ldots,n\},\ x_i\neq x_j \text{ for } i\neq j\big\}$. The adjacencies are as follows:

$$x_1x_2\ldots x_n \;\sim\; \begin{cases} x_1\ldots x_{n-2}x_nx_{n-1},\\ x_1\ldots x_{n-3}x_nx_{n-1}x_{n-2},\\ x_1\ldots x_{n-4}x_nx_{n-1}x_{n-2}x_{n-3},\\ \qquad\vdots\\ x_nx_{n-1}\ldots x_2x_1. \end{cases} \tag{5}$$

The pancake graph $P(n)$ is a vertex-transitive $(n-1)$-regular graph with $n!$ vertices. The pancakes graphs $P(2)$, $P(3)$, and $P(4)$ are shown in Figure 2. Their exact diameters $k=k(n)$ are only known for $n\leq 17$, as shown in Table 2 (see Cibulka [3] and Sloane [14]).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19

Table 2: The known values of the diameter k of the pancake graph $P(n)$.

The best results to our knowledge were given by Gates and Papadimitriou [9], who proved that

$$\frac{17}{16}n \leq k(n) \leq \frac{5n+5}{3}, \tag{6}$$

and by Heydari and Sudborough [10], who improved the lower bound to

$$k(n) \geq \frac{15}{14}n.$$

2 A new family of mixed graphs

In this section we introduce a new family of mixed graphs, which generalizes both the cycle prefix digraphs and the pancake graphs. In some cases we compute their exact diameters, showing that they are asymptotically optimal for their number of vertices. See an example of these new mixed graphs in Figure 3.

Definition 1. *Given the integers $n \geq 2$ and $d, r \geq 1$, with $r < n \leq d+1$, the mixed graph $\Gamma(d,n,r)$ has as vertex set the n-permutations of the $d+1$ symbols $0,1,2,\ldots,d$. Moreover, vertex $x_1x_2\ldots x_n$ is adjacent, through edges, to the r vertices*

$$x_1x_2\ldots x_n \;\sim\; \left\{ \begin{array}{l} x_1x_2\ldots x_{n-2}x_nx_{n-1} \\ x_1x_2\ldots x_{n-3}x_nx_{n-1}x_{n-2} \\ \qquad\vdots \\ x_1\ldots x_{n-r-1}x_nx_{n-1}\ldots x_{n-r} \end{array} \right. \tag{7}$$

and adjacent, through arcs, to the $z = d-r$ vertices

$$x_1x_2\ldots x_n \;\rightarrow\; \left\{ \begin{array}{ll} x_2x_3\ldots x_ny, \quad y \neq x_i,\ i=1,\ldots,n & (d-n+1 \textit{ vertices}) \\ \left.\begin{array}{l} x_1\ldots x_{n-r-2}x_{n-r}\ldots x_nx_{n-r-1} \\ x_1\ldots x_{n-r-3}x_{n-r-1}\ldots x_nx_{n-r-2} \\ \qquad\vdots \\ x_2\ldots x_nx_1 \end{array}\right\} & (n-r-1 \textit{ vertices}). \end{array} \right. \tag{8}$$

Thus, the number of vertices of the mixed graph $\Gamma(d,n,r)$ is the number of n-permutations of $d+1$ elements:

$$N = \frac{(d+1)!}{(d+1-n)!} = (d+1)d(d-1)\cdots(d+2-n). \tag{9}$$

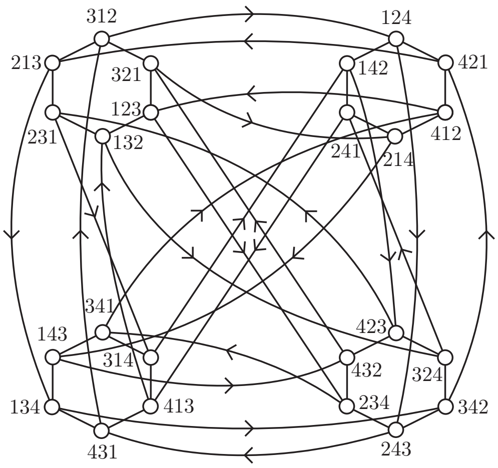

Figure 3: The new mixed graph $\Gamma(3,3,2)$.

Moreover, $\Gamma(d,n,r)$ is a totally (r,z)-regular mixed graph, it is also vertex-transitive, and it is isomorphic to its converse (obtained by changing the directions of all the arcs).

The following are some particular cases:

- For the extremal case $n = d+1$, and $r \leq d-1$, we have the isomorphism

$$\Gamma(d,d+1,r) \simeq \Gamma(d,d,r). \tag{10}$$

  Indeed, note first that (9) yields in both cases the same number of vertices (since the permutations of n symbols are in correspondence with the variations of n symbols taken $n-1$ at a time. Then, the required isomorphism is simply given by the mapping $x_1x_2\dots x_{d+1} \mapsto x_2\dots x_{d+1}$.

- If $n = d+1$ and $r = d$, then $\Gamma(n-1,n,n-1)$ is the pancake graph $P(n)$.

- If $r = 1$, then $\Gamma(d,n,1)$ coincides with the cycle prefix digraph $\Gamma_d(n)$, since, from the above, $\Gamma(d,d+1,1) \cong \Gamma(d,d,1)$. Note that, in the definition of cycle prefix digraph $\Gamma_d(n)$, we require that $d \geq n$. In particular, $\Gamma(d,2,1) = K(d,2)$, the Kautz digraph with diameter 2, which attains the Moore bound for mixed graphs with this diameter and $r = 1$.

Remark 1. *Due to the isomorphism in* (10)*, from now on, and if not otherwise stated, we assume that* $(r <)n \leq d$.

Proposition 1. *Let* $k(d,n,r)$ *be the diameter of the mixed graph* $\Gamma(d,n,r)$ *with* $r < n \leq d$. *Then,*

$$k(d,n,r) = k(d,r+1,r) + n - r - 1. \tag{11}$$

Proof. We want to show that there is a path between two generic vertices of $\Gamma(d,n,r)$, say $\boldsymbol{x} = x_1 \dots x_{n-r-1}x_{n-r} \dots x_n$ and $\boldsymbol{y} = y_1 \dots y_{r+1}y_{r+2} \dots y_n$, of length at most $k(d,r+1,r)+n-r-1$. To this end, we know that in $\Gamma(d,r+1,r)$ there is a path of length at most $k(d,r+1,r)$ from vertex $\boldsymbol{x}' = x_{n-r} \dots x_n$ to vertex $\boldsymbol{y}' = y_1 \dots y_{r+1}$. Of course, this path go, in general, through edges of type (7) and arcs of type (8). (This depends on the intersection between the sets $\{x_{n-r}, \dots, x_n\}$ and $\{y_1, \dots, y_{r+1}\}$.) Moreover, such a path gives us a path in $\Gamma(d,n,r)$, from vertex $\boldsymbol{x}$ to vertex $\boldsymbol{x}'' = x_{r+2} \dots x_n y_1 \dots y_{r+1}$. Then, since all digits $y_{r+2}, \dots, y_n$ are different from $y_1, \dots, y_{r+1}$, we can use $n-r-1$ arcs of type (8) to go from vertex $\boldsymbol{x}''$ to vertex $\boldsymbol{y}$. This gives the bound $k(d,n,r) \leq k(d,r+1,r)+n-r-1$. To prove equality, note that, in a shortest path from a vertex $\boldsymbol{x}$ to another vertex $\boldsymbol{y}$, the last $n-r-1$ steps (arcs) must be such that, starting from a vertex $\boldsymbol{x}''$ with last digits $y_1 \dots y_{r+1}$, we are adding successively (on the right) the remaining $n-r-1$ digits $y_{r+2}, y_{r+3}, \dots, y_n$. But, if in $\Gamma(d,r+1,r)$ the distance from $x_{n-r} \dots x_n$ to $y_1 \dots y_{r+1}$ equals its diameter, we need at least $k(d,r+1,r)$ steps to go from $\boldsymbol{x}$ to $\boldsymbol{x}''$ and, then, (11) holds. □

We also give an example of this proof as follows: Let us assume that, in $\Gamma(9,7,3)$ (that is, $d=9$, $n=7$, $r=3$, and $z=d-r=6$), we want to find a path from the vertex $\boldsymbol{x} = 0871234$ to the vertex $\boldsymbol{y} = 1826035$, with $\boldsymbol{x}' = 1234$ and $\boldsymbol{y}' = 1826$. From the table in Figure 5, we see that in $\Gamma(9,4,3)$ we have the following path $\boldsymbol{x}' \to \boldsymbol{y}'$:

$$\boldsymbol{x}' = 1234 \sim 1432 \sim 2341 \to 3418 \to 4182 \to 1826 = \boldsymbol{y}'.$$

Then, in $\Gamma(9,7,3)$, we have the following path $\boldsymbol{x} \to \boldsymbol{y}$:

$$\begin{aligned} \boldsymbol{x} &= 0871234 \sim 0871432 \sim 0872341 \to 0723418 \to 0734182 \to \\ & 7341826 = \boldsymbol{x}'' \to 3418260 \to 4182603 \to 1826035 = \boldsymbol{y}. \end{aligned}$$

As a consequence of this proposition, the following result gives the exact value of the diameter of $\Gamma(d,n,r)$ when $r = 1,2,3$, and an upper bound for general r.

Theorem 1. *For $r = 1,2,3$, the diameter of a mixed graph $\Gamma(d,n,r)$, with $r < n \leq d$, is:*

(*a*) $k(d,n,1) = n$,

(*b*) $k(d,n,2) = n+1$,

(*c*) $k(d,n,3) = n+1$.

Proof. (*a*) As commented before, the mixed graph $\Gamma(d,n,1)$, seen as a digraph, is isomorphic to the cycle prefix digraph $\Gamma_d(n)$, which is known to have diameter n.

(*b*) The rooted tree of Figure 4 shows that the diameter of $\Gamma(d,3,2)$ is $k=4$. Then, the result follows from Proposition 1.

(*c*) The table in Figure 5 shows that the diameter of $\Gamma(d,4,3)$ is $k=5$, and the result follows again from Proposition 1.

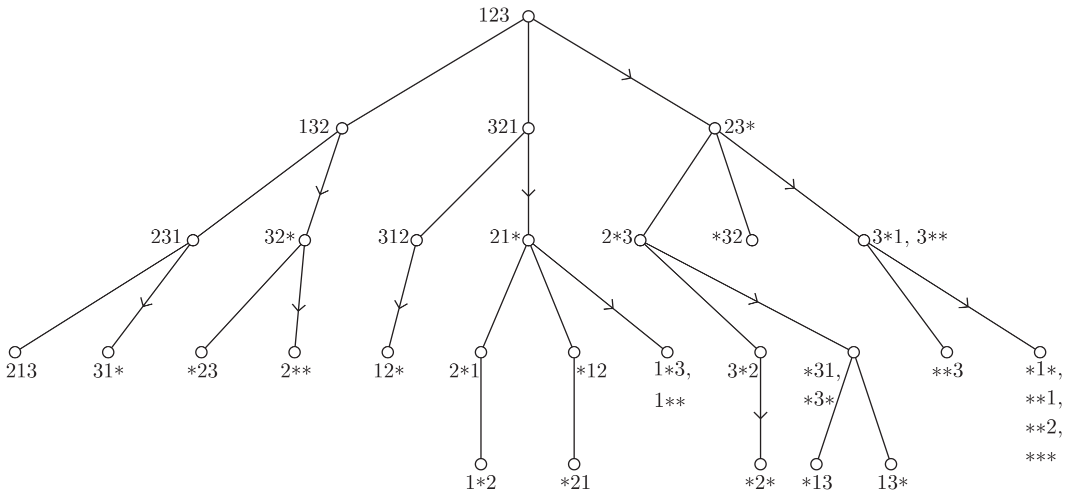

Figure 4: The rooted tree of depth four in the case $r = 2$.

$\square$

Note that the symbol '$*$' represents a character different from 1, 2, and 3 in Figure 4, and different from 1, 2, 3, and 4 in Figure 5. Moreover, if there are two symbols $*$ in a vertex, they are different. With the symbol '-', we mean that these vertices have been already reached in a previous step. For instance, in Figure 5 we see that vertex 3421 (at distance 2 from 1234) is adjacent to the vertices 3412, 3124, and 421$*$ (at distance 3 from 1234).

From this theorem we have the following result.

Corollary 1. (a) *The mixed graphs* $\Gamma(d,n,1) = \Gamma_d(n)$*, with* $r < n \leq d$*, is a family of* $(1,z)$*-regular mixed graphs with number of vertices of the same (optimal) order* $O(d^k)$ *as the Moore bound in* (1) *for large values of* d*.*

(b) *The mixed graphs* $\Gamma(d,n,r)$*, for* $r = 2,3$ *and* $r < n \leq d$*, are families of* (r,z)*-regular mixed graphs with number of vertices of the (quasi-optimal) order* $O(d^{k-1})$ *for large values of* d*.*

Until now, only an infinite family of type (a) was cited in the literature. Namely, the case of Kautz digraphs $\Gamma(d,2,1) = K(d,2)$ with diameter $k = 2$.

2.1 The case $z = 1$

Let us now consider the particular case when the directed degree is just $z = 1$. Then, the undirected degree is $r = d - 1$ and $n = d + 1 (= r + 2)$ (there are no adjacencies of the first class in (8)). In other words, the vertex set of the mixed graph $\Gamma(n-1, n, n-2)$ is the set the permutations of n elements (as the pancake graph), and adjacencies given by $n-2$ suffix reversals (edges), and one cyclic permutation (arcs). Alternatively, the isomorphism in (10) yields $\Gamma(n-1,n,n-2)\cong\Gamma(n-1,n-1,n-2)$, so that that the same graph is obtained by taking equal values for z and d with $n=r+1$ (there are no adjacencies of the second class in (8)). For instance, for $n=5$, the table (with 209 classes of vertices) in Figure 5 shows the vertices at a given distance (from 0 to 5) from 51234 in $\Gamma(4,5,3)\simeq\Gamma(4,4,3)$. For example, to go from 51234 to 34512, we look at the table the path

$$1234\sim 1243\sim 3421\rightarrow 421{*}\sim 4{*}12,$$

which corresponds to the path (with $*=5$, and adding the missing digit on the left)

$$51234\sim 51243\sim 53421\rightarrow 34215\sim 34512.$$

So, the diameter of this mixed graph is $D=5$.

For $z=1$ we have the following general result (recall that $k(n)$ denotes the diameter of the pancake graph $P(n)$).

Proposition 2. *The diameter D of the mixed graph $\Gamma(n-1,n-1,n-2)$ satisfies*

$$D\le k(n-1)+2. \tag{12}$$

Proof. By convenience, we use the isomorphism (10), that is, $\Gamma(n-1,n-1,n-2)\simeq\Gamma(n-1,n,n-2)$. Let us show a path from vertex $12\dots n$ to a generic vertex $\boldsymbol{x}=x_1x_2\dots x_n$. There are two cases:

- If $x_1=1$ we only need to consider a shortest path from $23\dots n$ to $x_2x_3\dots x_n$ in $P(n-1)$, which gives $D=k(n-1)$.
- Otherwise, vertex $\boldsymbol{x}$ is of the form $x_1\boldsymbol{A}1\boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ represent some (possibly void) sequences of a given length. Then, from vertex $12\dots n$ we can go, with a maximum of $k(n-1)$ steps (edges), to $1x_1\boldsymbol{A}\overline{\boldsymbol{B}}$ (where $\overline{\boldsymbol{B}}$ denotes the corresponding reversed sequence). Now, using the cyclic permutation (arc) we reach the vertex $x_1\boldsymbol{A}\overline{\boldsymbol{B}}1$ and, finally, with one more step (edge), we reach $\boldsymbol{x}=x_1\boldsymbol{A}1\boldsymbol{B}$. Schematically,

  $$12\dots n\quad\overset{k(n-1)}{\sim}\quad 1x_1\boldsymbol{A}\overline{\boldsymbol{B}}\quad\overset{1}{\rightarrow}\quad x_1\boldsymbol{A}\overline{\boldsymbol{B}}1\quad\overset{1}{\sim}\quad x_1\boldsymbol{A}1\boldsymbol{B}.$$

  This makes the upper bound in (12).

□

We give an example of this proof as follows. Let us assume that $n=5$. We want to find a path in $\Gamma(n-1,n,n-2)=\Gamma(4,5,3)$ from vertex 12345 to vertex $x_1\boldsymbol{A}1\boldsymbol{B}=42135$, with $\boldsymbol{A}=2$ and $\boldsymbol{B}=35$. This path is

$$12345\sim 12354\sim 14532\sim 14235\sim 14253=1x_1\boldsymbol{A}\overline{\boldsymbol{B}}\rightarrow 42531\sim 42135=\boldsymbol{x}.$$

From this last result and the upper bound in (6), we get a new infinite family of mixed graphs $\Gamma(n-1, n-1, n-2)$ for any value of $d = n-1$ and $z = 1$, with diameter

$$D \leq \frac{5}{3}n + 2.$$

2.2 The case $z = 2$

Now we consider the case $\Gamma(d, n, r) = \Gamma(n-1, n-1, n-3)$.

Proposition 3. *The diameter D of the mixed graph $\Gamma(n-1, n-1, n-3)$ satisfies*

$$D \leq k(n-2) + 4. \tag{13}$$

Proof. Similarly to the proof of Proposition 2, we consider the isomorphism $\Gamma(n-1, n-1, n-3) \simeq \Gamma(n-1, n, n-3)$. We want to find a path from vertex $12\ldots n$ to $\boldsymbol{x} = x_1x_2\ldots x_n$. We only study the more complex case, that is, when $\{x_1, x_2\} \neq \{1, 2\}$. We get two possibilities:

(*a*) $\boldsymbol{x} = x_1x_2\boldsymbol{A}1\boldsymbol{B}2\boldsymbol{C}$, where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ denote (possibly void) sequences. The path is the following, where $k(n)$ denotes the diameter of the pancake graph P_n:

$$\begin{aligned} 12\ldots n &\overset{k(n-2)}{\sim} 12x_1x_2\boldsymbol{A}\boldsymbol{C}\overline{\boldsymbol{B}} \overset{1}{\to} 2x_1x_2\boldsymbol{A}\boldsymbol{C}\overline{\boldsymbol{B}}1 \overset{1}{\sim} \\ 2x_1x_2\boldsymbol{A}1\boldsymbol{B}\overline{\boldsymbol{C}} &\overset{1}{\to} x_1x_2\boldsymbol{A}\overline{\boldsymbol{C}}1\overline{\boldsymbol{B}}2 \overset{1}{\sim} x_1x_2\boldsymbol{A}2\boldsymbol{B}1\boldsymbol{C}. \end{aligned}$$

(*b*) $\boldsymbol{x} = x_1x_2\boldsymbol{A}2\boldsymbol{B}1\boldsymbol{C}$. The path is the following:

$$\begin{aligned} 12\ldots n &\overset{k(n-2)}{\sim} 12x_1x_2\boldsymbol{A}\overline{\boldsymbol{C}}\boldsymbol{B} \overset{1}{\to} 2x_1x_2\boldsymbol{A}\overline{\boldsymbol{C}}\boldsymbol{B} \overset{1}{\sim} \\ 2x_1x_2\boldsymbol{A}\overline{\boldsymbol{C}}1\boldsymbol{B} &\overset{1}{\to} x_1x_2\boldsymbol{A}1\boldsymbol{B}\overline{\boldsymbol{C}}2 \overset{1}{\sim} x_1x_2\boldsymbol{A}1\boldsymbol{B}2\boldsymbol{C}. \end{aligned}$$

□

3 Open problem

We end this paper with the following conjecture.

Conjecture 1. *The diameter of the mixed graph $\Gamma(d, n, r)$ with $r < n \leq d$ is*

$$k(d, n, r) = k(r+1) + n - r,$$

where $k(r+1)$ is the diameter of the pancake graph $P(r+1)$.

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dist=0	dist=1	dist=2	dist=3	dist=4	dist=5
1234	1243	1342	1324	4231	231*
				324*	24**
			2431	2413	413*
				431*	31**
			342*	3*24	*241
					24
				*243	-
				42*1	1*24
					2*13
				42**	**24
					2**3
		3421	3412	412*	*214
			3124	3142	142*
				124*	*421
			421*	*124	*142
		243*	24*3	3*42	*42*
				4*31	13*4
					*312
					31
				4*3*	*3*4
					*3*1
					*3*2
					*3**
			2*34	*341	*314
				34	-
			*342	*324	-
			43*1	41*3	1*3*
				1*34	-
				3*1*	-
			43**	4**3	**31
					**32
					**3*
				**34	-
				3**1	**14
				3**2	-
				3***	-
	1432	1423	3241	241*	-
			423*	42*3	2*31
					2*3*
				23*1	-
				23**	-
		2341	2314	314*	14*2
					14**
			2143	143*	-
			341*	3*14	*14*
				*143	-
				41*2	1*2*
				41**	1**2
		432*	43*2	3*21	*21*
				3*2*	*2*3
					*2*1
					*2*4
					*2**
			4*23	32*4	2*41
					2*4*
				*231	-
				23	-
			*234	-	-
			32*1	31*2	-
				1*23	-

dist=0	dist=1	dist=2	dist=3	dist=4	dist=5
				2*14	-
				2*1*	-
			32**	**23	-
				2**1	-
				2**4	-
				2***	-
	4321	4312	4213	213*	13**
			2134	134*	-
			312*	*213	-
				12*4	-
				12**	-
		4123	4132	132*	13*2
			3214	214*	-
			123*	12*3	-
				1*32	-
				*321	-
		321*	3*12	21*3	-
				12	-
			*123	*132	-
			21*4	24*1	-
				1*4*	-
			21**	-	-
				**12	-
				1**3	-
				1**4	-
				1***	-
	234*	23*4	4*32	*32*	-
			3*41	14*3	-
				*412	-
				41	-
			3*4*	*4*1	-
				*4*2	-
				*4**	-
		2*43	34*2	4*21	-
				4*2*	-
			*431	*413	-
				*134	-
			43	*4*3	-
		*432	*423	-	-
		34*1	31*4	4*13	*13*
				1*42	-
			1*43	-	-
			4*12	-	-
			4*1*	*1*4	-
				*1*2	-
				*1*3	-
				*1**	-
		34**	3**4	**41	-
				**42	-
				**4*	-
			**43	-	-
			4**1	**13	-
				**1*	-
			4**2	**21	-
				**2*	-
			4***	***4	-
				***1	-
				***2	-
				***3	-
				****	-

Figure 5: The vertices at distance $0, 1, \ldots, 5$ from vertex 1234 in $\Gamma(d, 4, 3)$.