Universality Class of Absorbing Phase Transitions with a Conserved Field

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(Received 17 March 2000)

We investigate the critical behavior of systems exhibiting a continuous absorbing phase transition in the presence of a conserved field coupled to the order parameter. The results obtained point out the existence of a new universality class of nonequilibrium phase transitions that characterizes a vast set of systems including conserved threshold transfer processes and stochastic sandpile models.

PACS numbers: 05.70.Ln, 05.50.+q, 05.65.+b

Absorbing phase transitions (APT) are a category of critical nonequilibrium phase transitions, widespread in condensed matter physics and population and epidemics modeling [1]. Directed percolation (DP) [1] has been recognized as the paradigmatic example of a system exhibiting a transition from an active to a unique absorbing phase. DP defines a precise universality class (theoretically described by the Reggeon field theory [2,3]) which is very robust with respect to the introduction of microscopic modifications. This field theory is at the heart of a strong claim of universality, summarized in the following conjecture [4]: *Continuous absorbing phase transitions to a unique absorbing state fall generically into the universality class of directed percolation*. This conjecture is expected to hold for models with short range interactions that, most importantly, do not possess additional symmetries.

Examples of APT subject to extra symmetries, and thus out of the DP class, have been identified in recent years. Among them, we find systems with symmetric absorbing states [5], models of epidemics with perfect immunization (the so-called dynamic percolation class) [6], and systems with an infinite number of absorbing states [7]. Very recently, it has been pointed out that the critical point of self-organized critical (SOC) [8,9] sandpile models can also be interpreted as a continuous phase transition with many absorbing states [10,11]. What distinguishes sandpile models from other models with absorbing states is that the control parameter, represented by the global density of particles, is a conserved quantity.

Given the large class of systems whose dynamics involves conserved fields, it becomes particularly interesting to explore in general the effect of conservation rules in APT. With this purpose in mind, in this Letter we report the critical behavior of several models showing absorbing transitions that strictly conserve the number of particles. In particular, we introduce a conserved lattice gas (CLG) [12] with short range stochastic microscopic dynamical rules that undergoes a continuous phase transition to an absorbing state at a critical value of the particle density. We have also considered a conserved threshold transfer process (CTTP) [13] and several fixed energy sandpile models with stochastic rules [9–11]. All models show critical exponents compatible with a single and broad universality class that embraces all APT in stochastic models with a conserved field. This evidence leads us to conjecture that, in the absence of additional symmetries, *absorbing phase transitions in stochastic models with infinite absorbing states and activity coupled to a nondiffusive conserved field define a unique and per se universality class* [14]. This result is relevant in the understanding of several reaction-diffusion systems, sandpile models, and activated processes which could share the same theoretical description.

The CLG model is defined on a d-dimensional hypercubic lattice. To each site i, it is associated with a binary variable n_i that assumes the values $n_i = 1$, if the site is occupied by a particle, or $n_i = 0$, if the site is empty. Double occupancy is strictly forbidden. Nearest neighbor particles repel each other via repulsive short range interactions. As a product of this interaction, at each time step the *active* particles (i.e., particles with nearest neighbors) jump into one of their empty nearest neighbor sites, selected at random. The only dynamics in the model is due to these active particles; isolated particles do not move. The dynamics can be implemented with either sequential or parallel updating. In the latter case, an exclusion principle is applied so that a particle cannot move into an occupied site. Parallel updating [2] has been used in all of our simulations. We impose periodic boundary conditions, and, since the dynamics admits neither input nor loss, the total number of particles $N = \sum_i n_i(t)$ is a conserved quantity. It is clear that the model allows an infinite number (in the thermodynamic limit) of absorbing configurations, in which there are no active particles.

In the CLG model, the constant particle density $n = N/L^d$ acts as a tuning parameter. Initial conditions are generated by placing, at random in the lattice, nL^d particles. For small densities, the system will very likely fall into an absorbing configuration with only isolated particles. For large densities, the system reaches a stationary active state with everlasting activity. We shall see in the following that, as we vary n, the CLG model exhibits a continuous transition separating an absorbing phase from an active phase. The phase transition occurs at a nontrivial density n_c. APT are characterized by the order parameter ρ_a measuring the density of dynamical entities, in our case the density of active particles. The order parameter is null for $n < n_c$, and follows a power law $\rho_a \sim (n - n_c)^\beta$ for $n > n_c$. The system correlation length ξ and time τ both diverge as $n \to n_c^+$. In the critical region the system is characterized by power law behavior, namely, $\xi \sim (n - n_c)^{-\nu_\perp}$ and $\tau \sim (n - n_c)^{-\nu_\parallel}$. The dynamical critical exponent is defined as $\tau \sim \xi^z$, with $z = \nu_\parallel/\nu_\perp$.

In order to study the critical point of the CLG model, we performed numerical simulations in $d = 2$ for systems with size ranging from $L = 64$ to $L = 512$, averaging over 10^4–10^5 independent initial configurations. Very close to the critical point we have $\xi \gg L$, so that the actual characteristic length of the system is the lattice size L. Because of its finite size, the system will sometimes enter an absorbing configuration even for values of n in the supercritical region. It is then convenient to introduce averages over a set of independent trials and calculate the quasistationary properties in the active phase from a restricted average over surviving trials with nonzero final activity.

As shown in Fig. 1, after a transient which depends on the system size L and $\Delta n \equiv n - n_c$, the surviving average of the density of active sites reaches a stationary state $\rho_a(\Delta n, L)$. Close to the critical point, the finite size scaling ansatz tells us that all quantities depend on the system size through the ratio L/ξ, and the order parameter follows the finite size scaling form [15],

$$\rho_a(\Delta n, L) = L^{-\beta/\nu_\perp} \mathcal{G}(L^{1/\nu_\perp} \Delta n), \qquad (1)$$

where $\mathcal{G}$ is a scaling function with $\mathcal{G}(x) \sim x^\beta$ for large x. For $\Delta n = 0$ the stationary density follows the pure power law behavior $\rho_a \sim L^{-\beta/\nu_\perp}$. On the other hand, for values of n in the supercritical regime ρ_a should be independent of L for $L \gg \xi$, while in the subcritical regime ρ_a should decay faster than a power law. This allows us to locate the critical value n_c of the particle density as the only value of n at which we recover a nontrivial power law scaling for the density of active sites. In Fig. 1 we observe power law scaling for $n = 0.23875$, but clearly not for 0.2387 or 0.2388, indicating that $n_c = 0.23875(5)$ (numbers in parentheses indicate the statistical uncertainty in the last digit). From the power law decay we find the exponent ratio $\beta/\nu_\perp = 0.81(3)$. An independent estimate of the exponent β can be obtained by looking at the scaling of the active-site density with respect to Δn for the size $L = 320$. The resulting power law behavior yields $\beta = 0.63(1)$, where the error is mainly due to the uncertainty in the critical point n_c. A consistency test can be performed by considering the active-site density away from the critical point. In Fig. 2 we plot $\rho_a(\Delta n, L)L^{\beta/\nu_\perp}$ versus $\Delta n L^{1/\nu_\perp}$ for $\nu_\perp = 0.78$, $\beta/\nu_\perp = 0.81$, and $n_c = 0.23875$. As one would expect, all of the data collapse onto a single curve, following the scaling form of Eq. (1). A further check is provided by the direct fitting of the large x behavior of the scaling function $\mathcal{G}(x)$ that gives $\beta = 0.63$, recovering the independent measurement at $L = 320$.

To determine the dynamical exponents we turn our attention to the scaling properties of time dependent quantities. In particular, we can define a characteristic time by studying the decay of the probability $P(t)$ that a random initial configuration has survived up to time t. At the critical point this probability decays as a power law. However, because of finite size effects, at large times we have $P(t) \sim \exp(-t/\tau)$, where the only characteristic length is the system size L, and we have $\tau(L) \sim L^z$. We can access the value of $\tau(L)$ by a direct fitting of the $P(t)$ exponential tail; z is then estimated from the behavior of $\tau(L)$ for different L. Again, a clean power law behavior is obtained for $n_c = 0.23875$, yielding $z = 1.52(6)$. Also in

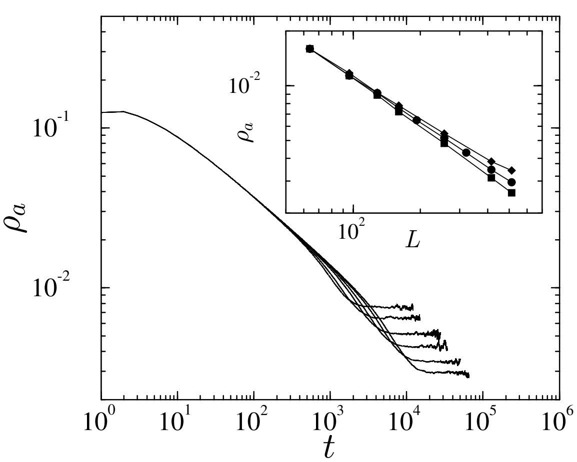

FIG. 1. Active-site density in surviving trials at the critical point for the CLG. From top to bottom, the system sizes are $L = 160, 192, 256, 320, 416$, and 512. Inset: stationarity active-site density as a function of L for a critical (center), subcritical (bottom), and supercritial (top) particle density.

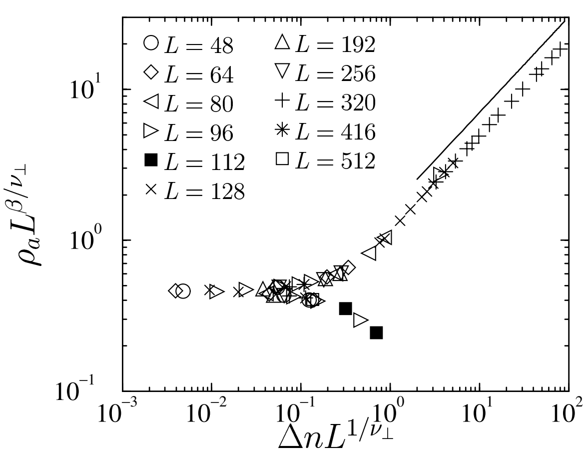

FIG. 2. Collapse of the stationary density $\rho_a(\Delta n, L)L^{\beta/\nu_\perp}$ as a function of $\Delta n L^{1/\nu_\perp}$ for the CLG. The slope of the line is 0.63.

this case a consistency check can be performed by studying the time decay of the active-site density $\rho_{a,\mathrm{all}}(t)$, averaging over all trials, even those that have reached an absorbing state. Assuming a single characteristic time scaling as L^z, we have, at $\Delta n = 0$, $\rho_{a,\mathrm{all}}(t, L) = t^{-\theta}\mathcal{F}(tL^{-z})$ [15], where $\mathcal{F}(x)$ is a constant for $x \ll 1$, and decays faster than any power law for $x \gg 1$. Data from simulations with different L can be collapsed onto a universal curve by plotting $\rho_{a,\mathrm{all}}(t, L)t^{\theta}$ versus tL^{-z}. The best collapse is obtained for $\theta = 0.43$ and $z = 1.52$, confirming the value obtained for the dynamical critical exponent. The exponent $\theta = 0.43(1)$ is recovered also from a direct fitting of the decay of the stationary density averaged over surviving trials (see Fig. 1). In usual APT, the latter exponent obeys $\theta = \beta/\nu_{\parallel}$. This relation assumes a standard scaling behavior at $\Delta n = 0$ for $\rho_a(t)$. In our model, however, the simple scaling behavior is broken by an anomalous scaling regime (visible in Fig. 1 by the sharp drop just before the stationary state) that seems to grow steeper with increasing L. It follows that data collapse in time is not achievable with standard scaling forms, and that θ violates the usual scaling relation. Albeit its origin is not clear, it is noteworthy that this anomaly is common to all APT with conserved fields inspected so far [10,11], irrespective of the updating rules employed, either parallel or sequential.

In APT it is possible to obtain more information on the critical properties by studying the evolution (spread) of activity in systems which start close to an absorbing configuration [2]. In each *spreading* simulation, a small perturbation is added to an absorbing configuration. It is then possible to measure the spatially integrated activity $N(t)$, averaged over all runs, and the survival probability $P(t)$ of the activity after t time steps. We have power law behavior for these magnitudes only at the critical point.

Here, we will follow a procedure equivalent to the definition of slowly driven simulations in sandpiles that enlightens the connections with these models. Instead of fixing the density n by working with periodic boundary conditions, and thus studying the system at a given distance below the critical point, we impose open boundary conditions and start each spreading experiment by adding a new particle. Under these conditions, the system flows to a stationary state with balance between the input of particles and the boundary dissipation. In the limit in which the particle addition is infinitely slow with respect to the spreading of activity, the system reaches a critical state with density n_c (in the thermodynamic limit) [16]. The infinitely slow drive is implemented by adding a new active particle only when the system falls into an absorbing configuration. The system thus jumps between absorbing states via avalanchelike rearrangements, and we can associate each spreading experiment with an avalanche. The probability distribution $P_s(s)$ of having a spreading event involving s sites, as well as the quantities $N(t)$ and $P(t)$, can be measured. The only characteristic length is the system size L, and we can write the scaling forms $N(t) = t^{\eta}\mathcal{F}_1(t/L^z)$, $P(t) = t^{-\delta}\mathcal{F}_2(t/L^z)$, and $P_s(s) = s^{-\tau_s}\mathcal{F}_3(s/L^D)$ [2], where the scaling functions $\mathcal{F}_i$ are decreasing exponentially for $x \gg 1$, and we have considered that the spreading characteristic time and size are scaling as L^z and L^D, respectively. Simulations were performed for a system of size between $L = 64$ and $L = 1024$, averaging over at least 5×10^6 spreading experiments. The extrapolation of the measured densities at infinite L yields a critical density $n_c = 0.2388(1)$, in perfect agreement with steady state simulations. The scaling exponents are measured using the now-standard moment analysis technique [17,18]. The resulting exponents are summarized in Table I. In particular, the dynamical exponent $z = 1.53(2)$ is in excellent agreement with the stationary state simulations, confirming the presence of a single critical behavior for both cases. It is interesting to note that the exponents of the model fulfill all scaling relations in standard APT. In respect to hyperscaling relations, some of them, such as $D = d + z - \beta/\nu_{\perp}$, are fulfilled, while others, such as $\eta + \delta + \theta = d/z$, are not. This is again due to the θ exponent anomaly.

In order to provide further evidence for the existence of a general universality class, we have simulated several other models exhibiting an APT in the presence of a conserved field. The first is a conserved threshold transfer process. In the CTTP, the sites of a lattice can be vacant, singly occupied, or doubly occupied by particles, corresponding to a dynamic variable $n_i = 0, 1$, or 2, respectively. Values $n_i > 2$ are strictly forbidden. Dynamics affects only doubly occupied (active) sites: every active site tries to transfer both of its particles to randomly selected nearest neighbors with $n_j < 2$. Singly occupied sites are, on the other hand, inert. The total number of particles $N = \sum_i n_i$ is thus constant in time. Results from simulations are obtained along the lines shown for the CLG and are reported in Table I; in this case, the largest sizes used are $L = 512$

TABLE I. Critical exponents for spreading and steady state experiments. Numbers in parentheses indicate the statistical uncertainty in the last digit. Steady state Manna exponents are from Ref. [11]. DP exponents are from Ref. [19].

	Steady state exponents			
	β	$\beta/\nu_{\perp}$	z	θ
CLG	0.63(1)	0.81(3)	1.52(6)	0.43(1)
CTTP	0.64(1)	0.78(3)	1.55(5)	0.43(1)
Manna	0.64(1)	0.78(2)	1.57(4)	0.42(1)
DP	0.583(4)	0.80(1)	1.766(2)	0.451(1)

	Spreading exponents				
	τ_s	D	z	η	δ
CLG	1.29(1)	2.75(1)	1.53(2)	0.29(1)	0.49(1)
CTTP	1.28(1)	2.76(1)	1.54(1)	0.30(3)	0.49(1)
Manna	1.28(1)	2.76(1)	1.55(1)	0.30(3)	0.48(2)
DP	1.268(1)	2.968(1)	1.766(2)	0.230(1)	0.451(1)

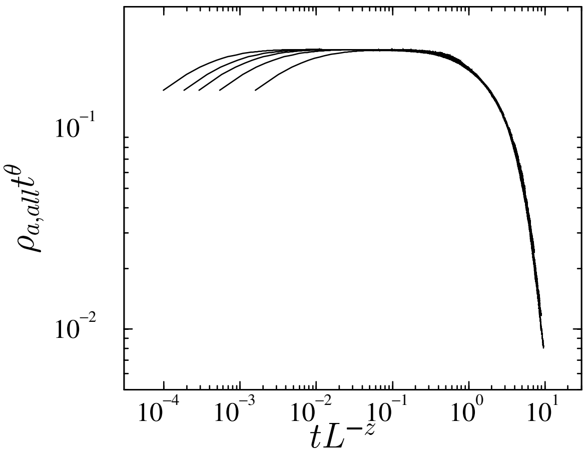

FIG. 3. Collapse of the active-site density $\rho_{a,\text{all}}(t,L)t^{\theta}$ as a function of tL^{-z} for the CTTP. System sizes range from $L = 64$ to $L = 384$.

for the stationary exponents and $L = 1024$ for the spreading exponents. As an example of our simulations, in Fig. 3 we plot $\rho_{a,\text{all}}(t,L)t^{\theta}$ as a function of tL^{-z} at the critical point, which shows remarkable data collapse. We have also investigated the Manna sandpile model [9] and its variations by the inclusion of a stochastic threshold [20]. In this case, an absorbing phase transition is obtained by using periodic boundary conditions and a fixed number of sand grains (energy), as reported in [10,11]. All models lead invariably to the same universality class as the Manna model [21] (complete results on these models will be reported elsewhere).

Our results provide striking evidence for a common critical behavior which is incompatible with the DP universality class (see Table I) [22]. More noticeably, the models also share the same scaling anomaly in the exponent θ [11], signaling a common behavior in the transient regime to the stationary state in the active phase. This uniformity of results confirms the hypothesis of a unique universality class for all models with the same conservation symmetry, and lead us to conjecture that, in the absence of additional symmetries, *absorbing phase transitions in systems with stochastic dynamics in which the order parameter is locally coupled to a nondiffusive conserved field define a single and new universality class*. This conjecture is supported by noticing that the sandpile model has the same structure and basic symmetries of the present CLG, once the field $n(x,t)$ is replaced by the local energy (sand grains) field [10,11]. Indeed, in all models presented here, a conserved noncritical field is dynamically coupled to the nonconserved order parameter field ρ_a. Very likely this basic structure will be reflected in a unique theoretical description (a field theory with the same relevant terms and symmetries) that accounts for the shared critical properties of these models.

This work has been supported by the European Network under Contract No. ERBFMRXCT980183. We thank D. Dhar, R. Dickman, P. Grassberger, M. A. Muñoz, and S. Zapperi for helpful comments and discussions.

[1] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, UK, 1999).
[2] P. Grassberger and A. de la Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).
[3] J.L. Cardy and R.L. Sugar, J. Phys. A **13**, L423 (1980); H.K. Janssen, Z. Phys. B **42**, 151 (1981).
[4] P. Grassberger, Z. Phys. B **47**, 365 (1982).
[5] J.L. Cardy and U.C. Täuber, Phys. Rev. Lett. **77**, 4780 (1996).
[6] P. Grassberger, Math. Biosci. **63**, 157 (1982); H.K. Janssen, Z. Phys. B **58**, 311 (1985).
[7] E.V. Albano, J. Phys. A **25**, 2557 (1992); I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993).
[8] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); H.J. Jensen, *Self-Organized Criticality* (Cambridge University Press, Cambridge, UK, 1998).
[9] S.S. Manna, J. Phys. A **24**, L363 (1991); D. Dhar, Physica (Amsterdam) **263A**, 4 (1999).
[10] R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998); A. Vespignani, R. Dickman, M.A. Muñoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998).
[11] A. Vespignani, R. Dickman, M.A. Muñoz, and S. Zapperi, cond-mat/0003285.
[12] The CLG can be considered the stochastic variant of a lattice gas introduced in the study of $1/f$ noise in driven systems [H.J. Jensen, Phys. Rev. Lett. **64**, 3103 (1990)].
[13] The CTTP is the conserved counterpart of the threshold transfer process introduced by J.F.F. Mendes, R. Dickman, M. Henkel, and M.C. Marques [J. Phys. A **27**, 3019 (1994)].
[14] Absorbing phase transitions coupled to a diffusive conserved field belong to a different universality class [F. van Wijland, K. Oerding, and H.J. Hilhorst, Physica (Amsterdam) **251A**, 179 (1998)].
[15] I. Jensen and R. Dickman, Phys. Rev. E **48**, 1710 (1993).
[16] R. Dickman, M.A. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000).
[17] M. De Menech, A.L. Stella, and C. Tebaldi, Phys. Rev. E **58**, R2677 (1998); C. Tebaldi, M. De Menech, and A.L. Stella, Phys. Rev. Lett **83**, 3952 (1999).
[18] A. Chessa, A. Vespignani, and S. Zapperi, Comput. Phys. Commun. **121–122**, 299 (1999); S. Lübeck, Phys. Rev. E **61**, 204 (2000).
[19] M.A. Muñoz, R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **59**, 6175 (1999).
[20] K. Christensen, A. Corral, V. Frette, J. Feder, and T. Jøssang, Phys. Rev. Lett. **77**, 107 (1996).
[21] It is worth noticing that the Bak, Tang, and Wiesenfeld sandpile model [8] has deterministic dynamics and does not belong to this universality class (see Ref. [11]).
[22] Preliminary results for the present universality class in $d = 1$ are reported in Ref. [16]. Noticeably, the CLG model in $d = 1$ with sequential updating has been analytically solved by D. Dhar (private communication).