UP-TO-HOMOTOPY ALGEBRAS WITH STRICT UNITS

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Abstract. We prove the existence of minimal models à la Sullivan for operads with non trivial arity zero. So up-to-homotopy algebras with strict units are just operad algebras over these minimal models. As an application we give another proof of the formality of the *unitary* n-little disks operad over the rationals.

Contents

1. Introduction

1.1. In the beginning, in Stasheff's seminal papers [Sta63], A_∞-spaces (algebras) had points (units) in what was subsequently termed the zero arity of the operad $\mathcal{A}ss$.[1] Stasheff called them *degenerations*. After that, points or units disappeared and for a while people working with operads assumed as a starting point $P(0) = \emptyset$, in the topological setting, or $P(0) = 0$ in the algebraic one: see for instance [GK94]. This may have been caused because of the problems posed by those points (units), including

(1) [Hin03] had to correct his paper [Hin97] about the existence of a model structure in the category of operads of complexes over an arbitrary commutative ring, excluding the arity zero of the operads—or considering just the case of characteristic zero.
(2) [Bur18] explains how the bar construction of a dg associative algebra with unit is homotopy equivalent to the trivial coalgebra, thus destroying the usual bar-cobar construction through which one usually builds minimal models for operads in the Koszul duality theory.
(3) [Mar96] (see also [MSS02]) constructs minimal models for operads of chain complexes over a field of zero characteristic, carefully excluding operads with non-trivial arity zero, which allows him to implicitly replace the somewhat "wild" general free operad $\Gamma(M)$ for the tamer one that we denote by $\Gamma_{01}(M)$.

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[1] Therefore, in our notation, we should write it as $u\mathcal{A}ss$, or $\mathcal{A}ss_+$: the *unitary* associative operad.

More recently, the situation changed and people have turned their efforts to problems involving non-trivial arity zero. In the topological context, we have the works [MT14], or [FTW18], for instance. In the algebraic context we can mention [FOOO09a], [FOOO09b], [Pos11], [Lyu11], [HM12], [Bur18]... And coping with both, [Mur16], or [Fre17a] and [Fre17b].

In introducing points (units) back in the theory of up-to-homotopy things, there are two main possibilities: either you consider *strict* ones, as in Stasheff's original papers [Sta63], or in [Fre17a], [Fre17b], [FTW18], [Bur18], or you consider *up-to-homotopy* ones, or other relaxed versions of them: [FOOO09b], [Pos11], [Lyu11], [HM12], [MT14]... Or you can do both: [KS09].

In this paper, we work in the algebraic and strict part of the subject. The contribution we add to the present panorama is to prove the existence of minimal models à la Sullivan P_∞ for operads P on cochain complexes over a characteristic zero field $\mathbf{k}$, with non-trivial arity zero in cohomology, $HP(0) = \mathbf{k}$. In doing so, we extend the works of Markl [Mar96], [Mar04] (see also [MSS02]) which proved the existence of such models for non-unitary operads, $P(0) = 0$. Our models include the one of [Bur18] for the unitary associative operad $\mathcal{A}ss_+ = u\mathcal{A}ss$. More precisely, our main result says:

Theorem 5.3. Every cohomologically connected operad $P \in \mathbf{Op}, HP(1) = \mathbf{k}$, and cohomologically non-unitary, $HP(0) = 0$ (resp., cohomologically unitary, $HP(0) = \mathbf{k}$) has a Sullivan minimal model $P_\infty \xrightarrow{\sim} P$. This minimal model is connected $P_\infty(1) = \mathbf{k}$ and non-unitary $P_\infty(0) = 0$ (resp., unitary, $P_\infty(0) = \mathbf{k}$).

In the non-unitary case, the importance of such minimal models is well-known. They provide, for instance, a *strictification* of up-to-homotopy algebras. That is, for an operad P (with hypotheses), up-to-homotopy P-algebras are the same as strict, regular P_∞-algebras. We show how A_∞-algebras with strict units are exactly $(\mathcal{A}ss_+)_\infty = su\mathcal{A}ss_\infty$-algebras too.

As an application too, we offer another proof of the formality of the *unitary* n-little disks operad $\mathcal{D}_{n+}$ over the rationals. This fills the gap in our paper [GNPR05] noticed by Willwacher in his speech at the 2018 Rio International Congress of Mathematicians [Wil18].

1.2. Markl's mimicking of the Sullivan's original algorithm for dg commutative algebras to non-unitary operads relies on the fact that, when restricted to operads which are non-unitary $P(0) = 0$[2] and cohomologically connected $HP(1) = \mathbf{k}$, their minimal model is a free graded operad $P_\infty = \Gamma(M)$ over a Σ-module M which is trivial in arities 0 and 1, $M(0) = M(1) = 0$.

In this situation, the free graded operad $\Gamma(M)$ has tamer behavior than the "wild" general one. We call it $\Gamma_{01}(M)$ and we prove for it lemma 3.4, which allows the Sullivan algorithm to work inductively on the arity of the operads. The precise statement, also containing our unitary case, is the following:

Lemma 3.4. For every module M with $M(0) = M(1) = 0$, and every homogeneous module E of arity $p > 1$, Γ_{01} and Γ_{+1} verify:

(a)

$$\Gamma_{01}(M)(l) = \begin{cases} 0, & \text{if } l = 0, \\ \mathbf{k}, & \text{if } l = 1, \\ \Gamma(M)(l), & \text{if } l \neq 0, 1 \end{cases} \qquad \text{and} \qquad \Gamma_{+1}(M)(l) = \begin{cases} \mathbf{k}, & \text{if } l = 0, \\ \mathbf{k}, & \text{if } l = 1, \\ \Gamma(M)(l), & \text{if } l \neq 0, 1\ . \end{cases}$$

(b) For $\Gamma = \Gamma_{01}, \Gamma_{+1}$,

$$\Gamma(M \oplus E)(l) = \begin{cases} \Gamma(M)(l), & \text{if } l < p, \\ \Gamma(M)(p) \oplus E, & \text{if } l = p. \end{cases}$$

[2] In fact, we show how there is only the need to assume *cohomologically* non-unitary operads, $HP(0) = 0$, in his case.

Part (a) of the lemma just says that the minimal models P_∞ we are going to construct for cohomologically non-unitary $HP(0) = 0$ (resp. unitary, $HP(0) = \mathbf{k}$), and cohomologically connected $HP(1) = \mathbf{k}$ will be non unitary $P_\infty(0) = 0$ (resp., unitary $P_\infty(0) = \mathbf{k}$) and connected, $P_\infty(1) = \mathbf{k}$.

The possibility of doing the Sullivan algorithm arity-wise relies on this part (b), which shows that, under these restrictions, the new generators E you add in arity p don't produce anything other than themselves in the arity p of the free operad $\Gamma(M \oplus E)$ and don't change what you had in previous arities. In case $M(0)$ or $M(1)$ were non-trivial, the situation would be much more involved. This was clearly the situation in Markl's case.

Now the point is that, if we want to construct the minimal model à la Sullivan for cohomologically unitary and cohomologically connected operads $HP(0) = HP(1) = \mathbf{k}$, *keeping the units strict*, we can also assume that the generating module M also has trivial arities 0 and 1. This possibility has been recently made feasible thanks to Fresse's Λ-modules and operads, [Fre17a].

We recall the definitions of Λ modules and operads in section 2, but to put it succinctly, we strip out of the operad all the structure carried by the elements of $P(0)$ and add it to the underlying category of Σ-modules. For instance, the action of a unit $1 \in \mathbf{k} = P(0)$ on a an arbitrary element $\omega \in P(m)$, $\omega \mapsto \omega \circ_i 1 \in P(m-1)$ becomes part of the structure of the underlying module as a *restriction operation* $\delta^i : P(m) \longrightarrow P(m-1)$. The enhanced category of Σ-modules with these operations is the category of Λ-modules, and the free operad Γ_{+1} of our lemma 3.4 is the left adjoint of the forgetful functor from operads to Λ-modules.

Notice that, as a consequence, the Λ-structure, or which is the same, the action of the units, becomes fixed and is inherited by the free operad Γ_{+1}. As a consequence, the units of our minimal models and their algebras are strict: up-to-homotopy units are not included in them.

1.3. As with our paper [CR19], a comparison with the minimal models of operads obtained thanks to the *curved* Koszul duality [Bur18], [HM12] might be in order. Of course, since both share the property of being minimal, they must give isomorphic models when applied to the same operads. Nevertheless, let us point out a slight advantage of our approach: in order to construct the minimal model of an operad P through the Sullivan algorithm, P does *not* need to fulfill any Koszul duality, curved or otherwise; not even to be quadratic. You just need the simpler conditions on its cohomology $HP(0) \in \{0, \mathbf{k}\}$ and $HP(1) = \mathbf{k}$.

1.4. The contents of the paper are as follows. In section two, we recall some general definitions and facts about Σ and Λ modules and operads. Section three does the same with trees, free operads and the two particular instances of them we use in the present paper. Here we prove lemma 3.4, which allows the Sullivan algorithm to work arity-wise in both cases that are studied in this paper, non-unitary and unitary ones. Section four contains the basic homotopy theory of operads we need: extensions and their cofibrant properties, and homotopies between morphisms of operads. The results are well known, at least in the non-unitary case (see [MSS02]). Here we check that everything works also in the unitary case. Section five is devoted to the proof of our main results: the existence and uniqueness of minimal models for dg operads in the non-unitary and unitary case. We show how, once we choose the right free operad, the proof is formally the same in both cases. In section five we prove the aforementioned formality result and check different issues rised by our main results, namely, the relationships between: (1) the minimal model of a unitary operad P_+ and the one of its non-unitary truncation P; (2) the minimal model of a unitary operad P_+ and up-to-homotopy P-algebras with strict units and (3) the minimal models of the unitary associative operad $u\mathcal{A}ss$ *with up-to-homotopy units* $hu\mathcal{A}ss_\infty$ and ours *with strict units* $su\mathcal{A}ss_\infty$, giving greater accuracy to a remark in [HM12] about the latter not being cofibrant.

2. Notations and conventions

2.1. Throughout this paper, $\mathbf{k}$ denotes a field of zero characteristic.

Except for a brief appearance of the little disks operad at the end of the paper, all of our operads live in two categories: $\mathcal{C} = \mathbf{dgVect_k}$, or $\mathcal{C} = \mathbf{gVect_k}$, the categories or *dg vector spaces* (also, cochain complexes, differential of degree $+1$) and *graded vector spaces*, over $\mathbf{k}$. If necessary, we will use the notation $\mathbf{\Sigma Mod}^{\mathcal{C}}, \mathbf{Op}^{\mathcal{C}} \dots$ for the categories of Σ-modules and operads with coefficients in $\mathcal{C}$; otherwise, we will omit $\mathcal{C}$ everywhere. Alternatively, we will call their objects *dg operads* and *graded operads*, respectively.

We denote by 0 the initial object of $\mathcal{C}$ and also by $\mathbf{k}$ the unit object of the standard tensor product. $1 \in \mathbf{k}$ denotes the unit of the field $\mathbf{k}$ and id denotes the identity of an object in any category.

2.2. Let $C \in \mathcal{C}$ be dg vector space or a graded space. If $c \in C^n$, we will say that c has *degree* n and note it as $|c| = n$.

A morphism of complexes $\varphi : C \longrightarrow D$ is a *quasi-isomorphism*, *quis* for short, if it induces an isomorphism in cohomology $\varphi_* = H\varphi : HC \longrightarrow HD$. Given a morphism $\varphi : C \longrightarrow D$ of complexes, we denote by $C\varphi$ the *cone of* φ. This is the cochain complex given by $C\varphi^n = C^{n+1} \oplus D^n$ with differential

$$\begin{pmatrix} -\partial_C & 0 \\ -\varphi & \partial_D \end{pmatrix} .$$

We will also denote by $ZC\varphi$, $BC\varphi$ and $HC\varphi = H(C, D)$ the graded vector spaces of the *relative cocycles*, *relative boundaries* and *relative cohomology*, respectively. The morphism φ is a quasi-isomorphism if and only if $HC\varphi = 0$.

2.3. Σ-modules. Let us recall some definitions and notations about operads (see [KM95], [MSS02], [Fre17a]).

Let Σ be the *symmetric groupoid*, that is, the category whose objects are the sets $\underline{n} = \{1, \dots, n\}$ for $n \geq 1$. For $n = 0$, we put $\underline{0} = \emptyset$, the empty set. As for the morphisms,

$$\Sigma(\underline{m}, \underline{n}) = \begin{cases} \Sigma_n, & \text{if } m = n, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\Sigma_n = \operatorname{Aut}\{1, \dots, n\}$ are the symmetric groups. In the case $n = 0, 1$, we have $\Sigma_0 = \Sigma_1 = *$, the one-point set. We will also need to consider its full subcategories $\Sigma_{>1} \subset \Sigma_{>0} \subset \Sigma$, without the $\underline{0}, \underline{1}$ objects, and the $\underline{0}$ object, respectively.

The category of contravariant functors from Σ to $\mathcal{C}$ is called the category of Σ-*modules* (Σ-sequences in [Fre17a]) and it is denoted by $\mathbf{\Sigma Mod}$. We identify its objects with sequences of objects in $\mathcal{C}$, $M = (M(l))_{l\geq 0} = (M(0), M(1), \dots, M(l), \dots)$, with a right Σ_l-action on each $M(l)$. So, every $M(l)$ is a $\mathbf{k}[\Sigma_l]$-module, or Σ_l-module for short.

If ω is an element of $M(l)$, l is called the *arity* of ω. We will write $\operatorname{ar}(\omega) = l$, in this case. Also, we will say that a Σ-module E is of *homogeneous arity* p if $E(l) = 0$ for $l \neq p$. If $\omega \in M(l)^p$, we will say that ω has *arity-degree* (l, p).

If M and N are Σ-modules, a *morphism of* Σ-*modules* $f : M \longrightarrow N$ is a sequence of Σ_l-equivariant morphisms $f(l) : M(l) \longrightarrow N(l)$, $l \geq 0$. Such a morphism is called a *quasi-isomorphism* if every $f(l) : M(l) \longrightarrow N(l)$ is a quasi-isomorphism of complexes for all $l \geq 0$.

We will also use the categories $\mathbf{\Sigma_{>0}Mod}$ and $\mathbf{\Sigma_{>1}Mod}$ of contravariant functors from $\Sigma_{>0}$ and $\Sigma_{>1}$ to $\mathcal{C}$. We can also consider $\mathbf{\Sigma_{>1}Mod}$ and $\mathbf{\Sigma_{>0}Mod}$ as the full subcategories of $\mathbf{\Sigma Mod}$ of those Σ-modules M such that $M(0) = M(1) = 0$ and $M(0) = 0$, respectively.

Remark 2.1. We are going to resort quite frequently to the fact that over the group algebras $\mathbf{k}[\Sigma_n]$ all modules are projective. So, for any Σ-module M and any n, $M(n)$ is a projective Σ_n-module. This is a consequence of Maschke's theorem.

2.4. Σ-operads. The category of Σ-operads is denoted by $\mathbf{Op}$. Operads can be described as Σ-modules together with either *structure morphisms* [MSS02] (also called *full composition products* [Fre17a]),

$$\gamma_{l;m_1,\dots,m_l} : P(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l) \longrightarrow P(m) \ ,$$

or, equivalently, *composition operations* [MSS02] (also called *partial composition products* [Fre17a]),

$$\circ_i : P(l) \otimes P(m) \longrightarrow P(l+m-1) \ ,$$

and a *unit* $\eta : \mathbf{k} \longrightarrow P(1)$, satisfying equivariance, associativity and unit axioms (see [KM95], [MSS02], [Fre17a]).

If P and Q are operads, a *morphism of operads* $\varphi : P \longrightarrow Q$ is a morphism of Σ-modules which respects composition products and units. A morphism of operads is called a *quasi-isomorphism* if it is so by forgetting the operad structure.

We say that an operad $P \in \mathbf{Op}$ is:

(a) *Non-unitary* if $P(0) = 0$, and we denote by $\mathbf{Op}_0$ the subcategory of *non-unitary operads.*
(b) *Unitary* if $P(0) = \mathbf{k}$, and we denote by $\mathbf{Op}_+$ the subcategory of *unitary operads.*
(c) *Connected* if $P(1) = \mathbf{k}$, and we denote by $\mathbf{Op}_{01}$ and $\mathbf{Op}_{+1}$ the subcategories of $\mathbf{Op}$ of *non-unitary and connected* operads and *unitary and connected operads*, respectively.

Two basic operations we perform on our operads, when possible, are the following:

(a) Let P be a connected operad. Denote by $\overline{P}$ its *augmentation ideal.* It is the Σ-module

$$\overline{P}(l) = \begin{cases} 0, & \text{if } l = 0, 1, \\ P(l), & \text{otherwise.} \end{cases}$$

(b) We say that a non-unitary operad P admits a *unitary extension* when we have a unitary operad P_+ which agrees with P in arity $l > 0$ and composition operations extend the composition operations of P. In this case, the canonical imbedding $i_+ : P \longrightarrow P_+$ is a morphism in the category of operads.

Later on, we will recall when a non-unitary operad admits such a unitary extension.

2.5. Λ-modules. Following [Fre17a], in order to produce minimal models for our unitary operads, we split the units in $P(0)$ out of them. But we don't want to forget about this arity zero term, so we "include" the data of the units in the Σ-module structure as follows:

Let Λ denote the category with the same objects as Σ, but with morphisms

$$\Lambda(\underline{m}, \underline{n}) = \{\text{injective maps } \underline{m} \longrightarrow \underline{n}\} \ .$$

We will also consider its subcategories $\Lambda_{>1} \subset \Lambda_{>0} \subset \Lambda$, defined in the same way as the ones of Σ.

The category of contravariant functors from Λ to $\mathcal{C}$ is called the category of Λ-modules (Λ-sequences in [Fre17a]) and it is denoted by $\mathbf{\Lambda Mod}$.

We still have the obvious notions of arity, morphisms and the full subcategories $\mathbf{\Lambda_{>1}Mod}$ and $\mathbf{\Lambda_{>0}Mod}$ for Λ-modules, and it's clear that the cohomology of a Λ-module M naturally inherits the structure of a Λ-module. Maschke's theorem also applies for Λ_n-modules because $\mathbf{k}[\Lambda_n] = \mathbf{k}[\Sigma_n]$.

2.6. Λ-operads. Let P_+ be an unitary operad $P_+(0) = \mathbf{k}$. We can associate to P_+ a non-unitary one $P = \tau P_+$, its truncation,

$$P(l) = \begin{cases} 0, & \text{if } l = 0, \\ P_+(l), & \text{otherwise.} \end{cases}$$

together with the following data:

(1) The composition operations $\circ_i : P_+(m) \otimes P_+(n) \longrightarrow P_+(m+n-1)$ of P_+ , for $m, n > 0$.
(2) The restriction operations $u^* : P_+(n) \longrightarrow P_+(m)$, for every $u \in \Lambda(\underline{m}, \underline{n})$, for $m, n > 0$. These restrictions are defined as $u^*(\omega) = \omega(1, \dots, 1, \mathrm{id}, 1, \dots, 1, \mathrm{id}, 1, \dots, 1)$, with id placed at the $u(i)$-th variables, for $i = 1, \dots, m$.
(3) The augmentations $\varepsilon : P_+(m) \longrightarrow \mathbf{k} = P_+(0)$, $\varepsilon(\omega) = \omega(1, \dots, 1)$, for $m > 0$.

A non-unitary operad P together with the structures $(1), (2), (3)$ is called a Λ-operad.

According to [Fre17a], p. 58, every unitary operad P_+ can be recovered from its non-unitary truncation P with the help of these data, which define the category of Λ-operads, $\Lambda\mathbf{Op}_0$, and its corresponding variants (see [Fre17a], page 71). This can be written as isomorphisms of categories

$$\tau : \mathbf{Op}_+ = \Lambda\mathbf{Op}_0/\mathcal{C}om : (\ )_+ \ , \qquad \text{and} \qquad \tau : \mathbf{Op}_{+1} = \Lambda\mathbf{Op}_{01}/\mathcal{C}om : (\ )_+ \ .$$

Here, $(\ )_+$ denotes the *unitary extension* associated with any non-unitary and augmented Λ-operad (see [Fre17a], p. 81). Namely, if $P \in \Lambda\mathbf{Op}_0/\mathcal{C}om$, its unitary extension P_+ is the Σ-operad defined by

$$P_+(l) = \begin{cases} \mathbf{k}, & \text{if } l = 0, \\ P(l), & \text{otherwise.} \end{cases}$$

And the unitary operad structure is recovered as follows:

(1) Composition operations $\circ_i : P_+(m) \otimes P_+(n) \longrightarrow P_+(m+n-1)$ for $m, n > 0$ are those of P.
(2) For $n > 1$, the *restriction operation* $u^* = \delta^i : P(n) \longrightarrow P(n-1)$ gives us the partial composition operations $\circ_i : P_+(n) \otimes P_+(0) \longrightarrow P_+(n-1)$. Here $u = \delta_i : \{1 < \dots < n-1\} \longrightarrow \{1 < \dots < n\}$ is the injective map defined by $\delta_i(x) = x$, for $x = 1, \dots, i-1$ and $\delta_i(x) = x+1$, for $x = i, \dots, n-1$.
(3) The augmentation $\varepsilon : P(1) \longrightarrow \mathbf{k}$ gives the unique partial composition product $P_+(1) \otimes P_+(0) \longrightarrow P_+(0)$.

Let us end this section with a couple of easy remarks.

Lemma 2.2. *The unitary extension functor $(\ )_+$ commutes with cohomology and colimits. That is,*

$$H(P_+) = (HP)_+ \qquad \textit{and} \qquad \varinjlim_n (P_n)_+ = (\varinjlim_n P_n)_+ \ .$$

Proof. Commutation with cohomology is obvious. Commutation with colimits is a consequence of $(\ )_+$ having a right adjoint, namely the truncation functor τ. □

As a consequence, $(\ )_+$ is an exact functor.

Remark 2.3. The initial object of the category of general operads **Op** is the operad I

$$I(l) = \begin{cases} \mathbf{k}, & \text{if } l = 1, \\ 0, & \text{otherwise} \end{cases},$$

and the obvious operad structure. It's also the initial object of the subcategory of non-unitary connected operads $\mathbf{Op}_{01}$. We shall denote it also by I_0. Its unitary extension I_+

$$I_+(l) = \begin{cases} \mathbf{k}, & \text{if } l = 0, 1, \\ 0, & \text{otherwise} \end{cases},$$

with the only possible non-zero partial composition operation being the identity, is the initial object of the subcategory of unitary operads $\mathbf{Op}_+$ and its subcatecory of unitary and connected ones, $\mathbf{Op}_{+1}$.

3. Free operads

We recall in this sections the definition of the general free operad and of two of its particular instances we are going to use. We start with a review of trees. Trees are useful to represent elements (operations) of operads, its composition products and to produce an accurate description of the free operad.

3.1. Trees. When we speak of trees, we adhere to the definitions and conventions of [Fre17a], appendix I. We include a summary here, for the reader's convenience.

Definition 3.1. An *r-tree* T consists of:

(a) A finite set of *inputs*, $\underline{r} = \{i_1, \dots, i_r\}$ and an *output* 0.
(b) A set of *vertices* $v \in V(T)$.
(c) A set of *edges* $e \in E(T)$, oriented from the *source* $s(e) \in V(T) \sqcup \underline{r}$ towards a *target* $t(e) \in V(T) \sqcup \{0\}$.

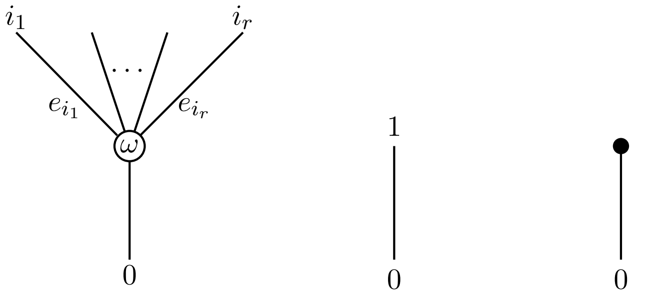
These items are subjected to the following conditions:

(1) There is one and only one edge $e_0 \in E(T)$, the *outgoing edge of the tree*, such that $t(e_0) = 0$.
(2) For each $i \in \underline{r}$, there is one and only one edge $e_i \in E(T)$, the *ingoing edge of the tree indexed by i*, such that $s(e_i) = i$.
(3) For each vertex $v \in V(T)$, there is one and only one edge $e_v \in E(T)$, *the outgoing edge of the vertex v*, such that $s(e_v) = v$.
(4) Each $v \in V(T)$ is connected to the output 0 by a chain of edges $e_v, e_{v_{n-1}}, \dots, e_{v_1}, e_{v_0}$ such that $v = s(e_v), t(e_v) = s(e_{v_{n-1}}), t(e_{v_{n-1}}) = s(e_{v_{n-2}}), \dots, t(e_{v_2}) = s(e_{v_1}), t(e_{v_1}) = s(e_{v_0})$ and $t(e_{v_0}) = 0$.

Some fundamental examples of trees:

(1) The *r-corolla*: the only tree having just one vertex, r inputs and one output. We will note it by Y_r
(2) The *unit tree*: the only tree without vertices; just one input and one output. We will note it by $|$.

(3) *Corks*, also called *units*, are trees without inputs, just one output and just one vertex. We will note them by  .



An r-corolla The unit tree A cork

Fig. 1

An operation of r variables (an element of arity r) $w \in P(r)$ can be depicted as a tree of r inputs. The unit tree represents the identity $\mathrm{id} \in P(1)$ and a cork can be thought as a unit $1 \in P(0)$.

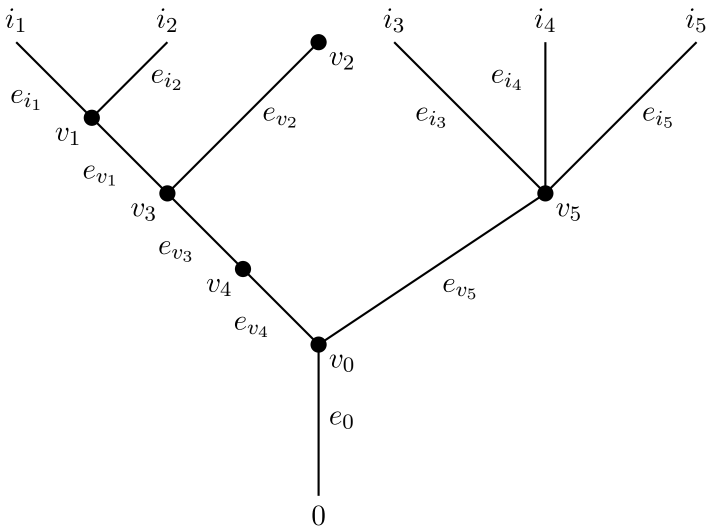


Fig. 2

A tree T with five inputs $\underline{r} = \{i_1, \dots, i_5\}$,
six vertices $V(T) = \{v_1, \dots, v_5, v_0\}$,
and eleven edges $E(T) = \{e_{i_1}, \dots, e_{v_5}, e_0\}$.

Composition operations can be represented as *grafting* of trees.

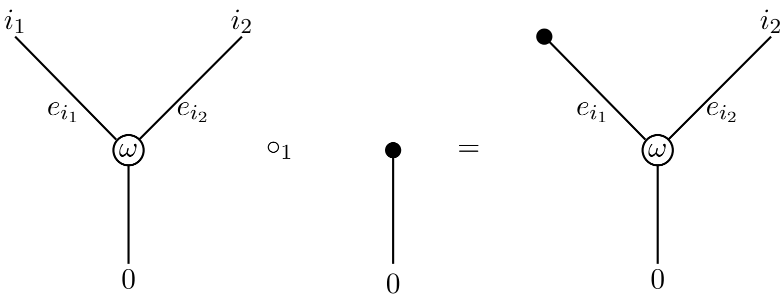


Fig. 3

The action of a unit (cork) on an arity two operation, $\omega \mapsto \delta^1(\omega) = \omega \circ_1 1$

We are going to consider trees that fulfill the following additional property

(5) For each vertex $v \in V(T)$, we have at least one edge $e \in E(T)$ such that $t(e) = v$,

In other words, except for the unit $1 \in \mathbf{k} = P(0)$, our trees won't have real *corks*, because every time a composition operation such as the one in Fig.3 is performed with a cork, we will not get a new operation with a cork $\omega \circ_1 1$, but it will be equal to an old one, without corks: for instance, $\omega \circ_1 1 = \mathrm{id}$. See remark 5.4.

To a vertex v we also associate a set of *ingoing edges*: those edges whose target is v. Let's denote its cardinal by

$$r_v = \sharp \{ e \in E(T) \mid t(e) = v \} \ .$$

The extra condition (5) is equivalent to the requirement that $r_v \geq 1$, for every $v \in V(T)$. In fact, in the constructions of our two particular instances of the free operad, we are going to find only vertices satisfying $r_v \geq 2$. A tree for which every vertex satisfies this extra condition is called *reduced.*

Example 3.2. So for the tree in Fig. 2, we have:

$$r_{v_2} = 0 \ , \quad r_{v_4} = 1 \ , \quad r_{v_1} = r_{v_3} = r_{v_0} = 2 \ , \quad r_{v_5} = 3 \ .$$

Hence, this is *not* a reduced tree, because of vertices v_2 and v_4.

Let us denote by $\mathbf{Tree}(r)$ the category whose objects are r-trees and whose morphisms are just isomorphisms. $\widetilde{\mathbf{Tree}}(r)$ will denote the full subcategory of *reduced* trees. For $r \geq 2$, $Y_r \in \widetilde{\mathbf{Tree}}(r)$.

3.2. The general free operad. The forgetful functor $U : \mathbf{Op} \longrightarrow \mathbf{\Sigma Mod}$ has a left adjoint, the *free operad functor*, $\Gamma : \mathbf{\Sigma Mod} \longrightarrow \mathbf{Op}$. Arity-wise it can be computed as

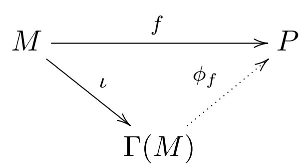
$$\Gamma(M)(l) = \underrightarrow{\mathrm{colim}}_{\, T \in \mathbf{Tree}(l)} M(T) \ ,$$

Here, $M(T)$ denotes the *treewise tensor product* of the Σ-module M over a tree T. It is the tensor product

$$M(T) = \bigotimes_{v \in V(T)} M(r_v) \ .$$

Of course, this free operad has the well-known universal property of free objects.

Proposition 3.3. *Any morphism of Σ-modules $f : M \longrightarrow P$, where P is an operad, admits a unique factorization*



such that ϕ_f is an operad morphism.

Proof. See [Fre17a], prop. 1.2.2. ∎

3.3. Two particular instances of the free operad. We will need two particular, smaller instances of the free operad.

First, because of [Fre17a], the restriction of the general free operad Γ to Σ-modules satisfying $M(0) = M(1) = 0$ is a unitary and connected operad $\Gamma(M) \in \mathbf{Op}_{01}$. So the general free operad functor restricts to a smaller one, which we note Γ_{01}. It is the left adjoint of the obvious forgetful functor:

$$\mathbf{Op}_{01} \underset{\Gamma_{01}}{\overset{U}{\rightleftarrows}} \mathbf{\Sigma_{>1}Mod} \ .$$

This is the free operad used by Markl in constructing his minimal models à la Sullivan of non-unitary and cohomologically connected operads (see [Mar96] and [MSS02]).

Second, if $M \in \mathbf{\Lambda_{>1}Mod}/\overline{\mathcal{C}om}$, then the general free operad $\Gamma(M)$ inherits the additional structure of an augmented, connected and unitary Λ-operad ([Fre17a], prop. A.3.12). Hence, because of the isomorphism of categories between Λ-operads and unitary Σ-operads, it has a unitary extension. Let's denote it by $\Gamma_{+1}(M) = \Gamma(M)_+$. It is the left adjoint of the forgetful functor $\overline{U}$ which sends each operad P to its augmentation ideal $\overline{P}$:

$$\mathbf{Op}_{+1} = \Lambda\mathbf{Op}_{01}/\mathcal{C}om \underset{\Gamma_{+1}}{\overset{\overline{U}}{\rightleftarrows}} \mathbf{\Lambda_{>1}Mod}/\overline{\mathcal{C}om} \ .$$

Here is a little road map for these categories and functors:

$$\begin{array}{ccccccccc}
\mathbf{\Sigma Mod} & \underset{U}{\overset{\Gamma}{\rightleftarrows}} & & \mathbf{Op} & & & & & \\
\uparrow\iota & & \nearrow\iota & & \nwarrow\iota & & & & \\
\mathbf{\Sigma_{>0}Mod} & \underset{U}{\overset{\Gamma_0}{\rightleftarrows}} & \mathbf{Op}_0 & \xleftarrow{\ \tau\ } & \mathbf{Op}_+ & = & \Lambda\mathbf{Op}_0/\mathcal{C}om & \underset{\Gamma_+}{\overset{U}{\rightleftarrows}} & \mathbf{\Lambda_{>0}Mod}/\mathcal{C}om \\
\uparrow\iota & & \uparrow\iota & & \uparrow\iota & & & & \uparrow\iota \\
\mathbf{\Sigma_{>1}Mod} & \underset{U}{\overset{\Gamma_{01}}{\rightleftarrows}} & \mathbf{Op}_{01} & \xleftarrow{\ \tau\ } & \mathbf{Op}_{+1} & = & \Lambda\mathbf{Op}_{01}/\mathcal{C}om & \underset{\Gamma_{+1}}{\overset{\overline{U}}{\rightleftarrows}} & \mathbf{\Lambda_{>1}Mod}/\overline{\mathcal{C}om}
\end{array} \ .$$

Here, ι denotes the natural inclusions. We are mainly interested in the bottom row.

The key point that encompases the possibility of constructing minimal models for operads in both cases we are studying, cohomologically non-unitary and unitary, is that, since $M(0) = M(1) = 0$, there will be no arity zero and one trees in the colimit defining the free operad and, since in this case the only morphisms in the subcategory of *reduced* trees $\widetilde{\mathbf{Tree}}(l)$ are trivial isomorphisms, this colimit is reduced to a direct sum [Fre17a], proposition A.3.14. Hence, for $\Gamma = \Gamma_{01}, \Gamma_{+1}$,

$$\Gamma(M)(l) = \bigoplus_{T \in \widetilde{\mathbf{Tree}}(l)} M(T) \ .$$

All this leads to the following

Lemma 3.4. *For every module M with $M(0) = M(1) = 0$, and every homogeneous module E of arity $p > 1$, Γ_{01} and Γ_{+1} verify:*

(a)

$$\Gamma_{01}(M)(l) = \begin{cases} 0, & \textit{if } l = 0, \\ \mathbf{k}, & \textit{if } l = 1, \\ \Gamma(M)(l), & \textit{if } l \neq 0, 1 \end{cases} \qquad \textit{and} \qquad \Gamma_{+1}(M)(l) = \begin{cases} \mathbf{k}, & \textit{if } l = 0, \\ \mathbf{k}, & \textit{if } l = 1, \\ \Gamma(M)(l), & \textit{if } l \neq 0, 1 \ . \end{cases}$$

(b) *For* $\Gamma = \Gamma_{01}, \Gamma_{+1}$,

$$\Gamma(M \oplus E)(l) = \begin{cases} \Gamma(M)(l), & \textit{if } l < p, \\ \Gamma(M)(p) \oplus E, & \textit{if } l = p. \end{cases}$$

Proof. Let's compute:

$$\begin{aligned} (M \oplus E)(T) &= \bigotimes_{v \in V(T)} (M \oplus E)(r_v) = \bigotimes_{v \in V(T)} (M(r_v) \oplus E(r_v)) \\ &= \left(\bigotimes_{\substack{v \in V(T) \\ r_v \neq p}} M(r_v) \right) \otimes \left(\bigotimes_{\substack{v \in V(T) \\ r_v = p}} (M(r_v) \otimes E) \right) . \end{aligned}$$

Hence,

$$\begin{aligned} \Gamma(M \oplus E)(m) &= \bigoplus_{T \in \mathbf{Tree}(m)} (M \oplus E)(T) \\ &= \bigoplus_{T \in \mathbf{Tree}(m)} \left[\left(\bigotimes_{\substack{v \in V(T) \\ r_v \neq p}} M(r_v) \right) \otimes \left(\bigotimes_{\substack{v \in V(T) \\ r_v = p}} (M(r_v) \oplus E) \right) \right] . \end{aligned}$$

If $m < p$, $\bigotimes_{\substack{v \in V(T) \\ r_v = p}} (M(r_v) \otimes E) = \mathbf{k}$, since there are no trees with $m < p$ ingoing edges and a vertex with p ingoing edges (there are no corks, since $M(0) = 0$). Hence, in this case, we simply have

$$\Gamma(M \oplus E)(m) = \bigoplus_{T \in \mathbf{Tree}(m)} \left(\bigotimes_{\substack{v \in V(T) \\ r_v \neq p}} M(r_v) \right) = \Gamma(M)(m) \ .$$

For $m = p$, we split our sum over all trees in two terms: one for the corolla Y_p and another one for the rest:

$$\begin{aligned} \Gamma(M \oplus E)(p) = & \left[\left(\bigotimes_{\substack{v \in V(Y_p) \\ r_v \neq p}} M(r_v) \right) \otimes \left(\bigotimes_{\substack{v \in V(Y_p) \\ r_v = p}} (M(r_v) \oplus E) \right) \right] \oplus \\ & \bigoplus_{\substack{T \in \mathbf{Tree}(m) \\ T \neq Y_p}} \left[\left(\bigotimes_{\substack{v \in V(T) \\ r_v \neq p}} M(r_v) \right) \otimes \left(\bigotimes_{\substack{v \in V(T) \\ r_v = p}} (M(r_v) \oplus E) \right) \right] \end{aligned}$$

And we have:

(1) The set of vertices $v \in V(Y_p)$, $r_v \neq p$ is empty. So for the first tensor product we get: $\bigotimes_{\substack{v\in V(Y_p)\\ r_v\neq p}} M(r_v) = \mathbf{k}$.
(2) There is just one vertex in $v \in V(Y_p)$ with $r_v = p$. Hence, $\bigotimes_{\substack{v\in V(Y_p)\\ r_v = p}} (M(r_v) \oplus E) = M(p) \oplus E$.
(3) We leave $\bigotimes_{\substack{v\in V(T)\\ r_v\neq p}} M(r_v)$ as it is.
(4) As for $\bigotimes_{\substack{v\in V(T)\\ r_v = p}} (M(r_v) \oplus E)$, since we are assuming $r_v \geq 2$, this set of vertices is empty for every tree. So we only get $\mathbf{k}$ for every vertex v.

All in all,

$$\Gamma(M \oplus E)(p) = (M(p) \oplus E) \ \oplus \bigoplus_{\substack{T\in\mathbf{Tree}(m)\\ T\neq Y_p}} \left[\bigotimes_{\substack{v\in V(T)\\ r_v\neq p}} M(r_v) \right] = \Gamma(M)(p) \oplus E \ .$$

□

Remark 3.5. So, for $M(0) = M(1) = 0$, and forgetting the Λ-structure if necessary, it's clear that both $\Gamma_{01}(M)$ and $\Gamma_{+1}(M)$ agree with the general free operad $\Gamma(M)$, outside arities 0 and 1. By definition, also $\Gamma_{+1}(M) = \Gamma_{01}(M)_+$ when M has a Λ-module structure.

4. Basic operad homotopy theory

We develop here the basic, standard homotopy theory for operads, non-unitary, unitary or otherwise. Since the results are the same for no matter which free operad we use, we will not make any distinctions, and call it just Γ. Nor we will bother to denote unitary operads by P_+: in this section, P stands for any kind of operad.

This basic homotopy theory can be formalized under the name of *Cartan-Eilenberg*, or *Sullivan* categories (see [GNPR10]) and emphasizes just three elements: weak equivalences, or *quis*, homotopy and cofibrant (minimal) objects.

Definition 4.1. (See [MSS02], cf [GNPR05]) Let $n \geq 1$ be an integer. Let $P \in \mathbf{Op}$ be free as a graded operad, $P = \Gamma(M)$, where M is a graded Σ-module. An *arity n principal extension* of P is the free graded operad

$$P \sqcup_d \Gamma(E) := \Gamma(M \oplus E) \ ,$$

where E is an arity-homogeneous Σ_n-module with zero differential and $d : E \longrightarrow ZP(n)^{+1}$ a map of Σ_n-modules of degree $+1$. The differential ∂ on $P \sqcup_d \Gamma(E)$ is built upon the differential of P, d and the Leibniz rule.

Remark 4.2. In the context of commutative dg algebras, the analogous construction is called a *Hirsch extension* [GM13], or a *KS-extension* [Hal83].

Lemma 4.3. *$P \sqcup_d \Gamma(E)$ is a dg operad and the natural inclusion $\iota : P \longrightarrow P \sqcup_d \Gamma(E)$ is a morphism of dg operads.*

Proof. This is clear. □

Lemma 4.4 (Universal property of principal extensions)**.** *Let $P \sqcup_d \Gamma(E)$ be a principal extension of a free-graded operad $P = \Gamma(M)$, and let $\varphi : P \to Q$ be a morphism of operads. A morphism $\psi : P \sqcup_d \Gamma(E) \longrightarrow Q$ extending φ is uniquely determined by a morphism of Σ_n-modules $f : E \to Q(n)$ satisfying $\partial f = \varphi d$.*

Proof. This is clear. $\square$

Lemma 4.5. *Let $\iota : P \longrightarrow P \sqcup_d \Gamma(E)$ be an arity n principal-extension and*

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & Q \\ {\scriptstyle \iota}\downarrow & \overset{\psi'}{\nearrow} & \downarrow{\scriptstyle \rho}\ \wr \\ P \sqcup_d \Gamma(E) & \xrightarrow{\ \psi\ } & R \end{array}$$

a solid commutative diagram of operad morphisms, where ρ is a surjective quasi-isomorphism. Then, there is an operad morphism ψ' making both triangles commute.

Proof. Consider the solid diagram of $\mathbf{k}[\Sigma_n]$-modules

$$\begin{array}{ccc} & & ZC\mathrm{id}_{Q(n)} \\ & \overset{\mu}{\nearrow} & \downarrow{\scriptstyle \mathrm{id}\oplus\rho(n)} \\ E & \xrightarrow{\ \lambda\ } & ZC\rho(n) \end{array} .$$

The given commutative square implies that the linear map $\lambda = (\varphi d \ \ \psi_{|E})^t$ has its image included in the relative cocycles of the morphism ρ:

$$\begin{pmatrix} -\partial_{Q(n)} & 0 \\ -\rho(n) & \partial_{Q(n)} \end{pmatrix} \begin{pmatrix} \varphi d \\ \psi_{|E} \end{pmatrix} = \begin{pmatrix} -\partial \varphi d \\ -\rho\varphi d + \partial \psi_{|E} \end{pmatrix} = \begin{pmatrix} -\partial^2 \psi_{|E} \\ -\varphi d + \partial \psi_{|E} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Also,

$$\mathrm{id}_{Q(n)^{+1}} \oplus \rho(n) = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} : Q(n)^{+1} \oplus Q(n) \longrightarrow Q(n)^{+1} \oplus R(n)$$

restricts to a linear map between the relative cocycles of $\mathrm{id}_{Q(n)^{+1}}$ and those of $\rho(n)$ because it commutes with the differentials of the respective cones:

$$\begin{pmatrix} -\partial & 0 \\ -\rho & -\partial \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} = \begin{pmatrix} -\partial & 0 \\ -\rho & \partial\rho \end{pmatrix} = \begin{pmatrix} -\partial & 0 \\ -\rho & \rho\partial \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} -\partial & 0 \\ -1 & \partial \end{pmatrix} .$$

Here and in the rest of this proof we will frequently drop the arity index n. To find our sought extension ψ' we just need to find a $\mathbf{k}[\Sigma_n]$-linear map μ making the triangle commute: if $\mu = (\alpha \ f)$, then we would take as ψ' the morphism induced by φ and $f : E \longrightarrow Q(n)$.

And this is because:

(a) According to the universal property of principal extensions of Lemma 4.4, in order to see that this defines a morphism of operads, all we have to check is that we get $\partial f = \varphi d$. And we would have it because, if the image of μ is included in the relative cocycles of $\mathrm{id}_{Q(n)}$ and makes the triangle commutative, we would have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial & 0 \\ -1 & \partial \end{pmatrix} \begin{pmatrix} \alpha \\ f \end{pmatrix} = \begin{pmatrix} \partial\alpha \\ -\alpha + \partial f \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{cases} \partial\alpha & = 0 \\ \alpha & = \partial f \end{cases} ,$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \alpha \\ f \end{pmatrix} = \begin{pmatrix} \varphi d \\ \psi_{|E} \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{cases} \alpha & = \varphi d \\ \rho f & = \psi_{|E} \end{cases} ;$$

hence, $\partial f = \alpha = \varphi d$.

(b) Also, such a ψ' would make the top triangle commute: $\psi' \iota = \psi'_{|P} \varphi$, by definition of ψ'.

(c) Also the lower triangle would commute: according to the universal property of KS-extensions, $\rho\psi' = \psi$ boils down to $\rho\psi'_{|P} = \psi_{|P}$ and $\rho\psi'_{|E} = \psi_{|E}$. The first equality is true because $\rho\psi'_{|P} = \rho\phi = \psi_{|P}$. The second one because $\rho\psi'_{|E} = \rho f = \psi_{|E}$.

So we only need to prove that the $\mathbf{k}[\Sigma_n]$-linear map μ exists. For which it is enough to see that $\mathrm{id} \oplus \rho$ is an epimorphism between the spaces of cocycles: let $(q, r) \in ZC\rho \subset Q(n)^{+1} \oplus R(n)$. We need to produce $(x, y) \in ZC\mathrm{id}_Q \subset Q(n)^{+1} \oplus Q(n)$ such that

$$x = q \ , \ \rho y = r \qquad \text{and} \qquad \partial x = 0 \ , \ x = \partial y \ .$$

So, there is no choice but to make $x = q$. As for y, since ρ is a *quis* $HC\rho = 0$. So, we have $(q', r') \in Q(n)^{+1} \oplus R(n)$ such that $q = \partial q'$ and $r = -\rho q' + \partial r'$. We try $y = q'$ and compute: $\rho y = \rho q' = \partial r' - r$. But ρ is an epimorphism, so we can find some $q'' \in Q(n)$ such that $r' = \rho q''$. So, finally, $r = \rho(q' + \partial q'')$ and we take $y = q' + \partial q''$. □

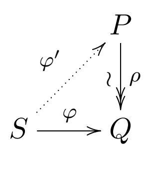
Definition 4.6. A *Sullivan operad* is the colimit of a sequence of principal extensions of arities $l_n \geq 2$, starting from the initial operad.

$$I_\alpha \longrightarrow P_1 = \Gamma(E(l_1)) \longrightarrow \cdots \longrightarrow P_n = P_{n-1} \sqcup_{d_n} \Gamma(E(l_n)) \longrightarrow \cdots \longrightarrow \varinjlim{}_n P_n = P_\infty \ ,$$

with $\alpha = 0, +$, depending on whether we are working with non-unitary or unitary operads, respectively.

The next result says that Sullivan operads are cofibrant objects in the Hinich model structure of the category of operads, [Hin97].

Proposition 4.7. *Let S be a Sullivan operad. For every solid diagram of operads*



in which ρ is a surjective quasi-isomorphism, there exists φ' making the diagram commute.

Proof. Induction and lemma 4.5. □

Similarly to the setting of commutative algebras, there is a notion of homotopy between morphisms of operads, defined via a functorial path (see Section 3.10 of [MSS02], cf. [CR19]), based on the following remark.

Remark 4.8. Let P be a dg operad and K a commutative dg algebra. Then $P\otimes K = \{P(n)\otimes K\}_{n\geq 0}$ has a natural operad structure given by the partial composition products

$$(\omega\otimes a)\circ_i(\eta\otimes b) = (-1)^{|a||\eta|}(\omega\circ_i\eta)\otimes(ab)\ .$$

In particular, let $K = \mathbf{k}[t,dt] = \Lambda(t,dt)$ be the free commutative dg algebra on two generators $|t|=0$, $|dt|=1$ and differential sending t to dt. We have the unit ι and evaluations δ^0 and δ^1 at $t=0$ and $t=1$ respectively, which are morphisms of $\mathcal{C}om$-algebras satisfying $\delta^0\circ\iota = \delta^1\circ\iota = \mathrm{id}$.

$$\mathbf{k}\xrightarrow{\ \iota\ }\mathbf{k}[t,dt]\underset{\delta^0}{\overset{\delta^1}{\rightrightarrows}}\mathbf{k}\quad ;\quad \delta^k\circ\iota = \mathrm{id}\ .$$

The following are standard consequences of Proposition 4.7. The proofs are adaptations of the analogous results in the setting of $\mathcal{C}om$-algebras (see Section 11.3 of [GM13]; see also [CR19] in the context of operad algebras).

Definition 4.9. A *functorial path* in the category of operads is defined as the functor

$$-[t,dt]:\mathbf{Op}\longrightarrow\mathbf{Op}$$

given on objects by $P[t,dt] = P\otimes\mathbf{k}[t,dt]$ and on morphisms by $\varphi[t,dt] = \varphi\otimes\mathbf{k}[t,dt]$, together with the natural transformations

$$P\xrightarrow{\ \iota\ }P[t,dt]\underset{\delta^0}{\overset{\delta^1}{\rightrightarrows}}P\quad ;\quad \delta^k\circ\iota = \mathrm{id}$$

given by $\delta^k = 1\otimes\delta^k : P[t,dt]\longrightarrow P\otimes\mathbf{k} = P$ and $\iota = 1\otimes\iota : P = P\otimes\mathbf{k}\to P[t,dt]$.

The map ι is a quasi-isomorphism of operads while the maps δ^0 and δ^1 are surjective quasi-isomorphisms of operads.

The functorial path gives a natural notion of homotopy between morphisms of operads:

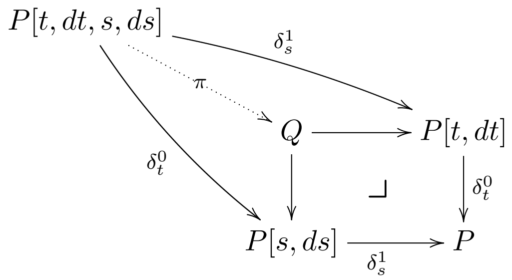
Definition 4.10. Let $\varphi,\psi : P\longrightarrow Q$ be two morphisms of operads. An *homotopy from φ to ψ* is given by a morphism of operads $H : P\longrightarrow Q[t,dt]$ such that $\delta^0\circ H = \varphi$ and $\delta^1\circ H = \psi$. We use the notation $H : f\simeq g$.

The homotopy relation defined by a functorial path is reflexive and compatible with the composition (see for example [KP97, Lemma I.2.3]. Furthermore, the symmetry of $\mathcal{C}om$-algebras $\mathbf{k}[t,dt]\longrightarrow\mathbf{k}[t,dt]$ given by $t\mapsto 1-t$ makes the homotopy relation into a symmetric relation. However, the homotopy relation is not transitive in general. As in the rational homotopy setting of $\mathcal{C}om$-algebras, we have:

Proposition 4.11. *The homotopy relation between morphisms of operads is an equivalence relation for those morphisms whose source is a Sullivan operad.*

Proof. It only remains to prove transitivity. Let S be a Sullivan operad and consider morphisms $\varphi,\varphi',\varphi'' : S\longrightarrow P$ together with homotopies $H:\varphi\simeq\varphi'$ and $H':\varphi'\simeq\varphi''$.

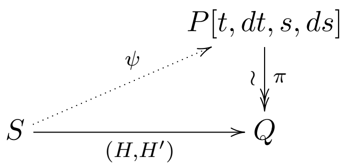
Consider the pull-back diagram in the category of operads



To see that the map π is surjective, note that if $p(s, ds)$ and $q(t, dt)$ are polynomials such that $p(1, 0) = q(0, 0)$, representing an element in $\mathcal{M}$, then

$$\pi(p(s, ds) + q(st, dt) - q(0, 0)) = (p(s, ds), q(t, dt)) \ .$$

It is straightforward to see that all the operads in the above diagram are quasi-isomorphic and that π is a quasi-isomorphism. Consider the solid diagram



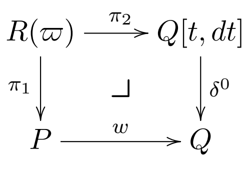
.

By Proposition 4.7, there exists a dotted arrow ψ such that $\pi\psi = (H, H')$. Let $H\widetilde{+}H' := \nabla\psi$, where $\nabla : P[t, dt, s, ds] \longrightarrow P[t, dt]$ is the map given by $t, s \mapsto t$. This gives the desired homotopy $H\widetilde{+}H' : \varphi \simeq \varphi''$. □

Denote by $[S, P]$ the set of homotopy classes of morphisms of operads $\varphi : S \longrightarrow P$.

Proposition 4.12. *Let S be a Sullivan operad. Any quasi-isomorphism $\varpi : P \longrightarrow Q$ of operads induces a bijection $\varpi_* : [S, P] \longrightarrow [S, Q]$.*

Proof. We first prove surjectivity: let $[\varphi] \in [S, Q]$. Consider the mapping path of ϖ, given by the pull-back



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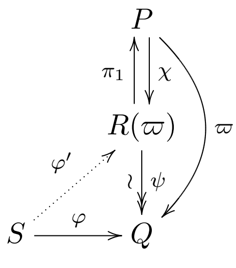
Define maps $\psi := \delta^1\pi_2 : R(\varpi) \longrightarrow Q$ and $\chi := (1, \iota\varpi) : P \longrightarrow R(\varpi)$. We obtain a solid diagram



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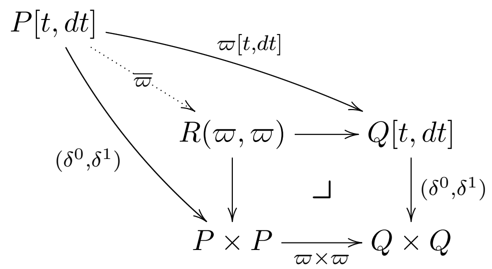
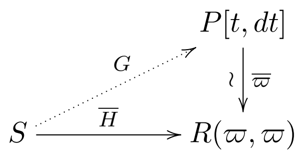
where ψ is a surjective quasi-isomorphism and $\psi\chi = \varpi$. By Proposition 4.7, there exists φ' such that $\psi\varphi' = \varphi$. Let $\phi := \pi_1\varphi'$. Then $\varpi\phi = \psi\chi\pi_1\varphi' \simeq \psi\varphi' = \varphi$. Therefore $[\varpi\phi] = [\varphi]$ and ϖ_* is surjective.

To prove injectivity, let $\varphi_0, \varphi_1 : S \longrightarrow Q$ be such that $H : \varpi\varphi_0 \simeq \varpi\varphi_1$. Consider the pull-back diagram



One may verify that $\overline{\varpi}$ is a surjective quasi-isomorphism. Let $\overline{H} = (\varphi_0, \varphi_1, H)$ and consider the solid diagram



Since $\overline{\varpi}_*$ is surjective, there exists a dotted arrow G such that $\overline{\varpi}G \simeq \overline{H}$. It follows that $\varphi_0 \simeq \delta^0 G \simeq \delta^1 G \simeq \varphi_1$. Thereby, $\varphi_0 \simeq \varphi_1$ by Proposition 4.11. ☐

5. Minimal models

Sullivan *minimal* operads are Sullivan operads for which the process of adding new generators E is done with strictly increasing arities. In this section we prove the existence and uniqueness of Sullivan minimal models for operads in our two aforementioned cases, (cohomologically) non-unitary and unitary.

Definition 5.1. A *Sullivan minimal operad* P_∞ is the colimit of a sequence of principal extensions starting from the initial operad, ordered by strictly increasing arities

$$I_\alpha \longrightarrow P_2 = \Gamma(E(2)) \longrightarrow \cdots \longrightarrow P_n = P_{n-1} \sqcup_{d_n} \Gamma(E(n)) \longrightarrow \cdots \longrightarrow \underrightarrow{\mathrm{colim}}_n P_n = P_\infty \ ,$$

with $E(n)$ an arity n homogeneous Σ_n-module with zero differential and $\alpha = 0, +$, depending on whether we are working with non-unitary, or unitary operads. A *Sullivan minimal model* for an operad P is a Sullivan minimal operad P_∞ together with a quasi-isomorphism $\rho : P_\infty \xrightarrow{\sim} P$.

Remark 5.2. In particular, a Sullivan minimal operad is a free graded operad $P_\infty = \Gamma(E)$, with $E = \bigoplus_n E(n)$, plus an extra condition on its differential ∂, usually called being *decomposable*. The interested reader can check that both definitions, as a colimit of principal extensions, or as a free graded operad plus a decomposable differential agree by looking at [GNPR05], proposition 4.4.1. Even though this second characterization is useful in practice to recognise a Sullivan minimal operad, we are not going to use it in this paper.

5.1. Existence.

Theorem 5.3. *Every cohomologically connected operad $P \in \mathbf{Op}, HP(1) = \mathbf{k}$, and cohomologically non-unitary, $HP(0) = 0$, (resp. cohomologically unitary, $HP(0) = \mathbf{k}$) has a Sullivan minimal model $P_\infty \longrightarrow P$. This minimal model is connected $P_\infty(1) = \mathbf{k}$ and non-unitary $P_\infty(0) = 0$ (resp., unitary, $P_\infty(0) = \mathbf{k}$).*

Proof. The non-unitary case. Let P be a cohomologically connected, cohomologically non-unitary operad. This is Markl's case [MSS02], with the slight improvement that we are just assuming $HP(0) = 0$ instead of $P(0) = 0$. We are going to first write down the proof for this case, and then comment on the modifications needed for the cohomologically unitary case.

Here, we use the free operad functor $\Gamma = \Gamma_{01}$ and start with $E = E(2) = HP(2)$. Take a $\mathbf{k}[\Sigma_2]$-linear section $s_2 : HP(2) \longrightarrow ZP(2) \subset P(2)$ of the projection $\pi_2 : ZP(2) \longrightarrow HP(2)$, which exists because $\mathbf{k}$ is a characteristic zero field, and define:

$$P_2 = \Gamma(E) \ , \quad \partial_{2|E} = 0 \ , \qquad \text{and} \qquad \rho_2 : P_2 \longrightarrow P \ , \quad \rho_{2|E} = s_2 \ .$$

It's clear that P_2 is a dg operad with differential $\partial_2 = 0$ and ρ_2 a morphism of dg operads. Also it is a *quis* in arities ≤ 2 because:

(0) $P_2(0) = \Gamma(E)(0) = 0 = HP(0)$, because of lemma 3.4 (a),
(1) $P_2(1) = \Gamma(E)(1) = \mathbf{k} = HP(1)$, because of lemma 3.4 (a), and
(2) $P_2(2) = \Gamma(E)(2) = E(2) = HP(2)$, because of lemma 3.4 (b).

Assume we have constructed a morphism of dg operads $\rho_{n-1} : P_{n-1} \longrightarrow P$ in such a way that:

(1) P_{n-1} is a minimal operad, and
(2) $\rho_{n-1} : P_{n-1} \longrightarrow P$ is a *quis* in arities $\leq n-1$.

To build the next step, consider the Σ_n-module of the relative cohomology of $\rho_{n-1}(n) : P_{n-1}(n) \longrightarrow P(n)$

$$E = E(n) = H(P_{n-1}(n), P(n)) \ .$$

Since we work in characteristic zero, every Σ_n-module is projective. So we have a Σ_n-equivariant section $s_n = (d_n \ f_n)$ of the projection

$$P_{n-1}(n)^{+1} \oplus P(n) \supset Z(P_{n-1}(n), P(n)) \longrightarrow H(P_{n-1}(n), P(n)) \ .$$

That is, $e = \pi_n s_n e = [s_n e]$. Let

$$\begin{pmatrix} -\partial_{n-1}(n) & 0 \\ -\rho_{n-1}(n) & \partial(n) \end{pmatrix}$$

be the differential of the mapping cone $C_{\rho_{n-1}(n)}$: the cocycle condition implies that

$$\partial_{n-1}(n) d_n = 0 \quad \text{and} \quad \rho_{n-1} d_n = \partial_n(n) f_n \ .$$

That is, d_n induces a differential ∂_n on $P_n = P_{n-1} \sqcup_{d_n} \Gamma(E)$ and f_n a morphism of operads $\rho_n : P_n \longrightarrow P$ such that $\rho_{n|P_{n-1}} = \rho_{n-1}$ and $\rho_{n|E'} = f_n$, because of lemmas 4.3 and 4.4.

Let us verify that ρ_n induces an isomorphism in cohomology

$$\rho_{n*} : HP_n(m) \longrightarrow HP(m)$$

in arities $m = 0, \dots, n$. First, if $m < n$,

$$\rho_n(m) = \rho_{n-1}(m)$$

by lemma 3.4 and so, by the induction hypothesis, we are done. Again by lemma 3.4 and its definition, in arity n, ρ_n is

$$\rho_n(n) = (\rho_{n-1}(n) \quad f_n) : P_{n-1}(n) \oplus E(n) \longrightarrow P(n)$$

Let us see that $\rho_n(n)$ is a *quis*.

• $\rho_n(n)_*$ *is a monomorphism.* Let $\omega + e \in P_{n-1}(n) \oplus E(n)$ be a cocycle such that $\rho_n(n)_*[\omega + e] = 0$. Note that being a cocyle means

$$\partial_{n-1}(n)\omega + d_n e = 0$$

and the fact that $\rho_n(n)$ sends its cohomology class to zero means that we have $\nu \in P(n)$ such that

$$d\nu = \rho_{n-1}(n)\omega + f_n e \ .$$

Hence the differential of $\omega + \nu \in P_{n-1}(n)^{+1} \oplus P(n)$ in the mapping cone $C^*_{\rho_{n-1}(n)}$ is

$$\begin{pmatrix} -\partial_{n-1}(n) & 0 \\ -\rho_{n-1}(n) & \partial(n) \end{pmatrix} \begin{pmatrix} \omega \\ \nu \end{pmatrix} = \begin{pmatrix} -\partial_{n-1}(n)\omega \\ -\rho_{n-1}(n)\omega + \partial(n)\nu \end{pmatrix} = \begin{pmatrix} d_n e \\ f_n e \end{pmatrix} = s_n(e) \ .$$

Therefore, $e = \pi_n s_n e = [s_n e] = 0$, and we are left with only ω in our cocyle, which means that ω must be a cocycle itself and

$$0 = \rho_n(n)_*[\omega] = [\rho_{n-1}(n)\omega] \ .$$

So, there must be some $\nu' \in P(n)$ such that

$$\rho_{n-1}(n)\omega = \partial(n)\nu' \ .$$

Which means that $\omega + \nu' \in P_{n-1}(n)^{+1} \oplus P(n)$ is a relative cocycle of $\rho_{n-1}(n)$. Let us call

$$e' = \pi_n(\omega + \nu') = [\omega + \nu'] \in H^*(P_{n-1}(n), P(n)) = E(n)$$

its cohomology class. By definition of s_n,

$$[\omega + \nu'] = [s_n e'] = [d_n e' + f_n e'] \ ,$$

so both relative cocycles have to differ on a relative boundary:

$$\begin{pmatrix} d_n e' - \omega \\ f_n e' - \nu' \end{pmatrix} = \begin{pmatrix} -\partial_{n-1}(n) & 0 \\ -\rho_{n-1}(n) & \partial(n) \end{pmatrix} \begin{pmatrix} \omega' \\ \nu'' \end{pmatrix} \ .$$

Which in particular implies

$$\omega = \partial_{n-1}(n)\omega' + d_n e' = \partial_n(n)(\omega' + e') \ .$$

Thus $[\omega] = 0$ in $HP_n(n)$ and we are done.

• $\rho_n(n)_*$ *is an epimorphism.* From any cocyle $\nu \in P(n)$ we can build a relative one:

$$0 + \nu \in P_{n-1}(n)^{+1} \oplus P(n) \ .$$

Let us denote its cohomology class by $e = [0+\nu] \in H(P_{n-1}, P)(n) = E(n)$. Then $s_n e = d_n e + f_n e$ and $0+\nu$ are relative cohomologous cocycles. This means that there is a primitive $\omega+\nu' \in P_{n-1}(n)^{+1} \oplus P(n)$ such that:

$$\begin{pmatrix} d_n e \\ f_n e - \nu \end{pmatrix} = \begin{pmatrix} -\partial_{n-1}(n) & 0 \\ -\rho_{n-1}(n) & \partial_n(n) \end{pmatrix} \begin{pmatrix} \omega \\ \nu' \end{pmatrix} = \begin{pmatrix} -\partial_{n-1}(n)\omega \\ -\rho_{n-1}(n)\omega + \partial_n(n)\nu' \end{pmatrix} \ .$$

Particularly,

$$\nu = f_n e + \rho_{n-1}(n)\omega - \partial_n(n)\nu' = \rho_n(n)(\omega + e) + \partial_n(n)(-\nu') \ .$$

So $\rho_n(n)_*[\omega + e] = [\nu]$ and we are done.

The unitary case. Let P be a cohomologically connected and cohomologically unitary operad. As for the non-unitary case, start with $E = E(2) = HP(2)$ and do exactly the same, but now using $\Gamma = \Gamma_{+1}$.

For this, we have to prove that indeed E is a Λ-module, or, in other words, that there is an action of the unit 1 on E. Indeed, this is induced by the one on P:

$$P(2) \otimes P(0) \longrightarrow P(1) \ ,$$

after passing to cohomology:

$$HP(2) \otimes HP(0) \longrightarrow HP(1) \ .$$

It is also clear we have a *quis* in arities ≤ 2 because:

(0) $P_2(0) = \Gamma(E)(0) = \mathbf{k} = HP(0)$, because of lemma 3.4 (a),
(1) $P_2(1) = \Gamma(E)(1) = \mathbf{k} = HP(1)$, because of lemma 3.4 (a), and
(2) $P_2(2) = \Gamma(E)(2) = E(2) = HP(2)$, because of lemma 3.4 (b).

Assume we have constructed the n-stage of the unitary minimal model as before, but using $\Gamma = \Gamma_{+1}$ instead of Γ_{01}. For the inductive step, we need a Λ-structure on

$$E = E(n) = H(P_{n-1}(n), P(n))$$

in order for the piece $\Gamma(E)$ we are adding to P_{n-1} to be a unitary operad. But, since E is concentrated in arity n, there is just one way to define it:

$$\Gamma(E)(n) \otimes \mathbf{k} \longrightarrow \Gamma(E)(n-1) = 0 \ .$$

□

Remark 5.4. Notice that the action of the units on $(P_+)_\infty$ we have built

$$\circ_i : (P_+)_\infty(n) \otimes (P_+)_\infty(0) \longrightarrow (P_+)_\infty(n-1) \ , n > 1 \ , i = 0, \dots, n-1$$

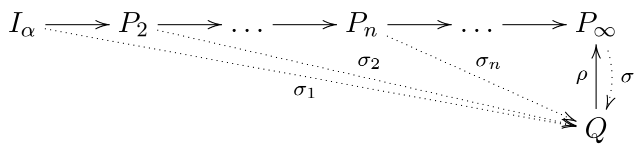
reduces to two cases:

(a) For $n = 2$, it is just the induced action from P_+: $HP_+(2) \otimes HP_+(0) \longrightarrow HP_+(1)$
(b) For $n > 2$, it is the trivial action $\omega \mapsto \omega \circ_i 1 = 0$.

5.2. Uniqueness. The following lemma will provide a proof of the uniqueness up to isomorphism of minimal models. It is inspired in [HT90], definition 8.3 and theorem 8.7. It also inspired a categorical definition of minimal objects: see [Roi93] and [Roi94b], cf [GNPR10].

Lemma 5.5. *Let P_∞ be a Sullivan minimal operad and $\rho : Q \longrightarrow P_\infty$ a* quis *of non-unitary operads. Then there is a section $\sigma : P_\infty \longrightarrow Q$, $\rho\sigma = \mathrm{id}_{P_\infty}$.*

Proof. We are going to build the section $\sigma : P_\infty \longrightarrow Q$ inductively on the arity:



in such a way that:

(1) $\rho\sigma_n = \mathrm{id}_{P_n}$ (note that, because of the minimality, $\mathrm{im}\,\rho\sigma_n \subset P_n$), and
(2) $\sigma_{n|P_{n-1}} = \sigma_{n-1}$.

In both cases, cohomologically non-unitary and cohomologically unitary, start with the universal morphism $\sigma_1 : I_\alpha \longrightarrow Q$ from the initial operad I_α to Q, $\alpha = 0, 1$. It's clear that $\rho\sigma_1 = I_\alpha$.

Let us assume that we have already constructed up to $\sigma_{n-1} : P_{n-1} \longrightarrow Q$ satisfying conditions above (1) and (2) and let us define $\sigma_n : P_n \longrightarrow Q$ as follows: first, take the Σ-module

$$Q_{n-1} := \mathrm{im}\,(\sigma_{n-1} : P_{n-1} \longrightarrow Q) \ .$$

By induction hypothesis, σ_{n-1} is a monomorphism, so $\sigma_{n-1} : P_{n-1} \longrightarrow Q_{n-1}$ is an isomorphism of Σ-modules and $\rho_{|Q_{n-1}}$ its inverse.

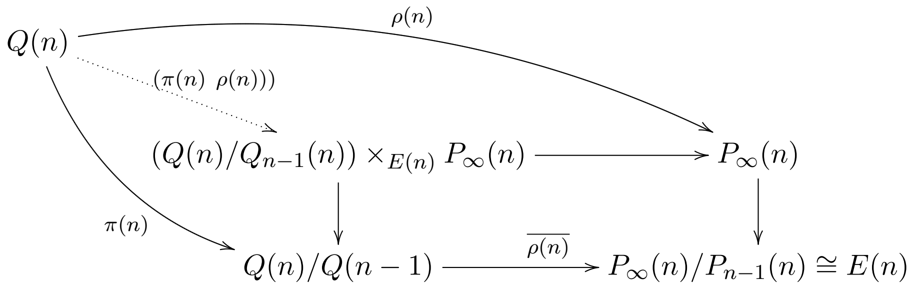
Next, consider the following commutative diagram of Σ_n-modules,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_{n-1}(n) & \longrightarrow & Q(n) & \longrightarrow & Q(n)/Q_{n-1}(n) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \rho(n)_{|Q_{n-1}}} & & \big\downarrow{\scriptstyle \rho(n)} & & \big\downarrow{\scriptstyle \overline{\rho}(n)} & & \\ 0 & \longrightarrow & P_{n-1}(n) & \longrightarrow & P_\infty(n) & \longrightarrow & P_\infty(n)/P_{n-1}(n) & \longrightarrow & 0 \end{array} .$$

in which the horizontal rows are exact. As we said, the first column is an isomorphism and the second a *quis*. So the third column is also a *quis*. By minimality and lemma 3.4, $P_\infty(n) = P_n(n) = P_{n-1} \oplus E(n)$. Hence, $P_\infty(n)/P_{n-1}(n) \cong E(n)$, with zero differential. So we have an epimorphism of Σ_n-modules,

$$Z(Q(n)/Q_{n-1}(n)) \longrightarrow H(Q(n)/Q_{n-1}(n)) \cong E(n) \ .$$

Take a section $s : E(n) \longrightarrow Z(Q(n)/Q_{n-1}(n))$ and consider the pull-back of Σ_n-modules



and the induced morphism $(\pi(n)\ \rho(n))$. This turns out to be an epimorphism: if

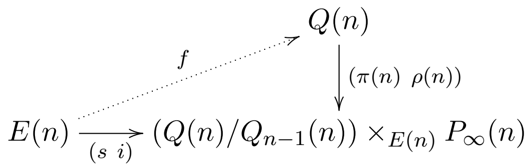
$$(\overline{\omega}, \nu) \in (Q(n)/Q_{n-1}(n)) \times_{E(n)} P_\infty(n)\ ,$$

it means that $\overline{\rho}(n)\overline{\omega} = \overline{\nu}$. That is to say, $\rho(n)\omega - \nu \in P_{n-1}(n)$. Then

$$\begin{aligned}(\pi(n)\ \rho(n))(\omega - \sigma_{n-1}(n)(\rho(n)\omega - \nu)) &= (\overline{\omega} - 0, \rho(n)\omega - \rho(n)\sigma_{n-1}(n)(\rho(n)\omega - \nu)) \\ &= (\overline{\omega}, \nu)\end{aligned}$$

by induction hypothesis.

Let $i : E(n) \hookrightarrow P_\infty(n)$ denote the inclusion. We can lift $(s\ i)$ in the diagram



to a morphism $f : E(n) \longrightarrow Q(n)$ such that $(\pi(n)\ \rho(n)) \circ f = (s\ i)$. Note that here we are using bare projectiviness to lift morphisms, since we don't need them to commute with any differentials at this stage. Finally, define $\sigma_n : P_n \longrightarrow Q$ by

$$\sigma_{n|P_{n-1}} = \sigma_{n-1} \qquad \text{and} \qquad \sigma_{n|E(n)} = f\ .$$

According to the universal property of principal extensions 4.4, in order to check that σ_n is a morphism of operads, we only need to prove that

$$\sigma_{n-1} d_n e = \partial_{Q(n)} f e$$

for every $e \in E(n)$. Very well: since $\pi(n) f e = se \in Z(Q(n)/Q_{n-1}(n))$, we have $\overline{0} = \overline{\partial} se = \overline{\partial}\pi(n) f e = \pi(n)\partial f e$. Hence, $dfe \in Q_{n-1}(n) = \operatorname{im}\sigma_{n-1}(n)$. Let $\omega \in P_{n-1}(n)$ be such that $\partial f e = \sigma_{n-1}(n)\omega$. Then, apply $\rho(n)$ to both sides of this last equality:

$$\rho(n)\overline{\partial} f e = \rho(n)\sigma_{n-1}(n)\omega = \omega\ ,$$

by induction hypothesis, and also

$$\rho(n)\partial f e = \partial\rho(n) f e = de\ ,$$

because f is a lifting of $(s\ d)$. So $de = \omega$ and hence $\sigma_{n-1}de = \sigma_{n-1}\omega = \partial fe$, as we wanted.

Finally, $\rho\sigma_n = \mathrm{id}_{P_n}$ because $\rho\sigma_{n|P_{n-1}} = \rho\sigma_{n-1} = \mathrm{id}_{P_{n-1}}$, by induction hypothesis, and $\rho fe = e$, because f lifts $(s\ i)$. □

From this lemma, uniqueness follows at once.

Proposition 5.6. *Let $\rho : P_\infty \longrightarrow P'_\infty$ be a* quis *between minimal Sullivan operads. Then, ρ is an isomorphism.*

Proof. Because of the previous lemma 5.5, ρ has a section σ which, by the two out of three property is also a *quis*. So σ also has a section and it's both a monomorphism and an epimorphism. □

Theorem 5.7. *Let $\varphi : P_\infty \longrightarrow P$ and $\varphi' : P'_\infty \longrightarrow P$ be two Sullivan minimal models of P. Then there is an isomorphism $\psi : P_\infty \longrightarrow P'_\infty$, unique up-to-homotopy, such that $\varphi'\psi \simeq \varphi$.*

Proof. The existence of the ψ follows from the up-to-homotopy lifting property 4.7. It is a *quis* because the 2 out of 3 property and so an isomorphism because of the previous proposition. □

6. Miscellanea

In this section we develop some corollaries relating the minimal models of P_+ and P, and stablishing their relationship with up-to-homotopy algebras. Namely:

(6.1) We compare the minimal model of an unitary operad P_+ and its non-unitary truncation P.
(6.2) We relate the minimal model of an unitary operad $(P_+)_\infty$ and the up-to-homotopy P_+-algebras with strict units.
(6.3) For the case of the unitary associative operad, we compare our minimal model $su\mathcal{A}ss_\infty = \mathcal{A}ss_{+\infty}$ with the one of up-to-homotopy algebras with up-to-homotopy units, $hu\mathcal{A}ss_\infty$.
(6.4) Here we extend some results of our previous paper [CR19], that we could not address there for the lack of minimal models for unitary operads.
(6.5) We complete the results of [GNPR05] concerning the formality of operads, so as to include unitary operads.

6.1. Minimal models of an operad and its unitary extension. Let P be an operad admitting a unitary extension P_+. We clearly have a split exact sequence of Σ-modules

$$0 \longrightarrow P \longrightarrow P_+ \longrightarrow \mathbf{k}[1] \longrightarrow 0\ .$$

Here, $P \longrightarrow P_+$ is the canonical embedding, $\mathbf{k}[1] = \mathbf{k}$ the Σ-module which is just a $\mathbf{k}$-vector space on one generator 1 in arity-degree $(0,0)$ and zero outside and $P_+ \longrightarrow \mathbf{k}[1]$ the projection of Σ-modules that sends $P(l)$, $l > 0$ to zero and the identity on $P_+(0) = \mathbf{k}$. We could also write

$$P_+ = P \oplus \mathbf{k}[1]$$

as Σ-modules.

Proposition 6.1. *For every (cohomologically) unitary (cohomologically) connected operad P_+ we have an isomorphism of operads*

$$(P_+)_\infty = (P_\infty)_+\ .$$

In particular, we have an isomorphism of Σ-modules,

$$(P_+)_\infty = P_\infty \oplus \mathbf{k}[1] \ .$$

Proof. As we have said, for every P_+, its truncation P is a Λ-operad and this structure passes to its minimal model P_∞. So, we have a unitary extension $(P_\infty)_+$. Let's see how this unitary extension agrees with the minimal model of P_+. Indeed, P_∞ is a colimit of principal extensions

$$I_0 \longrightarrow P_2 = \Gamma\left(E(2)\right) \longrightarrow \cdots \longrightarrow P_n = \Gamma\left(\bigoplus_{n\geq 2} E(n)\right) \longrightarrow \cdots \longrightarrow \varinjlim_n P_n = P_\infty$$

starting with the non-unitary initial operad I_0. For the same reasons we just remarked about P_∞, all these operads P_n have unitary extensions. So we can take the unitary extension of the whole sequence

$$I_+ \longrightarrow (P_2)_+ = \Gamma(E(2))_+ \longrightarrow \cdots \longrightarrow (P_n)_+ = \Gamma\left(\bigoplus_{n\geq 2} E(n)\right)_+ \longrightarrow \cdots \longrightarrow (\varinjlim_n P_n)_+ = (P_\infty)_+ \ .$$

But, as we noticed in lemma 2.2, the functor $(\ )_+$ commutes with colimits, so $(P_\infty)_+ = (\varinjlim_n P_n)_+ = \varinjlim_n (P_{n+}) = (P_+)_\infty$. □

6.2. Minimal models and up-to-homotopy algebras. In the non-unitary case, the importance of these minimal models P_∞ is well-known: they provide a *strictification* of up-to-homotopy P-algebras. That is, up-to-homotopy P-algebras are the same as regular, *strict* P_∞-algebras. One way to prove it is the following: first we have a commonly accepted definition for up-to-homotopy P-algebras, at least for *Koszul* operads. Namely, the one in [GK94]:

Definition 6.2. ([GK94], see also [LV12]) Let P a Koszul operad. Then an *up-to-homotopy* P-algebra is an algebra over the Koszul resolution (model) ΩP^{i} of P.

Then one proves that $\Omega P^{\text{i}} \stackrel{\sim}{\longrightarrow} P$ is a minimal model of P, in the sense that it is unique up to isomorphism ([LV12], corollary 7.4.3). Since Markl's minimal model *à la Sullivan* $P_\infty \stackrel{\sim}{\longrightarrow} P$ is also minimal and cofibrant, we necessarily have an isomorphism $P_\infty = \Omega P^{\text{i}}$ (see [Mar96]).

Then one has to check that this definition as ΩP^{i}-algebras also agrees with the definitions through "equations" in the particular cases. For instance, one has to check that $\mathcal{A}ss_\infty = \Omega \mathcal{A}ss^{\text{i}}$-algebras are the same as A_∞-algebras, defined as dg modules, together with a sequence of n-ary operations

$$\mu_n : A^{\otimes n} \longrightarrow A, n \geq 2, \quad |\mu_n| = 2 - n \ ,$$

satisfying the equations

$$\partial(\mu_n) = \sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{qr+p+1} \mu_m \circ_{p+1} \mu_q \ .$$

(See [LV12].)

We would like to say that the same is true in the unitary case, in other words, to prove a theorem such as

Theorem 6.3. *Up-to-homotopy P_+-algebras with strict unit are the same as $(P_+)_\infty$-algebras.*

But for this, one important ingredient is missing: we lack a common, accepted definition for up-to-homotopy P_+-algebras with strict units. To the best of our knowledge, such a definition exists only for the operad $u\mathcal{A}ss = \mathcal{A}ss_+$. For instance, the one in [KS09], definition 4.1.1 (cf. [Lyu11], [Bur18]):

Definition 6.4. An A_∞-algebra A is said to have a *strict unit* if there is an element $1 \in A$ of degree zero such that $\mu_2(1, a) = a = \mu_2(a, 1)$ and $\mu_n(a_1, \dots, 1, \dots, a_n) = 0$ for all $n \neq 2$ and $a, a_1, \dots, a_n \in A$.

So we prove our theorem for the only case currently possible: $P_+ = \mathcal{A}ss_+$.

Theorem 6.5. *Up-to-homotopy A_∞-algebras with strict unit are the same as $\mathcal{A}ss_{+\infty}$-algebras.*

Proof. We just have to prove that the unit $1 \in \mathcal{A}ss_{+\infty}(0)$ acts as described in the definition. Namely,

$$\mu_2 \circ_1 1 = \mathrm{id} = \mu_2 \circ_2 1 \ , \qquad \text{and} \qquad \mu_n \circ_i 1 = 0 \ , \quad \text{for all} \quad n \neq 2 \ , \quad \text{and} \quad i = 1, \dots, n$$

As for the first equations, because of remark 5.4, partial composition products

$$\circ_i : \mathcal{A}ss_{+\infty}(2) \otimes \mathcal{A}ss_{+\infty}(0) \longrightarrow \mathcal{A}ss_{+\infty}(1) \ , \quad i = 1, 2 \ ,$$

are induced by

$$\circ_i : \mathcal{A}ss_{+}(2) \otimes \mathcal{A}ss_{+}(0) \longrightarrow \mathcal{A}ss_{+}(1) \ , \quad i = 1, 2$$

which verify said identities.

As for the rest of the equations, for $n > 2$, again because of remark 5.4, partial composition products $\circ_i : \mathcal{A}ss_{+\infty}(n) \otimes \mathcal{A}ss_{+\infty}(0) \longrightarrow \mathcal{A}ss_{+\infty}(n-1)$ are trivial. □

6.3. Strict units and up-to-homotopy units. Here we compare two minimal models of a unitary operad P_+: the one with *strict units* that we developed $(P_+)_\infty$, and the one with *up-to-homotopy units* that we find for the case of the unitary associative operad in [HM12] or [Lyu11]. We will use the notations $su\mathcal{A}ss_\infty = \mathcal{A}ss_{+\infty}$ and $hu\mathcal{A}ss_\infty$, respectively.

As [HM12] mentions, $su\mathcal{A}ss_\infty$ *cannot* be cofibrant, nor *minimal* and cofibrant, since if it were, we would have two *quis* $su\mathcal{A}ss_\infty \xrightarrow{\sim} u\mathcal{A}ss \xleftarrow{\sim} hu\mathcal{A}ss_\infty$ and hence, by the up-to-homotopy lifting property and the fact that both are minimal, we would conclude that both operads $su\mathcal{A}ss_\infty$ and $hu\mathcal{A}ss_\infty$ should be isomorphic, which we know they clearly are not, just by looking at their presentations:

$$hu\mathcal{A}ss_\infty = \Gamma(\{\mu_n^S\}_{S, n\geq 2}) \ ,$$

(see [HM12], [Lyu11]) and

$$su\mathcal{A}ss_\infty = \Gamma_{+1}(\{\mu_n\}_{n\geq 2}) = \frac{\Gamma(1, \{\mu_n\}_{n\geq 2})}{\langle \mu_2 \circ_1 1 - \mathrm{id},\ \mu_2 \circ_2 1 - \mathrm{id},\ \{\mu_n \circ_i 1\}_{n\geq 2, i=1,\dots,n}\rangle}$$

Nevertheless, apparently we have indeed proven that $su\mathcal{A}ss_\infty$ is a minimal and cofibrant operad. And it is of course, but *as a unitary operad*, in $\mathbf{Op}_{+1}$. Even though *it is not as an operad* in $\mathbf{Op}$. Indeed, looking at its presentation, we see that it seems to lack the first condition of minimality; i.e., being free as a graded operad. Again, there is no contradiction at all: it is free graded *as a unitary operad*; that is, in $\mathbf{Op}_{+1}$.

So, summing up: $su\mathcal{A}ss_\infty$ is an honest minimal, cofibrant and graded-free operad *in the category of unitary operads* $\mathbf{Op}_+$, while it is none of the above in the category of all operads $\mathbf{Op}$.

This apparently paradoxical phenomenon of an object being minimal, or cofibrant, in a subcategory, and losing these properties in a bigger category containing it it is not so new and has already been observed (see, for instance, [Roi94b], remark 4.8). Here we present another example of this phenomenon, but in the category of dg commutative algebras.

Example 6.6. Let $\mathbf{Cdga}(\mathbb{Q})$ denote the category of dg commutative algebras, *without unit.* Let $\mathbf{Cdga}(\mathbb{Q})_1$ denote the category of algebras *with unit.* By forgetting the unit, we can consider $\mathbf{Cdga}(\mathbb{Q})_1$ as a subcategory of $\mathbf{Cdga}(\mathbb{Q})$.

$\mathbb{Q}$, being the initial object in $\mathbf{Cdga}(\mathbb{Q})_1$, is free, cofibrant and minimal in $\mathbf{Cdga}(\mathbb{Q})_1$. Indeed, if we denote by Λ_1 the free graded commutative algebra *with unit* functor, then $\Lambda_1(0) = \mathbb{Q}$: the free graded commutative algebra *with* unit on the $\mathbb{Q}$-vector space 0.

However, it is *neither* minimal, nor cofibrant, nor free as an object in the larger category $\mathbf{Cdga}(\mathbb{Q})$. To see this, let us denote by Λ the free graded-commutative algebra *without* unit. As an algebra *without* unit, $\mathbb{Q}$ has an extra relation. Namely, $1^2 = 1$. So, it is *not* a free algebra in $\mathbf{Cdga}(\mathbb{Q})$:

$$\mathbb{Q} = \Lambda_1(0) = \frac{\Lambda(1)}{(1^2 - 1)}$$

Next, consider the free graded-commutative algebra *without* unit $\Lambda(t, x)$ on two generators t, x in degrees $|t| = -1$ and $|x| = 0$ and differential $dx = 0$ and $dt = x^2 - x$. Hence, as a graded vector space,

$$\Lambda(t,x)^i = \begin{cases} (x), & \text{if } i = 0, \\ [t, tx, tx^2, \dots, tx^n, \dots], & \text{if } i = -1, \\ 0, & \text{otherwise.} \end{cases}$$

where:

(1) (x) is the ideal generated by x in the polynomial algebra $\mathbb{Q}[x]$. That is, the $\mathbb{Q}$-vector space $[x, x^2, \dots, x^n, \dots]$
(2) $[t, tx, tx^2, \dots, tx^n, \dots]$ means the $\mathbb{Q}$-vector space generated by those vectors.

Consider the morphism of algebras *without* unit

$$\varphi : \Lambda(t,x) \longrightarrow \mathbb{Q}$$

defined by $\varphi(x) = 1, \varphi(t) = 0$. It's clear that φ is a *quis* and an epimorphism. So, if $\mathbb{Q}$ were a minimal and cofibrant algebra *without unit*, we would have a section $\sigma : \mathbb{Q} \longrightarrow \Lambda(t, x)$, $\varphi\sigma = \mathrm{id}$. For degree reasons, we would then have $\sigma(1) = p(x)$, for some polynomial $p(x) \in (x)$. That is, a polynomial of degree ≥ 1. But, since $\sigma(1)\sigma(1) = \sigma(1^2) = \sigma(1)$, we would get $p(x)^2 = p(x)$, which is impossible for a polynomial of degree ≥ 1.

Hence, $\mathbb{Q}$ is graded-free, cofibrant and minimal *as an algebra with unit.* But it's neither of those things *as an algebra without unit.* In fact, we could argue that we have computed its minimal model $\Lambda(t, x)$ in $\mathbf{Cdga}(\mathbb{Q})$, but this would lead us to develop the theory of minimal dg commutative algebras without unit, possible generators in degree zero, and elements of negative degrees, which is beyond the scope of this paper.

6.4. Minimal models of operad algebras for tame operads. In [CR19] we proved the existence and uniqueness of Sullivan minimal models for operad algebras, for a wide class of operads we called "tame", and for operad algebras satisfying just the usual connectivity hypotheses.

Of particular importance was the fact that, if an operad P is tame, then its minimal model P_∞ is also tame: that is, P_∞-algebras also have Sullivan minimal models [CR19], proposition 4.10. This provides minimal models for $\mathcal{A}ss_\infty, \mathcal{C}om_\infty$ and $\mathcal{L}ie_\infty$-algebras, for instance. Since at that time we were not aware of the possibility of building minimal models for unitary operads, there was a gap in our statements, meaning we had to formulate them only for non-unitary operads (there called "reduced"). Now we can mend that gap.

Proposition 6.7. *Let $P \in$ **Op** be a cohomologically connected and cohomologically non-unitary, or unitary r-tame operad. Then its minimal model is a also r-tame.*

Proof. Indeed, the presence of a non-trivial arity zero $P(0)$ adds nothing to the condition of being tame or not. □

Corollary 6.8. *Every cohomologically connected $\mathcal{A}ss_{+\infty}$ or $\mathcal{C}om_{+\infty}$-algebra has a Sullivan minimal model. Also every 1-connected $\mathcal{G}er_{+\infty}$-algebras has a Sullivan minimal model.*

Then we went on to prove the same results for pairs (P, A), where P is a tame operad and A a P-algebra, thus providing a global invariance for our minimal models in the form of a minimal model $(P_\infty, \mathcal{M}) \stackrel{\sim}{\longrightarrow} (P, A)$ in the category of such pairs, the category of operad algebras *over variable operads*. We can add now unitary operads to that result too.

Theorem 6.9. *Let P be a cohomologically connected and cohomologically non-unitary, or unitary, r-tame operad and A an r-connected P-algebra. Then (P, A) has a Sullivan r-minimal model $(P_\infty, \mathcal{M}) \stackrel{\sim}{\longrightarrow} (P, A)$.*

6.5. Formality. It has been pointed out by Willwacher in his speech at the 2018 Rio's International Congress of Mathematicians, [Wil18], talking about the history of the formality of the little disks operad, that our paper [GNPR05] missed the arity zero. Here we complete the results of that paper for the unitary case.

Proposition 6.10. *Let P_+ be a unitary dg operad with $HP_+(0) = HP_+(1) = \mathbf{k}$. Then*

$$P_+ \text{ is a formal operad} \qquad \Longleftrightarrow \qquad P \text{ is a formal operad}$$

Proof. Since the truncation functor is exact, implication $\Longrightarrow$ is clear. In the opposite direction, because of the hypotheses, P and P_+ have minimal models P_∞ and $(P_\infty)_+$. Assume P is formal. Then we have a couple of *quis*

$$HP \stackrel{\sim}{\longleftarrow} P_\infty \stackrel{\sim}{\longrightarrow} P \ .$$

Applying the unitary extension functor to this diagram, and taking into account that it is an exact functor because of 2.2, we get

$$(HP)_+ \stackrel{\sim}{\longleftarrow} (P_\infty)_+ \stackrel{\sim}{\longrightarrow} P_+ \ .$$

Which, it is just

$$(HP_+) \stackrel{\sim}{\longleftarrow} (P_+)_\infty \stackrel{\sim}{\longrightarrow} P_+ \ .$$

Hence, P_+ is also a formal operad. □

Corollary 6.11. *(cf.* [Kon99], [Tam03], [GNPR05], [LV14], [FW18]*) The unitary n-little disks operad $\mathcal{D}_{n+}$ is formal over $\mathbb{Q}$.*

Proof. Follows from [GNPR05], corollary 6.3.3 and our previous proposition 6.10. □

We can also offer a unitary version of the main theorem 6.2.1 in *op.cit.* about the independence of formality from the ground field.

Corollary 6.12. *(cf.* [Sul77], [HS79], [Roi94a], [GNPR05]*) Let $\mathbf{k}$ be a field of characteristic zero, and let $\mathbf{k} \subset \mathbf{K}$ be a field extension. If P is a cohomologically connected and cohomologically unitary dg $\mathbf{k}$-operad with finite type cohomology, then the following statements are equivalent:*

(1) *P is formal.*
(2) *$P \otimes \mathbf{K}$ is formal.*

Proof. Because the statements only depend on the homotopy type of the operad, we can assume P to be minimal, and hence connected and unitary: let's call it P_+. Then, P_+ is formal if and only if P is so, because of previous proposition 6.10. Because of op.cit. theorem 6.2.1, P is formal if and only if $P \otimes \mathbf{K}$ is so. Because of previous proposition 6.10, this is true if and only if $(P \otimes \mathbf{K})_+ = P_+ \otimes \mathbf{K}$ is formal. □

The interested reader can easily check that the rest of the sections of [GNPR05] concerning non-unitary operads admit similar extensions to unitary ones. This is true, even for the finite type results like theorem 4.6.3 in *op.cit* from which the descent of formality hinges:

Theorem 6.13. *Let P be a cohomologically connected and cohomologically non-unitary, or unitary operad. If the cohomology of P is of finite type, then its minimal model P_∞ is of finite type.*

And this is so because, even in the unitary case, $P_\infty = \Gamma(E)$, with $E(0) = E(1) = 0$.

In particular, we have the celebrated Sullivan's criterium of formality based on the lifting of a *grading automorphism* also for unitary operads.

Definition 6.14. Let $\alpha \in \mathbf{k}^*$ to not be a root of unity and C a complex of $\mathbf{k}$-vector spaces. The *grading automorphism* ϕ_α of HC is defined by $\phi_\alpha = \alpha^i \mathrm{id}_{HC^i}$ for all $i \in \mathbb{Z}$. A morphism of complexes f of C is said to be a *lifting* of the grading automorphism if $Hf = \phi_\alpha$.

Proposition 6.15. *(cf.* [Sul77], [GNPR05], [Pet14]*) Let P be a cohomologically connected and cohomologically non-unitary or unitary operad with finite type cohomology. If for some nonroot of unity $\alpha \in \mathbf{k}^*$, P has a lifting of ϕ_α, then P is formal.*

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