Chaos in the hysteretic grazing-sliding codimension-one saddle-node bifurcation of piecewise dynamical systems.

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We present two ways of regularizing a parameter family of piecewise smooth dynamical systems undergoing a grazing-sliding bifurcation. We use the Sotomayor-Teixeira regularization and prove that the bifurcation is a saddle-node (see [**?**]). Then we perform a hysteretic regularization. However, in spite that the two regularization will give the same dynamics in the sliding modes (see [**?**]), when a tangency appears, so is in the case of grazing-sliding, the hysteretic process generate chaotic dynamics. Finally, we smooth the hysteresis by embedding the system in a higher dimension. Now the discontinuous control variable u is also a continuous time dependent variable although a fast-fast one. We then encounter loop feedback chaotic behaviour.

1 The Sotomayor-Teixeira regularization of a grazing-sliding bifurcation

Let variables $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$ satisfy the differential equation

$$\begin{array}{rcl} \dot{x} & = & f(x,y;u) \\ \dot{y} & = & g(x,y;u) \end{array} \tag{1}$$

where f and g are smooth functions of x, y, u, and where u (the control) is given by

$$u = \mathrm{sign}(y)\ . \tag{2}$$

The values of the vector field either side of the switch $y = 0$ can be written as

$$f^{\pm}(x,y) := f(x,y;\pm 1), g^{\pm}(x,y) := g(x,y;\pm 1)\ . \tag{3}$$

This is typical example of a piecewise smooth system. There are many ways of regularization, that is, ways of unfolding this system in a parametric family of smooth (in the switch also) vector fields system. Different unfolding will produce, if any, different dynamics in the switch. So the regularization used must be consistent with the dynamics we want in the switching manifold.

One of the most popular regularization is the Sotomayor-Teixeira regularization: (in two dimensional setting for simplicity)

$$\left.\begin{array}{rcl} \dot{x} & = & \frac{1+\varphi(\frac{y}{\epsilon})}{2} f^{+}(x,y) + \frac{1-\varphi(\frac{y}{\epsilon})}{2} f^{-}(x,y) \\ \dot{y} & = & \frac{1+\varphi(\frac{y}{\epsilon})}{2} g^{+}(x,y) + \frac{1-\varphi(\frac{y}{\epsilon})}{2} g^{-}(x,y) \end{array}\right\} \tag{4}$$

where φ is any function satisfying

$$\left.\begin{array}{r} \varphi = sign(w), |w| > 1 \\ \varphi \in [-1,1], |w| \leq 1 \\ \varphi'(w) \geq 0 \end{array}\right\} \tag{5}$$

So, the $sign(y)$ function, is approximated in an ϵ boundary layer of the switching manifold $y = 0$, by $\varphi(\frac{y}{\epsilon})$. The usual variable change $v = \frac{y}{\epsilon}$, transform it in:

$$\left.\begin{array}{rcl} \dot{x} & = & \frac{1+\varphi(v)}{2} f^{+}(x,\epsilon v) + \frac{1-\varphi(v)}{2} f^{-}(x,\epsilon v) \\ \epsilon\dot{v} & = & \frac{1+\varphi(v)}{2} g^{+}(x,\epsilon v) + \frac{1-\varphi(v)}{2} g^{-}(x,\epsilon v) \end{array}\right\} \tag{6}$$

It is a slow-fast system. Fenichel theory says that, ϵ-near the sliding parts of the switching, there exists an exponentially attracting manifold(s)and the flow there tends regularly to the so called Filippov sliding mode:

$$\dot{x} = f_F(x) := \frac{f^+g^- - f^-g^+}{g^- - g^+}(x,0) \tag{7}$$

Moreover, this manifold can be continued to non sliding (non hyperbolic) points, a fold, for instance. In [**?**] we have studied the behaviour when X^+ has a tangency to $y = 0$ (a fold) in $(0,0)$ and X^- is transversal there. The results can be summarized in Figure **??** and in Theorem (**??**)(see [**?**]).

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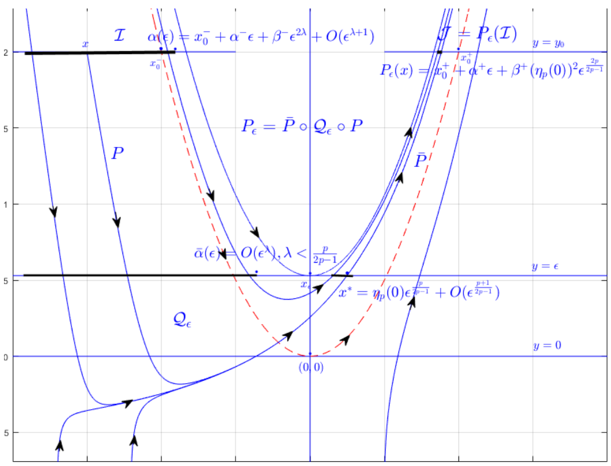

Figure 1: Dynamics of the Poincaré map P_ϵ for the regularized system (6). The large domain $\mathcal{I}$ is smashed to the small $\mathcal{J}$. The dotted red parabola is the trajectory of X^+ passing through the fold $(0,0)$.

Consider the regularized vector field (6) with φ of class $\mathcal{C}^{p-1}$ $(p \geq 2)$ and fix any $0 < \lambda < \frac{p}{2p-1}$.
Then there exist $\epsilon_0 > 0$, $L^- < 0$,constants $\alpha^- > 0$, $\beta^- < 0$, $\alpha^+ < 0$, $\beta^+ > 0$ depending on X^+ and $\alpha(\epsilon) = x_0^- + \alpha^-\epsilon + \beta^-\epsilon^{2\lambda} + O(\epsilon^{\lambda+1})$ such that, for $0 < \epsilon \leq \epsilon_0$, the map P_ϵ restricted to the interval $\mathcal{I} := [L^-, \alpha(\epsilon)]$ is a Lipschitz function with Lipschitz constant exponentially small in ϵ and satisfies:

$$P_\epsilon(x) = x_0^+ + \alpha^+\epsilon + \beta^+(\eta(0))^2\epsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\epsilon^{\frac{2p+1}{2p-1}}),\ \forall x \in \mathcal{I},$$

where $\eta(u)$ is the unique solution of equation:

$$\text{(8)} \qquad \frac{d\eta}{du} = \frac{2}{4\eta - \frac{\varphi^{(p)}(1)}{p!}u^p}$$

satisfying $\eta(u) - \frac{\varphi^{(p)}(1)}{4p!}u^p \to 0$ as $u \to -\infty$. Moreover the Poincaré map $\mathcal{Q}_\epsilon$ is defined in the set $[L^-, -\epsilon^\lambda] \times \{\epsilon\}$, its Lipschitz constant is exponentially small in ϵ and

$$\text{(9)} \quad \forall x \in [L^-, -\epsilon^\lambda], \quad \mathcal{Q}_\epsilon(x) = \eta_0(0)\epsilon^{\frac{p}{2p-1}} + \mathcal{O}(\epsilon^{\frac{p+1}{2p-1}}).$$

As the Fenichel solution attracts a wide region, this theorem is particularly useful to study global behaviour in case that system X^+ has any recurrence. As a direct application consider Z_μ, a family of non-smooth planar systems having a grazing sliding bifurcation of a hyperbolic attracting or repelling periodic orbit of the vector field X_μ at $\mu = 0$. Then we have:

Let Z_μ, $\mu \in$ be a family of non-smooth planar systems that undergoes a grazing sliding bifurcation of a hyperbolic periodic orbit Γ_μ of the vector field X_μ at $\mu = 0$. We assume that, for $\mu > 0$ the periodic orbit Γ_μ is entirely contained in $\mathcal{V}^+$, it becomes tangent to Σ for $\mu = 0$ and intersects both regions $\mathcal{V}^\pm$ for $\mu < 0$.

Consider the regularized family $Z_{\mu,\epsilon}$.

- If Γ_μ is attracting, the regularized system has a periodic orbit $\Gamma_{\mu,\epsilon}$ for any ϵ, μ small enough. No bifurcation occurs in the regularized system.
- If Γ_μ is repelling, the regularized system has a periodic orbit $\Gamma_{\mu,\epsilon}$ for any $\mu > 0$ and $0 < \epsilon < \epsilon_0(\mu)$ which co-exists with the periodic orbit Γ_μ contained in $\mathcal{V}^+ \cap \{(x,y),\ y > \epsilon\}$. This result is also true for $\mu = \mathcal{O}(\epsilon)$. For $\mu \leq 0$ small enough, the system has no periodic orbits near Γ_0 if ϵ is small enough. Therefore the family $Z_{\mu\epsilon}$ undergoes a bifurcation of periodic orbits near $\mu = 0$.

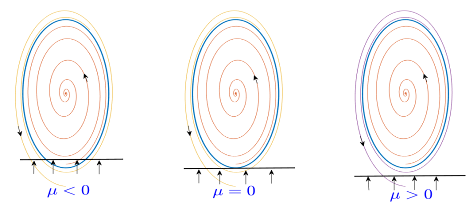

Figure 2: The position of fields X_μ^+ and $X^- = (0,1)$ with respect to the switching line $y = 0$ for the different values of μ

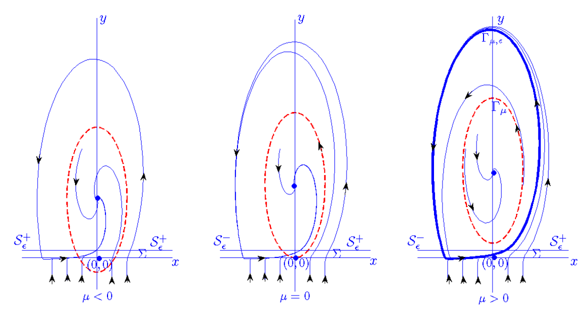

Figure 3: Periodic orbit of the regularized family $Z_{\mu,\epsilon}$ when X_μ^+ has a repelling periodic orbit Γ_μ (the red dotted orbit). For $\mu < 0$ and $\mu = 0$ the regularized system has no periodic orbits. For $\mu > 0$ the regularized system has a periodic orbit $\Gamma_{\mu,\epsilon}$ which coexists with Γ_μ. A bifurcation of periodic orbits in the regularized system corresponding with the grazing-sliding bifurcation in the Filippov system can occur.

So only in the repelling case there is a bifurcation. In [**?**] we prove it and also we present numerical evidence that the bifurcation is of saddle-node type, but not a rigourous proof. Now we have just ended a proof and will present it in a forthcoming paper.

In the hypothesis of Theorem (**??**), and if Γ_μ is repelling, the regularized system has a periodic orbit $\Gamma_{\mu,\epsilon}$ for any $\mu > 0$ and $0 < \epsilon < \epsilon_0(\mu)$ which co-exists with the periodic orbit Γ_μ contained in $\mathcal{V}^+ \cap \{(x,y),\ y > \epsilon\}$. This result is also true for $\mu = \mathcal{O}(\epsilon)$. For $\mu \leq 0$ small enough, the system has no periodic orbits near Γ_0 if ϵ is small enough.The family $Z_{\mu\epsilon}$ undergoes a saddle node bifurcation of periodic orbits near $\mu = 0$.

The steps to prove Theorem (**??**)are:

- As Theorem (**??**) says,the parameter of bifurcation must satisfy $0 < \mu < \tilde{\mu}\varepsilon$
- If the orbit of the regularized system enters to the interior of the unstable circle through $y = \epsilon$ then this orbit will be trapped by the focus. That's for two dimensional topological reasons of a focus and the fact that in $|y| \leq \epsilon$ the regularized system is bounded by X^+.
- Then, the bifurcation must occur when the Fenichel solution(s) and the upper segment of the periodic orbit collide. That is $\mu = \epsilon - \frac{\eta_0(0)^2}{2}\epsilon^{\frac{4}{3}} + \mathcal{O}(\epsilon^{\frac{5}{3}})$
- In the neighbourhood of this parameter the Poincare map $\mathcal{Q}_\epsilon$ is decreasing and concave, and the external P^e is convex. Then $P^e \circ \mathcal{Q}_\epsilon$ is convex
- Necessarily, the bifurcation must be a saddle-node.

As an example, let's take the family of vector fields $Z_\mu = (X^+_\mu, X^-_\mu)$ where X^+ is given by

$$\begin{array}{llll} \dot{x} & = f(x,y,\mu) = & -y + \mu + 1 + x(r-1) \\ \dot{y} & = g(x,y,\mu) = & x + (y - \mu - 1)(r-1), \end{array} \tag{10}$$

with $\quad r = \sqrt{x^2 + (y-\mu-1)^2}$ and $X^- = (0,1)$.

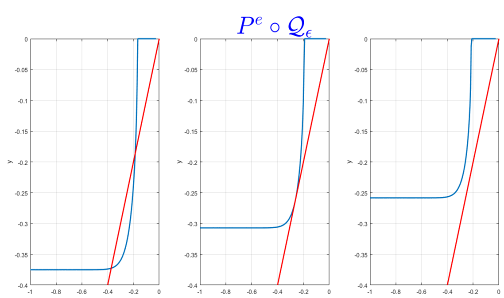

Figure 4: The Poincare map $P^e \circ \mathcal{Q}_\varepsilon$ defined in $[-1,0]$ and for $\varepsilon = .05$ and $\mu_{1,2,3} = \varepsilon - (.5, .5623, .6)\varepsilon^{\frac{4}{3}}$ has two, one and zero fixed points.

2 Hysteresis

But the regularization Sotomayor-Teixeira is a special one. We now introduce hysteresis as another way of regularizing discontinuous systems. In a 'negative' boundary layer we define an overlap in the non smooth system:

$$\left. \begin{array}{r} u = +1, y > -\alpha \\ u = -1, y < +\alpha \\ u \in [-1,1], |y| \leq \alpha \end{array} \right\} \tag{11}$$

In [**?**] we proved

Let's fixed $T > 0$, if the solution of the Filippov equation (**??**) satisfies $|x_F(t)| < M$ for $0 \leq t \leq T$, then the histeretic solution $(x_h(t), y_h(t))$ tends to the the Filippov solution $x_F(t)$ in $y = 0$ as

$$|x_h(t) - x_F(t)| \leq L\alpha \quad 0 \leq t \leq T \tag{12}$$

We can illustrate the hysteretic regularization method with a simple example proposed by Utkin in order to illustrate the disparity between Filippov's and Utkin's methods to define a flow in the switching manifold.

Consider the planar piecewise smooth system

$$\dot{x} = 0.3 + u^3 \qquad \dot{y} = -0.5 - u \qquad u = \text{sign}(y)\ . \tag{13}$$

If we perform the hysteretic regularization we obtain the trajectories shown in Figure (**??**).

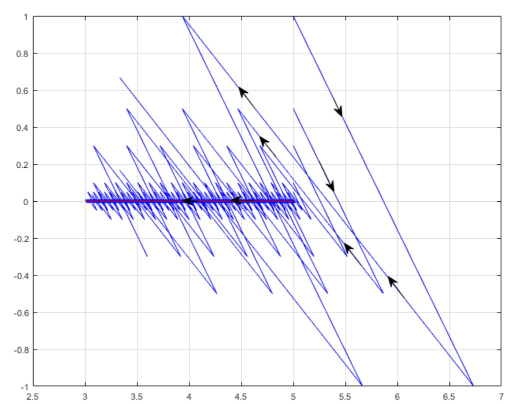

Figure 5: The hysteretic behaviour for the example for diminishing values of α. The line in red is the solution of Filippov system $x_F(t)$ (**??**) with $x_F(0) = 5$ and $0 \leq t \leq 10$

In spite Theorem(**??**) says that both hysteretic and Sotomayor-Teixeira regularizations tend to Filippov sliding mode,the way of approximation is different. And in the non-sliding regions can differ totally. For instance, if we regularize by hysteresis the grazing sliding family (**??**) the trajectories have a chaotic behaviour as shows Figure(**??**).

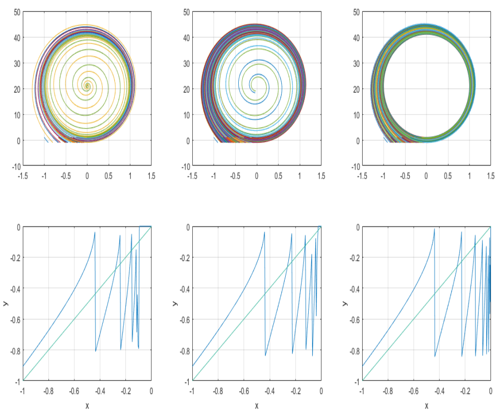

Figure 6: A chaotic motion similar to "spiral type chaos", produced when the μ and α parameters are near. Here $\alpha = .05$ and $\mu = .04968....$ Note the many discontinuity points of the Poincaré map

3 Smoothing the hysteresis

Of course, the hysteretic process we have defined is not smooth. In [**?**] we faced the problem of "smoothing" the hysteresis. And this requires to embed the system in a higher dimension, where the control u is also a time dependent variable. To do so, we can write the frame of the hysteretic process as a differential-algebraic system

$$
\begin{array}{rcl}
\dot{x} & = & f(x,y;u) \ , \\
\dot{y} & = & g(x,y;u) \ , \\
0 & = & \Phi(y+\alpha u) - u \ ,
\end{array}
\tag{14}
$$

where Φ is a set-valued step function defined as

$$
\left.\begin{array}{c}
\Phi(z) = sign(z), z \neq 0 \\
\Phi(z) \in [-1,1], z = 0.
\end{array}\right\}
\tag{15}
$$

This embeds, formally, the u-parameterized problem in variables (x, y), inside a surface $u = \Phi(y + \alpha u)$ in the higher dimensional space of variables (x, y, u). The $(x, y; u)$ space is divided by the plane $y + \alpha u = 0$. On the right the equation $0 = \Phi(y + \alpha u) - u$ has the solution $u = +1$. On the left the solution is $u = -1$. This express the form of the discontinuous system. Inside the plane $y+\alpha u = 0$ the solution is $u \in [-1, 1]$, that is, $|y| \leq \alpha$. This express the overlapping for $\alpha > 0$. See Figure (**??**).

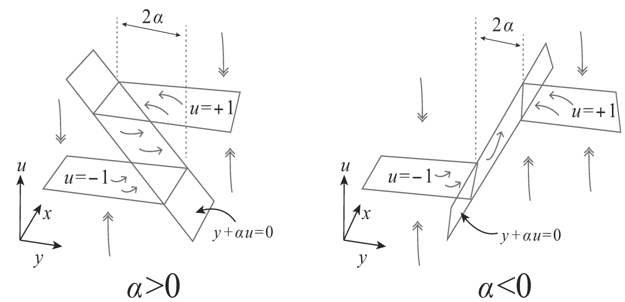

Figure 7: The formal differential-algebraic system. The vertical double arrows express the fast dynamics to be defined later (see Figure (**??**)). Note that for $\alpha < 0$, the system has also a sense. This will fit Utkin convention in this setting. See [**?**]

But this is a static representation. To introduce hysteretic dynamics, the form of this differential-algebraic system suggest the singular perturbation system:

$$
\begin{array}{rcl}
\dot{x} & = & f(x,y;u) \ , \\
\dot{y} & = & g(x,y;u) \ , \\
\epsilon\dot{u} & = & \varphi(\frac{y+\alpha u}{\epsilon}) - u \ ,
\end{array}
\tag{16}
$$

By the definition of φ, we have

$$
\lim_{\epsilon\to 0}\varphi(\tfrac{y+\alpha u}{\epsilon}) \in \Phi(y+\alpha u) \ . \tag{17}
$$

Then, for $\epsilon = 0$ the system (**??**) is formally equivalent to the differential-algebraic system, but we had embedded a (x, y) problem with a parameter u, in the higher dimensional space (x, y, u), where u is now a fast variable that relaxes quickly to $u = \pm 1$. And by letting α be either positive or negative, we can consider both a hysteretic case for $\alpha > 0$, and non-hysteretic case for $\alpha < 0$, due to the resulting shape of the surface $u = \varphi(\frac{y+\alpha u}{\epsilon})$, shown in Figure(**??**). (See also [**?**])

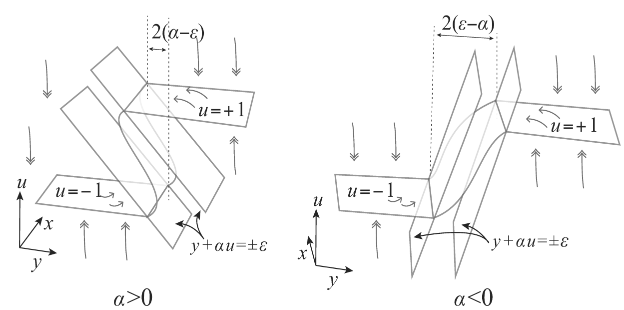

Figure 8: The system (**??**) for $\epsilon \neq 0$

As we also want α be small,we introduce a scaled variable $v = y/|\alpha|$ and defining $\kappa \equiv \epsilon/\alpha$, the system transforms to

$$
\begin{array}{rcl}
\dot{x} & = & f(x,|\alpha|v;u) \ , \\
|\alpha|\dot{v} & = & g(x,|\alpha|v;u) \ , \\
\kappa\alpha\dot{u} & = & \varphi(\frac{u+\mathrm{sign}(\alpha)v}{\kappa}) - u \ .
\end{array}
\tag{18}
$$

As we can suppose that the ϵ relaxation is faster than the α switching, we assume

(19) $$0 < \epsilon \ll |\alpha| \ll 1$$

which implies $0 < |\kappa| \ll 1$. Now there are three time scales, but we shall treat κ as fixed and nonzero to guarantee that the embedded system is 'sufficiently' smooth to apply standard singular perturbation theory. That is, when we let $\alpha \to 0$ (and by implication $\epsilon = \alpha\kappa \to 0$), the system remains smooth provided κ remains bounded away from zero. This allows us to use standard singular perturbation theory for small α. We have the following result for the hysteresis:

Fix $T > 0$, consider $x_F(t)$ the solution of the Filippov System in Σ,

and assume that $|x_F(t)| < M$ for $0 \leq t \leq T$. Then there exist constants $C, L, \alpha_0 > 0$, such that, for any $0 < \alpha \leq \alpha_0$, if we take $0 < \kappa < \frac{1}{4}$ and δ_0 satisfying

$$2\mathrm{e}^{-\frac{1}{2\kappa C}} < \delta_0 \leq \kappa\alpha_0,$$

the solution $(x(t), y(t), u(t))$ of the system with initial condition (x_0, y_0, u_0), with $|x_0| < M$, $|y_0| < \alpha$ and $||u_0| - 1| < \delta_0$, satisfies, for all $t \in [0, T]$,

$$|x(t) - x_F(t)| < L(\kappa + \frac{\delta_0}{\kappa} + \kappa \left|\log \frac{\delta_0}{2}\right| + \alpha), |y(t)| < \alpha.$$

Taking $\kappa = \alpha$ and $\delta_0 = \alpha^2$, then $(x(t), y(t), u(t))$ satisfies:

$$|x(t) - x_F(t)| < L\alpha \left|\log(\alpha)\right|, |y(t)| < \alpha.$$

4 Route to chaos

Then we can embed the system (**??**)in a hysteretic relaxation, and we will obtain chaotic behaviour. Figure(**??**) shows the bifurcation of a trajectory for $\alpha = .05$ with $\kappa = .5$ and $\mu = .055, .057, .06$ and its projections in the (x, y) plane.

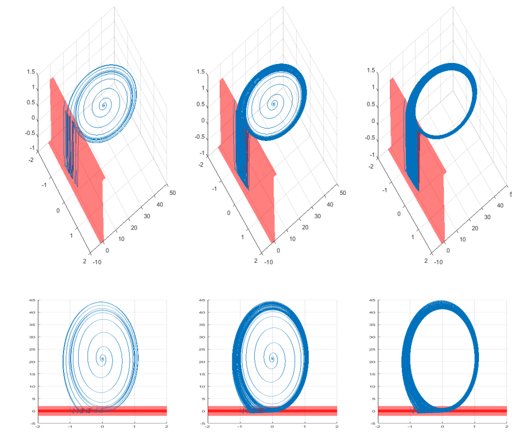

Figure 9: The route to chaos by embedding the saddle-node bifurcation (**??**) in a hysteretic relaxation. Note the small size of the regularizing strip compared with the large loops of the orbits. This obeys to the different orders between α and y .

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