QUALITATIVE RESULTS FOR A MIXTURE OF GREEN-LINDSAY THERMOELASTIC SOLIDS

ANTONIO MAGAÑA, JAIME E. MUÑOZ RIVERA, MARIA GRAZIA NASO, AND RAMON QUINTANILLA

ABSTRACT. We study qualitative properties of the solutions of a PDE system modeling thermomechanical deformations for mixtures of thermoelastic solids when the theory of Green and Lindsay for the heat conduction is considered. Three dissipation mechanisms are introduced in the system: thermal dissipation, viscosity effects on one constituent of the mixture and damping in the relative velocity of the two displacements of both constituents. We prove the existence and uniqueness of the solutions and their stability over the time. We use the semigroup arguments to establish our results.

1. Introduction

Thermoelastic mixtures of solids have been an important issue of study for mathematicians and engineers in the last decades (see, e.g., [5, 6, 7, 8, 12, 13, 33, 34]). In particular, a lot of effort has been done to find qualitative properties of the solutions of the partial differential equations (PDE) related to the mixture of materials problems. Several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [1, 2, 3, 20, 25, 26, 27, 31, 32]. In these contributions the Fourier law is used to describe the heat propagation. However, it is known that the classical Fourier theory gives rise to several paradoxes. Perhaps the most known is the infinite velocity of propagation, which is not compatible with the causality principle. Since the decade of the 1960's, other thermomechanic theories that allow heat to propagate as a wave with finite speed have been stated to overcome the aforementioned paradox. These new theories are mainly based on the heat conduction model of Cattaneo and Maxwell [28] (for a deeper knowledge about these theories see Hetnarski and Ignaczack [17, 18], the books by Ignaczack and Ostoja-Starzewski [22], Straughan [35] and the references cited therein). In 1972, Green and Lindsay [11] presented another thermoelasticity theory by adding restrictions on the constitutive equations; in fact, they used and entropy production inequality proposed by Green and Laws [10]. In the last decade of the twentieth century, Green and Naghdi [14, 15, 16] also proposed a set of theories which have been deeply studied lately.

In this paper we want to study several qualitative properties of the solutions of the PDE that arise for thermoviscoelastic mixtures in the three dimensional case when the heat conduction is modeled using the theory of Green and Lindsay. To be precise, we will analyze the theory proposed by Iesan and Scalia [21] following the works of Green and Lindsay. We want to point out that, recently, the asymptotic behavior of the solutions for the mixtures problem when the Lord-Shulman theory [24] is considered has been studied by Alves *et al.* [4].

The structure of the paper is the following. First of all we recall the field equations, impose the initial and boundary conditions and set the assumptions over the constitutive coefficients. In Section 3 we prove the existence and uniqueness of the solutions using semigroup arguments. In Section 4 we analyze the time behavior of the solutions and we prove their exponential stability.

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2. The system of equations and the basic assumptions

We consider a mixture of two continua and suppose that, at a fixed time, the body occupies a bounded and regular region B of the three-dimensional Euclidean space with boundary smooth enough to apply the divergence theorem. The field equations for isotropic and homogeneous bodies with a center of symmetry are given by the system (see [21], p.238)

$$\begin{aligned}
&A_1\triangle u_i + A_2 u_{j,ji} + B_1\triangle w_i + B_2 w_{j,ji} - \xi(u_i - w_i) - m(T_{,i} + \alpha\dot{T}_{,i})\\
&-\xi^*(\dot{u}_i - \dot{w}_i) - b^* T_{,i} + \mu^*\triangle\dot{u}_i + (\lambda^* + \mu^*)\dot{u}_{j,ji} + \tau^*\dot{T}_{,i} = \rho_1\ddot{u}_i\\
&B_1\triangle u_i + B_2 u_{j,ji} + C_1\triangle w_i + C_2 w_{j,ji} + \xi(u_i - w_i) - \beta(T_{,i} + \alpha\dot{T}_{,i}) + \xi^*(\dot{u}_i - \dot{w}_i) + b^* T_{,i} = \rho_2\ddot{w}_i\\
&k\triangle T - T_0(d\dot{T} + h\ddot{T} + m\dot{u}_{i,i} + \beta\dot{w}_{i,i}) = 0.
\end{aligned} \tag{2.1}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are the displacements of each constituent, T is the temperature, $A_1, A_2, B_1, B_2, C_1,\ C_2, \xi, m, \xi^*, b^*, \mu^*, \lambda^*, \beta, k, d, h, \tau^*, \rho_1, \rho_2, \alpha$ and T_0 are the constitutive coefficients. T_0 is usually the temperature at the reference configuration and, from now on and without loose of generality, we will assume that $T_0 = 1$. Coefficient α is a relaxation parameter in the temperature.

We are going to consider three different dissipation mechanisms in the system: thermal dissipation, viscosity effects on the first constituent and damping in the relative velocity.

To have a well posed problem we need to impose initial and boundary conditions to system (2.1). As initial conditions we take

$$\begin{aligned}
u_i(x,0) &= u_i^0(x), \quad \dot{u}_i(x,0) = v_i^0(x) \quad \text{in } B,\\
w_i(x,0) &= w_i^0(x), \quad \dot{w}_i(x,0) = z_i^0(x) \quad \text{in } B,\\
T(x,0) &= T^0(x), \quad \dot{T}(x,0) = \theta^0(x) \quad \text{in } B,
\end{aligned} \tag{2.2}$$

for some given functions. And as boundary conditions we consider

$$u_i(x,t) = w_i(x,t) = T(x,t) = 0, \text{ in } \partial B. \tag{2.3}$$

In order to obtain results of existence and uniqueness for the solutions of the problem determined by system (2.1) with initial conditions (2.2) and boundary conditions (2.3) we need some basic assumptions over the coefficients.

First of all, we impose that the energy of the system has to be positive. To this end, we will suppose that the matrices

$$\begin{pmatrix} A_1 & B_1 \\ B_1 & C_1 \end{pmatrix} \text{ and } \begin{pmatrix} A_2 & B_2 \\ B_2 & C_2 \end{pmatrix} \tag{2.4}$$

are definite positive. We will also assume that

$$\rho_1 > 0, \quad \rho_2 > 0, \quad d\alpha - h > 0, \ \xi > 0, \tag{2.5}$$

conditions given by the entropy production law.

Secondly, we want the dissipation to be also positive and, therefore, we impose that the following inequalities hold:

$$3\lambda^* + 2\mu^* \geq 0, \ \mu^* \geq 0, \quad 4(d\alpha - h)(3\lambda^* + 2\mu^*) - \tau^* \geq 0, k \geq 0, \text{ and } 4k\xi^* - (b^*)^2 \geq 0. \tag{2.6}$$

3. Existence and uniqueness of solutions

The aim of this section is to prove the existence and the uniqueness of solutions for problem (2.1)–(2.3). It is worth noting that for the problem arising when the Fourier heat conduction law is used, existence and uniqueness of the solutions have been already proved [31]. The arguments we use here are similar to the ones used before. Nevertheless, we think suitable to sketch at least the more important features.

For the system coefficients we assume the conditions proposed at (2.5) and (2.6).

With the usual notation, we introduce the spaces $L^2(B)$, $H_0^1(B)$ and $H^{-1}(B)$ acting on a bounded domain B. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the L^2-inner product and the L^2-norm, respectively.

In the case of Dirichlet thermal boundary conditions, let us consider the Hilbert space

$$\mathcal{H} = H_0^1 \times H_0^1 \times H_0^1 \times L^2 \times L^2 \times L^2.$$

We will denote the elements of $\mathcal{H}$ by $U = (\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)$, with $\mathbf{v} = \dot{\mathbf{u}}$, $\mathbf{z} = \dot{\mathbf{w}}$ and $\theta = \dot{T}$.

We define an inner product in $\mathcal{H}$ by

$$\langle(\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta), (\widetilde{\mathbf{u}}, \widetilde{\mathbf{w}}, \widetilde{T}, \widetilde{\mathbf{v}}, \widetilde{\mathbf{z}}, \widetilde{\theta})\rangle_{\mathcal{H}} = \frac{1}{2}\int_B \Pi dV, \tag{3.1}$$

where

$$\Pi = A_1 u_{i,j}\overline{\widetilde{u}_{i,j}} + A_2 u_{i,i}\overline{\widetilde{u}_{j,j}} + B_1(u_{i,j}\overline{\widetilde{w}_{i,j}} + w_{i,j}\overline{\widetilde{u}_{i,j}}) + B_2(u_{i,i}\overline{\widetilde{w}_{j,j}} + w_{i,i}\overline{\widetilde{u}_{j,j}}) + C_1 w_{i,j}\overline{\widetilde{w}_{i,j}} + C_2 w_{i,i}\overline{\widetilde{w}_{j,j}}$$
$$+\rho_1 v_i\overline{\widetilde{v}_i} + \rho_2 z_i\overline{\widetilde{z}_i} + \xi(u_i - w_i)(\overline{\widetilde{u}_i - \widetilde{w}_i}) + \frac{h}{\alpha}(T + \alpha\theta)(\overline{\widetilde{T} + \alpha\widetilde{\theta}}) + (d - \frac{h}{\alpha})T\overline{\widetilde{T}} + \alpha k T_{,i}\overline{\widetilde{T}_{,i}}. \tag{3.2}$$

Its corresponding norm is given by

$$\|(\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)\|_{\mathcal{H}}^2 = \frac{1}{2}\int_B \Pi^* dV, \tag{3.3}$$

where

$$\Pi^* = A_1 u_{i,j}\overline{u_{i,j}} + A_2 u_{i,i}\overline{u_{j,j}} + B_1(u_{i,j}\overline{w_{i,j}} + w_{i,j}\overline{u_{i,j}}) + B_2(u_{i,i}\overline{w_{j,j}} + w_{i,i}\overline{u_{j,j}}) + C_1 w_{i,j}\overline{w_{i,j}} + C_2 w_{i,i}\overline{w_{j,j}}$$
$$+\rho_1 v_i\overline{v_i} + \rho_2 z_i\overline{z_i} + \xi(u_i - w_i)(\overline{u_i - w_i}) + \frac{h}{\alpha}(T + \alpha\theta)(\overline{T + \alpha\theta}) + (d - \frac{h}{\alpha})T\overline{T} + \alpha k T_{,i}\overline{T_{,i}}. \tag{3.4}$$

In particular there exists a positive constant c such that the inequality

$$\|(\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)\|_{\mathcal{H}}^2 \geq c\left(\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{w}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{z}\|^2 + \|\theta\|^2 + \|\nabla T\|^2\right), \tag{3.5}$$

is satisfied. For the sake of simplicity, here and in what follows we will employ the same symbol c for different constants, even in the same formula.

We will rewrite system (2.1) in matricial terms and, afterwards, we will use the technique of contractive semigroups to prove the existence of solutions.

In order to obtain a written synthetic expression for our problem, we introduce the following operators:

$$\begin{array}{ll}
\mathbf{A}_1\mathbf{u} = \frac{1}{\rho_1}(A_1 u_{i,jj} + A_2 u_{j,ji} - \xi u_i), & \mathbf{A}_2\mathbf{w} = \frac{1}{\rho_1}(B_1 w_{i,jj} + B_2 w_{j,ji} + \xi w_i), \\
\mathbf{A}_3\mathbf{v} = \frac{1}{\rho_1}(\mu^* v_{i,jj} + (\lambda^* + \mu^*)v_{j,ji} - \xi^* v_i), & \mathbf{A}_4\mathbf{z} = \frac{1}{\rho_1}(\xi^* z_i), \\
\mathbf{A}_5 T = \frac{1}{\rho_1}(-mT_{,i} - b^* T_{,i}), & \mathbf{A}_6\theta = -\frac{1}{\rho_1}((m\alpha - \tau^*)\theta_{,i}), \\
\mathbf{A}_7\mathbf{u} = \frac{1}{\rho_2}(B_1 u_{i,jj} + B_2 u_{j,ji} + \xi u_i), & \mathbf{A}_8\mathbf{w} = \frac{1}{\rho_2}(C_1 w_{i,jj} + C_2 w_{j,ji} - \xi w_i), \\
\mathbf{A}_9\mathbf{v} = \frac{1}{\rho_2}(\xi^* v_i), & \mathbf{A}_{10}\mathbf{z} = -\frac{1}{\rho_2}(\xi^* z_i), \\
\mathbf{A}_{11} T = \frac{1}{\rho_2}(-\beta T_{,i} + b^* T_{,i}), & \mathbf{A}_{12}\theta = -\frac{1}{\rho_2}(\beta\alpha\theta_{,i}), \\
A_{13}\mathbf{v} = \frac{1}{h}(-m v_{i,i}), & A_{14}\mathbf{z} = -\frac{1}{h}(\beta z_{i,i}), \\
A_{15} T = \frac{1}{h}(k\triangle T), & A_{16}\theta = -\frac{d}{h}\theta.
\end{array}$$

Therefore, system (2.1) with initial conditions (2.2) and boundary conditions (2.3) can be written as

$$\frac{d}{dt}U(t) = \mathcal{A}U(t), \quad U(0) = U_0, \tag{3.6}$$

where $U_0 = \left(\mathbf{u}^0, \mathbf{w}^0, T^0, \mathbf{v}^0, \mathbf{z}^0, \theta^0\right)$, and $\mathcal{A}$ is the following matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_5 & \mathbf{A}_3 & \mathbf{A}_4 & \mathbf{A}_6 \\ \mathbf{A}_7 & \mathbf{A}_8 & \mathbf{A}_{11} & \mathbf{A}_9 & \mathbf{A}_{10} & \mathbf{A}_{12} \\ 0 & 0 & A_{15} & A_{13} & A_{14} & A_{16} \end{pmatrix}. \tag{3.7}$$

The domain of the operator $\mathcal{A}$ is $D(\mathcal{A}) = \{U \in \mathcal{H} : \quad \mathcal{A}U \in \mathcal{H}\}$. It is clear that $D(\mathcal{A})$ is dense in the Hilbert space $\mathcal{H}$.

Lemma 3.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions, denoted by $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and that 0 belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$). Then our conclusion will follow by using the Lumer-Phillips theorem (see, e.g., [29]).

If $U \in D(\mathcal{A})$ then, direct calculations give

$$\operatorname{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_B D^+ dV \leq 0, \tag{3.8}$$

where

$$D^+ = \mu^* v_{i,j}\overline{v_{i,j}} + (\lambda^* + \mu^*) v_{i,i}\overline{v_{j,j}} + \xi^*(v_i - z_i)(\overline{v_i} - \overline{z_i}) + \tau^* \operatorname{Re}\theta\overline{v}_{i,i}$$
$$+ b^* \operatorname{Re} T_{,i}(\overline{v_i} - \overline{z_i}) + (d\alpha - h)\theta\overline{\theta} + kT_{,i}\overline{T_{,i}}$$

and, therefore, the operator $\mathcal{A}$ is dissipative.

Given $F = (\mathbf{f}, \mathbf{g}, h, \mathbf{p}, \mathbf{q}, r) \in \mathcal{H}$, we must show that there exists a unique $U = (\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)$ in $D(\mathcal{A})$ such that $\mathcal{A}U = F$, that is

$$\begin{aligned} &\mathbf{v} = \mathbf{f} && \text{in } \mathbf{H}_0^1, \\ &\mathbf{z} = \mathbf{g} && \text{in } \mathbf{H}_0^1, \\ &\theta = h && \text{in } H_0^1, \\ &\mathbf{A_1 u} + \mathbf{A_2 w} + \mathbf{A_5} T + \mathbf{A_3 v} + \mathbf{A_4 z} + \mathbf{A_6}\theta = \mathbf{p} && \text{in } \mathbf{L}^2, \\ &\mathbf{A_7 u} + \mathbf{A_8 w} + \mathbf{A_{11}} T + \mathbf{A_9 v} + \mathbf{A_{10} z} + \mathbf{A_{12}}\theta = \mathbf{q} && \text{in } \mathbf{L}^2, \\ &A_{15}T + A_{13}\mathbf{v} + A_{14}\mathbf{z} + A_{16}\theta = r && \text{in } L^2. \end{aligned} \tag{3.9}$$

Using $(3.9)_1$-$(3.9)_3$ in $(3.9)_6$ we have

$$A_{15}T = r - (A_{13}\mathbf{f} + A_{14}\mathbf{g} + A_{16}h). \tag{3.10}$$

Thus, the existence of a unique $T \in H^2$ satisfying (3.10) is clear. Now, from $(3.9)_4$ and $(3.9)_5$, we obtain the following system of equations with unknowns $\mathbf{u}$ and $\mathbf{w}$.

$$\begin{cases} \mathbf{A_1 u} + \mathbf{A_2 w} = \mathbf{p} - \mathbf{A}_5 T - \mathbf{A}_3\mathbf{f} - \mathbf{A}_4\mathbf{g} - \mathbf{A}_6 h & \text{in } \mathbf{H}^{-1,2}, \\ \mathbf{A_7 u} + \mathbf{A_8 w} = \mathbf{q} - \mathbf{A}_{11} T - \mathbf{A}_9\mathbf{f} - \mathbf{A}_{10}\mathbf{g} - \mathbf{A}_{12} h & \text{in } \mathbf{H}^{-1,2}. \end{cases} \tag{3.11}$$

The sesquilinear form $\mathbf{B} : H_0^1 \times H_0^1 \to \mathbb{C}$ given by

$$\mathbf{B}[(\mathbf{u}, \mathbf{w}), (\widetilde{\mathbf{u}}, \widetilde{\mathbf{w}})] = \langle \mathbf{A_1 u} + \mathbf{A_2 w}, \overline{\widetilde{\mathbf{u}}}\rangle + \langle \mathbf{A_7 u} + \mathbf{A_8 w}, \overline{\widetilde{\mathbf{w}}}\rangle$$

is continuous and coercive. As the right hand side of (3.11) is in the dual, using the Lax-Milgram theorem (see, e.g., [9]), it follows that there exists a unique vector $(\mathbf{u}, \mathbf{w})$ satisfying system (3.11).

Therefore, there exists also a unique vector U satisfying (3.9). It is easy to show that $\|U\|_{\mathcal{H}} \leq c\|F\|_{\mathcal{H}}$, for a positive constant c. Hence, we conclude that $0 \in \rho(\mathcal{A})$. □

As a consequence of the above established results, we can state the following theorem.

Theorem 3.2. *The operator $\mathcal{A}$ generates a contraction semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$, and for $U_0 \in D(\mathcal{A})$ there exists a unique solution $U(t) \in \mathcal{C}^1([0,\infty),\mathcal{H}) \cap \mathcal{C}^0([0,\infty), D(\mathcal{A}))$ of system (2.1) with initial conditions (2.2) and boundary conditions (2.3).*

4. Exponential stability

In this section we analyze the asymptotic behavior of the solutions with respect the time variable. We consider thermal dissipation, viscosity effects on the first constituent of the mixture and damping effects on the relative velocity of the two displacements of both constituents.

To enforce the dissipation mechanisms act we have to assume that

$$3\lambda^* + 2\mu^* > 0,\ \ \mu^* > 0, \quad 4(d\alpha - h)(3\lambda^* + 2\mu^*) - \tau^* > 0,\ \ k > 0 \text{ and } 4k\xi^* - (b^*)^2 > 0. \tag{4.1}$$

Besides the assumptions for the coefficients given by (4.1), we assume also that

$$\begin{aligned} &\int_B (B_1 w_{i,j}w_{i,j} + B_2 w_{i,i}w_{j,j})\, dV \geq C\int_B w_{i,j}w_{i,j}\, dV \text{ or} \\ &\int_B (B_1 w_{i,j}w_{i,j} + B_2 w_{i,i}w_{j,j})\, dV \leq -C\int_B w_{i,j}w_{i,j}\, dV. \end{aligned} \tag{4.2}$$

It is useful to recall the following known result (see [30]):

Theorem 4.1. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R} \subset \rho(\mathcal{A})$,
(ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Lemma 4.2. *The operator $\mathcal{A}$ defined at (3.7) satisfies that $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. Following the arguments given by Liu and Zheng [23], the proof consists of the following steps:

(i) Since 0 is in the resolvent of $\mathcal{A}$, using the contraction mapping theorem, we have that for any real λ such that $|\lambda| < ||\mathcal{A}^{-1}||^{-1}$, the operator $i\lambda\mathcal{I} - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - \mathcal{I})$ is invertible. Moreover, $||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

(ii) If $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I} - \mathcal{A} = (i\lambda_0\mathcal{I} - \mathcal{A})\Big(\mathcal{I} + i(\lambda - \lambda_0)(i\lambda_0\mathcal{I} - \mathcal{A})^{-1}\Big),$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that, by choosing λ_0 as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\lambda, |\lambda| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1} - M^{-1}, ||\mathcal{A}^{-1}||^{-1} + M^{-1})$.

(iii) Let us assume that the intersection of the imaginary axis and the spectrum is not empty, then there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore, there exists a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (\mathbf{u}_n, \mathbf{w}_n T_n, \mathbf{v}_n, \mathbf{z}_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0. \tag{4.3}$$

If we write (4.3) in components, we obtain the following conditions:

$$i\lambda_n \mathbf{u}_n - \mathbf{v}_n \to \mathbf{0}, \text{ in } \mathbf{H}^1 \tag{4.4}$$

$$i\lambda_n \mathbf{w}_n - \mathbf{z}_n \to \mathbf{0}, \text{ in } \mathbf{H}^1 \tag{4.5}$$

$$i\lambda_n T_n - \theta_n \to 0, \text{ in } H^1 \tag{4.6}$$

$$i\lambda_n \mathbf{v}_n - \mathbf{A}_1\mathbf{u}_n - \mathbf{A}_2\mathbf{w}_n - \mathbf{A}_3\mathbf{v}_n - \mathbf{A}_5 T_n - \mathbf{A}_6\theta_n \to \mathbf{0}, \text{ in } \mathbf{L}^2 \tag{4.7}$$

$$i\lambda_n \mathbf{z}_n - \mathbf{A}_7\mathbf{u}_n - \mathbf{A}_8\mathbf{w}_n - \mathbf{A}_{11} T_n - \mathbf{A}_{12}\theta_n \to \mathbf{0}, \text{ in } \mathbf{L}^2 \tag{4.8}$$

$$i\lambda_n \theta_n - A_{15} T_n - A_{13}\mathbf{v}_n - A_{14}\mathbf{z}_n - A_{16}\theta_n \to 0, \text{ in } L^2 \tag{4.9}$$

with $\lambda_n \in \mathbb{R}$.

In view of the dissipative term for the operator, we see that

$$\theta_n, \, \nabla\mathbf{v}_n, \, \mathbf{v}_n - \mathbf{z}_n, \, \nabla T_n \to 0. \tag{4.10}$$

From (4.4) we also have that $\nabla\mathbf{u}_n \to 0$.

If we multiply (4.7) by $\mathbf{w}_n$ we obtain that

$$\int_B (B_1 w_{i,j} w_{i,j} + B_2 w_{i,i} w_{j,j}) \, dV \to 0,$$

and, from the assumptions (4.2), we conclude that $\nabla\mathbf{w}_n$ tends to zero. Therefore, $\mathbf{z}_n$ also tends to zero and, in consequence, we have seen that the imaginary axis in contained in the resolvent of $\mathcal{A}$. □

Lemma 4.3. *The operator $\mathcal{A}$ defined at (3.7) satisfies that* $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. Let us assume the existence of $\lambda_n \to \infty$ and a sequence of unit norm vectors U_n such that the relations (4.4) – (4.9) hold. Again, our aim is to prove that U_n tends to zero. We proceed as in the previous case. In fact, we obtain again that θ_n, $\nabla\mathbf{v}_n$, $\mathbf{v}_n - \mathbf{z}_n$ and $\nabla T_n \to 0$. And, hence $\nabla\mathbf{u} \to 0$. Now, from (4.7), we see that

$$\lambda_n^{-1}(\mathbf{A}_1\mathbf{u}_n + \mathbf{A}_2\mathbf{w}_n + \mathbf{A}_3\mathbf{v}_n + \mathbf{A}_6\theta_n) \to 0.$$

From (4.5) we see that $\lambda_n\mathbf{w}_n$ is bounded. Then, we multiply the above expression by $\lambda_n\mathbf{w}_n$ and, using an argument similar to the one used in the proof of Lemma 4.2, we obtain that $\nabla\mathbf{w}_n$ tends to zero.

Finally, multiplying (4.8) by $\mathbf{w}_n$ we obtain

$$\langle i\lambda_n\mathbf{z}_n, \frac{\mathbf{z}_n}{i\lambda_n}\rangle - \langle \mathbf{A}_7\mathbf{u}_n, \mathbf{w}_n\rangle - \langle \mathbf{A}_8\mathbf{w}_n, \mathbf{w}_n\rangle - \langle \mathbf{A}_{11}T_n, \mathbf{w}_n\rangle - \langle \mathbf{A}_{12}\theta_n, \mathbf{w}_n\rangle \to 0.$$

Applying the integration by parts, we see that $\mathbf{z}_n$ also tends to zero. □

The two previous lemmas give rise to the following result.

Theorem 4.4. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable.*

The proof is a direct consequence of Lemma 4.2, Lemma 4.3 and Theorem 4.1.

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Departament de Matemàtiques, UPC, C. Colón 11, 08222 Terrassa, Barcelona, Spain.
E-mail address: antonio.magana@upc.edu

National Laboratory for Scientific Computation
Rua Getulio Vargas 333, Quitadinha-Petrópolis 25651-070,
Rio de Janeiro, RJ, Brasil,
and
Instituto de Matemática - UFRJ
Av. Horácio Macedo, Centro de Tecnologia,
Cidade Universitária - Ilha do Fundão 21941-972,
Rio de Janeiro, RJ, Brasil.
E-mail address: rivera@lncc.br

Dipartimento di Matematica, Facoltà di Ingegneria, Università di Brescia
Via Valotti 9, 25133 Brescia, Italy.
E-mail address: naso@ing.unibs.it

Departament de Matemàtiques, UPC C. Colón 11, 08222 Terrassa, Barcelona, Spain.
E-mail address: ramon.quintanilla@upc.edu