Hiring Expert Consultants in E-Healthcare: An Analytics-Based Two Sided Matching Approach

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Abstract. Very often in some censorious healthcare scenario, there may be a need to have some expert consultancies (especially by doctors) that are not available in-house to the hospitals. Earlier, this interesting healthcare scenario of hiring the expert consultants (mainly doctors) from outside of the hospitals had been studied with the robust concepts of mechanism design with money and mechanism design without money. In this paper, we explore the more realistic two sided matching market in our healthcare set-up. In this, the members of the two participating communities, namely the patients and the doctors are revealing the strict preference ordering over the members of the opposite community for a stipulated amount of time. We assume that the patients and doctors are strategic in nature. With the theoretical analysis, we demonstrate that the TOMHECs, that results in the stable allocation of doctors to the patients, satisfies the several economic properties such as strategy-proofness (or truthfulness) and optimality. Further, the analytically based analysis of our proposed mechanisms *i.e.* RAMHECs and TOMHECs are carried out on the ground of the expected distance of the allocation done by the mechanisms from the top most preference. The proposed mechanisms are also validated with the help of exhaustive experiments.

Keywords: E-Healthcare, Analytics, Collective Intelligence, Hiring, Mechanism Design, Stable Allocation.

1 Introduction

The *expert advices* or *consultancies* provided by the expert consultants (ECs) mainly by doctors, can be thought of as one of the most indispensable events that occurs in the hospital(s) or medical unit(s) on a regular basis. It is to be noted that, the *expert advices* could be either in-house (inter departmental) or from outside of the hospital. There had been a spate of research work in the direction

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of handling the issues of scheduling the in-house ECs especially doctors [1, 2] and nurses [3] in an efficient and effective manner. In [1, 2, 4, 5] different techniques are discussed and presented to schedule the physicians that are in-house to the hospitals. The work in [6, 7, 8] focuses on the question of: *how to effectively and efficiently plan and schedule the operation theatres (OTs)?* In [9, 10, 8] the work has been done for allocating OTs on time to increase operating room efficiency. Interesting situation of taking expert consultancy from outside of the in-house medical unit during some censorious medical scenario (mainly *surgical* process) was taken care by [11, 12, 13, 14, 15]. Moreover, the introduction of such a pragmatic field of study in the healthcare domain by [11, 12, 13, 14, 15] has given rise to several open questions for the researchers, such as: (a) *which ECs are to be considered as the possible expertise provider in the consultancy arena*? (b) *What incentive policies in the form of perks and facilities are to be presented in-front of the ECs, so as to drag as many ECs as possible in the consultancy arena*?

As opposed to the money involved hiring of ECs as mentioned in [16, 17], another market of hiring ECs can be thought of where the ECs are providing their expertise free of cost. Recently, *Singh et. al.* [18] have addressed this idea by considering a one sided matching market. In this paper, we have tried to model the *ECs hiring problem* as a two sided matching market in healthcare domain motivated by [19, 20, 21, 22]. In this environment, the members present in two different communities have the privilege to provide the strict preference ordering over all or on the subset of the available members of the opposite community. Now, several questions may arise in ones mind: (1) How the patients gives strict preference ordering over the doctors? (2) Why the doctors will be giving preferences over the patients? Is it practical? (3) If yes, how the doctors will be providing the preferences over the patients? Answering to the issue raised in point 1, the members of the patient party provides the strict preference ordering over the available doctors or the subset of doctors. The preferences over the doctors maybe provided by aggregating some of the factors such as: *qualification* of the doctors, *organization* to which the doctors belong, doctors work experience, and may be the feedback from the patients.

Considering the issues mentioned in point 2 and 3, in case of reputed and busy doctors the patients may be strictly ordered based on: the reputation of the hospital to which the patients are admitted. For e.g. say a patient admitted to Massachusetts General Hospital, Boston is preferred over a patient admitted to Johns Hopkins Hospital, Baltimore and is preferred over a patient admitted to All India Institute of Medical Sciences (AIIMS), New Delhi, India and so on. One reasoning that could be given is, the doctors providing consultancy to the patient admitted to reputed hospitals will be projected more to the outside world. In other words, it will increase their reputation by huge amount as compared to the consultancy provided to the patients in low rated hospitals. On the other hand, if the doctors are not bothered about the fame and are more socially motivated then in that case they may prefer to serve the patients admitted to low rated hospitals over the patients in reputed hospitals. So, whatever may be the circumstances, the doctors may give preferences (strict) over the patients. In our proposed model, in order to generate the preferences from the *patient party* and the *doctor party*, the above discussed criterias are taken into consideration.

Our Contributions

The main contributions of our work are as follows.

- We have tried to model the *ECs hiring problem* as a two sided matching problem in healthcare domain.
- We propose two mechanisms: a naive approach *i.e.* ***ra**ndomized **m**echanism for **h**iring **e**xpert **c**onsultants* (RAMHECs) and a *truthful* and *optimal* mechanism motivated by [21, 22, 27]; namely ***t**ruthful **o**ptimal **m**echanism for **h**iring **e**xpert **c**onsultants* (TOMHECs).
- We have also proved that for any instance of n patients and m doctors the allocation done by TOMHECs results in *stable*, *truthful*, and *optimal* allocation for requesting party.
- TOMHECs establish an upper bound of $O(kn^2)$ on the number of iterations required to determine a *stable* allocation for any instance of n patients and n doctors.
- A substantial amount of analysis and simulation are done to validate the performance of RAMHECs and TOMHECs via *optimal* allocation measure.

The remainder of the paper is structured as follows. Section 2 describes our proposed model. Some required definitions are discussed in Section 3. Section 4 illustrates the proposed mechanisms. Further analytic-based analysis of the mechanisms are carried out in Section 5. A detailed analysis of the experimental results is carried out in Section 6. Finally, conclusions are drawn and some future directions are depicted in Section 7.

2 System model

We consider the scenario, where there are multiple hospitals *say* n given as $\hbar = \{\hbar_1, \hbar_2, \ldots, \hbar_n\}$. In each hospital $\hbar_i \in \hbar$, there exists several patients with different diseases (signifying categories), belonging to different income groups. The patients requires somewhat partial or complete expert consultancies from outside of the admitted hospitals. By partial expert consultancies it is meant that, the part of expertise from the overall required expert consultancies. $\boldsymbol{\mathcal{C}} = \{c_1, c_2, \ldots, c_k\}$ is the set of k different categories. The set of all the admitted patients in different categories to different hospitals is given as: $\boldsymbol{\mathcal{P}} = \bigcup\limits_{\hbar_k \in \hbar} \bigcup\limits_{c_i \in \boldsymbol{\mathcal{C}}} \bigcup\limits_{j=1}^{\mathring{\hbar}_k^i} p_{i(j)}^{\hbar_k}$ where, $p_{i(j)}^{\hbar_k}$ is the patient j belonging to c_i category admitted to $\hbar_k$ hospital. The expression $\mathring{\hbar}_k^i$ in term $p_{i(\mathring{\hbar}_k^i)}^{\hbar_k}$ indicates the total number of patients in hospital $\hbar_k$ belonging to c_i category. On the other hand, there are several doctors having different expertise (signifies categories) associated with different hospitals say $\boldsymbol{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n\}$. The set of all the available doctors in different categories associated with different hospitals is given as: $\boldsymbol{\mathcal{D}} = \bigcup\limits_{\mathcal{H}_k \in \mathcal{H}} \bigcup\limits_{c_j \in \boldsymbol{\mathcal{C}}} \bigcup\limits_{i=1}^{\mathring{\mathcal{H}}_k^j} d_{j(i)}^{\mathcal{H}_k}$ where $d_{j(i)}^{\mathcal{H}_k}$ is the doctor i belonging to c_j category associated to $\mathcal{H}_k$ hospital. The expression $\mathring{\mathcal{H}}_k^j$ in term $d^{\mathcal{H}_k}_{j(\mathring{\mathcal{H}}_k^j)}$ indicates the total number of doctors associated with hospital $\mathcal{H}_k$ in c_j category. Our model captures only a single category say c_i but it works well for the system consisting of multiple categories. In that case, we have to repeat the process k times as k categories are existing. In more general setting, one can think of the situation where there are n number of patients and m number of doctors in a category such that $m \neq n$ ($m > n$ or $m < n$). The members of the participating parties may provide the strict preference ordering over the subset of the members of the opposite party.

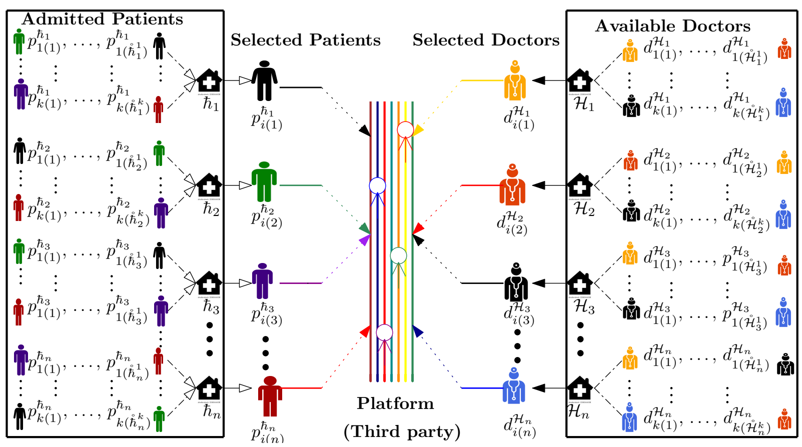


Fig. 1: System model

In Fig. 1 for representation purpose, from a category c_i, one doctor is selected from all the interested doctors from each hospital. But in general one can think of the situation where, in a particular category c_i, multiple doctors can be selected from the available doctors from a particular hospital. More formally, $|\boldsymbol{\mathcal{P}}_i| = \sum_{\mathcal{H}_j \in \boldsymbol{\mathcal{H}}} \dot{\mathcal{H}}_j^i$; where $0 \leq \dot{\mathcal{H}}_j^i \leq n$ is the number of doctors selected from hospital $\mathcal{H}_j$ in c_i category and placed into the consultancy arena. Following the above discussed criteria, the third party selects n doctors out of all the doctors in a particular category c_i as a possible expert consultant and is given as $\boldsymbol{\mathcal{D}}_i = \{d_{i(1)}^{\mathcal{H}_1}, d_{i(2)}^{\mathcal{H}_2}, \ldots, d_{i(n)}^{\mathcal{H}_n}\}$. The set of selected patients from c_i category is given as $\boldsymbol{\mathcal{P}}_i = \{p_{i(1)}^{\hbar_1}, p_{i(2)}^{\hbar_2}, \ldots, p_{i(n)}^{\hbar_n}\}$. The strict preference ordering of the patient $p_{i(k)}^{\hbar_j}$ over the set of doctors $\boldsymbol{\mathcal{D}}_i$ is denoted by $\succ_k^i$. More formally, the significance of

$d^{\mathcal{H}_j}_{i(\ell)} \succ^i_k d^{\mathcal{H}_k}_{i(m)}$ is that the patient $p^{\hbar_t}_{i(k)}$ ranks doctor $d^{\mathcal{H}_j}_{i(\ell)}$ above the doctor $d^{\mathcal{H}_k}_{i(m)}$. The preference profile of all the patients for k different categories is denoted as $\succ = \{\succ^1, \succ^2, \ldots, \succ^k\}$, where $\succ^i$ denotes the preference profile of all the patients in category c_i over the doctors in $\boldsymbol{\mathcal{D}}_i$. $\succ^i$ is given as $\succ^i = \{\succ^i_1, \succ^i_2, \ldots, \succ^i_n\}$. The strict preference ordering of the doctor $d^{\mathcal{H}_k}_{j(t)}$ over the patients in $\boldsymbol{\mathcal{P}}_j$ is denoted by $\succcurlyeq^t_j$. More formally, the significance of $p^{\hbar_k}_{j(\ell)} \succcurlyeq^t_j p^{\hbar_i}_{j(m)}$ is that doctor $d^{\mathcal{H}_k}_{j(t)}$ ranks $p^{\hbar_k}_{j(\ell)}$ above $p^{\hbar_i}_{j(m)}$. The set of preferences of all the doctors in k different categories is denoted as $\succcurlyeq = \{\succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_k\}$, where $\succcurlyeq_j$ contains the strict preference ordering of all the doctors in c_j category over the patients in set $\boldsymbol{\mathcal{P}}_j$. $\succcurlyeq_j$ is given as $\succcurlyeq_j = \{\succcurlyeq^1_j, \succcurlyeq^2_j, \ldots, \succcurlyeq^n_j\}$. It is to be noted that the allocation of the doctors to the patients for category c_i under consideration is captured by the allocation function $\mathcal{A}_i$: $\succ \times \succcurlyeq \rightarrow \boldsymbol{\mathcal{P}}_i \times \boldsymbol{\mathcal{D}}_i$. The resulting allocation vector is given as $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$; where each $a^i_{lm} \in \mathcal{A}_i$ is a pair $\{p^{\hbar_k}_{i(l)}, d^{\mathcal{H}_j}_{i(m)}\}$. The matching between the patients and doctors for any category c_i is captured by the mapping function $\boldsymbol{\mathcal{M}} : \boldsymbol{\mathcal{P}}_i \cup \boldsymbol{\mathcal{D}}_i \rightarrow \boldsymbol{\mathcal{D}}_i \cup \boldsymbol{\mathcal{P}}_i$.

3 Required definitions

Definition 1. ***Blocking pair****: Fix a category c_k. We say that a pair $p^{\hbar_t}_{k(i)}$ and $d^{\mathcal{H}_\ell}_{k(j)}$ form a* blocking pair *for matching $\boldsymbol{\mathcal{M}}$, if the following three conditions holds: (i) $\boldsymbol{\mathcal{M}}(p^{\hbar_t}_{k(i)}) \neq d^{\mathcal{H}_\ell}_{k(j)}$, (ii) $d^{\mathcal{H}_\ell}_{k(j)} \succ^k_i \boldsymbol{\mathcal{M}}(p^{\hbar_t}_{k(i)})$, and (iii) $p^{\hbar_t}_{k(i)} \succcurlyeq^j_k \boldsymbol{\mathcal{M}}(d^{\mathcal{H}_\ell}_{k(j)})$.*

Definition 2. ***Stable matching****: Fix a category c_k. A matching $\boldsymbol{\mathcal{M}}$ is* stable *if there is no pair $p^{\hbar_t}_{k(i)}$ and $d^{\mathcal{H}_\ell}_{k(j)}$ such that it satisfies the conditions mentioned in (i)-(iii) in Definition 1.*

Definition 3. ***Perfect matching****: Fix a category c_k. A matching $\boldsymbol{\mathcal{M}}$ is* perfect matching *if there exists one-to-one matching between the members of $\boldsymbol{\mathcal{P}}_k$ and $\boldsymbol{\mathcal{D}}_k$.*

Definition 4. ***Patient-optimal stable allocation****: Fix a category c_k. A matching $\boldsymbol{\mathcal{M}}$ is* patient optimal*, if there exists no stable matching $\boldsymbol{\mathcal{M}}'$ such that $\boldsymbol{\mathcal{M}}'(p^{\hbar_t}_{k(j)}) \succ^k_j \boldsymbol{\mathcal{M}}(p^{\hbar_t}_{k(j)})$ or $\boldsymbol{\mathcal{M}}'(p^{\hbar_t}_{k(j)}) =^k_j \boldsymbol{\mathcal{M}}(p^{\hbar_t}_{k(j)})$ for at least one $p^{\hbar_t}_{i(j)} \in \boldsymbol{\mathcal{P}}_i$. Similar is the situation for* doctor-optimal stable allocation*.*

Definition 5. ***Strategy-proof for requesting party****: Fix a category c_k. Given the preference profile $\succ^k$ and $\succcurlyeq_k$ of the patients and doctors in c_k category, a mechanism $\mathbb{M}$ is* strategy-proof *(truthful) for the requesting party if for each members of the requesting party $\mathcal{A}_k$ is preferred over $\hat{\mathcal{A}}_k$; where $\hat{\mathcal{A}}_k$ is the allocation when at least one member in requesting party is misreporting.*

4 Proposed mechanisms

The idea behind proposing randomized mechanism *i.e.* RAMHECs is to better understand the more robust and philosophically strong *optimal* mechanism

TOMHECs. The further illustration of the mechanisms are done under the consideration that patient party is requesting. Moreover, one can utilize the same road map of the mechanisms by considering doctors as the requesting party. This can easily be done by just interchanging their respective roles in the mechanisms.

4.1 RAMHECs

The idea lies behind the construction of *initialization phase* is to handle the system consisting of k different categories. The detailed algorithm is depicted in Algorithm 1.

Algorithm 1 RAMHECs ($\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{C}}$, $\succ$, $\succcurlyeq$)

```
Output: 𝓐 = {𝒜_1, 𝒜_2, ..., 𝒜_k}
begin
/* Initialization phase */
𝓐 ← ϕ
for each c_i ∈ 𝓒 do
    k ← 0, i ← 0, d* ← ϕ, p* ← ϕ, 𝒜_i ← ϕ, 𝒫* ← ϕ, 𝒟* ← ϕ
    i ← select(𝓟)            ▷ Return the index of patient from patient set.
    𝒫* ← 𝒫_i
    i ← select(𝓓)            ▷ Return the index of doctor from doctor set.
    𝒟* ← 𝒟_i
    /* Allocation phase */
    while |𝒜_i| ≠ min{m, n} do    ▷ m and n are the no. of doctors and patients.
        t ← rand(𝒫*)          ▷ Return index of randomly selected patient.
        p* ← p_{i(t)}^{ħ_k}
        k ← rand(≻_t^i, 𝒟*)   ▷ Return index of random doctor from patient t list.
        d* ← d_{i(k)}^{𝓗_ℓ}
        𝒜_i ← 𝒜_i ∪ {(p*, d*)}
        𝒫_i ← 𝒫_i \ p*       ▷ Remove the allocated patient from available patient list.
        𝒟_i ← 𝒟_i \ d*       ▷ Remove the allocated doctor from available doctor list.
    end while
    𝓐 ← 𝓐 ∪ 𝒜_i
end for
return 𝓐
end
```

Upper bound analysis of RAMHECs The overall running time of RAMHECs is $O(1) + O(kn) = O(kn)$ if $m = n$.

4.2 TOMHECs

Our main focus is to propose a mechanism that satisfy the two important economic properties: *truthfulness*, and *optimality*. The proposed mechanism is motivated by [21, 22, 25, 26]. The detailed algorithm is depicted in Algorithm 2.

Algorithm 2 TOMHECs ($\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{C}}$, $\succ$, $\succcurlyeq$)

Output: $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$

begin
/* **Initialization phase** */
$i \leftarrow 0$, $\boldsymbol{\mathcal{A}} \leftarrow \phi$
/* **Allocation phase** */
for each $c_i \in \boldsymbol{\mathcal{C}}$ **do**
    $\mathcal{A}_i \leftarrow \phi$
    $i \leftarrow \mathit{select}(\boldsymbol{\mathcal{P}})$ ▷ Return the index of patient from patient set.
    $\mathcal{P}^* \leftarrow \boldsymbol{\mathcal{P}}_i$
    $i \leftarrow \mathit{select}(\boldsymbol{\mathcal{D}})$ ▷ Return the index of doctor from doctor set.
    $\mathcal{D}^* \leftarrow \boldsymbol{\mathcal{D}}_i$
    for each $d_{i(j)}^{\mathcal{H}_k} \in \mathcal{D}^*$ **do**
        $\Pi(d_{i(j)}^{\mathcal{H}_k}) \leftarrow \phi$ ▷ $\Pi(d_{i(j)}^{\mathcal{H}_k})$ *keeps track of set of* $p_{i(j)}^{\hbar_k} \in \boldsymbol{\mathcal{P}}_i$ *requesting to* $d_{i(j)}^{\mathcal{H}_k}$.
    end for
    while $|\mathcal{A}_i| \neq \min\{m,\ n\}$ **do** ▷ m and n are the no. of doctors and patients.
        for each free patient $p_{i(j)}^{\hbar_k} \in \boldsymbol{\mathcal{P}}_i$ **do**
            $d^* \leftarrow$ select most preferred doctor from $\succ_j^i$ not approached till now.
            $\Pi(d^*) \leftarrow \Pi(d^*) \cup p_{i(j)}^{\hbar_k}$
        end for
        for each engaged doctor $d_{i(j)}^{\mathcal{H}_k} \in \boldsymbol{\mathcal{D}}_i$ **do**
            if $|\Pi(d_{i(j)}^{\mathcal{H}_k})| > 1$ **then**
                $p^* \leftarrow \text{select\_best}(\succcurlyeq_i^j,\ \Pi(d_{i(j)}^{\mathcal{H}_k}))$ ▷ Selecting the best patient among the multiple requests to the doctor $d_{i(j)}^{\mathcal{H}_k}$.
                if $(p_{i(j)}^{\hbar_k}, d_{i(j)}^{\mathcal{H}_k}) \in \mathcal{A}_i$ **and** $p^* \succcurlyeq_i^j p_{i(j)}^{\hbar_k}$ **then**
                    $\mathcal{A}_i \leftarrow \mathcal{A}_i \setminus (p_{i(j)}^{\hbar_k}, d_{i(j)}^{\mathcal{H}_k})$ ▷ Removes the already allocated less preffered patient to $d_{i(j)}^{\mathcal{H}_k}$ from $\mathcal{A}_i$.
                    $\mathcal{A}_i \leftarrow \mathcal{A}_i \cup (p^*, d_{i(j)}^{\mathcal{H}_k})$
                    $\Pi(d_{i(j)}^{\mathcal{H}_k}) \leftarrow \Pi(d_{i(j)}^{\mathcal{H}_k}) \setminus \Pi(d_{i(j)}^{\mathcal{H}_k}) - \{p^*\}$
                else if $(p_{i(j)}^{\hbar_k}, d_{i(j)}^{\mathcal{H}_k}) \notin \mathcal{A}_i$ **then**
                    $\mathcal{A}_i \leftarrow \mathcal{A}_i \cup (p^*, d_{i(j)}^{\mathcal{H}_k})$
                    $\Pi(d_{i(j)}^{\mathcal{H}_k}) \leftarrow \Pi(d_{i(j)}^{\mathcal{H}_k}) \setminus \Pi(d_{i(j)}^{\mathcal{H}_k}) - \{p^*\}$
                end if
            else if $|\Pi(d_{i(j)}^{\mathcal{H}_k})| == 1$ **then**
                if $(\Pi(d_{i(j)}^{\mathcal{H}_k}), d_{i(j)}^{\mathcal{H}_k}) \notin \mathcal{A}_i$ **then**
                    $\mathcal{A}_i \leftarrow \mathcal{A}_i \cup (\Pi(d_{i(j)}^{\mathcal{H}_k}), d_{i(j)}^{\mathcal{H}_k})$
                end if
            end if
        end for
    end while
    $\boldsymbol{\mathcal{A}} \leftarrow \boldsymbol{\mathcal{A}} \cup \mathcal{A}_i$
end for
return $\boldsymbol{\mathcal{A}}$
end

Running time The total running time of TOMHECs is given as: $T(n) = \sum_{i=1}^{k}(O(1) + (\sum_{i=1}^{n} O(n))) = O(kn^2)$ if $m = n$.

Correctness of the TOMHECs The correctness of the TOMHECs is proved with the *loop invariant* technique [23, 24]. For simplicity, the correctness is shown for the case with n patients and n doctors. In the similar fashion, one can interrogate the case with n patients and m doctors such that $m \neq n$. The *loop invariant*: Fix a category c_i. At the start of ℓ^{th} iteration of the *while* loop, the number of patient-doctor pairs held by $\mathcal{A}_i$ is given as: $| \cup_{j=1}^{\ell-1} \mathcal{A}'_j|$, where $\mathcal{A}'_j$ is the net patient-doctor pairs temporarily maintained in the set $\mathcal{A}'_j$ at the j^{th} iteration. So, now $n - | \cup_{j=1}^{\ell-1} \mathcal{A}'_j|$ number of patients or doctors (whomsoever is greater) are to be explored in further iterations. From the construction of the TOMECs after any ℓ^{th} iteration $0 \leq n - | \cup_{i=1}^{\ell} \mathcal{A}'_j| \leq n$ holds; where $1 \leq \ell \leq n^2$. Hence, inequality $0 \leq n - | \cup_{i=1}^{\ell} \mathcal{A}'_j| \leq n$ is always *true*. We must show three things for this *loop invariant* to be true.
Initialization: It is true prior to the first iteration of the *while* loop. Just before the first iteration of the *while* loop, the inequality $0 \leq n - | \cup_{i=1}^{\ell} \mathcal{A}'_j| \leq n$ boils down to $0 \leq n - 0 \leq n \Rightarrow 0 \leq n \leq n$ *i.e.* $\mathcal{A}_i \leftarrow \phi$.
Maintenance: For the *loop invariant* to be true, if it is true before each iteration of the *while* loop, it remains true before the next iteration of the *while* loop. In each iteration of the *while* loop, the cardinality of $\mathcal{A}_i$ is either incremented by some amount or remains similar to previous iteration. Just before the ℓ^{th} iteration the patient-doctor pairs temporarily added to $\mathcal{A}_i$ are $\cup_{i=1}^{\ell-1} \mathcal{A}'_j$. So, the number of patient-doctor pairs that are left is given by inequality: $0 \leq n - | \cup_{i=1}^{\ell-1} \mathcal{A}'_j| \leq n$. After the $(\ell-1)^{th}$ iteration, the remaining patient-doctor pair $n - | \cup_{i=1}^{\ell-1} \mathcal{A}'_j| \geq 0$ can be captured by two cases:
Case 1: If $|\mathcal{A}_i| = n$: This case will lead to exhaust all the remaining patient-doctor pair in the current ℓ^{th} iteration and no patient-doctor pair is left for the next iteration. The inequality $n - (| \cup_{i=1}^{\ell-1} \mathcal{A}'_i \cup \mathcal{A}'_\ell|) = n - (| \cup_{i=1}^{\ell} \mathcal{A}'_i|) = n - |\mathcal{A}_i|$ $= 0$. Hence, it means that all the remaining patient-doctor is absorbed in this iteration and no patient-doctor pair is left for processing.
Case 2: If $|\mathcal{A}_i| < n$: This case captures the possibility that there may be the scenario when few patient-doctor pairs from the remaining patient-doctor pairs may still left out; leaving behind some of the pairs for further iterations. So, the inequality $n - (| \cup_{i=1}^{\ell-1} \mathcal{A}'_i \cup \mathcal{A}'_\ell|) > 0 \Rightarrow n > n - (| \cup_{i=1}^{\ell} \mathcal{A}'_i|) > 0$ is satisfied. From Case 1 and Case 2, at the end of ℓ^{th} iteration the loop invariant is satisfied.
Termination: In each iteration, the cardinality of $\mathcal{A}_i$ is either incremented by some amount or remains as the previous iteration. This indicates that at some ℓ^{th} iteration the loop terminates when $|\mathcal{A}_i| = n$. We can say $n - | \cup_{i=1}^{\ell} \mathcal{A}'_i| = 0 \Rightarrow 0 \leq n$. Thus, this inequality indicates that all the n patient and doctors in c_i category are processed.
If the TOMHECs is true for the $c_i \in \boldsymbol{\mathcal{C}}$ category it will remain true when all category in $\boldsymbol{\mathcal{C}}$ taken simultaneously. Hence, the TOMHECs is correct.

Illustrative example For understanding purpose, let the category be c_3 (say *eye surgery*). The set of patients is given as: $\boldsymbol{\mathcal{P}}_3 = \{p^{\hbar_2}_{3(1)}, p^{\hbar_3}_{3(2)}, p^{\hbar_4}_{3(3)}, p^{\hbar_1}_{3(4)}\}$. The set of available doctors is given as: $\boldsymbol{\mathcal{D}}_3 = \{d^{\mathcal{H}_3}_{3(1)}, d^{\mathcal{H}_1}_{3(2)}, d^{\mathcal{H}_4}_{3(3)}, d^{\mathcal{H}_2}_{3(4)}\}$. The preference profile of patient set $\mathcal{P}_3$ is given as: $p^{\hbar_2}_{3(1)} = [d^{\mathcal{H}_2}_{3(4)} \succ^3_1 d^{\mathcal{H}_4}_{3(3)} \succ^3_1 d^{\mathcal{H}_3}_{3(1)} \succ^3_1 d^{\mathcal{H}_1}_{3(2)}]$, $p^{\hbar_3}_{3(2)} = [d^{\mathcal{H}_4}_{3(3)} \succ^3_2 d^{\mathcal{H}_2}_{3(4)} \succ^3_2 d^{\mathcal{H}_1}_{3(2)} \succ^3_2 d^{\mathcal{H}_3}_{3(1)}]$, $p^{\hbar_4}_{3(3)} = [d^{\mathcal{H}_2}_{3(4)} \succ^3_3 d^{\mathcal{H}_1}_{3(2)} \succ^3_3 d^{\mathcal{H}_3}_{3(1)} \succ^3_3 d^{\mathcal{H}_4}_{3(3)}]$, $p^{\hbar_1}_{3(4)} = [d^{\mathcal{H}_1}_{3(2)} \succ^3_4 d^{\mathcal{H}_4}_{3(3)} \succ^3_4 d^{\mathcal{H}_2}_{3(4)} \succ^3_4 d^{\mathcal{H}_3}_{3(1)}]$. Similarly, the preference profile of doctor set $\mathcal{D}_3$ is given as: $d^{\mathcal{H}_3}_{3(1)} = [p^{\hbar_2}_{3(1)} \succeq^1_3 p^{\hbar_3}_{3(2)} \succeq^1_3 p^{\hbar_1}_{3(4)} \succeq^1_3 p^{\hbar_4}_{3(3)}]$, $d^{\mathcal{H}_1}_{3(2)} = [p^{\hbar_3}_{3(2)} \succeq^2_3 p^{\hbar_1}_{3(4)} \succeq^2_3 p^{\hbar_2}_{3(1)} \succeq^2_3 p^{\hbar_4}_{3(3)}]$, $d^{\mathcal{H}_4}_{3(3)} = [p^{\hbar_4}_{3(3)} \succeq^3_3 p^{\hbar_2}_{3(1)} \succeq^3_3 p^{\hbar_3}_{3(2)} \succeq^3_3 p^{\hbar_1}_{3(4)}]$, $d^{\mathcal{H}_2}_{3(4)} = [p^{\hbar_1}_{3(4)} \succeq^4_3 p^{\hbar_4}_{3(3)} \succeq^4_3 p^{\hbar_2}_{3(1)} \succeq^4_3 p^{\hbar_3}_{3(2)}]$. Each of the patients $p^{\hbar_2}_{3(1)}$, $p^{\hbar_3}_{3(2)}$, $p^{\hbar_4}_{3(3)}$, and $p^{\hbar_1}_{3(4)}$ are requesting to the most preferred doctor from their respective preference list *i.e.* $d^{\mathcal{H}_2}_{3(4)}$, $d^{\mathcal{H}_4}_{3(3)}$, $d^{\mathcal{H}_2}_{3(4)}$, and $d^{\mathcal{H}_1}_{3(2)}$ respectively. In the next step, we will check if any requested doctor among $d^{\mathcal{H}_3}_{3(1)}$, $d^{\mathcal{H}_1}_{3(2)}$, $d^{\mathcal{H}_4}_{3(3)}$, and $d^{\mathcal{H}_2}_{3(4)}$ has got the multiple request from the patients in $\boldsymbol{\mathcal{P}}_3$. The competitive environment between patient $p^{\hbar_2}_{3(1)}$, and $p^{\hbar_4}_{3(3)}$ can be resolved by considering the strict preference ordering of doctor $d^{\mathcal{H}_2}_{3(4)}$ over the available patients in $\boldsymbol{\mathcal{P}}_3$. From the strict preference ordering of doctor $d^{\mathcal{H}_2}_{3(4)}$ it is clear that patient $p^{\hbar_4}_{3(3)}$ is preferred over patient $p^{\hbar_2}_{3(1)}$. Hence, patient $p^{\hbar_2}_{3(1)}$ is rejected. So, for the meanwhile $p^{\hbar_3}_{3(2)}$ gets a doctor $d^{\mathcal{H}_4}_{3(3)}$, $p^{\hbar_4}_{3(3)}$ gets a doctor $d^{\mathcal{H}_2}_{3(4)}$, and $p^{\hbar_1}_{3(4)}$ gets a doctor $d^{\mathcal{H}_1}_{3(2)}$. Now, as the patient $p^{\hbar_2}_{3(1)}$ do not get his/her (henceforth his) most preferred doctor *i.e.* $d^{\mathcal{H}_2}_{3(4)}$ from his preference list. So, he will request the second best doctor *i.e.* $d^{\mathcal{H}_4}_{3(3)}$ from his preference list. In the similar fashion, the remaining allocation is done. The final allocation is:$\{(p^{\hbar_2}_{3(1)}, d^{\mathcal{H}_4}_{3(3)}), (p^{\hbar_3}_{3(2)}, d^{\mathcal{H}_3}_{3(1)}), (p^{\hbar_4}_{3(3)}, d^{\mathcal{H}_2}_{3(4)}), (p^{\hbar_1}_{3(4)}, d^{\mathcal{H}_1}_{3(2)})\}$.

4.3 Several properties

The proposed TOMHECs has several compelling properties. These properties are discussed next.

Proposition 1. *The matching computed by the Gale-Shapley mechanism [21, 22, 27] results in a stable matching.*

Proposition 2. *A stable matching computed by Gale-Shapley mechanism [21, 22, 27] is requesting party optimal.*

Proposition 3. *Gale-Shapley mechanism [21, 22, 27] is truthful for the requesting party.*

Following the above mentioned propositions and motivated by [21, 22, 27] we are proving that the TOMHECs results in *stable*, *optimal*, and *truthful* allocation for c_i category. Our proof holds when all the k different categories are taken simultaneously.

Lemma 1. *TOMHECs results in a stable allocation for the requesting party (patient party or doctor party).*

Proof. Fix a category $c_i \in \boldsymbol{\mathcal{C}}$. Let us suppose for the sake of contradiction there exists a *blocking pair* $(p_{i(j)}^{\hbar_k}, d_{i(j)}^{\mathcal{H}_l})$ that results in an unstable matching $\boldsymbol{\mathcal{M}}$ for the requesting party. As their exists a *blocking pair* $(p_{i(j)}^{\hbar_k}, d_{i(j)}^{\mathcal{H}_l})$ it may be due to the case that $(p_{i(j)}^{\hbar_k}, d_{i(k)}^{\mathcal{H}_j})$ and $(p_{i(k)}^{\hbar_j}, d_{i(j)}^{\mathcal{H}_l})$ are their in the resultant matching $\boldsymbol{\mathcal{M}}$. This situation will arise only when $d_{i(j)}^{\mathcal{H}_l} \succ_j^i d_{i(k)}^{\mathcal{H}_j}$ *i.e.* in the strict preference ordering of patient $p_{i(j)}^{\hbar_k}$ doctor $d_{i(j)}^{\mathcal{H}_l}$ is preferred over doctor $d_{i(k)}^{\mathcal{H}_j}$. From the matching result $\boldsymbol{\mathcal{M}}$ obtained, it can be seen that in-spite the fact that $d_{i(j)}^{\mathcal{H}_l} \succ_j^i d_{i(k)}^{\mathcal{H}_j}$; $d_{i(j)}^{\mathcal{H}_l}$ is not matched with $p_{i(j)}^{\hbar_k}$ by the TOMHECs. So, this upset may happen only when doctor $d_{i(j)}^{\mathcal{H}_l}$ received a proposal from a patient $p_{i(k)}^{\hbar_j}$ to whom $d_{i(j)}^{\mathcal{H}_l}$ prefers over $p_{i(j)}^{\hbar_k}$ *i.e.* $p_{i(k)}^{\hbar_j} \succeq_i^j p_{i(j)}^{\hbar_k}$. Hence, this contradicts the fact that the $(p_{i(j)}^{\hbar_k}, d_{i(j)}^{\mathcal{H}_l})$ is a *blocking pair*. As their exists no *blocking pair*, it can be said that the resultant matching by TOMHECs is *stable*. □

Lemma 2. *A stable allocation resulted by TOMHECs is requesting party (patient or doctor) optimal.*

Proof. Fix a category c_i. Let us suppose for the sake of contradiction that the allocation set $\boldsymbol{\mathcal{M}}$ obtained using TOMHECs is not an *optimal* allocation for requesting party (say *patient party*). Then, from *Lemma* 1 there exists a *stable* allocation $\boldsymbol{\mathcal{M}}'$ such that $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k}) \succ_j^i \boldsymbol{\mathcal{M}}(p_{i(j)}^{\hbar_k})$ or $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k}) =_j^i \boldsymbol{\mathcal{M}}(p_{i(j)}^{\hbar_k})$ for at least one patient $p_{i(j)}^{\hbar_k} \in \boldsymbol{\mathcal{P}}_i$. Therefore, it must be the case that, some patient $p_{i(j)}^{\hbar_k}$ proposes to $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k})$ before $\boldsymbol{\mathcal{M}}(p_{i(j)}^{\hbar_k})$ since $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k}) \succ_i^j \boldsymbol{\mathcal{M}}(p_{i(j)}^{\hbar_k})$ and is rejected by $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k})$. Since doctor $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k})$ rejects patient $p_{i(j)}^{\hbar_k}$, the doctor $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k})$ must have received a better proposal from a patient $p_{i(k)}^{\hbar_j}$ to whom doctor $\boldsymbol{\mathcal{M}}'(p_{i(j)}^{\hbar_k})$ prefers over $p_{i(j)}^{\hbar_k}$ *i.e.* $p_{i(k)}^{\hbar_j} \succeq_i^j p_{i(j)}^{\hbar_k}$. Since, this is the first iteration at which a doctor rejects a patient under $\boldsymbol{\mathcal{M}}'$. It follows that the allocation $\boldsymbol{\mathcal{M}}$ is preferred over allocation $\boldsymbol{\mathcal{M}}'$ for the patient $p_{i(j)}^{\hbar_k}$. Hence, this contradicts the fact that the allocation set $\boldsymbol{\mathcal{M}}$ obtained using TOMHECs is not an *optimal* allocation. As their exists an optimal allocation $\boldsymbol{\mathcal{M}}$. □

Lemma 3. *A stable allocation resulted by TOMHECs is requesting party (patient or doctor) truthful.*

Proof. Fix a category c_i. Let us suppose for the sake of contradiction that the matching set $\boldsymbol{\mathcal{M}}$ obtained using TOMHECs is not a *truthful* allocation for requesting party (say *patient party*). The TOMHECs results in *stable* matching $\boldsymbol{\mathcal{M}}$ when all the members of the proposing party reports their true preferences. Now, let's say a patient $p_{i(j)}^{\hbar_k}$ misreport his preference list $\succ_j^i$ and getting better off in the resultant matching $\boldsymbol{\mathcal{M}}'$. Let $\mathcal{P}_i'$ be the set of patients who are getting better off in $\boldsymbol{\mathcal{M}}'$ as against $\boldsymbol{\mathcal{M}}$. Let $\mathcal{D}'_i$ be the set of doctors matched to patients in $\mathcal{P}'_i$ in matching $\boldsymbol{\mathcal{M}}'$. Let $d^{\mathcal{H}_\ell}_{i(k)}$ be the doctor that $p^{\hbar_k}_{i(j)}$ gets in $\boldsymbol{\mathcal{M}}'$. Since $\boldsymbol{\mathcal{M}}$ is *stable*, we know that $d^{\mathcal{H}_\ell}_{i(k)}$ cannot prefer $p^{\hbar_k}_{i(j)}$ to the patient got in $\boldsymbol{\mathcal{M}}$, because this would make $(p^{\hbar_k}_{i(j)}, d^{\mathcal{H}_\ell}_{i(k)})$ a *blocking pair* in $\boldsymbol{\mathcal{M}}$ (see **Lemma 1**). In other words, doctor $\boldsymbol{\mathcal{M}}(d^{\mathcal{H}_\ell}_{i(k)}) \succeq^k_i p^{\hbar_k}_{i(j)}$. Now, if $\boldsymbol{\mathcal{M}}(d^{\mathcal{H}_\ell}_{i(k)})$ patient would not improve in $\boldsymbol{\mathcal{M}}'$ then $\boldsymbol{\mathcal{M}}(d^{\mathcal{H}_\ell}_{i(k)}) \succeq^k_i p^{\hbar_k}_{i(j)}$. Hence, $d^{\mathcal{H}_\ell}_{i(k)}$ can not be matched with $p^{\hbar_k}_{i(j)}$ in $\boldsymbol{\mathcal{M}}'$, a contradiction. Therefore, patient in $\boldsymbol{\mathcal{M}}$ also improves in $\boldsymbol{\mathcal{M}}'$. That is, $\mathcal{D}'_i$ is not the only set of doctors in $\boldsymbol{\mathcal{M}}'$ of those patient who are getting better off in $\boldsymbol{\mathcal{M}}$; but also the set of doctors where patient in $\boldsymbol{\mathcal{M}}$ improve in $\boldsymbol{\mathcal{M}}'$. In other words, each doctor in $\mathcal{D}_i$ is matched to two different patient from $\mathcal{P}_i$ in match $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}'$, being better off in $\boldsymbol{\mathcal{M}}$ than in $\boldsymbol{\mathcal{M}}'$. It can also be proved using **Lemma 1** that $\boldsymbol{\mathcal{M}}'$ is not stable; a contradiction that terminates the proof. $\square$

5 Further analytics-based analysis

In order to provide sufficient reasoning to our simulation results presented in section 6, the two proposed mechanisms are in general analyzed on the ground of the expected distance of allocation done by the mechanisms from the top most preference. As a warm up, first the the analysis is done for any patient j, to estimate the expected distance of allocation from the top most preference. After that the analysis is extended to more general setting where all the patients present in the system are considered. It is to be noted that the results revealed by the simulations can easily be verified by the lemmas below.

Lemma 4. *The allocation resulted by RAMHECs for any patient (or doctor) j being considered first is on an average $\frac{n}{2}$ distance away from its most preferred doctor (or patient) i.e. $E[Z] \simeq \frac{n}{2}$; where Z is the random variable measuring the distance from the top most preference.*

Proof. Fix a category $c_i \in \boldsymbol{\mathcal{C}}$, and an arbitrary patient j being considered first. In RAMHECs, for any arbitrary patient (AP) being considered first are allotted a random doctor from his preference list. The index position of the doctor in the preference list is decided by k, where $k = 1, 2, \ldots, n$. Now, when a doctor is selected randomly from the preference list any of these k $(1 \leq k \leq n)$ may be selected. So any index k could be the outcome of the experiment (allocation of a doctor) and it is to be noted that selection of any such k is equally likely. Therefore, for each k such that $1 \leq k \leq n$ any k^{th} doctor can be selected with probability $\frac{1}{n}$. For $k = 1, 2, \ldots, n$, we define indicator random variable X_k where $X_k = I\{k^{th}\, doctor\ selected\ from\ patients'\ preference\ list\}$. Here, $X_k = 1$ if k^{th} doctor is selected and 0 otherwise.

$$E[X_k] = E[I\{k^{th}\ doctor\ selected\ from\ patients'\ preference\ list\}]$$

As always with the *indicator random variable*, the expectation is just the probability of the corresponding event [23]:

$$E[X_k] = 1 \cdot Pr\{X_k = 1\} + 0 \cdot Pr\{X_k = 0\} = 1 \cdot Pr\{X_k = 1\} = \frac{1}{n}$$

For a given call to RAMHECs, the indicator random variable X_k has the value 1 for exactly one value of k, and it is 0 for all other k. For $X_k = 1$, we can measure the distance of k^{th} allocated doctor from the most preferred doctor in the patient j's preference list. So, let d_k be the distance of k^{th} allocation from the best preference. More formally, it can be represented in the case analytic form as:

$$Z = \begin{cases} d_0 : & \textit{If } 1^{st} \textit{ agent is selected from the preference list } (k=1) \\ d_1 : & \textit{If } 2^{nd} \textit{ agent is selected from the preference list } (k=2) \\ \vdots & \vdots \\ d_{n-1} : & \textit{If } n^{th} \textit{ agent is selected from the preference list } (k=n) \end{cases}$$

Where Z is the random variable measuring the distance of the allocation from the patient's top most preference. Here, $d_0 = 0$, $d_1 = 1$, $d_2 = 2$, ..., $d_{n-1} = n-1$. It is to be observed that, once the doctor k is selected from the patient j's preference list, the value calculation of d_k is no way dependent on k. Now, observe that the random variable Z that we really care about can be formulated as: $Z = \sum_{k=1}^{n} X_k \cdot d_{k-1}$.

$$E[Z] = E\left[\sum_{k=1}^{n} X_k \cdot d_{k-1}\right] = \sum_{k=1}^{n} E[X_k \cdot d_{k-1}] = \sum_{k=1}^{n} E[X_k] \cdot E[d_{k-1}]$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot E[d_{k-1}] = \frac{1}{n} \sum_{k=1}^{n} E[d_{k-1}] = \frac{1}{n} \sum_{k=1}^{n} d_{k-1} = \frac{1}{n} \cdot \frac{(n-1)(n)}{2} = \frac{(n-1)}{2} \simeq \frac{n}{2}$$

as claimed. □

Lemma 5. *In RAMHECs, $E[D] \simeq \frac{n^2}{16}$; where D is the total distance of all the patients in the system from the top most preference.*

Proof. Fix a category $c_i \in \boldsymbol{\mathcal{C}}$. We are analysing, the expected distance of the allocations done to the patients by RAMHECs from the top most preferences. For this purpose, as there are n patients, the index of these patients are captured by i such that $i = 1, 2, \ldots, n$. Without loss of generality, the patients are considered in some order. The index position of the doctor in any patient j's preference list is decided by k, where $k = 1, 2, \ldots, n$. For any patient i $(1 \leq i \leq n)$ selected first, when a doctor is selected randomly from the preference list any of the available k $(1 \leq k \leq n)$ doctors can be selected. So, any index k could be the outcome of the experiment (allocation of doctor) and any such k is equally likely. But what could the case, if instead of considering the patient in the first place, say a patient is selected in i^{th} iteration. In that case, from the construction of RAMHECs the length of the preference list of the patient under consideration would be $n - i + 1$. So, when a doctor is selected randomly from the preference list, any of the $(n - i + 1)$ doctors may be selected. It is to be noted that the selection of any of the $(n-i+1)$ doctors is equally likely. Therefore, for a patient under consideration in i^{th} iteration, for each k such that $1 \leq k \leq n-i+1$ any k^{th}

doctor can be selected with probability $\frac{1}{n-i+1}$. Here, we are assuming that each agent's top preferences are still remaining when that agent is considered by the RAMHECs. To get the lower bound this is the best possible setting. If an agent is not provided that list, he will be further away from his top most preference. For each patient i and for $k = 1, 2, \ldots, n$, we define indicator random variable X_{ik} where $X_{ik} = I\{k^{th}\ doctor\ selected\ from\ patient\ i's\ preference\ list\}$. Here, $X_{ik} = 1$ if k^{th} doctor is selected from patient i's preference list and 0 otherwise.

$$E[X_{ik}] = E[I\{k^{th}\ doctor\ selected\ from\ patient\ i\ preference\ list\}]$$

As always with the *indicator random variable*, the expectation is just the probability of the corresponding event:

$$E[X_{ik}] = 1 \cdot Pr\{X_{ik} = 1\} + 0 \cdot Pr\{X_{ik} = 0\} = 1 \cdot Pr\{X_{ik} = 1\} = \frac{1}{n-i+1}$$

For a given call to RAMHECs, the indicator random variable X_{ik} has the value 1 for exactly one value of k, and it is 0 for all other k. For $X_{ik} = 1$, we can measure the distance of k^{th} allocated doctor from the most preferred doctor in the patient j's preference list. So, let d_{ik} be the distance of k^{th} allocation from the best preference. More formally, it can be represented in the case analytic form as:

$$D = \begin{cases} d_{i0}: & \textit{If } 1^{st} \textit{ agent is selected from the preference list } (k=1) \\ d_{i1}: & \textit{If } 2^{nd} \textit{ agent is selected from the preference list } (k=2) \\ \vdots & \vdots \\ d_{i(n-1)}: & \textit{If } n^{th} \textit{ agent is selected from the preference list } \ (k{=}n) \end{cases}$$

Where D is the total distance of all the patients in the system from the top most preference. It is to be observed that, once the doctor k is selected from the patient j's preference list, the value calculation of d_k is no way dependent on k. Now, observe that the random variable D that we really care about is given as: $D \geq \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} X_{ik} \cdot d_{ik}$.

$$= E[D] \geq E[\sum_{i=1}^{n} \sum_{k=1}^{n-i+1} X_{ik} \cdot d_{ik}] = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} E[X_{ik} \cdot d_{ik}] = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} E[X_{ik}] \cdot E[d_{ik}]$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{1}{n-i+1} \cdot d_{ik} \geq \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{1}{n} \cdot d_{ik} = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} d_{ik}$$

$$= \frac{1}{n}\left[\sum_{i=1}^{\frac{n}{2}} \sum_{k=1}^{n-i+1} d_{ik} + \sum_{i=\frac{n}{2}}^{n} \sum_{k=1}^{n-i+1} d_{ik}\right] \geq \frac{1}{n}\left[\sum_{i=1}^{\frac{n}{2}} \sum_{k=1}^{n-i+1} d_{ik}\right] + \left[\sum_{i=\frac{n}{2}}^{n} \sum_{k=1}^{n-i+1} 0\right]$$

$$\geq \frac{1}{n}\left[\sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} d_{ik}\right] \geq \frac{1}{n}\left[\sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} d_{i\frac{n}{2}}\right] = \frac{1}{n}\left[\sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} \frac{n}{2}\right] = \frac{1}{2}\left[\sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} 1\right]$$

$$\geq \left(\frac{1}{2}\sum_{j=1}^{\frac{n}{2}} j\right) - 1 = \frac{1}{2}\left[\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\right] - 1 = \frac{n^2+2n-16}{16} \simeq \frac{n^2}{16}$$

as claimed. It is to be observed that for each agent, the expected distance of allocation done by RAMHECs from the top preference in an amortized sense is $\frac{n}{16}$. □

Lemma 6. *The expected number of rejections for any arbitrary patient (or doctor) j resulted by TOMHECs is constant. If the probability of any k length rejection is considered as $\frac{1}{2}$ i.e. $Pr\{Y_k = 1\} = \frac{1}{2}$ then $E[Y] = 2$; where Y is the random variable measuring the total number of rejections made to the patient (or doctor) under consideration.*

Proof. Fix a category $c_i \in \boldsymbol{\mathcal{C}}$, and an arbitrary patient j. To analyze the expected number of rejections suffered by the patient under consideration in case of TOMHECs, we capture the total number of rejections done to any patient j by a random variable Y. So, the expected number of rejections suffered by any patient j is given as $E[Y]$. It is considered that the rejection by any member $k = 0, \ldots, n-1$, present on the patients' j preference list is an independent experiment. It means that, the m length rejections suffered by an arbitrary patient j is no way dependent on any of the previous $m-1$ rejections. Let us suppose for each $0 \leq k \leq n-1$, the probability of rejection by any k^{th} doctor be $\frac{1}{2}$ (it can be any value between 0 and 1 depending on the scenario). For $k = 0, \ldots, n-1$, we define indicator random variable Y_k where $Y_k = I\{k\ length\ rejection\}$. Here, Y_k is if k length rejection and 0 otherwise.

$$E[Y_k] = E[I\{k\ length\ rejection\}]$$

As always with the *indicator random variable*, the expectation is just the probability of the corresponding event:

$$E[Y_k] = 1 \cdot Pr\{Y_k = 1\} + 0 \cdot Pr\{Y_k = 0\} = 1 \cdot Pr\{Y_k = 1\} = \left(\frac{1}{2}\right)^k$$

Observe that the random variable Y that we really care about is given by $Y = \sum_{k=0}^{n-1} Y_k$.

$$E[Y] = E\left[\sum_{k=0}^{n-1} Y_k\right] = \sum_{k=0}^{n-1} E[Y_k] = \sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k < \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-(\frac{1}{2})} = 2$$

as claimed. Moreover, if we consider the probability of k^{th} rejection as $\frac{2}{3}$ then, the expected number of rejections will be given as 3 *i.e* $E[Y] = 3$. Similarly, $E[Y] = 10$ if the probability of k^{th} rejection is taken as $\frac{9}{10}$. It means that, even with the high probability of rejection to any arbitrary patient j by the members of the proposed party, there is a chance that after constant number of rejections patient j will be allocated a good doctor according to his choice. Hence, we can say that each agent's allocation is not far away from his top most preference. □

Lemma 7. *In TOMHECs, $E[R] = 2n$, where R is the random variable measuring the total number of rejections made to all the patients.*

Proof. Fix a category $c_i \in \boldsymbol{\mathcal{C}}$. We are analysing the total number of rejections suffered by all the patients in expectation. For this purpose, as there are n patients, the index of these patients are captured by i such that $i = 1, 2, \ldots, n$. The index position of the doctor in any patient $j's$ preference list is decided by k, where $k = 1, 2, \ldots, n$. We capture the total number of rejections done to all patients by a random variable R. So, the expected number of rejections suffered by all the patients is given as $E[R]$. It is considered that the rejection by any member $k = 1, \ldots, n-1$, present on the patients' i preference list is an independent experiment. It means that, the m length rejections suffered by an arbitrary patient i is no way dependent on any of the previous $m-1$ rejections. Let us suppose for each patient i and for each $1 \leq k \leq n-1$, the probability of rejection by any k^{th} doctor be $\frac{1}{2}$ (it can be any value between 0 and 1 depending on the scenario). For $k = 1, \ldots, n-1$, we define indicator random variable R_{ik} where $R_{ik} = I\{k \ length \ rejection \ of i^{th} \ patient\}$. Here, $R_{ik} = 1$ if k length rejection of i^{th} patient and 0 otherwise.

$$E[R_{ik}] = E[I\{k \ length \ rejection \ of \ i^{th} \ patient\}]$$

As always with the *indicator random variable*, the expectation is just the probability of the corresponding event:

$$E[R_{ik}] = 1 \cdot Pr\{R_{ik} = 1\} + 0 \cdot Pr\{R_{ik} = 0\} = 1 \cdot Pr\{R_{ik} = 1\} = \left(\frac{1}{2}\right)^k$$

Observe that the random variable R that we really care about is given as $R = \sum_{i=1}^{n} \sum_{k=1}^{n-i} R_{ik}$.

$$E[R] = E\left[\sum_{i=1}^{n} \sum_{k=1}^{n-i} R_{ik}\right] = \sum_{i=1}^{n} \sum_{k=1}^{n-i} E[R_{ik}] = \sum_{i=1}^{n} \sum_{k=1}^{n-i} \left(\frac{1}{2}\right)^k < \sum_{i=1}^{n} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$
$$= \sum_{k=0}^{n-1} \frac{1}{1 - (\frac{1}{2})} = 2n$$

as claimed. □

Corollary 1. *It is to be observed that for each patient, the expected number of rejections in case of TOMHECs in an amortized sense is $O(1)$. As we have shown that for all n agents, the expected number of rejection are $O(n)$.*

6 Experimental findings

The experiments are carried out in this section to compare the efficacy of the TOMHECs based on the preference lists of the doctors and patients generated randomly using *Random* library in Python. RAMHECs is considered as the benchmark mechanism.

6.1 Simulation setup

For creating a real world healthcare scenario we have considered 10 different categories of patients and doctors for our simulation purpose. One of the scenarios that is taken into consideration is, say there are equal number of patients and doctors present in each of the categories. Each of the members of participating communities are providing strict preference over all the members of the opposite community. In second scenario with $m = n$ case, each of the members in the respective parties are providing the strict preference ordering over the subset of the members of the opposite community.

6.2 Performance metrics

The efficacy of TOMHECs is measured under the banner of two important parameters: (a) **Satisfaction level ($\boldsymbol{\eta_\ell}$):** It is defined as the sum over the difference between the index of the doctor (patient) allocated from the patient's (doctor's) preference list to the index of the most preferred doctor (patient) by the patient (doctor) from his/her preference list. Considering the *requesting party*, the $\boldsymbol{\eta_\ell^j}$ for c_j category is defined as: $\boldsymbol{\eta_\ell^j} = \sum_{i=1}^{n} \left(\overline{\xi}_i - \xi_i\right)$; where, $\overline{\xi}_i$ is the index of the doctor (patient) allocated from the initially provided preference list of the patients (doctors) i, and ξ_i is the index of the most preferred doctor (patient) in the initially provided preference list of patient (doctor) i. For k categories, $\boldsymbol{\eta_\ell} = \sum_{j=1}^{k} \sum_{i=1}^{n} \left(\overline{\xi}_i - \xi_i\right)$. It is to be noted that lesser the value of satisfaction level higher will be the satisfaction of patients or doctors. (b) **Number of preferable allocation ($\boldsymbol{\zeta}$):** The term "*preferable allocation*" refers to the allocation of most preferred doctor or patient from the revealed preference lists by the patients or the doctors respectively. For a particular patient or doctor the *preferable allocation* is captured by the function $f : \boldsymbol{\mathcal{P}}_i \rightarrow \{0, 1\}$. For the category c_i, the number of preferable allocation (NPA) is defined as the number of patients (doctors) getting their first choice from the initially provided preference list. So, $\boldsymbol{\zeta_i} = \sum_{j=1}^{n} f(p_{i(j)}^{\hbar_\ell})$. For k categories $\boldsymbol{\zeta} = \sum_{i=1}^{k} \sum_{j=1}^{n} f(p_{i(j)}^{\hbar_\ell})$.

6.3 Simulation directions

The three directions are seen for measuring the performance of TOMHECs, they are: (1) All the patients and doctors are reporting their true preference list. (2) When fraction of total available members of the *requesting party* are misreporting their preference lists. (3) When fraction of total available members of the *requested party* are misreporting their preference lists.

6.4 Result analysis

In this section, the result is simulated for the above mentioned three directions and discussed. Our result analysis is broadly classified into two categories:

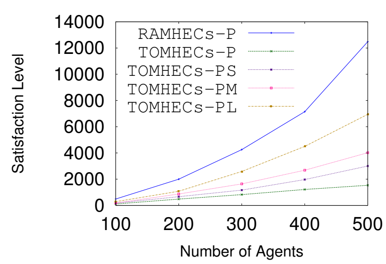
Expected amount of patients/doctors deviating The following analysis motivated by [23] justifies the idea of choosing the parameters of variation. Let χ_j be the random variable associated with the event in which j^{th} patient in c_i category varies its true preference ordering.

Table 1: Abbreviations used in simulation

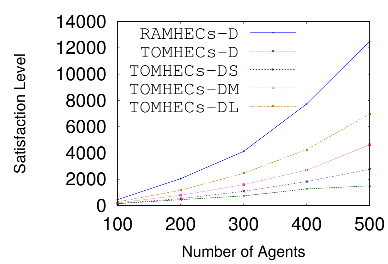
Abbreviation	**Description**
RAMHECs-P	Patients allocation using RAMHECs without variation.
TOMHECs-P	Patients allocation using TOMHECs without variation.
RAMHECs-D	Doctors allocation using RAMHECs without variation.
TOMHECs-D	Doctors allocation using TOMHECs without variation.
TOMHECs-PS	Patients allocation using TOMHECs with small variation.
TOMHECs-DS	Doctors allocation using TOMHECs with small variation.
TOMHECs-PM	Patients allocation using TOMHECs with medium variation.
TOMHECs-DM	Doctors allocation using TOMHECs with medium variation.
TOMHECs-PL	Patients allocation using TOMHECs with large variation.
TOMHECs-DL	Doctors allocation using TOMHECs with large variation.

Thus, $\chi_j = \{j^{th}$ patient varies preference ordering$\}$. $\chi = \sum_{j=1}^{n} \chi_j$. We can write $E[\chi] = \sum_{j=1}^{n} E[\chi_j] = \sum_{j=1}^{n} 1/8 = n/8$. Here, $\Pr\{j^{th}$ *patient varies preference ordering*$\}$ is the probability that given a patient whether he will vary his true preference ordering. The probability of that is taken as 1/8 (small variation).

• **Case 1: Requesting party with full preference (FP) and partial preference (PP)** In Fig. 2a, Fig. 2b and Fig. 3a, Fig.3b, the $\boldsymbol{\eta_\ell}$ and $\boldsymbol{\zeta}$ of the requesting party respectively are more for TOMHECs. Further, it is seen that higher the manipulation lower will be the $\boldsymbol{\eta_\ell}$ and $\boldsymbol{\zeta}$ for the requesting party in case of TOMHECs.

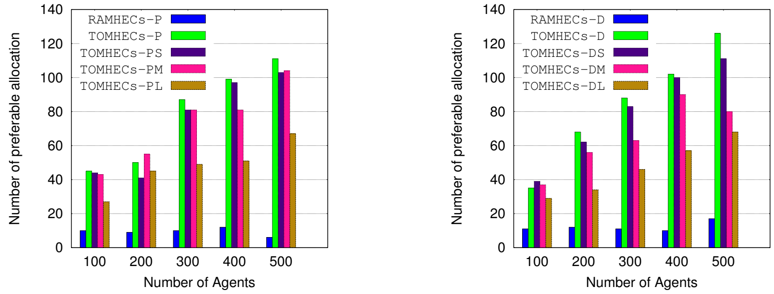


(a) $\boldsymbol{\eta_\ell}$ of patients



(b) $\boldsymbol{\eta_\ell}$ of doctors

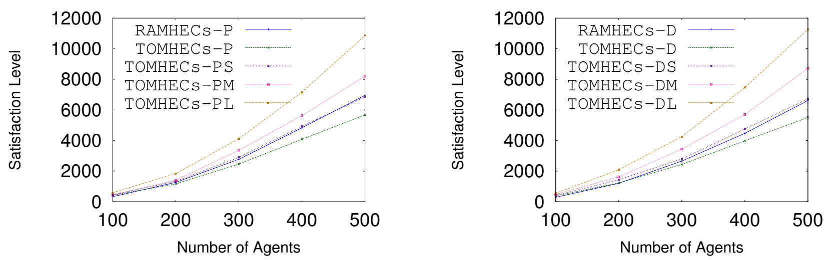
Fig. 2: $\boldsymbol{\eta_\ell}$ of requesting party with FP $(m == n)$



(a) $\boldsymbol{\zeta}$ of patients

(b) $\boldsymbol{\zeta}$ of doctors

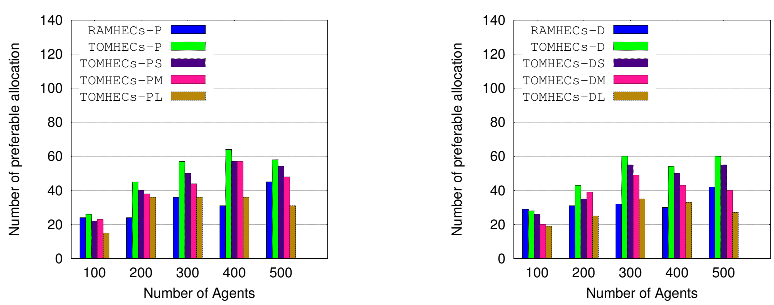
Fig. 3: $\boldsymbol{\zeta}$ of requesting party with FP $(m == n)$



(a) $\boldsymbol{\eta_\ell}$ of patients

(b) $\boldsymbol{\eta_\ell}$ of doctors

Fig. 4: $\boldsymbol{\eta_\ell}$ of requesting party with PP ($m == n$)



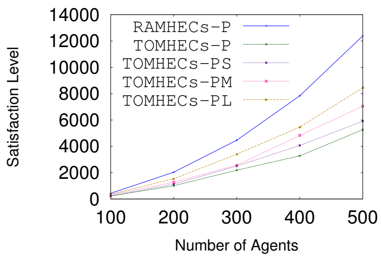
(a) $\boldsymbol{\zeta}$ of patients

(b) $\boldsymbol{\zeta}$ of doctors

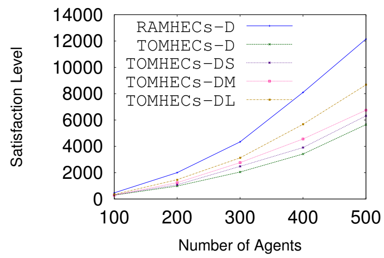
Fig. 5: $\boldsymbol{\zeta}$ of requesting party with PP $(m == n)$

Similar argument can be given for the partial preference case shown in Fig. 4a, Fig. 4b and Fig. 5a, Fig. 5b.

• **Case 2: Requested party with full preference (FP) and partial preference (PP)** In Fig. 6a, Fig. 6b and Fig. 7a, Fig. 7b, the $\boldsymbol{\eta_\ell}$ and $\boldsymbol{\zeta}$ of the requesting party respectively are more for TOMHECs. Further, it is seen that higher the manipulation lower will be the $\boldsymbol{\eta_\ell}$ and $\boldsymbol{\zeta}$ for the requesting party in case of TOMHECs.

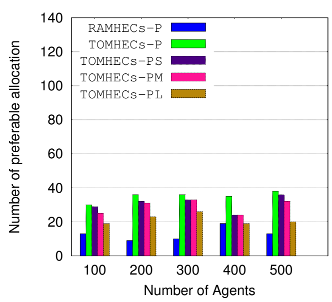


(a) $\boldsymbol{\eta_\ell}$ of patients

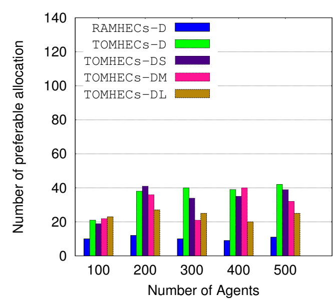


(b) $\boldsymbol{\eta_\ell}$ of doctors

Fig. 6: $\boldsymbol{\eta_\ell}$ of requested party with FP $(m == n)$



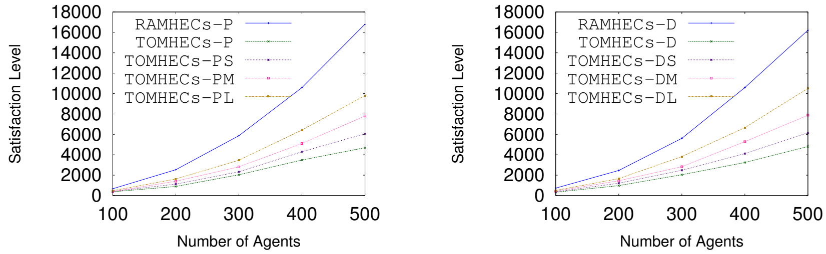
(a) $\boldsymbol{\zeta}$ of patients



(b) $\boldsymbol{\zeta}$ of doctors

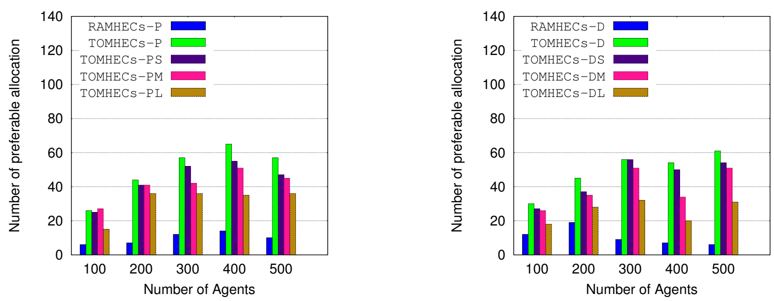
Fig. 7: $\boldsymbol{\zeta}$ of requested party with FP $(m == n)$

Similar argument can be given for the partial preference case shown in Fig. 8a, Fig. 8b and Fig. 9a, Fig. 9b.



(a) $\boldsymbol{\eta_\ell}$ of patients (b) $\boldsymbol{\eta_\ell}$ of doctors

Fig. 8: $\boldsymbol{\eta_\ell}$ of requested party with PP $(m == n)$



(a) $\boldsymbol{\zeta}$ of patients (b) $\boldsymbol{\zeta}$ of doctors

Fig. 9: $\boldsymbol{\zeta}$ of requested party with PP $(m == n)$

7 Conclusions and future works

We have tried to model the *ECs hiring problem* as a two sided matching problem in healthcare domain. This paper proposed an *optimal* and *truthful* mechanism, namely TOMHECs to allocate the ECs to the patients. The immediate future work could be the more general setting with n patients and m doctors ($m \neq n$ or $m == n$). In this, the additional constraint is that, the members of the *patient party* and *doctor party* can provide the preference ordering (not necessarily strict) over the subset of the members of the opposite party.

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