Stabilisation of state and input constrained nonlinear systems via diffeomorphisms. A Sontag's formula approach with an actual application

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Abstract

In this work we provide a different and constructive outlook for the control of state and control constrained nonlinear systems. Explicit solutions have been mainly focused in the finding of a barrier-like Lyapunov function, unlike here where we propose the construction of a (*proper*) diffeomorphism to map all the trajectories of the constrained dynamics into an unconstrained one. The careful analysis has revealed that only some foundations of differential geometry and a couple of technical assumptions are necessary to construct the proposed methodology based on the well-established theories of control Lyapunov functions and Sontag's universal formulae. Altogether it allows us to obtain an explicit solution that even includes bounded constraints in the control action, giving the designer a way to decide—to some extent—the tradeoff between control saturations and robustness. Moreover, this approach does not rely in the own structure of the system dynamics, covering thereby a broad class of nonlinear systems. The result has been successfully applied to solve the dynamic positioning of an actual ship where the nonlinear state constraints describe a strait. This approach allowed us the design of a control Lyapunov function and thereby use Sontag's formula to solve the stabilisation problem. Realistic simulations have been executed in a real scenario on the simulator owned by an international shipbuilding company.

1 Introduction

This work deals with the stabilisation of nonlinear systems with (nonlinear) state and input constraints. The underlying idea is the construction of a differentiable and bijective map which transforms the dynamics living in a constrained "world" into a dynamics living in an unconstrained one, so that the control designer relies only on Lyapunov-based techniques in the transformed dynamics and somehow "forgets" about the constraints. Firstly motivated by the works with barrier Lyapunov functions for strick-feedback cascade systems of [1], [2], [3] and [4], with our approach we provide a different outlook giving a constructive result for a broader class of nonlinear systems. This work is also partially related to [5] where an output error transformation was proposed to fulfil some prescribed performance for feedback linearisable nonlinear systems. Either way, both approaches rely on the structure of the system—either strick-feedback or feedback linearisable nonlinear systems–which is instrumental for the proposed solutions. The main advantages of this approach are: to split the mathematical treatment of the constraint and the Lyapunov design and, cover a broader class of nonlinear systems.

Although out of the scope of this work, it is also worth mentioning some related works in the opimisation-based context. Thus, in [6] a weighted barrier function based was included in the objective function of the model predictive control approach, so that in each step an unconstrained optimisation problem has to be solved to obtain the computational control input. On the other hand, in [7] a barrier function for first-order kinematical model of nonholonomic agents is proposed to encode collision avoidance in decision making of multi-agent systems. In [8] the notion of barrier stopping points is analysed. Finally, we mention a closer related work given in [9] where the authors introduced the notion of barrier certificates to certify that all trajectories of a system starting from a given initial set do not enter in unsafe regions. The latter approach becomes computationally tractable with sum-of-squares optimisation approach for polynomial vector fields, covering hybrid and stochastic systems. The main difference between that approach and ours is that the work of [9] could be seen as a procedure to check that all trajectories converge safely for a given closedloop system. Here we propose a procedure to design a stabiliser such that, *a priori*, we know that all the trajectories will converge safely to the equilibrium providing an explicit solution and so, it does not rely on optimisation-based numerical computations.

This work continues, refines and completes the preliminary work of [10] where we stated part of the results, without proofs, and just for the state constrained case. Thus, in here we refine the theory removing some unnecessary assumptions, prove all the statements and provide new results that include also the case of input constraints. Moreover, we show that once the equivalence is established, the combination of the proposed mapping between the constrained and unconstrained dynamics with Sontag's formulae based on control Lyapunov functions [11, 12] (closely related with Arstein's work in [13]) provides a constructive approach to obtain a solution to stabilise state and input (bounded) constrained nonlinear systems. As a result, the mapping point of view provides a breakthrough to deal with input constraints (input saturations), unlike neither of the aforementioned approaches, and hence easing its use in practice as it is demonstrated in the application section. In fact, with the new developments provided here we redesign the controller in [14] complementing the numerical results on a realistic simulator of an actual ship–succinctly described in [14]– owned by an internationally renowned shipbuilding company [15] being, to the best of authors' knowledge the first implementation of Sontag's formula in a real problem at that level.

Although there is still work to be done to solve this difficult nonlinear control problem, this work goes a step forward in the finding of explicit solutions for stabilisation of constrained nonlinear systems, generalising to some extent the approaches based on nonlinear transformations.

The paper is organised as follows. In Section 2, we formulate the problem and provide all the necessary previous results and definitions. The state constrained case is treated in Section 3 and, in Section 4, we extend the result to use Sontag's universal formulae to deal with the state and input constraints. In Section 5 we provide the Sontag-based solution for the dynamic positioning of a ship along a strait. The paper closes with a conclusion Section.

Notation. Unless otherwise indicated, all vectors are defined as column vectors including the gradient. The jacobian of a vector function $\Phi(x)$ is denoted by $\partial_x \Phi$, $x \in \mathbb{R}^n$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x| = \sqrt{x^\top x}$. Acronyms: GAS (GES) means Globally Asymptotically Stable (Globally Exponentially Stable), *i.e.* means that is, w.r.t. means with respect to.

2 Background

In this section we introduce and formalise the problem and recall some foundations. Thus, consider an autonomous and affine-in-control system given by

$$\Sigma_x^u : \quad \dot{x} = f(x) + g(x)\, u, \quad x \in \mathcal{X} \subseteq \mathbb{R}^n, \tag{1}$$

where f and g are smooth vector fields, with $f(0) = 0$, $g(0) \neq 0$ and $u \in \mathcal{U} \subseteq \mathbb{R}^m$ the input. Let $\mathcal{X} = \{\mathcal{X}_c, \mathcal{X}_u\}$ be a partition of the state vector, where $\mathcal{X}_c$ and $\mathcal{X}_u$ are the constrained and unconstrained states, respectively, so that if $\dim(\mathcal{X}_c) = l$

then $\dim(\mathcal{X}_u) = n - l$. We consider here the stabilisation problem at the origin of (1) where the states $x_i(t)$, $i = 1, ..., l$, are required to remain in the open and connected set $\mathcal{X}_c$ for all time $t \geq 0$ with $x_i(0) \in \mathcal{X}_c$. In this approach we consider first a control design with $\mathcal{U} \equiv \mathbb{R}^m$ and, then with input bounded taken values in an open ball.

In short, this approach relies on mapping all the trajectories of (1) through a diffeomorphism, such that the constrained state is embedded into the transformed dynamics. For the sake of completeness, we recall in the appendix the formal definition of a diffeomorphism from [18]. Essentially, a diffeomorphism $\Phi : \mathcal{X} \mapsto \mathcal{Z}$ with $\mathcal{X}, \mathcal{Z} \subseteq \mathbb{R}^n$, is a one-to-one continuous map with continuous inverse Φ^{-1} where both Φ and Φ^{-1} are $\mathcal{C}^1$—i.e. a bijective and differentiable map. Thus, for a given diffeomorphism, say $z = \Phi(x)$, the transformed dynamics (1) through it become

$$\Sigma_z^{\bar{u}} : \quad \dot{z} \quad = \quad F(z) + G(z)\ \bar{u}, \quad z \in \mathcal{Z} \subseteq \mathbb{R}^n, \tag{2}$$

with $F(z) := \partial_x \Phi\ f(x)\big|_{x=\Phi^{-1}(z)}, G(z) := \partial_x \Phi\ g(x)\big|_{x=\Phi^{-1}(z)}$, and where for $u = u(x)$ then $\bar{u} := u \circ \Phi^{-1}(z) \in \mathcal{U}$. However, to be able to map any trajectory between both open subsets $\mathcal{X}$ and $\mathcal{Z}$ the diffeomorphism has to be well defined everywhere, requirement covered by the *proper* property. For the sake of completeness, in the appendix we collect its formal definition and a Lemma adapted from [19] and [20], which provides the equivalence between a (*proper*) diffeomorphism and its jacobian for open subsets of $\mathbb{R}^n$. The following assumption formalises the map required in this approach.

Assumption 1. *The* $\mathcal{C}^1$ *map* $\Phi : \mathcal{X} \mapsto \mathbb{R}^n$*,* $\Phi(0) = 0$*, is a diffeomorphism.*

Remark 1. *All the machinery recalled above and collected in the appendix is necessary because the* local *characterisation of a diffeomorphism is not enough. In particular, we cannot make use of the Inverse Function Theorem, which provides a sufficient condition between neighbourhoods.*

3 Lyapunov-based control design by diffeomorphism equivalence with barrier functions. State-constrained case

In this section we establish the equivalence between some known Lyapunov stability results and barrier functions. First, we recall from [1] the definition of a Barrier Lyapunov Function (henceforth BLF). We believe that it is essential to leave the definition given in [1] untouched for the sake of comparison thereafter, even knowing that some extra machinery is needed for our approach. Thus, the way we proceed is keeping that definition as it is and adding the necessary extra assumption/property to be able to conclude Lyapunov stability. To make this work self-contained we reproduce below such definition.

Definition 1. *A Barrier Lyapunov Function is a scalar function* $W(x)$*, defined w.r.t. the system* $\dot{x} = \mathcal{F}(x) := f(x, u(x))$ *on an open region* $\mathcal{X}$ *containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of* $\mathcal{X}$*, has the property* $W(x) \to \infty$ *as* x *approaches* *the boundary of $\mathcal{X}$, and satisfies $W(x(t)) \leq c$ for all $t \geq 0$ along any solution of $\dot{x} = \mathcal{F}(x)$ with $x(0) \in \mathcal{X}$ and some positive constant c.*

In [1] the controller design relies on a BLF proposed *ad hoc* for systems in strict feedback form. Here we propose to transform the dynamics (1) through the aforementioned diffeomorphism such that the controller design for the constrained dynamics becomes a standard unconstrained design. Thus, although its derivation is straightforward, in the following lemma we state the equivalence between BLFs defined on transformed dynamics through a map satisfying Assumption 1.

Lemma 1. *Let $W : \mathcal{X} \mapsto \mathbb{R}_+$ be a BLF for (1), for some $u(x)$ and $x \in \mathcal{X}$. Then, for any map $z = \Phi(x)$ verifying Assumption 1 the function $V(z) := W \circ \Phi^{-1}(z)$ is a BLF on $\mathbb{R}^n$ w.r.t. (2) and with $\bar{u} = u \circ \Phi^{-1}(z)$.*

Proof. By assumptions W is a BLF and Φ a diffeomorphism, hence $V(z) = W(\Phi^{-1}(z))$ is continuous, positive definite and with continuous first-order partial derivatives at every point of $\mathbb{R}^n$. Moreover, the BLF assumption and z living in $\mathbb{R}^n$ imply that $V(z) = W(\Phi^{-1}(z)) \to \infty$ whenever $\Phi^{-1}(z)$ approaches to the boundary of $\mathcal{X}$. Finally, $V(z(t)) = W(\Phi^{-1}(z(t))) \leq c$, $z(0) \in \mathbb{R}^n$, for all $t \geq 0$ along any solution of (2) and some positive constant c[1], with $\bar{u} = u(\Phi^{-1}(z))$. □

To clarify the Lemma 1, in Fig. 1 we depict the corresponding relational diagram of Lemma 1 where BLF functions and control are at the upper and lower level, respectively. Notice that Lemma 1 does not conclude anything about

[1]The constant c is a function of the initial conditions.

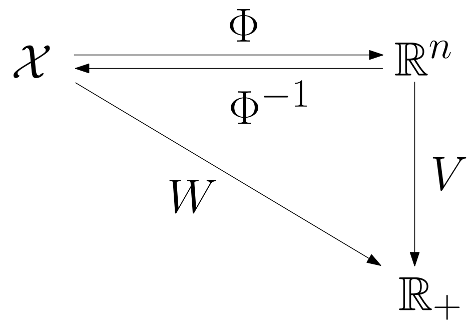

Figure 1: Relational diagram.

stability neither convergence, in the Lyapunov sense. This fact stems from the own Definition 1 that ensures only boundedness and confinement of trajectories. Thus, the following assumption adds the required property to qualify a BLF as a Lyapunov function. We hightlight the removal of some unnecessary conditions made in [10].

Assumption 2. *Let* $W : \mathcal{X} \mapsto \mathbb{R}^n$ *be a BLF as stated in Definition 1 that additionally satisfies* $\dot{W}(x(t)) < 0$*, for all* $x \in \mathcal{X}$ *and* $t \geq 0$*.*

Now we are in position to state a preliminary stability result in the following theorem which states the equivalence between Lyapunov theory and BLFs. This comes directly from properties of maps satisfying Assumption 1 together with BLF-like functions verifying Assumption 2.

Theorem 1. *Let* $z = \Phi(x)$ *be a map verifying Assumption 1.* $W : \mathcal{X} \mapsto \mathbb{R}_+$ *is a BLF for (1), for some* $u = u_l(x)$ *and* $x \in \mathcal{X}$*, verifying Assumption 2 if and only if* $V(z) := W \circ \Phi^{-1}(z)$ *is a Lyapunov function for (2), with* $\bar{u} = u_l \circ \Phi^{-1}(z)$ *and* $z \in \mathbb{R}^n$*. Moreover, the origin of (1) and (2) are Lyapunov stable.*

Proof. $[W \Rightarrow V]$. By Definition 1 the function W is positive definite, $\mathcal{C}^1$ and by Assumption 2 $\dot{W} \leq 0$, for all $x \in \mathcal{X}$ and some $u_l(x)$. Hence, W is a Lyapunov function for (1) and Lyapunov theory ensures that $x = 0$ is a stable equilibrium of (1). On the other hand, the map Φ of Assumption 1 is a $\mathcal{C}^1$ bijection from $\mathcal{X}$ to $\mathbb{R}^n$ (diffeomorphism) with $z \in \mathbb{R}^n$ and so, both $V(z) = W \circ \Phi^{-1}(z)$ and $\bar{u} = u_l \circ \Phi^{-1}(z)$ for (2) inherit all the smoothness properties. Additionally, the bijection guarantees positiveness of V and by Assumption 2 its derivative along the trajectories of (2) becomes $\dot{V}(z) = \dot{W}(\Phi^{-1}(z)) \leq 0$, for all $z \in \mathbb{R}^n$. Altogether proves that V is a Lyapunov function for (2) and its origin is Lyapunov stable.

$[W \Leftarrow V]$. Replace Φ by Φ^{-1} and flip V and W to obtain the reciprocal implication. □

The result of Theorem 1 allows control designers to use any Lyapunov-based technique for the stabilisation of the state-constrained problem.The idea consists in constructing a new unconstrained state z and designing a control algorithm $\bar{u}(z)$ with any available design method. The only requirement is that for this control law, $\bar{u}(z)$, there should be a Lyapunov function V radially unbounded. Finally, the control law is given by $u(x) := \bar{u} \circ \Phi(x)$ and—although unnecessary—for the design the BLF can be obtained from $W(x) := V \circ \Phi(x)$. Moreover, there is not restriction in the structure of the nonlinear system (1) for example, unlike in [1] a cascade structure is not longer needed. It is worth to mention that the transformed dynamics might be of a higher complexity, but not necessarily as it is shown in the benchmark example below. In [10] a simple and nonlinear example to illustrate the use of the result of Theorem 1 is thoroughly solved and where a fair comparison with the approach in [1] is also provided. For future reference on the new development in this paper w.r.t. [10] we summarise it below.

Benchmark example [10]

Consider the stabilisation at the origin $x = 0$ of the system given by

$$\Sigma_x^u : \qquad f(x) := \begin{bmatrix} x_2 - x_1^2 x_2 \\ 0 \end{bmatrix}, \quad g(x) := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

together with forcing, firstly a partial-constrained state where $\mathcal{X} = \{|x_1| < 1,\ x_2 \in \mathbb{R}\}$ and secondly, a full-constrained state where $\mathcal{X} = \{|x_1| < 1,\ |x_2| < 1\}$ and $u \in \mathcal{U} \equiv \mathbb{R}$.

Partial-constrained state: $\mathcal{X} = \{\mathcal{X}_c; \mathcal{X}_u\} = \{|x_1| < 1;\ x_2 \in \mathbb{R}\}$. The following diffeomorphism was defined to transform the constrained state to an unconstrained one

$$z = \Phi(x) := \begin{bmatrix} \tanh^{-1} x_1 \\ x_2 \end{bmatrix}, \ \Phi^{-1}(z) = \begin{bmatrix} \tanh z_1 \\ z_2 \end{bmatrix},$$

with $\partial_x \Phi(x) > 0$, for all $x \in \mathcal{X}$. The diffeomorphism is not unique and another possible choice is provided in [10] so that a tradeoff between robustness and simplicity of the dynamics should make the designer to decide the “best”. From (2), the dynamics in the new coordinates become

$$\Sigma_z^{\bar{u}} : \quad F(z) = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}, \quad G(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The design step for the transformed (linear) system is straightforward. Thus, the stabiliser $\bar{u} = -k_1 z_1 - k_2 z_2$, $k_1, k_2 > 0$, makes $z = 0$ GAS, and GES or undoing the change of coordinates $x = 0$ with $u(x) = -k_1 \tanh^{-1} x_1 - k_2 x_2$. A Lyapunov function V was proposed in [10] to prove the stability result that completes the design without needing to know the associated BLF on x which, for the sake of completeness, was also provided there. For a fair comparison and since x_1 is scalar, in [10] we also provided the solution with the approach of [1] resulting more involved.

Full-constrained state: $\mathcal{X} = \{\mathcal{X}_c; \mathcal{X}_u\} = \{|x_1| < 1,\ |x_2| < 1; \emptyset\}$. For that Φ was defined as

$$\Phi(x) := \begin{bmatrix} \tanh^{-1} x_1 \\ \tanh^{-1} x_2 \end{bmatrix}, \ \Phi^{-1}(z) = \begin{bmatrix} \tanh z_1 \\ \tanh z_2 \end{bmatrix},$$

where $\partial_x \Phi(x) > 0$ for all $|x_i| < 1$, $i = 1, 2$. The dynamics in the new coordinates become

$$\Sigma_z^{\bar{u}} :\ F(z) = \begin{bmatrix} \tanh z_2 \\ 0 \end{bmatrix}, \ G(z) = \begin{bmatrix} 0 \\ \dfrac{1}{1 - (\tanh z_2)^2} \end{bmatrix},$$

and a stabiliser that makes $z = 0$ GAS became $\bar{u} := (-z_1 - 2z_2 - 2\tanh z_2)(1 - (\tanh z_2)^2)$. In [10] we provide further details and, in particular, in Fig. 2 there, we show the level curves of the resulting Lyapunov functions for both designs which help to the understanding.

4 CLF-based control design. State and input constrained case

In this section, we go forward in the design stage making use of control Lyapunov functions with a twofold objective: being constructive and including control constraints (bounds). A weak point of the approach proposed in the previous section and also in [1], and [2], is that the barrier for the state is made through a (nonlinear) "high-gain" feedback. The closer the state is to the barrier the higher (nonlinear) gain is provided, so that in practice this might cause undesired saturations and, in turn, instabilities. The point-of-view given here allows us to force additionally to the input to be constrained in some bounded set, unlike in [1], and [2]. At first view this is very surprising since it looks rather unintuitive, but the underlying idea is the tradeoff between robustness and a bounded input. Toward this end, we make use of the control Lyapunov functions approach and constructive Sontag's formulae. Let us brief the control Lyapunov function approach from the unconstrained transformed dynamics side (2). Thus, this approach relies on the equivalence between the *stabilisability* and the existence of Lyapunov functions with some special properties introduced in [13] and later on in [11]. In fact, in [13] Arstein proved that there exists a stabiliser $\bar{u}(z)$, $z \in \mathbb{R}^n$ that makes the origin of (2) *asymptotically stable* if and only if there exists a $\mathcal{C}^1$ Lyapunov function $V(z) > 0$, with $V(0) = 0$, satisfying the inequality

$$\inf_{u \in \mathcal{U}} \left\{ \partial_z V^\top (F + G\bar{u}) \right\} < 0, \tag{3}$$

for at least an $\bar{u}$ if $z \neq 0$. Moreover, the origin is GAS if $V(z) \to \infty$ as $|z| \to \infty$, i.e. radially unbounded. In a related work [11] Sontag called such Lyapunov function *Control Lyapunov Function* (henceforth CLF). Arstein also proved that, although smooth elsewhere such stabiliser $\bar{u}$ does not have to be continuous at the origin. However, he also provided a *necessary and sufficient* condition on V to make $\bar{u}$ continuous, which is: *for every* $\delta > 0$ *there is an* $\epsilon > 0$ *such that, whenever* $|z| < \delta$, $z \neq 0$, *there is some* $\bar{u}$ *with* $|\bar{u}| < \delta$ *such that the inequality* $\partial_z V^\top (F + G\bar{u}) < 0$ *holds.* Sontag in [11] called such condition the *small control property.*

Coming back to our approach, it becomes apparent that a BLF satisfying Assumption 2 might or might not be a CLF, according with its aforementioned definition (3). Thus, for the sake of formality and comparison let us define a BLF that additionally satisfies the conditions to be a CLF.

Definition 2. *A Barrier Control Lyapunov Function (henceforth BCLF) is a BLF that additionally is a CLF endowed with the* small control property*, w.r.t. some dynamics.*

With Definition 2 the next theorem establishes the equivalence between both constrained (1) and unconstrained (2) worlds via CLFs and diffeomorphisms that, in turn, simplifies drastically the controller design stage, upon having a map Φ satisfying Assumption 1.

Theorem 2. *Let* $z = \Phi(x)$, $x \in \mathcal{X}$ *and* $z \in \mathbb{R}^n$, *be a map verifying Assumption 1. Then,* $W(x)$ *is a BCLF w.r.t.* Σ_x^u if and only if $V(z)$ *is a BCLF w.r.t.* $\Sigma_z^{\bar{u}}$, *with* $W(x) := V \circ \Phi(x)$ *and* $u(x) := \bar{u} \circ \Phi(x)$.

Proof. $[W(x) \in \text{BCLF} \Leftarrow V(z) \in \text{BCLF}]$. Since $V(z)$ is a BCLF for (2), $z \in \mathbb{R}^n$, there is a smooth stabiliser $\bar{u}$ such that the following inequality holds

$$\begin{aligned} \dot{V}(z) &= \partial_z V^\top \, \dot{z} \\ &= \partial_z V^\top \left(F(z) + G(z)\bar{u}(z)\right) < 0, \quad z \neq 0. \end{aligned} \tag{4}$$

On the one hand, by Assumption 1 the function $W(x) := V \circ \Phi(x)$ is positive definite, smooth and $\mathcal{C}^1$, and the stabiliser $u = \bar{u} \circ \Phi(x)$ is smooth as well, for all $x \in \mathcal{X}$ and hence W is a BLF. On the other hand, the derivative of W along the trajectories of the system (1), for all $x \in \mathcal{X}$, becomes

$$\begin{aligned} \dot{W}(x) &= \dot{V}(\Phi(x)) \\ &= \partial_z V^\top \Big|_{z=\Phi(x)} \partial_x \Phi(x) \, \dot{x} \\ &= \partial_z V^\top \Big|_{z=\Phi(x)} \left(\partial_x \Phi \; f(x) + \partial_x \Phi \; g(x) u(x)\right) \\ &= \partial_z V^\top \left(F(z) + G(z)\bar{u}(z)\right) \Big|_{z=\Phi(x)} < 0, \end{aligned}$$

where the last inequality holds from (4) and so the BLF W is a CLF as well, i.e. a BCLF.

$[W(x) \in \text{BCLF} \Rightarrow V(z) \in \text{BCLF}]$. Since Φ is a diffeomorphism, replacing Φ by Φ^{-1} and flipping V and W in the previous proof one obtains the reciprocal implication. □

Remark 2. *As it was aforementioned the use of a diffeomorphism eases the control-design stage at the level of Lyapunov functions avoiding* ad-hoc *cross terms in BLFs. In fact, in an unstructured nonlinear system (neither feedback nor feedforward structure) the design of a cross-term might become a daunting* *task. In [10] we made a comparison with the numerical "feasibility check" needed in [1] and related references therein.*

4.1 Sontag-formulae based control design

The equivalence established in Theorem 2 allows the use of Sontag's universal formulae for stabilisation. The design methodology can be described into two steps: first, describe the constraints in x with Φ and map the dynamics to z; and secondly find a BCLF in z and make use of Sontag's formulae. Moreover, we also propose the use of the formula for bounded control which solves constrained state problems with bounded control. Notice that altogether it provides a methodology to solve the hard problem of stabilisation of nonlinear systems with state and control constraints. Thus, let us brief those formulae below, w.r.t. the control-design side $\Sigma_z^{\bar{u}}$ (2). For that, recall from [11] definitions of the (vector) functions $\mathbf{a} := \partial_z V^\top F$, $\mathbf{b} := \partial_z V^\top G$ and denoting the Euclidean norm as $|\cdot|$ let us define $\overrightarrow{\mathbf{b}} := \mathbf{b}^\top/|\mathbf{b}|^2$.

Unbounded control formula $\Leftrightarrow \mathcal{U} \equiv \mathbb{R}^m$. In [11] a universal formula for stabilisation was provided upon knowing a CLF. Thus, our control design reduces to find a CLF w.r.t. $\Sigma_z^{\bar{u}}$ that globally stabilises its origin, say $V(z)$, and then the stabiliser formula reads

$$\bar{u}(z) := \begin{cases} -\left(\mathbf{a} + \sqrt{\mathbf{a}^2 + |\mathbf{b}|^4}\right)\overrightarrow{\mathbf{b}} & , \quad \mathbf{b} \neq 0 \\ 0 & , \quad \mathbf{b} = 0. \end{cases} \tag{5}$$

Recall that (5) is continuous at $z = 0$ if and only if the CLF satisfies the aforementioned *small control property* [11]. In fact, if $V(z)$ is a BCLF according to Definition 2 that property holds.

Bounded control formula $\Leftrightarrow \mathcal{U} \equiv \mathcal{B}_r^m$. The above formula of [11] is essentially a (nonlinear) "high-gain" feedback which might cause saturation, in practice. In fact, the tradeoff between robustness and actuator saturation was highlighted in [12] and, it is well known the interaction between instability and saturating actuators elegantly described by Stein in [16]. Thus, in [12] the authors provided the following formula for bounded control

$$\bar{u}(z) := \begin{cases} -\dfrac{\mathbf{a} + \sqrt{\mathbf{a}^2 + |\mathbf{b}|^4}}{1 + \sqrt{1 + |\mathbf{b}|^2}} r \overrightarrow{\mathbf{b}} & , \quad \mathbf{b} \neq 0 \\ 0 & , \quad \mathbf{b} = 0, \end{cases} \tag{6}$$

and where as before the formula (6) has also been defined on the unconstrained dynamics. The formula (6) was defined to confine the control in the unit ball, i.e. $r = 1$. For practical issues, in here we have just replaced the unit ball ($\mathcal{B}_1^m$) for one of radius r defined as

$$\mathcal{B}_r^m := \{u \in \mathbb{R}^m : |u|^2 < r\}, r > 0.$$

In fact, as it is shown in the application section, in practise the formulae (5) and (6) need to be endowed with an Euclidean weighted norm so that the designer is able to tune the controller according with the performances required.

All the technical properties of (6), such as K-continuity[2], analyticity and boundedness, were provided in [12] and so we refer the reader there for details. Those properties were developed for a controller taking values in the unit ball,

[2] Definition of K-continuity, from [12], is also collected in the appendix.

and it is not difficult to see that all those properties hold for a ball of radius r, although the "scaling" factor r has to be inserted accordingly. In this way, the following theorem establishes the way to proceed in this framework to design controllers with BCLFs. This result is a direct consequence of the join of both a BCLF and a diffeomorphism of Assumption 1. However, its proof needs a (non-linear) re-scaled version of the Lemma 2.3 in [12] that, although straightforward, we streamline along it, for clarity.

Theorem 3. *Let $z = \Phi(x)$, $x \in \mathcal{X}$ and $z \in \mathbb{R}^n$, be a map verifying Assumption 1 and V be a BCLF for the system (2) in with $\mathcal{U} \equiv \mathbb{R}^m$ ($\mathcal{U} \equiv \mathcal{B}_r^m$). Then, the origin of (2) is GAS with Sontag's formula (5) (formula (6)), and so is the origin of (1) on $\mathcal{X}$, with $x(t) \in \mathcal{X}$, $t \geq 0$.*

Proof. $[\mathcal{U} \equiv \mathbb{R}^m]$. For the unbounded control case a BCLF is just a CLF with *small control property* and, hence by the result of [11] the formula (5) makes the origin of (2) GAS, which in turn makes the origin of (1) GAS on $\mathcal{X}$ by the equivalence established in Theorem 2.

$[\mathcal{U} \equiv \mathcal{B}_r^m]$. As it has been aforementioned, in the case of bounded control $\mathcal{U} \equiv \mathcal{B}_r^m$, for clarity we streamline a "re-scaled" version of Lemma 2.3 in [12]. Thus, following the same line of arguments as in [12], the fact that V is a BCLF with $\bar{u} \in \mathcal{U} \equiv \mathcal{B}_r^m$ is equivalent to $a + r|b| < 0$. It is straightforward to see that Lemma 2.3 of [12] holds with a rescaled definition of the open set $\mathcal{D}$ from [12] (involved definitions collect in the appendix), as $\mathcal{D}_r := \{(a, b) : a < r|b|,\ a, b \in \mathbb{R}\}$ and also the function $\alpha_r := r\alpha(a, b)$. In particular, properties (a) and (b) remain unchanged, (c) becomes $|\alpha_r| < r$ and (d) $a + b\alpha_r < 0$

for all $(a,b) \in \mathcal{D}_r$. Thus, the *small control property* assumption of V implies that $(\mathbf{a}, |\mathbf{b}|) \in \mathcal{D}_r$ and hence, (d) implies that V is a BCLF and (c) from (6) implies that $\mathcal{U} \equiv \mathcal{B}_r^m$. It is straightforward to see that Theorem 1 of [12] with $\mathcal{U} \equiv \mathcal{B}_r^m$ guarantees the the origin of (2) is GAS. Finally, stabilisability of (1) is also guaranteed recalling the equivalence established by Theorem 2, with $u(x) := \bar{u} \circ \Phi(x) = \bar{u}(\Phi(x)) \in \mathcal{U} \subseteq \mathcal{B}_r^m$. □

Benchmark example cont'd

We redesign the controller for the example of Section 3 by using the results of Theorems 2 and 3 together with the universal formulae. On the one hand, it is straightforward to verify that $V(z) = z_1^2 + z_1 z_2 + z_2^2$ proposed in [10] is a BCLF. To see this notice that

$$\begin{aligned} \mathbf{a} &= (2z_1 + z_2)\tanh z_2, \\ \mathbf{b} &= \frac{z_1 + 2z_2}{1 - (\tanh z_2)^2}, \end{aligned}$$

and after some straightforward manipulations the derivative reads

$$\dot{V}(z) = -3z_2 \tanh z_2 + (z_1 + 2z_2)\left(2\tanh z_2 + \frac{\bar{u}}{1 - (\tanh z_2)^2}\right).$$

Thus, V is a BCLF and, moreover, is also a BCLF with $\mathcal{U} \subseteq \mathcal{B}_1^1$. Hence, both formulae (5) and (6) guarantee the GAS property of the origin of (2) GAS and, with Theorem 2 of the origin of (1) with $u = \bar{u} \circ \Phi(x)$. Comparison of Sontag's formula in a batch of 100 simulations are shown in Fig. 2 in the original coordinates x. Fig. 2 at the top left (right) shows the simulations for

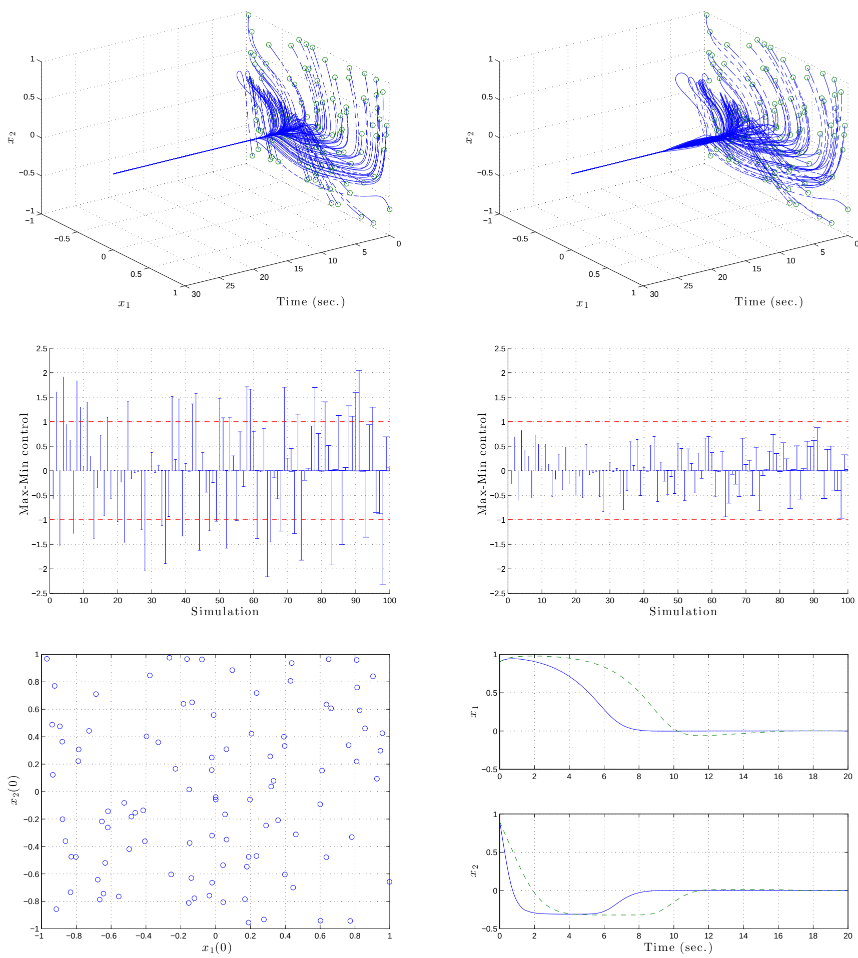

Figure 2: Batch of 100 simulations with Sontag's formulas. Initial conditions $x(0) \in \mathcal{X} = (-1, 1) \times (-1, 1)$.

unbounded control (5) (bounded control (6)). In the middle left (right) the minimum (maximum) of the input in each simulation with unbounded control (5) (bounded control (6)). Notice that in the bounded control case the $|u| < 1$, as expected by design. At the bottom left it shows the cloud of initial conditions and, finally, at the bottom right the tradeoff of unbounded and bounded performances for $x(0) = (0.9, 0.9)$, noting that the stabilisation takes about 30% longer in the bounded case.

Remark 3. *Even though the constructive Sontag's result is powerful providing analytic formulae, for the sake of generality we underscore that Arstein's result goes further because it considers non smoothness at the origin, even for non-affine nonlinear systems. Moreover, Arstein's result guarantees the existence of a feedback, i.e. stabilisability. In that regard, let us redesign the stabiliser in the benchmark example without the use of Sontag's formula (6) and ensuring* $|\bar{u}| < 1$*. To this end, let design stabiliser ensuring GAS with that bound for the input with the following state feedback*

$$\bar{u} = -2(k \tanh(z_1 + 2z_2) + \tanh z_2)(1 - (\tanh z_2)^2),$$

where k *is a control gain such that* $0 < k < 1/2 - 2/(3\sqrt{3})$ *in order to fulfil the constraint* $|\bar{u}| < 1$*.*

5 Application: Dynamic positioning control of marine craft

This application entails a more complicated scenario due to the nonlinearities introduced by the use of two different reference frames. It consists in the dynamic positioning problem of a ship[3]. The Lyapunov-based design mentioned in Section 3 is used in [14] to solve the constrained problem. In here, we make use of a BCLF control design making use of the result of Theorems 2 and 3. Thus, consider the widely accepted low-speed ship dynamics (see [17]) as

$$\dot{x}_c = J(x_c)x_u, \tag{7}$$

$$M\dot{x}_u = -Dx_u + u + J^\top(x_c)\mathbf{d}, \tag{8}$$

where $x_c \in \mathcal{X}_c \subseteq \mathbb{R}^3$ and $x_u \in \mathcal{X}_u \subseteq \mathbb{R}^3$ are the position coordinate vector in the Earth-fixed reference frame and the relative vessel-frame velocity coordinate vector, respectively, $u \in \mathbb{R}^3$ is the vector forces and torque applied to the vessel in the fixed reference frame and $\mathbf{d} \in \mathbb{R}^3$ represents the environmental disturbances due to the sea currents, waves, and wind (which is assumed known). The rotation matrix relating the Earth-fixed frame to the relative-frame of the reference is $J(x_c)$ and matrices $M = M^\top > 0$ and $D + D^\top > 0$ represent constant regime damping/drag and inertia, respectively (see [14] for more details).

Remark 4. *It is important to highlight that the model (7)-(8) is just for control design purposes and all the simulations have been made in the aforementioned*

[3] According to the Encyclopædia Britannica, a ship is any large floating vessel capable of crossing open waters, and a displacement ≥ 500 tons.

realistic shipbuilding-company simulator keeping all the marine-craft nonlinearities as wind, ocean currents and waves.

The control objective is twofold: 1) the stabilisation of the ship at a desired position x_c^d, i.e. $\lim_{t\to\infty}(x_c(t) - x_c^d) = 0$ and $\lim_{t\to\infty} x_u(t) = 0$, $t \geq 0$; and, 2) reach the desired position with the state confined in a predefined region on the sea, i.e. $x_c(t) \in \mathcal{X}_c$, where $\mathcal{X}_c := \left\{(x_{c1}, x_{c2}) \in \mathbb{R}^2 : C(x_c) < 0\right\}$ and $(x_{c1}(0), x_{c2}(0)) \in \mathcal{X}_c$. Notice that, this control problem is partially constrained because only two of the states $\mathcal{X}_c$ are constrained. In summary, it is a regulation control problem with nonlinear state constrains. In fact, it is an interesting problem for dynamic positioning of ships since, among others, it alleviates the computational burden of motion planning.

On the other hand, to shift the desired equilibrium to the origin and ease the calculations, we define the partition $z_c := \phi(x_c)$, $z_u := x_u$ and the error coordinate $\tilde{z}_c := z_c - z_c^d$, with $z_c^d := \phi(x_c^d)$ the desired position, so that the corresponding *diffeomorphism* according to the notation becomes $\Phi := [\phi(x_c)^\top, x_u^\top]^\top$. Thus, the unconstrained error dynamics from (7)-(8) yield

$$\dot{\tilde{z}}_c = \partial\bar{\Psi} \cdot \bar{J} z_u, \tag{9}$$

$$M\dot{z}_u = -Dz_u + \bar{u} + \bar{J}^\top \mathbf{d}, \tag{10}$$

where for compactness we have defined $\partial\bar{\Psi}(\tilde{z}_c) := \partial_{x_c}\phi \circ \phi^{-1}(\tilde{z}_c)$ and $\bar{J} = J \circ \phi^{-1}(\tilde{z}_c)$. For the sake of simplicity, we consider the case $\mathbf{d} = 0$ because in practice, their good estimation allow to compensate them with a feedforward action.

Unlike in the Lyapunov-based design of [14], the design here is based on a CLF, and so we need to find a suitable CLF. Unfortunately, the Lyapunov function used in [14] is not a CLF. However, that design paves the way to find a CLF defined as

$$V := \frac{1}{2}\begin{bmatrix} \tilde{z}_c \\ z_u \end{bmatrix}^\top \begin{bmatrix} 2I_3 & \bar{J} \\ \bar{J}^\top & I_3 \end{bmatrix} \begin{bmatrix} \tilde{z}_c \\ z_u \end{bmatrix}. \qquad (11)$$

Thus, we are in position to state the proposition below.

Proposition 1. *Consider a map* $\Phi := [\phi(x_c)^\top, x_u^\top]^\top$ *satisfying Assumption 1 and the dynamics given by* (9)-(10)*. Then, for any* ϕ *such that*

$$\partial\bar{\Psi} + \partial\bar{\Psi}^\top > 0, \qquad (12)$$

the function (11) is a CLF and hence, Sontag's formula (5) makes the origin GAS.

Proof. The function (11) is radially unbounded on $\mathbb{R}$ and positive definite recalling the fact that $J^\top J = I_3$. The derivative of (11) along the trajectories of (9)-(10) reads

$$\begin{aligned} \dot{V} &= \left(2\tilde{z}_c^\top + z_u^\top \bar{J}^\top\right)\partial\bar{\Psi}\bar{J}z_u + \tilde{z}_c^\top \dot{\bar{J}} z_u + \left(z_c^\top \bar{J} + z_u^\top\right) M^{-1}(-Dz_u + \bar{u}) \\ &= -\frac{1}{2}\tilde{z}_c^\top\left(\partial\bar{\Psi} + \partial\bar{\Psi}^\top\right)\tilde{z}_c + \left(z_c^\top \bar{J} + z_u^\top\right)\left[M^{-1}(-Dz_u + \bar{u}) + \partial\bar{\Psi}z_u + \partial\bar{\Psi}^\top \bar{J}^\top z_c + Sz_u\right], \end{aligned}$$

where we have made use of the property of $\dot{J}$ with S its corresponding skew-symmetric matrix. Defining the set $\Omega := \{(\tilde{z}_c, z_u) \in \mathbb{R}^6 : z_u = -\bar{J}^\top \tilde{z}_c\}$, by definition of CLF we have

$$\dot{V}\big|_\Omega = -\frac{1}{2}\tilde{z}_c^\top\left(\partial\bar{\Psi} + \partial\bar{\Psi}^\top\right)\tilde{z}_c,$$

which under the positivity condition (12) becomes negative definite restricted to the set Ω, and hence V is a CLF. Since (11) is a CLF and satisfies *small control property*, Sontag's formula (5) is a global *smooth* stabiliser for the origin of (9)-(10) and, Theorem 2 establishes the equivalence needed. $\square$

Remark 5. *We underscore that the positivity condition (12) is not difficult to be satisfied. The own construction of the diffeomorphism makes in many applications its jacobian sign definite. In the following controller design for a typical scenario this fact becomes apparent.*

Let us consider the strait[4] scenario proposed in [14] where the position constraints are set as

$$\mathcal{X}_c := \begin{cases} x_{c1} \in \mathbb{R}: & x_{c1}^{\min} < x_{c1} < x_{c1}^{\max}, \\ x_{c2} \in \mathbb{R}: & x_{c2}^{\min}(x_{c1}) < y < x_{c2}^{\max}(x_{c1}), \\ x_{c3} \in \mathbb{R}, & \end{cases}$$

with functions $x_{c2}^{\min}(x_{c1}) := a_m x_{c1}^2 + b_m$ and $x_{c2}^{\max}(x_{c1}) := a_M x_{c1}^2 + b_M$, becoming apparent the quadratic boundaries for the x_{c2} coordinate, and constants $x_{c1}^{\min}, a_m, b_m < 0$ and $x_{c1}^{\max}, a_M, b_M > 0$. Thus, a suitable diffeomorphism fulfilling the required properties becomes

$$z_c = \phi(x_c) := \begin{bmatrix} \dfrac{k_1 x_{c1}}{(x_{c1}^{\min} - x_{c1})(x_{c1}^{\max} - x_{c1})} \\ \dfrac{k_2 x_{c2}}{(x_{c2}^{\min}(x_{c1}) - x_{c2})(x_{c2}^{\max}(x_{c1}) - x_{c2})} \\ x_{c3} \end{bmatrix}.$$

[4]A strait is a narrow area of sea connecting larger areas.

It is straightforward to check that (12) is satisfied so that

$$\partial_{x_c}\phi = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with

$$\begin{aligned}
\alpha_{11} &= k_1 \frac{{x_{c1}}^2 - x_{c1}^{\min} x_{c1}^{\max}}{(x_{c1} - x_{c1}^{\min})^2 (x_{c1} - x_{c1}^{\max})^2} \\
\alpha_{21} &= 2k_2 x_{c1} x_{c2} \frac{2a_m a_M {x_{c1}}^2 + a_m b_M - a_m x_{c2} + a_M b_m - a_M x_{c2}}{(x_{c2}^{\min}(x_{c1}) - x_{c2})(x_{c2}^{\max}(x_{c1}) - x_{c2})} \\
\alpha_{22} &= k_2 \frac{{x_{c2}}^2 - a_m {x_{c1}}^4 a_M - a_m {x_{c1}}^2 b_M - b_m a_M {x_{c1}}^2 - b_m b_M}{(x_{c2}^{\min}(x_{c1}) - x_{c2})(x_{c2}^{\max}(x_{c1}) - x_{c2})}.
\end{aligned}$$

and where k_1, k_2 are positive control gains, $\partial_{x_c}\phi$ is positive definite and consequently $\partial\bar{\Psi}$, in $\mathcal{X}_c$. In Fig. 3 we show a representative simulation result. To explore the performance and capabilities of the controller all the simulations have been made without wind, current and waves disturbances, but we keep all the nonlinearities of the model so that we can test the robustness to parametric uncertainties. A complete simulation analysis including all those real data and disturbances is pending for the approval and certification of the company. The simulation was made starting from the initial condition $x_c(0) = (-150, 30, 1^\circ)$ with a destination at $x_c^d = (150, 30, 0^\circ)$, and where the arrow points to the course in an obvious way. In Fig. 3 three regions have been differentiated: the green is the ground and the dotted line on the sea is splitting the ground and the continental sea area of relatively shallow water. The diffeomorphism definition is based on the safe margin with dotted line. Notice that the initial condition is forced to be in the direction of the prohibited area and the controller confines

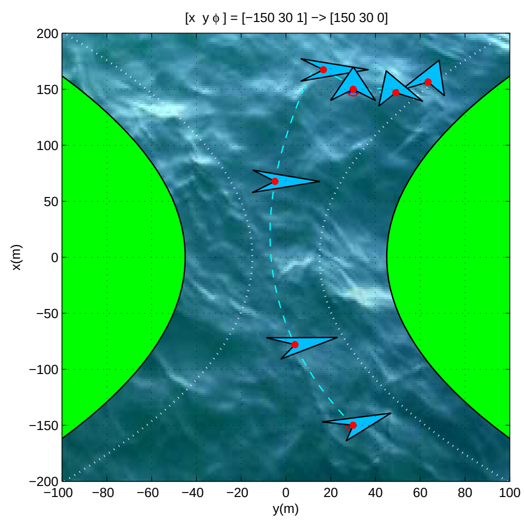

Figure 3: Position and orientation of the ship along the path. Dotted line: safe constraint programmed; green patch: earth; dashed line: trajectory described by the center of gravity.

the trajectory fulfilling the objective. However, due to the high nonlinearity the tuning of the controller gains is not straightforward. For those that they do not know ship thrust configurations, it is also worth to mention that this ship is overactuated, i.e. it can be drove in any direction with its thrusters. In this way, since there is not restriction on the orientation of the ship the controller first tries to follow the constraints requirement leaving the orientation for the end. Current research is under way to analyse the relationship between the controller gains and the performance.

Remark 6. *After analysing a massive amount of data from simulations we realised that the mathematical model (7)–(8) considered for the design is not precise enough, in the sense that by physical considerations the ship dynamics are obviously bounded. The latter means that we were able to compute the necessary bounds to identify the minimum r that qualifies the function (11) as a BCLF as well, and hence the design with bounded input is also guaranteed with formula (6), although r depends on the actual ship configuration and disturbances considered.*

Remark 7. *Recall that the realistic simulator succinctly described in [14] includes environmental disturbances and uncertainties giving insight of the robustness of the controller. However, we had to reshape the norms in the formulae (5) and (6) to achieve good performances. To the best of authors' knowledge it is the first implementation of Sontag's formula in a real problem at that level.*

6 Conclusions

A methodology to find explicit solutions for stabilisation of constrained nonlinear systems is provided. The approach relies on the construction of a (*proper*) diffeomorphism so that standard Lyapunov-based techniques and can be applied. Moreover, the proposed methodology makes use of Sontag's universal formulae as a constructive alternative to the controller design, and therefore being able to deal with bounded input constraints. To wrap up the paper, we provide the solution of the dynamic positioning of a ship along a strait making use of Sontag's formula, and reporting representative realistic simulations on the simulator owned by a renowned shipbuilding company.

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A Appendix

Definition. Diffeomorphism (From [18])

If $\mathcal{X}$ and $\mathcal{Z}$ are open sets in $\mathbb{R}^n$, a differentiable function $\Phi : \mathcal{X} \mapsto \mathcal{Z}$ with a differentiable inverse $\Phi^{-1} : \mathcal{Z} \mapsto \mathcal{X}$ will be called a diffeomorphism.

Definition. Proper map

A continuous map $\Phi : \mathcal{X} \mapsto \mathbb{R}^n$ is proper if $|\Phi(x)| \to \infty$ as $|x|$ approaches to the boundary of $\mathcal{X}$, with $x \in \mathcal{X}$.

Lemma. Diffeomorphism vs. jacobian equivalence (From [19] and [20])

Let $\mathcal{X}$ and $\mathcal{Z}$ be open subsets of $\mathbb{R}^n$. A $\mathcal{C}^1$ map $\Phi : \mathcal{X} \mapsto \mathcal{Z}$ is a diffeomorphism if and only if Φ is proper and the jacobian $\partial_x \Phi$ never vanishes.

Definition. K-continuous (From [11])

Let $\mathcal{D} \subseteq \mathbb{R}^2$ denote the open set $\mathcal{D} := \{(a, b) : a < |b|, a, b \in \mathbb{R}\}$. A function $\sigma : \mathcal{D} \mapsto \mathbb{R}$ is K-continuous if for each $\epsilon > 0$, there is a $\delta > 0$ such that: $|b| < \delta$ and $a < \delta|b| \Rightarrow |\sigma(a, b)| < \epsilon$.

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