A Parameterization for a class of Complete Games with Abstention

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Abstract: Voting games with abstention are voting systems in which players can cast not only yes and no vote, but are allowed to abstain. This paper centers on the structure of a class of complete games with abstention. We obtain, a parameterization that can be useful for enumerating these games, up to isomorphism. Indeed, any I-complete game is determined by a vector of matrices with non-negative integers entries. It also allows us determining whether a complete game with abstention is a strongly weighted (3,2) game or not, and for other purposes of interest in game theory.

Keywords: (3,2) games · Abstention · Desirability relations · Weighted games and Complete games ·

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1 Introduction

Simple games have been intensively used as models of collective choice and, especially, for situations arising from political science. Many of these situations are described by weighted majority games, the most interesting class of simple games. In a simple game, a single alternative, such as a bill or an amendment, is pitted against the status quo, the players or voters vote in favor of the alternative or against it and the motion is passed or not depending of the collective strength of members who vote “”yes”.” The motion passes if and only if the set of all those who vote ”yes” is a winning coalition. Abstention plays a key role in many of the real voting systems that have been modeled by these games (such as the United Nations Security Council, or the United States federal system), yet simple games, by their very definition, do not take the possibility of abstention into account; those who do not vote “yes” are presumed to vote “no.” Felsenthal and Machover [6] define ternary voting games (TVGs), a generalization of simple voting games. This class of games is a particular case of the more general class of (j, k) games introduced by Freixas and Zwicker [11]. Taking $j = 3$ and $k = 2$ leads to $(3, 2)$ games that are equivalent to TVGs. In either of these models of games, abstention is treated as a level of approval intermediate to “yes” and “no”.

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Defining a simple game requires to list winning coalitions. The representation is rather simple if it is a weighted game, but many decisions rules do not admit such a representation. A characterization of games that admit a representation as weighted game is due to Taylor and Zwicker [21] (see [22] for a complete description of weighted games and related games). Looking for a more convenient representation is another motivation of the work by Carreras and Freixas [3]. Since completeness is a necessary condition for a simple game to be representable as a weighted game, they argued that complete games constitute a natural framework for discussing the characterization of weighted voting. This paper deals with the class of complete simple games and centers on their structure. These authors showed that a complete simple game is determined uniquely, up to isomorphism, by a vector with positive integers components and a matrix with non-negative integers entries. Clearly, this is simpler and more intuitive than setting all the winning coalitions of the game. The present paper is a generalization of the former to simple games with abstention or (3,2) simple games. We obtain in this larger class of vote a similar result as that by Carreras and Freixas [3].

Our results allow us to obtain some enumerations of I-complete (3,2) games. They also allow us to simplify the task to determine whether a given I-complete $(3,2)$ game is strongly weighted or not. This is a very important step for the resolution of voting game design problems, one of which is the well known inverse problem, (see Alon and Edelman [1], Kurz [15] and Dragan [5]). In this sort of problem, we look for a weighted voting game that minimizes the distance between the distribution of power [1] among the players and a given target distribution of power (according to a given distance measure). In [14], Keijzer *et al* provide algorithms that solve voting game design problems by enumerating all games of interest. The algorithm has been improved in the subclass of weighted games. Our result is a preparation for the extension of the work by Keijzer *et al* [14], to voting games with abstention.

The enumeration of I-complete (3,2) games we obtain is very restrictive. Indeed, unlike Kurz and Tautenhahn [16] who describe an approach to determine enumeration formulas for the number of complete simple games, as for I-complete (3,2) games, we are able to achieve this only for very small values of n the number of players. The parameterization for I-complete (3,2) games we obtain in this paper combined with the application of some enumerating techniques may potentially serve for achieving further enumerations of I-complete (3,2) games and strongly weighted games.

In order to achieve the results mentioned above, we follow the same methodology as Carreras and Freixas [3]. The main tool used in this paper is the desirability relation introduced by Isbell [13]. We consider the natural extension of this relation in $(3,2)$ games, introduced by Tchantcho *et al* [23] and reconsidered in Pongou *et al* [19] and Freixas *et al* ([9], [10]). This extension is termed influence relation.

The rest of the paper is organized as follows. In Section 2, we recall basic definitions on $(3,2)$ games. We also extend the well known influence relation introduced by Tchantcho *et al* [23] to tripartitions and provide a characterization of indifference classes. In Section 3, we associate with any complete $(3,2)$ game a multilattice of tripartition models which are represented by matrices since they are clearly easier to manipulate. The main result is presented in Section 4 in which we show that any $(3,2)$ complete game is characterized, up to an isomorphism by a vector of

[1]See [7] for a full description on power measurement problem.

matrices, entries of which are non-negative integers. Such a representation is clearly simpler and more intuitive than enumerating all the winning tripartitions of the game. We apply it to the United Nations Security Council and show that this game is strongly weighted. Section 5 discusses our results and concludes the paper. All the proofs are presented in the Appendix.

2 Preliminaries : (3,2) simple games

The materials on this section are essentially taken from Freixas and Zwicker [11], Tchantcho *et al* [23] and Freixas *et al* ([9], [10]). In [11], Freixas and Zwicker introduced (j, k) simple games, we consider the particular case where $j = 3$ and $k = 2$. Before the main notions are introduced we need some preliminary definitions.

Throughout the paper, N denotes the non-empty and finite set of voters or players. An ordered tripartition of N is a sequence $S = (S_1, S_2, S_3)$ of mutually disjoint subsets of N whose union is N. In S, S_1 stands for the set of yes voters, S_2 for abstainers and S_3 stands for no voters. We denote by 2^N the set of all subsets of N or the set of all ordered bipartitions of N and by 3^N the set of all ordered tripartitions of N. For any subset C of N and any $a \in N$, we simply write $C \cup a$ for $C \cup \{a\}$ while $C \setminus a$ stands for $C \setminus \{a\}$. For $S, S' \in 3^N$ we write $S \subseteq^3 S'$ to mean that either $S = S'$ or S may be transformed into S' by shifting 1 or more voters to higher levels of approval. Formally $S \subseteq^3 S' \Leftrightarrow S_1 \subseteq S'_1$ and $S_2 \subseteq S'_1 \cup S'_2$; we write $S \subset^3 S'$ if $S \subseteq^3 S'$ and $S \neq S'$.
The $\subseteq^3$ order defined in 3^N has minimum: the tripartition $(\emptyset, \emptyset, N)$, and maximum: the tripartition $(N, \emptyset, \emptyset)$. Hence for every tripartition S, $(\emptyset, \emptyset, N) \subseteq^3 S \subseteq^3 (N, \emptyset, \emptyset)$.

Definition 2.1 *A simple game (or (2,2) game) is a pair (N, V) where N is the non-empty but finite set of voters and V is a value function defined from 2^N to $\{0, 1\}$ such that for all coalitions C, C', if $C \subset C'$ then $V(C) = 1$ implies $V(C') = 1$.*

It is often demanded that V be exhaustive, which leads to $V(\emptyset) = 0$ and $V(N) = 1$.

Definition 2.2 *A $(3, 2)$ game $G = (N, V)$ consists of a finite set N of voters together with a value function $V : 3^N \longrightarrow \{0, 1\}$ such that for all ordered tripartitions S, S', if $S \subset^3 S'$ then $V(S) = 1$ implies $V(S') = 1$.*

A tripartition S such that $V(S) = 1$ is said to be *winning*. A $(3, 2)$ game can be defined by its set of winning tripartitions, $\mathcal{W} = \{S \in 3^N : V(S) = 1\}$. In that case we denote the game by $(N, \mathcal{W})$. In voting, it is often demanded that V be exhaustive, then from the monotonicity demanded to V, $V(\emptyset, \emptyset, N) = 0$ and $V(N, \emptyset, \emptyset) = 1$. This enable us to obtain the equivalent definition below.

Definition 2.3 *A (3,2) game $G = (N, \mathcal{W})$ consists of a finite set N of voters together with a set $\mathcal{W}$ verifying the following conditions:*

- $(\emptyset, \emptyset, N) \notin \mathcal{W}$,
- $(N, \emptyset, \emptyset) \in \mathcal{W}$,

- *If $S \subset^3 T$ and $S \in \mathcal{W}$ then $T \in \mathcal{W}$ (monotonicity).*

Standard notions on simple games naturally extend for tripartitions in $(3,2)$ games : S is a *losing tripartition* whenever $S \notin \mathcal{W}$, S is a *minimal winning tripartition* provided that $S \in \mathcal{W}$ and for all $T \in 3^N$ such that $T \subset^3 S$, $T \notin \mathcal{W}$. Let $\mathcal{W}^m$ denote the set of minimal winning tripartitions. It is clear that $\mathcal{W}$ or $\mathcal{W}^m$ uniquely determine the $(3,2)$ game. Similarly, S is a maximal losing tripartition if $S \notin \mathcal{W}$ and for all T such that $S \subset^3 T$, $T \in \mathcal{W}$. Furthermore, $\mathcal{L}$ or $\mathcal{L}^M$ uniquely determine the $(3,2)$ game, where $\mathcal{L}$ is the set of *losing tripartitions* and $\mathcal{L}^M$ the set of *maximal losing* tripartitions. *Anonymous* $(3,2)$ games are games for which for all tripartition S, S is winning *if and only if* for all permutation $\varphi : N \to N$, $\varphi(S) = (\varphi(S_1), \varphi(S_2), \varphi(S_3))$ is winning.

Next, we introduce weighted $(3,2)$ games, which is a special type of weighted (j,k) games introduced in [11].

Definition 2.4 *Let $G = (N, \mathcal{W})$ be a (3,2) game. A representation of G as a weighted (3,2) game consists of a vector $w = (w_1, w_2, w_3)$ where $w_i : N \to \mathbb{R}$ for each i together with a real number quota q such that for every $S \in 3^N$, $S \in \mathcal{W} \Leftrightarrow w(S) \geq q$ where $w(S)$ denotes $\sum_{i=1}^{3} \sum_{a \in S_i} w_i(a)$ and $w_1(a) \geq w_2(a) \geq w_3(a)$ for each $a \in N$.*

We say that $G = (N, \mathcal{W})$ is a weighted (3,2) game if it has such a representation.

According to the definition above, we can normalize, i.e. assign a zero weight, to any level of approval. Here we are mainly concerned with games with abstention for which we can normalize the weights at any of the three input levels, but it seems to be quite natural to choose the "abstention" level . If a null weight is assigned to abstainers, then a non-negative weight is assigned to "yes" voters and a non-positive weight to "no" voters. Thus, a weight [2] $w(a) = (w^+(a), 0, w^-(a))$ with $w^+(a) \geq 0$ and $w^-(a) \leq 0$ is assigned to each $a \in N$. The only requirement for the threshold q, if the $(3,2)$ game is demanded to be exhaustive, is that:

$$w(\emptyset, \emptyset, N) = \sum_{a \in N} w^-(a) < q \leq \sum_{a \in N} w^+(a) = w(N, \emptyset, \emptyset).$$

The previous definition can now be rewritten as follows : G is weighted if there exists a sequence of weight functions $(w^+, 0, w^-)$ with $w^-(a) \leq 0 \leq w^+(a)$ for all $a \in N$, and a quota q such that for all $S = (S_1, S_2, S_3) \in 3^N$, $S \in \mathcal{W} \Longleftrightarrow w(S) = \sum_{a \in S_1} w^+(a) + \sum_{a \in S_3} w^-(a) \geq q$.

Two consecutive stronger conditions of a weighted $(3,2)$ game are the following which were introduced in Freixas and Zwicker [11]:

Definition 2.5 *A* strongly weighted $(3,2)$ *game is a weighted* $(3,2)$ *game that admits a representation such that for every pair of voters a and b,*

$$[w^+(a) \geq w^+(b), -w^-(a) \geq -w^-(b)] \text{ or } [w^+(a) \leq w^+(b), -w^-(a) \leq -w^-(b)].$$

The influence relation defined in simple games were extended to (3,2) games by Tchantcho et *al* [23] as follows.

[2]We are identifying w^+ with w_1, 0 with w_2 and w^- with w_3 in Definition 2.4.

Definition 2.6 *Let $G = (N, \mathcal{W})$ be a $(3, 2)$ game, $a, b \in N$:*

- *a is said to be at least as* influential *as b, denoted $a \geq_I b$, if $a \geq_{D^+} b$, $a \geq_{D^-} b$ and $a \geq_{D^\pm} b$ where :*

  1) D^+-desirability : for all $(S_1, S_2, S_3) \in 3^N$ such that $a, b \in S_2$,
  $(S_1 \cup b, S_2 \setminus b, S_3) \in \mathcal{W} \Rightarrow (S_1 \cup a, S_2 \setminus b, S_3) \in \mathcal{W}$

  2) D^--desirability: for all $(S_1, S_2, S_3) \in 3^N$ such that $a, b \in S_3$,
  $(S_1, S_2 \cup b, S_3 \setminus b) \in \mathcal{W} \Rightarrow (S_1, S_2 \cup b, S_3 \setminus b) \in \mathcal{W}$

  3) $D^\pm$-desirability: for all $(S_1, S_2, S_3) \in 3^N$ such that $a, b \in S_3$,
  $(S_1 \cup b, S_2, S_3 \setminus b) \in \mathcal{W} \Rightarrow (S_1 \cup a, S_2, S_3 \setminus a) \in \mathcal{W}$

- *a is said to be strictly more* influential *than b, denoted $a >_I b$ if $a \geq_{D^+} b$, $a \geq_{D^-} b$, $a \geq_{D^\pm} b$ and at least one of the three relations is strict.*

- *a is said to be as* influential *as b, denoted $a \equiv_I b$ if $a \geq_I b$ and $b \geq_I a$. In this case, players a and b are said to be I-equivalent.*

- *G is a I-complete (3,2) game if either $a \geq_I b$ or $b \geq_I b$ for all pair $a, b \in N$.*

It is straightforward to check that $\equiv_I$ is an equivalence relation on N. In the sequel, the equivalence classes will be denoted by $N_1, N_2, \ldots, N_t$; the quotient set for $\equiv_I$ is then denoted and given by $N/{\equiv_I} = \{N_1, \ldots, N_t\}$. That is, $a \equiv_I b$ *if and only if* a and b belong to the same equivalence class. Furthermore, $>_I$ induces a ranking $\succ_I$ on the set of $\equiv_I$-classes. If $a >_I b$, $a \in N_u$ and $b \in N_v$ then $N_u \succ_I N_v$ and we convey $u < v$.

The I-influence relation, which is reflexive is neither complete nor transitive in general. However, it has been proved in [23] that it is transitive whenever it is I-complete. Particularly, in I-complete games, the influence relation is a complete preorder on the set of voters.

We illustrate the I-influence relation through the following example that will be very useful in the sequel. From now on we will refer as $\overline{n}$ the vector defined by : $\overline{n} = (n_1, n_2, ..., n_t)$ where for all $i = 1, ..., t$, $n_i = |N_i|$. In I-complete (3,2) games, we have : $N_1 \succ_I N_2 \succ_I \cdots \succ_I N_t$.

Example 2.7 Let us consider the 4-player game defined by : $N = \{1, 2, 3, 4\}$ and
$\mathcal{W}^m = \left\{ \begin{array}{l} (12, 3, 4), (12, 4, 3), (13, 2, 4), (14, 2, 3), (23, 1, 4), \\ (24, 1, 3), (23, 4, 1), (13, 4, 2), (14, 3, 2), (24, 3, 1) \end{array} \right\}$ where for instance, 12 represents $\{1, 2\}$.

The game is I-complete and there are two equivalence classes: the highest is $N_1 = \{1, 2\}$ and the other class is $N_2 = \{3, 4\}$. More precisely, we have : $1 \equiv_I 2 >_I 3 \equiv_I 4$ or equivalently $N_1 \succ_I N_2$.

It is well known that in $(2, 2)$ games, weighted games are complete and there exist, when $n \geq 6$, complete games that are not weighted. Unlike in the $(2, 2)$ simple games, a weighted $(3, 2)$ game may not be I-complete. As well, I-completeness does not imply weightedness. However, if a $(3, 2)$ game is strongly weighted then it is I-complete but the converse is not true. Although for $n = 2$, I-completeness implies strongly weightedness. When $n > 2$, one may find for every n an I-complete game not being strongly weighted, see Freixas et *al* [9] for these known results.

We recall below the definition of transposition operation of two players within a given tripartition. Given a tripartition $S = (S_1, S_2, S_3)$ of N and two players a and b, $\pi_{ab}(S)$ is defined by : $\pi_{ab}(S) = (\pi_{ab}(S_1), \pi_{ab}(S_2), \pi_{ab}(S_3))$ where for any coalition C of N,

$$\pi_{ab}(C) = \begin{cases} C & \text{if } \{a,b\} \cap C = \emptyset \text{ or } \{a,b\} \subseteq C \\ (C \setminus a) \cup b & \text{if } a \in C \text{ and } b \notin C \\ (C \setminus b) \cup a & \text{if } a \notin C \text{ and } b \in C \end{cases}$$

The following definition will be useful in the sequel.

Definition 2.8 *Let $G = (N, \mathcal{W})$ be an I-complete (3,2) game. A tripartition S is said to be shift-minimal winning if : S is winning and $\pi_{ab}(S)$ is losing for all $a \in S_i$, $b \in S_j$, $i < j$ and $a >_I b$. The set of shift-minimal winning tripartitions is denoted by $\mathcal{W}^{sm}$.*

As in simple games, for any $(3,2)$ game $(N, \mathcal{W})$ it is straightforward to see that $\mathcal{W}^{sm} \subseteq \mathcal{W}^m \subseteq \mathcal{W}$ and these inclusions can be strict.

In the sequel we would like to extend relations $\geq_I$ and $\equiv_I$ which are defined on the set of players to the set of tripartitions. Given two tripartitions S and T, consider the following binary relations on 3^N.

- $T \perp S$ means that there exist $a, b \in N$ with $a \equiv_I b$ such that $\pi_{ab}(S) = T$;
- $T \dashv S$ means that either $S \subseteq^3 T$ or there exist two players a and b such that $a \geq_I b$, $a \in S_j$, $b \in S_i$ with $i \leq j$ and $\pi_{ab}(S) \subseteq^3 T$.

Definition 2.9 *Let G be an I-complete (3,2) game, $S, T \in 3^N$, then :*

- *S is said to be equivalent to T denoted $S \sim_I T$ if $S \perp R_1 \perp \cdots \perp R_h = T$ for some integer number h.*
- *T is said to dominate S denoted $T \succsim_I S$ if $T \dashv R_h \dashv \cdots \dashv R_1 = S$ for some integer number h.*

It can be easily checked that $\sim_I$ is an equivalence relation on 3^N. Furthermore, $\succsim_I$ is a preordering in the set 3^N with $\sim_I$ as associated equivalence relation.

Proposition 2.10 *For all $S, T \in 3^N$, $T \sim_I S$ if and only if $T \succsim_I S$ and $S \succsim_I T$.*

In the sequel, the $\sim_I$-class of a tripartition $S \in 3^N$ will be denoted by $\overline{S}$.

Proposition 2.11 *Let $(N, \mathcal{W})$ be an I-complete (3,2) simple game, $N_1, \ldots, N_t$, be the equivalence classes of $\equiv_I$, with $n_i = |N_i|$ for all $1 \leq i \leq t$. Then,*

1) for all $R, S \in 3^N$, $S \sim_I R \Leftrightarrow |N_i \cap S_j| = |N_i \cap R_j|$ for all $i = 1, 2, ..., t$ and all $j = 1, 2, 3$.

2-a) for all $R, S \in 3^N$, if $\overline{S} = \overline{R}$ then $\overline{s} = \overline{r}$ where $\overline{s} = (s_{i,j})$ and $\overline{r} = (r_{i,j})$ for all $i = 1, ..., t$ and all $j = 1, 2, 3$, with $s_{i,j} = |N_i \cap S_j|$; furthermore, $0 \leq s_{i,j}$ and $\sum_{j=1}^{3} s_{i,j} = n_i$ for all $i = 1, 2, ..., t$.

2-b) Conversely, for any matrix $\overline{s} = (s_{i,j})_{\substack{i=1,\dots,t\\ j=1,2,3}}$ such that: $0 \leq s_{i,j}$ and $\sum_{j=1}^{3} s_{i,j} = n_i$ for all $i = 1, ..., t$ defines a unique $\sim_I$-class $\overline{S} \in \overline{3^N}$.

3) for any $S \in 3^N$, the cardinality of $\overline{S}$ is given by : $|\overline{S}| = \prod_{i=1}^{t} \binom{n_i}{s_{i,1}}\binom{n_i - s_{i,1}}{s_{i,2}}$.

In the sequel, we shall call $\overline{s} = (s_{i,j})_{\substack{i=1,\dots,t\\ j=1,2,3}}$ the *matrix of indices* associated with the class $\overline{S}$: it provides the common model, in terms of equivalent players of all tripartitions belonging to $\overline{S}$.

3 The multilattice associated with an I-complete (3,2) game

It is clearly easier to manipulate models that are represented by matrices $\overline{s} = (s_{i,j})$ rather than tripartitions themselves. Given an I-complete (3,2) game $(N, \mathcal{W})$, we recall the notation $\overline{n} = (n_1, \dots, n_t)$ and denote by $\Lambda(\mathcal{W})$ the set of all admissible models of tripartitions of the game, that is, $\Lambda(\mathcal{W}) = \{\overline{s} = (s_{i,j}) : 0 \leq s_{i,j}$ and $\sum_{j=1}^{3} s_{i,j} = n_i$ for all $i = 1, ..., t\}$ and $\overline{\mathcal{W}} = \{\overline{S} \in \overline{3^N} : S \in \mathcal{W}\}$ be the set of classes of winning tripartitions.

We shall define a (dominance) relation in the set of $\Lambda(\mathcal{W})$ in the spirit of Carreras and Freixas [3]. For this purpose, the following results are fundamental.

Proposition 3.1 *Let $G = (N, \mathcal{W})$ be an I-complete (3,2) game. If $S \in \mathcal{W}$, $a >_I b$, $a \in S_j$, $b \in S_i$ with $i < j$, then $\pi_{ab}(S) \in \mathcal{W}$.*

Let $G = (N, \mathcal{W})$ be an I-complete (3,2) game and assume that G is not anonymous, which implies that $t \geq 2$ that is, it has at least two types of equivalent players. Let $\overline{s} = (s_{i,j})$ be an element of $\Lambda(\mathcal{W})$. For any fixed i', i'', j', j'' such that $1 \leq i' < i'' \leq t$ and $1 \leq j' < j'' \leq 3$, we define (when possible) the following matrix $\overline{s}' = (s'_{i,j})$ where

$$s'_{i,j} = \begin{cases} s_{i,j} + 1 & \text{if } i = i' \text{ and } j = j' \\ s_{i,j} - 1 & \text{if } i = i' \text{ and } j = j'' \\ s_{i,j} - 1 & \text{if } i = i'' \text{ and } j = j' \\ s_{i,j} + 1 & \text{if } i = i'' \text{ and } j = j'' \\ s_{i,j} & \text{otherwise} \end{cases}$$

This simply means that, for instance if $t = 3$ and $\overline{s} = \begin{pmatrix} s_{1,1} & s_{1,2} & s_{1,3} \\ s_{2,1} & s_{2,2} & s_{2,3} \\ s_{3,1} & s_{3,2} & s_{3,3} \end{pmatrix}$,

for $i' = 1$, $i'' = 2$ and $j' = 1$, $j'' = 3$ then we have $\overline{s}' = \begin{pmatrix} s_{1,1}+1 & s_{1,2} & s_{1,3}-1 \\ s_{2,1}-1 & s_{2,2} & s_{2,3}+1 \\ s_{3,1} & s_{3,2} & s_{3,3} \end{pmatrix}$

From the proposition above, we have the following corollary.

Corollary 3.2 *Let $G = (N, \mathcal{W})$ be an I-complete (3,2) game. Let S be a tripartition represented by a matrix $\overline{s} = (s_{i,j})$. Let i', i'', j', j'' with $1 \leq i' < i'' \leq t$ and $1 \leq j' < j'' \leq 3$ such that the matrix $\overline{s}' = ({s'}_{i,j})$ is well defined.*

If $\overline{S} \in \overline{\mathcal{W}}$ then $\overline{S}' \in \overline{\mathcal{W}}$, for all tripartition S' represented by the matrix $\overline{s}'$.

Let S and S' be two tripartitions represented by the models $\overline{s} = (s_{i,j})$ and $\overline{s}' = (s'_{i,j})$ respectively :

- If $S \subset^3 S'$ then we say that $\overline{s}'$ is a *monotonic shift* of $\overline{s}$.
- If there exist $1 \leq i' < i'' \leq t$ and $1 \leq j' < j'' \leq 3$, such that $s'_{i,j}$ are all well-defined as above, then we will say that $\overline{s}'$ is an *elementary positive shift* of $\overline{s}$. When necessary, we will say that $\overline{s}'$ is an elementary positive shift of $\overline{s}$ for rows (i', i'') and columns (j', j'').
- We will say that $\overline{s}'$ dominates $\overline{s}$ in $\Lambda(\mathcal{W})$ if $\overline{s}'$ can be obtained from $\overline{s}$ by a sequence of monotonic and/or elementary positive shifts.

We provide below an equivalent formulation for the dominance relation in the set $\Lambda(\mathcal{W})$.

Definition 3.3 *Given $\overline{r} = (r_{i,j}), \overline{s} = (s_{i,j}) \in \Lambda(\mathcal{W})$, we have $\overline{s}\ \delta\ \overline{r}$ if : $\sigma(\overline{s}) \succcurlyeq \sigma(\overline{r})$ where $\sigma(\overline{s}) = (\sigma^s_{i,j})$, with $\sigma^s_{i,j} = \sum\limits_{i' \leq i\ ;\ j' \leq j} s_{i',j'}$*

In the example 2.7, the matrix $\overline{s} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ dominates the matrix $\overline{r} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ by δ because $\sigma(\overline{s}) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$, $\sigma(\overline{r}) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 4 \end{pmatrix}$ and hence $\sigma(\overline{s}) \succcurlyeq \sigma(\overline{r})$.

In the same example, $\overline{s} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\overline{u} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ are not comparable by δ.

It is easy to check that δ is a partial ordering as stated below.

Proposition 3.4 *The binary relation δ is a partial ordering on $\Lambda(\mathcal{W})$.*

We shall also note $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3)$ where any $\overline{s}_j$ is the column number j of $\overline{s}$. Given two column vectors $\overline{s}_j$ and $\overline{r}_j$ for any $j = 1, 2, 3$ of $\overline{s}$ and $\overline{r}$, we denote by:
$\overline{s}_j\ \delta'\ \overline{r}_j$ if $\Sigma_i(\overline{s}_j) \geq \Sigma_i(\overline{r}_j), \forall i = 1, 2, \ldots, t$ with $\Sigma_i(\overline{s}_j) = s_{1,j} + s_{2,j} + \cdots + s_{i,j}$.
In words, $\overline{s}_j\ \delta'\ \overline{r}_j$ if for any row i, the sum of $\overline{s}_j$-components up to i is greater than or equal to the corresponding sum of $\overline{r}_j$.

δ' is an ordering that need not be complete.

Proposition 3.5 *Given $\overline{s}, \overline{r} \in \Lambda(\mathcal{W})$, the following two statements are equivalent:*

- $\overline{s}\ \delta\ \overline{r}$
- $\begin{cases} \textit{either } \overline{s}_1 = \overline{r}_1 \textit{ and } \overline{s}_2\ \delta'\ \overline{r}_2 \\ \textit{or } \overline{s}_1 \neq \overline{r}_1 \textit{ and } (\overline{s}_1\ \delta'\ \overline{r}_1 \textit{ and } (\overline{s}_1 + \overline{s}_2)\ \delta'\ (\overline{r}_1 + \overline{r}_2)) \end{cases}$

In the sequel we shall show that the pair $(\Lambda(\mathcal{W}), \delta)$ is a multilattice that is, for all $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$, if we denote by $Maj(\overline{r}, \overline{s})$ the set of upper bounds of $\{\overline{s}; \overline{r}\}$ and by $Min(\overline{r}, \overline{s})$ the set of lower bounds of $\{\overline{s}; \overline{r}\}$ then, both of those sets are non-empty with a minimal and maximal element respectively.

For all matrices $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$, define the following matrix $\overline{u}(\overline{r}, \overline{s})$ or simply $\overline{u}$ as follows.

Definition of $\overline{u}$ given $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$

Let $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$: Consider the following matrix M defined by:
$M = (M_{i,j})$ where, $M_{i,j} = \max(\sigma^r_{i,j}, \sigma^s_{i,j})$ and M' defined as follows:

- $M'_{i,1} = M_{i,1}$ for all $1 \leq i \leq t$;
- $M'_{1,2} = M_{1,2}$ and
$M'_{i,2} = \begin{cases} M_{i,1} + M_{i-1,2} - M_{i-1,1} & \text{if } M_{i,2} + M_{i-1,1} - M_{i,1} - M_{i-1,2} < 0 \\ M_{i,2} & \text{otherwise} \end{cases}$ for all $1 < i \leq t$;
- $M'_{i,3} = M_{i,3}$ for all $1 \leq i \leq t$.

We define $\overline{u} = (u_{i,j})$ to be the matrix such that $\sigma(\overline{u}) = M'$, that is,

- $u_{1,1} = M'_{1,1}$ and $u_{i,1} = M'_{i,1} - M'_{i-1,1}$ for all $1 < i \leq t$.
- $u_{1,2} = M'_{1,2} - M'_{1,1}$ and $u_{i,2} = M'_{i,2} + M'_{i-1,1} - M'_{i-1,2} - M'_{i,1}$ for all $1 < i \leq t$.
- $u_{i,3} = n_i - u_{i,1} - u_{i,2}$ for all $1 \leq i \leq t$.

It is easy to check that $\overline{u}$ is an element of $\Lambda(\mathcal{W})$. We state below that in general, $\overline{u}$ is minimal in $Maj(\overline{r}, \overline{s})$ with $Maj(\overline{r}, \overline{s}) = \{\overline{v} \in \Lambda(\mathcal{W}) : \ \overline{v}\ \delta\ \overline{s} \text{ and } \overline{v}\ \delta\ \overline{r}\}$.

Lemma 3.6 *For all* $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$

- $\overline{u} \in Maj(\overline{r}, \overline{s})$ *and*
- $\overline{u}$ *is minimal in* $Maj(\overline{r}, \overline{s})$.

We now define a matrix $\overline{d}$ like $\overline{u}$ using instead a matrix $m = (m_{ij})$ where : $m_{i,j} = \min(\sigma^r_{i,j}, \sigma^s_{i,j})$. We also consider the matrix m' defined as follows:

- $m'_{i,1} = m_{i,1}$ for all $1 \leq i \leq t$;
- $m'_{i,2} = \begin{cases} m_{i+1,2} + m_{i,1} - m_{i+1,1} & \text{if } m_{i+1,2} + m_{i,1} - m_{i+1,1} - m_{i,2} < 0 \\ m_{i,2} & \text{otherwise} \end{cases}$
for all $1 \leq i < t$ and $m'_{t,2} = m_{t,2}$;
- $m'_{i,3} = m_{i,3}$ for all $1 \leq i \leq t$.

Definition of $\overline{d}$ given $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$

The matrix $\overline{d} = (d_{ij})$ such that $\sigma(\overline{d}) = m'$ is define as follows:

- $d_{1,1} = m'_{1,1}$ and $d_{i,1} = m'_{i,1} - m'_{i-1,1}$ for all $1 < i \leq t$;
- $d_{1,2} = m'_{1,2} - m'_{1,1}$ and $d_{i,2} = m'_{i,2} + m'_{i-1,1} - m'_{i,1} - m'_{i-1,2}$ for all $1 < i \leq t$; and
- $d_{i,3} = n_i - d_{i,1} - d_{i,2}$ for all $1 \leq i \leq t$.

It is easy to check that $\overline{d}$ is an element of $\Lambda(\mathcal{W})$. We state below that in general, $\overline{d}$ is maximal in $Min(\overline{r},\overline{s})$ with $Min(\overline{r},\overline{s}) = \{\overline{v} \in \Lambda(\mathcal{W}) : \overline{s}\ \delta\ \overline{v} \text{ and } \overline{r}\ \delta\ \overline{v}\}$.

Lemma 3.7 *For all* $\overline{r},\overline{s} \in \Lambda(\mathcal{W})$

- $\overline{d} \in Min(\overline{r},\overline{s})$ *and*
- $\overline{d}$ *is maximal in* $Min(\overline{r},\overline{s})$.

The two matrices $\overline{s} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ and $\overline{r} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ are not comparable. Let us compute $\overline{u}$ and $\overline{d}$ for these two matrices.

$\sigma(\overline{s}) = \begin{pmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \end{pmatrix}$ and $\sigma(\overline{r}) = \begin{pmatrix} 3 & 4 & 4 \\ 4 & 5 & 8 \end{pmatrix}$, thus, $m = \begin{pmatrix} 1 & 4 & 4 \\ 4 & 5 & 8 \end{pmatrix}$, $m' = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 8 \end{pmatrix}$ and $M = M' = \begin{pmatrix} 3 & 4 & 4 \\ 4 & 7 & 8 \end{pmatrix}$. It then follows that $\overline{u} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and $\overline{d} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$.

It can be easily seen in this example that $\overline{u}\ \delta\ \overline{s}$ and $\overline{u}\ \delta\ \overline{r}$ whereas $\overline{s}\ \delta\ \overline{d}$ and $\overline{r}\ \delta\ \overline{d}$.

It follows from the lemmas above that $(\Lambda(\mathcal{W}),\delta)$ is a multilattice. This will later be referred to as the multilattice associated to the I-complete (3,2) game $\mathcal{W}$. The maximum and the minimum of the multilattice $(\Lambda(\mathcal{W}),\delta)$ are respectively given by :

$$\varepsilon = \begin{pmatrix} n_1 & 0 & 0 \\ n_2 & 0 & 0 \\ \cdots & \cdots & \cdots \\ n_t & 0 & 0 \end{pmatrix} \text{ and } \theta = \begin{pmatrix} 0 & 0 & n_1 \\ 0 & 0 & n_2 \\ \cdots & \cdots & \cdots \\ 0 & 0 & n_t \end{pmatrix}$$

Example 3.8 The multilattice associated with the 4-player game considered in Example 2.7 is shown in Figure 1.

As we can see, in the first example above there are only two equivalence classes for the I-influence relation. In the following example in which the game has 3 equivalence classes, the construction of the multilattice associated to this game becomes less obvious.

Example 3.9 Let $N = \{1,2,3,4\}$ and $\mathcal{W}^m = \{(12,3,4),(13,2,4),(14,2,3),(12,4,3)\}$. The game is I-complete with the following classes :

$N_1 = \{1\} \succ_I N_2 = \{2\} \succ_I N_3 = \{3,4\}$.

The multilattice associated with this I-complete game is given in Figure 2.

We conclude this section by showing that $(\Lambda(\mathcal{W}),\delta)$ and $(3^N/{\sim_I}, \succeq_I)$ are isomorphic, implying in particular that any tripartition class $\overline{S}$ in $3^N/{\sim_I}$ (where $\succeq_I$ is the dominance relation induced on $3^N/{\sim_I}$ by the dominance relation $\succsim_I$ on 3^N: that is $\overline{S} \succeq_I \overline{R}$ if and only if $S \succsim_I R$) can be identified with a unique model $\overline{s}$ in $\Lambda(\mathcal{W})$. In this respect we need a preliminary result.

Lemma 3.10 *Let* $(N,\mathcal{W})$ *be an I-complete (3,2) simple game.*

For all $S, R \in 3^N$, *if* $S \dashv R$ *then* $\overline{s}\ \delta\ \overline{r}$.

The mentioned result states as follows.

Theorem 3.11 *Let* $(N,\mathcal{W})$ *be an I-complete (3,2) simple game and let* $N_1 \succ_I N_2 \succ_I \cdots \succ_I N_t$ *be the linear ordering of the classes with respect to* $\equiv_I$. *Then the map*

$$\begin{array}{cccc} \Phi: & (3^N/{\sim_I}, \succeq_I) & \longrightarrow & (\Lambda(\mathcal{W}),\delta) \\ & \overline{S} & \longmapsto & \overline{s} = (s_{i,j})_{\substack{i=1,\ldots,t \\ j=1,2,3}} \end{array}$$ *is an isomorphism of ordered sets.*

4 Characteristic invariant of an I-complete (3,2) game

In this section we shall introduce a component that will allow us to classify and determine I-complete (3,2) games. In order to achieve this, we need some additional notations and definitions.

Let $(N,\mathcal{W})$ be a (3,2) game. We denote by :

- $\overline{\mathcal{W}} = \{\overline{S} \in \overline{3^N} : S \in \mathcal{W}\}$ the set of classes of winning tripartitions,
- $\overline{\mathcal{W}^m} = \{\overline{S} \in \overline{3^N} : S \in \mathcal{W}^m\}$ the set of classes of minimal winning tripartitions, and
- $\overline{\mathcal{W}}^m = \{\overline{S} \in \overline{3^N} : S \in \mathcal{W}^{sm}\}$ the set of classes of shift-minimal winning tripartitions.

Note that $S \in \mathcal{W}$ *if and only if* $\overline{s}\ \delta\ \overline{r}$ for some $R \in \mathcal{W}^{sm}$. In other words,
$\overline{\mathcal{W}} = \{\overline{S} \in \overline{3^N} : \overline{S} \succsim_I \overline{R}$ for some $R \in \mathcal{W}^{sm}\}$.

The following proposition shows the inclusion relation of the sets $\overline{\mathcal{W}}$, $\overline{\mathcal{W}^m}$ and $\overline{\mathcal{W}}^m$.

Proposition 4.1 *Let $(N,\mathcal{W})$ be an I-complete (3,2) game ; then, $\overline{\mathcal{W}}^m \subseteq \overline{\mathcal{W}^m} \subseteq \overline{\mathcal{W}}$.*

We know that whenever $R \subset^3 S$, it follows that $\overline{s}\ \delta\ \overline{r}$. We recall that $N_1 \succ_I N_2 \succ_I \cdots \succ_I N_t$ is the strict linear ordering of classes according to the relation $\equiv_I$ and $\overline{n} = (n_1, n_2, \ldots, n_t)$ is the vector defined by their cardinalities.

Without loss of the generality, we assume that there are r models associated to the different classes of shift-minimal winning tripartitions. Let $\overline{m}^1$, $\overline{m}^2$, ... ,$\overline{m}^r$ be these models (where $\overline{m}^p = (m^p_{i,j})_{\substack{i=1,\ldots,t \\ j=1,2,3}}$). Again, without loss of the generality we can assume that these models are ordered lexicographically with respect to rows, as follows.

Given x and y two vectors of r components each, we say that x is lexicographically greater or equal to y if $x = y$ or (there exists $h < r$ such that $x_u = y_u$ for all $u \leq h$ and $x_{h+1} > y_{h+1}$).

Now we order the $\overline{m}^p$'s as follows. $\overline{m}^{p_1}$ is greater than $\overline{m}^{p_2}$ if the 3^{th}-component vector $(m^{p_1}_{1,1}, m^{p_1}_{1,2}, m^{p_1}_{1,3}, \ldots, m^{p_1}_{t,1}, m^{p_1}_{t,2}, m^{p_1}_{t,3})$ is lexicographically greater than the 3^{th}-component vector $(m^{p_2}_{1,1}, m^{p_2}_{1,2}, m^{p_2}_{1,3}, \ldots, m^{p_2}_{t,1}, m^{p_2}_{t,2}, m^{p_2}_{t,3})$.

As from now we will assume, without loss of the generality that these models are ordered lexicographically and this leads to the sequence denoted $\mathcal{M} = (\overline{m}^1, \overline{m}^2, \ldots, \overline{m}^r)$. We provide below some useful properties for $\mathcal{M}$.

Theorem 4.2 *Let $G = (N,\mathcal{W})$ be an I-complete (3,2) simple game. The vector $\mathcal{M}$ associated with G satisfies the following properties :*

1) $0 \leq m^p_{i,j}$ and $0 < \sum_{j=1}^{3} m^p_{i,j} = n_i$ for all $p = 1,\ldots,r$ and all $i = 1,\ldots,t$ where $n_i = |N_i|$ with $N_1 \succ_I N_2 \succ_I \cdots \succ_I N_t$ are the I-classes of G;

2) If $r > 1$ then $\overline{m}^p$ and $\overline{m}^q$ are not δ-comparable if $p \neq q$; and

3) (i) If $t = r = 1$, then $m^1_{1,3} < n$;

(ii) If $t > 1$ then for every $i < t$ there exists some p such that ($m^p_{i,1} > 0$ and $m^p_{i+1,1} < n_{i+1}$) or ($m^p_{i,3} < n_i$ and $m^p_{i+1,3} > 0$).

Condition (3) reflects that for every $i < t$, $a \in N_i$ and $b \in N_{i+1}$, some tripartition witnesses that $b \not\gtrsim_I a$.

The next theorem shows that :

- The vector $\mathcal{M}$ is left invariant by any isomorphism of I-complete (3,2) games. We recall to this end that two (3,2) games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ are said to be isomorphic if there exists a bijective map $f : N \to N'$ such that $S \in \mathcal{W}$ *if and only if* $f(S) \in \mathcal{W}'$; f is called an isomorphism of (3,2) games and is also denoted by $f : (N, \mathcal{W}) \to (N', \mathcal{W}')$.

- The vector $\mathcal{M}$ determines the game, in the sense that we are able to define a (unique up to isomorphism) I-complete (3,2) game that possesses this invariant.

- Thanks to the above points, and to the fact that the vector $\mathcal{M}$ also allows us to classify the game, i.e. to distinguish it from any other non-isomorphic game; we shall refer to $\mathcal{M}$ as the *characteristic invariant* of the I -complete (3,2) game $(N, \mathcal{W})$.

Theorem 4.3

(a) *Two I-complete (3,2) games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ are isomorphic if and only if $\mathcal{M} = \mathcal{M}'$.*

(b) *Given a vector $\mathcal{M}$ satisfying the conditions of Theorem 4.2, there exists an I -complete (3,2) game the characteristic invariant of which is $\mathcal{M}$.*

Now let us illustrate this result through examples.

Example 4.4

1) The $(3,2)$ game defined by $\mathcal{M}$ where $\mathcal{M} = \left(\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\right)$ is an I-complete $(3,2)$ game with $n = 6$ players and 24 shift-minimal winning tripartitions. Note that $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ corresponds to 12 minimal winning tripartitions. Thus, this game has 36 minimal winning tripartitions.

2) The voting procedure in the Security Council of the United Nations Organization is described as follows : $N = N_1 \cup N_2$ where $N_1 = \{1, 2, 3, 4, 5\}$ represents the set of permanent members and $N_2 = \{6, 7, \ldots, 15\}$ the set of non permanent members, $\mathcal{W}^m = \{(S_1, S_2, S_3) \in 3^N : |S_1| = 9$ and $S_3 \cap N_1 = \emptyset\}$. It can be checked that this game may be simply described by :
$\mathcal{M} = \left(\begin{pmatrix} 5 & 0 & 0 \\ 4 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 0 \\ 5 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 0 \\ 6 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 0 \\ 7 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 0 \\ 8 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}\right)$.

Note however that this game has 5005 shift-minimal winning tripartitions, and there are not minimal winning tripartitions not being shift-minimal winning.

3) Now let us describe the unique I-complete (3,2) game up to isomorphism defined by the invariant:
$\mathcal{M} = \left(\begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \end{pmatrix}\right)$.

- The vector $\mathcal{M}$ satisfies the conditions in Theorem 4.2.

- We let $N = \{1, 2, \ldots, 8\}$ and form the classes: $N_1 = \{1, 2, 3\} > N_2 = \{4, 5, 6, 7, 8\}$.

The shift-minimal winning tripartitions are given by the models: $\begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \end{pmatrix}$. And the remaining minimal winning tripartitions by the models $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$. By using Proposition 2.11, we determine how many tripartitions are associated with each model. We should need 126 minimal winning tripartitions to describe the game $(N, \mathcal{W})$ in classical form.

Remark 4.5 : *It is well known that for $n = 2$, there are exactly 12 I-complete (3,2) games. It is also known that for $t = 1$ and n arbitrary there are $2^{n+1} - 2$ I-complete (3,2) games (Freixas and Zwicker [12]). The representation above allows us to compute, for $n = 3$, the number of I-complete (3,2) games is 162. We have also computed the number of these games for $n = 4,\ t = 2,3$ and $n = 5,\ t = 2$. It is a challenging problem to obtain more enumerations for other combinations of n and t. These numbers are depicted in the following table in terms of the number of I-classes $t = 1, 2, 3$.*

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$t=1$	2	6	14	30	62
$t=2$	-	6	80	888	12752
$t=3$	-	-	68	7292	?
$t=4$	-	-	-	?	?
$t=5$	-	-	-	-	?

Table 1:

? means we are unable to enumerate and $-$ non-possible combination for n and t. The previous table does not contain the number of I-complete $(3,2)$ games for $t = 4$ and $t = 5$ if $n \leq 5$ because the computer wasn't able to give us these numbers. It is an interesting computational problem to obtain these numbers, which would allow to know the number of I-complete $(3,2)$ games up to isomorphism for $n \leq 5$. Of course, enumerations for other combinations of n and t are equally interesting.

In the following part, we prove that the representation above is also useful for determining whether a given I-complete $(3,2)$ game is strongly weighted or not. First, recall that any weighted $(3,2)$ game can be represented as a normalized weighted $(3,2)$ game, that is, a weighted $(3,2)$ game for which, for $a, b \in N$, $a \equiv_I b$ if and only if $w(a) = w(b)$ with $w(a) = (w_1(a), w_2(a), w_3(a))$ and $w_1(a) \geq w_2(a) \geq w_3(a)$. Moreover, it is feasible to normalize by 0 at any level of approval. As the intermediate level usually stand for abstention, we may assume $w_2(a) = 0$ for all $a \in N$. A strongly weighted (3,2) game is a weighted (3,2) game such that for every pair $a, b \in N$ it yields either $[w_1(a) \geq w_1(b)$ and $w_3(a) \leq w_3(b)]$ or $[w_1(b) \geq w_1(a)$ and $w_3(b) \leq w_3(a)]$. We prove the following result.

Proposition 4.6 *Let $(N, \mathcal{W})$ be an I-complete (3,2) game.*

$(N, \mathcal{W})$ is a strongly weighted (3,2) game if and only if there is a vector $w = (w(1), w(2), \ldots, w(t))$, such that $w(i) = (w_1(i), w_2(i), w_3(i)) = (w_1(i), 0, w_3(i))$ with $w_1(i) \geq 0 \geq w_3(i)$ $(i = 1, \ldots, t)$ which satisfies the system of inequalities:

$\displaystyle\sum_{i=1}^{t} (\overline{m}_i^p - \overline{\alpha}_i^q) \cdot w(i) > 0$ *for $p = 1, 2, \ldots, r;\ q = 1, 2, \ldots, s,$ where $\overline{m}^1, \ldots, \overline{m}^r$ are the models of shift-minimal winning tripartitions and $\overline{\alpha}^1, \ldots, \overline{\alpha}^s$ are those of shift-maximal losing tripartitions and "$\cdot$" is the inner product.*

The above result can be illustrated with the UNSC game. Recall that the UNSC voting rule is clearly an I-complete (3,2) game and its characteristic invariant $\mathcal{M}$ is given by the shift-minimal models $\overline{m}^1 = \begin{pmatrix} 5 & 0 & 0 \\ 4 & 0 & 6 \end{pmatrix}$, $\overline{m}^2 = \begin{pmatrix} 4 & 1 & 0 \\ 5 & 0 & 5 \end{pmatrix}$, $\overline{m}^3 = \begin{pmatrix} 3 & 2 & 0 \\ 6 & 0 & 4 \end{pmatrix}$, $\overline{m}^4 = \begin{pmatrix} 2 & 3 & 0 \\ 7 & 0 & 3 \end{pmatrix}$, $\overline{m}^5 = \begin{pmatrix} 1 & 4 & 0 \\ 8 & 0 & 2 \end{pmatrix}$ and

$\overline{m}^6 = \begin{pmatrix} 0 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$. The models of shift-maximal losing tripartitions are given by $\overline{\alpha}^1 = \begin{pmatrix} 4 & 0 & 1 \\ 10 & 0 & 0 \end{pmatrix}$, $\overline{\alpha}^2 = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 7 & 0 \end{pmatrix}$.

We note that, since there are 125 shift-maximal losing tripartitions and 5005 shift-minimal winning ones, so a naïve attempt to study the weightedness of this game would lead to a system of $5005 \times 125 = 625625$ inequalities with 30 unknowns. However, the result above allows us to solve only $6 \times 2 = 12$ inequalities corresponding to: $\sum_{i=1}^{2} (\overline{m}_i^p - \overline{\alpha}_i^q) \cdot w(i) > 0$ for $p = 1, 2, \ldots, 6;\ q = 1, 2$. Solving these inequalities gives the following vectors of weights : $(1, 0, -6)$ for a permanent member and $(1, 0, 0)$ for a non permanent, and the quota is $q = 9$.

5 Conclusion and future work

The main contribution of this paper was to provide a simpler and more intuitive representation of a significant subclass of (3, 2) simple games, that of I-complete (3,2) games. Any such game can be represented by a finite list of matrices with non negative entries fulfilling some simple algebraic properties. We give some enumerations of I-complete (3,2) games for combinations of the parameters n and t, where t is the number of equivalent classes on players. As a consequence of our enumerations we know that there are 162 I-complete (3,2) games for $n = 3$, 8210 I-complete (3,2) games for $n = 4$ with $t \leq 3$ and 12814 I-complete (3,2) games for $n = 5$ and $t \leq 2$. A computer-savvy researcher should be able to obtain enumerations for other relatively combinations of n and t, than those we obtained in the paper. We apply it to the United Nations Security Council and show that this game is strongly weighted.

Many power indices for (3,2) games are easily computed in the class of weighted (3,2) games by using generating functions, as shown in [8]. Thus, in this respect, it is very important to determine if a given (3,2) game is weighted. If the (3,2) game is I-complete, then it could be (strongly) weighted and proposition 4.6 gives us the answer. If the game is strongly weighted then it is easy, by means of the generating function methods, as shown in Feixas *et al* [9], to compute some power indices for a reasonable large number of players.

Many significant subclasses of I-complete (3,2) games are more easily tractable by using our numerical parameterization. For instance, I-complete (3,2) games being either constant-sum games or games with consensus or homogeneous are now susceptible of being studied, classified or enumerated. Herewith, we point out some possible lines of future research related to our work.

The starting point was the extension of the desirability relation defined on individuals to coalitions. We used the I-influence relation defined by Tchantcho *et al* [23]. However, as raised in Freixas *et al* [9], there are weighted games not being complete for the influence relation, something different to what occurs for simple games. This leads to the introduction of several extensions of the desirability relation for simple games. From the completeness of these extensions, follows the consistent link with weighted games. In a future work it could be interesting to analyze the replacement of the I-influence used in this paper, with any of these extensions.

As pointed out above, we obtained some enumerations for I-complete (3,2) games for some combinations of n and t. It seems computationally tractable to get enumerations for $n = 4$ and $n = 5$. It would be very interesting to get further enumerations for other combinations of these two parameters. Following Kurz and Tautenhahn [16], it would be interesting to find an algorithm to determine formulas for the number of I-complete (3,2) games. Proposition 4.6 serves for determining if a given I-complete (3,2) game is a strongly weighted (3,2) game. It is a challenging question to determine the number of strongly weighted (3,2) games for small combinations of the two parameters n and t.

We know that weighted (3,2) games can be written (in a suitable way) as a threshold function. In this respect, one can wonder if the work by Bohossian and Bruck [2] who developed algebraic techniques for constructing minimal weight threshold functions can be extended to (3,2) games setup.

The notion of game with consensus has been extensively considered in the litterature (see Peleg [18] and Carreras and Freixas [4]). These are game which are obtained by intersecting a linear game with a symmetric weighted game. For example, Carreras and Freixas investigate the behaviour of the Shapley-Shubik [20] power index when passing from one such game to another. The analogous of these notions can be obtained for (3,2) games raising the problem of extension of the results obtained in simple games to (3,2) games.

6 Appendix: Proofs

Proof of Proposition 2.10

$\Rightarrow$) $S \perp T$ means that there exist $a, b \in N$ such that $\pi_{ab}(S) = T$ and $a \equiv_I b$ (or, $a \geq_I b$ and $b \geq_I a$)

With no loss to generality, assume that $a \in S_i$ and $b \in S_j$ with $i \leq j$: then we have $b \in T_i$ and $a \in T_j$. Thus, ($\pi_{ab}(S) = T, a \geq_I b$ and $a \in T_i$, $b \in T_j$ with $i \leq j$) and ($\pi_{ab}(T) = S, b \geq_I a$ and $b \in S_i$, $a \in S_j$ with $i \leq j$) and hence $T \dashv S$ and $S \dashv T$.

$\Leftarrow$) Conversely, assume that $S \dashv T$ and $T \dashv S$.

- If $S = T$ then $S \perp T$.
- If $S \neq T$ then : $S \dashv T$ and $T \dashv S$ means that:

  - There exists $\{a, b\} \subseteq N$ such that $\pi_{ab}(T) \subseteq^3 S$, $a \geq_I b$, and $b \in S_j$ and $a \in S_i$ with $i \leq j$

  - There exists $\{a, b\} \subseteq N$ such that $\pi_{cd}(S) \subseteq^3 T, c \geq_I d$, and $d \in T_j$ and $c \in T_i$ with $i \leq j$.
  We shall first prove that in this conditions, it holds $(a, b) = (c, d)$.
  Assume by contradiction that $(a, b) \neq (c, d)$. We claim that $|S_p| = |T_p|$ for all $p = 1, 2, 3$. Indeed, assume that there exists $p_0 \in \{1, 2, 3\}$ such that $|S_{p_0}| \neq |T_{p_0}|$. With no loss to the generality, assume that $|S_{p_0}| < |T_{p_0}|$.

- If $p_0 = 1$, then $|S_1| < |T_1|$, but $\pi_{ab}(T) \subseteq^3 S$ so, $|T_1| \leq |S_1|$ which is impossible.

- If $p_0 = 2$, then $|S_2| < |T_2|$, but $\pi_{cd}(S) \subseteq^3 T$ so, $|T_1| \geq |S_1|$ and $|T_1 \cup T_2| \geq |S_1 \cup S_2|$ then $|T_1 \cup T_2| = |T_1| + |T_2| > |S_1| + |S_2| = |S_1 \cup S_2|$. But $\pi_{ab}(T) \subseteq^3 S$ thus, $|T_1 \cup T_2| \leq |S_1 \cup S_2|$ which is a contradiction.

- If $p_0 = 3$ then $|S_3| < |T_3|$, but $\pi_{cd}(S) \subseteq^3 T$ so $|S_3| \geq |T_3|$ which is impossible.

We can conclude at this point that $|S_p| = |T_p|$ for all $p \in \{1, 2, 3\}$ and hence $\pi_{ab}(T_p) = S_p$ and $\pi_{cd}(S_p) = T_p$ for all $p = 1, 2, 3$.

Since $S \neq T$, there exists p_0 such that $S_{p_0} \neq T_{p_0}$. Either ($c \notin S_{p_0}$ and $d \in S_{p_0}$) or ($d \notin S_{p_0}$ and $c \in S_{p_0}$). With no loss of generality, assume that $c \notin S_{p_0}$ and $d \in S_{p_0}$. Then, $\pi_{ab}(T_{p_0}) = S_{p_0}$ and $\pi_{cd}(S_{p_0}) = T_{p_0}$ imply that $\pi_{ab}(\pi_{cd}(S_{p_0})) = S_{p_0}$. As $d \in S_{p_0}$, it follows that $c \in \pi_{cd}(S_{p_0})$ and thus $c \in \pi_{ab}(\pi_{cd}(S_{p_0})) = S_{p_0}$, which is a contradiction, hence, $(a, b) = (c, d)$.
As $(a, b) = (c, d)$, we then have, ($\pi_{ab}(T) \subseteq^3 S,\ a \geq_I b$), and ($\pi_{ab}(S) \subseteq^3 T,\ b \geq_I a$) so, $\pi_{ab}(T) \subseteq^3 S$ and $T \subseteq^3 \pi_{ab}(S) \subseteq^3 T$: consequently, $\pi_{ab}(S) = T$, which together with $a \equiv_I b$ yield $S \perp T$. ■

Proof of Proposition 2.11

Let $(N, \mathcal{W})$ be an I-complete (3,2) simple game, $N_1, N_2, \ldots, N_t$, be the equivalence classes of $\equiv_I$, with $|N_i| = n_i$ for all $1 \leq i \leq t$.

1. $\Rightarrow$) Since $S \sim_I R$, $R = f(S)$ with $f : N \to N$ a product of transpositions of equivalent players; therefore $f(N_i) = N_i$ for all i. It follows that for all $j = 1, 2, 3$ and for all $i = 1, \ldots, t$, $R_j \cap N_i = f(S_j) \cap f(N_i) = f(S_j \cap N_i)$ so, $|R_j \cap N_i| = |S_j \cap N_i|$ for all i and all j because f is bijective.
   $\Leftarrow$) Conversely, assume that $|S_j \cap N_i| = |R_j \cap N_i|$ for all $j = 1, 2, 3$ and all $i \in \{1, 2, ..., t\}$. Let $A = \{j : S_i = R_j\}$: then $|A| \in \{0, 1, 2, 3\}$.

   (a) If $|A| \geq 2$ then it is obvious that $S = R$ and it follows that $S \sim_I T$.

   (b) If $|A| = 1$ then assuming with no loss of the generality that $S_1 = R_1$, we have: $S_2 \neq R_2$ and $S_3 \neq R_3$.
   It follows from $|S_j \cap N_i| = |R_j \cap N_i|$ for all i, j that $|S_2| - |S_2 \cap R_2| = |R_2| - |S_2 \cap R_2|$ and $|S_3| - |S_3 \cap R_3| = |R_3| - |S_3 \cap R_3|$; thus, $|S_2 \setminus R_2| = |R_2 \setminus S_2|$ and $|S_3 \setminus R_3| = |R_3 \setminus S_3|$.
   Furthermore, we have :
   For all $i = 1, ..., t$, $\begin{cases} |(S_2 \setminus R_2) \cap N_i| = |(R_2 \setminus S_2) \cap N_i| \\ |(S_3 \setminus R_3) \cap N_i| = |(R_3 \setminus S_3) \cap N_i| \end{cases}$ $(*)$
   $$\begin{aligned} |(S_2 \cup R_2) \setminus (S_2 \cap R_2)| &= |(S_2 \setminus R_2) \cup (R_2 \setminus S_2)| \\ &= |\bigcup_{i=1}^{t}((S_2 \setminus R_2) \cap N_i)| + |\bigcup_{i=1}^{t}((R_2 \setminus S_2) \cap N_i)| \\ &= \sum_{i=1}^{t} |(S_2 \setminus R_2) \cap N_i| + \sum_{i=1}^{t} |(R_2 \setminus S_2) \cap N_i| \\ &= 2\sum_{i=1}^{t} |(S_2 \setminus R_2) \cap N_i| \end{aligned}$$
   since $|(S_2 \setminus R_2) \cap N_i| = |(R_2 \setminus S_2) \cap N_i|$ for all i.
   Now let $m = \sum_{i=1}^{t} |(S_2 \setminus R_2) \cap N_i|$: $m \neq 0$ because $S_2 \neq R_2$.
   We shall now proceed by induction on m in order to "transform" S into R .
   Let $a \in (S_2 \setminus R_2) \cap N_i$ and $b \in (R_2 \setminus S_2) \cap N_i$ and let us consider the transposition $\pi^{(1)} = \pi_{ab}$ such that: $S'_2 = \pi^{(1)}(S_2)$.

Then $m' = \sum_{i=1}^{t} |(S'_2 \setminus R_2) \cap N_i| = m - 1$ and by induction we obtain a sequence $(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(m)})$ of transpositions of equivalent players such that: $\pi^{(m)} \circ \pi^{(m-1)} \circ \cdots \circ \pi^{(1)}(S_2) = R_2$. We consider the application $f : N \to N$ which is the product of the m transpositions of indifferent players that leads to : $f(S_2) = R_2$. We then have: $f(S) = (f(S_1), f(S_2), f(S_3)) = (R_1, R_2, f(N \setminus (S_1 \cup S_2))) = R$ and it follows that $S \sim_I R$.

(c) If $|A| = 0$ then $S_j \neq R_j$, for all j.

- Since $S_1 \neq R_1$ we consider $m = \sum_{i=1}^{t} |(S_1 \setminus R_1) \cap N_i|$. Using induction on n and a similar proof to the one used in $(1.b)$ we show that there exists a mapping $g : N \to N$ product of transpositions of indifferent players such that $g(S) = (R_1, S'_2, S'_3)$; hence $S \sim_I g(S)$. If $g(S_2) = R_2$ then $g(S_3) = R_3$ and it follows that $g(S) = R$ and hence $S \sim_I R$. If $g(S_2) \neq R_2$, then $g(S) = (R_1, S'_2, S'_3)$ with $S'_2 \neq R_2$, that is $|\{j : \ g(S_j) = T_j\}| = 1$.
- Note that $g(S)$ satisfies $(*)$. We can now refer to $(1.b)$ to deduce that $g(S) \sim_I R$. The conclusion $S \sim_I R$ then follows.

(d) The two equalities and the inequality are obvious.

(e) Any tripartition T of the $\sim_I$-class $\overline{S}$ is obtained by choosing for every i, $s_{i,1}$ players in N_i to form T_1 and choosing for any i, $s_{i,2}$ players among the $n_i - s_{i,1}$ remaining players in N_i to form T_2. The remaining players form T_3.

2. This comes directly from the procedure above and merely states the number of ways S can be formed. ■

Proof of Proposition 3.1

Consider R the tripartition obtained from S by moving player b from the j-th level to the i-th level. Then both a and b belong to R_i

If R is winning, then by monotonicity, $\pi_{ab}(S)$ is winning.

If R is losing, as $a >_I b$, we have $a \geq_{D^+} b$, $a \geq_{D^-} b$ and $a \geq_{D^\pm} b$.

If $j = 1$ and $i = 2$, $a \geq_{D^+} b$ implies that $\pi_{ab}(S)$ is winning.

If $j = 1$ and $i = 3$, $a \geq_{D^\pm} b$ implies that $\pi_{ab}(S)$ is winning.

If $j = 2$ and $i = 3$, $a \geq_{D^-} b$ implies that $\pi_{ab}(S)$ is winning. ■

Proof of Corollary 3.2

As $\overline{s}'$ is well defined, $s_{i',j''} > 0$ and $s_{i'',j'} > 0$. It follows that there exists $a, b \in N$ such that, $a \in S_{j''} \cap N_{i'}$ and $b \in S_{j'} \cap N_{i''}$. We have $a \in N_{i'}$ and $b \in N_{i''}$ with $i' < i''$: thus $a >_I b$. At the same time $a \in S_{j''}$ and $b \in S_{j'}$ with $j' < j''$ so it follows from Proposition 3.1 that $\pi_{ab}(S) \in \mathcal{W}$, since $S \in \mathcal{W}$. As the tripartition $\pi_{ab}(S)$ is represented by the matrix $\overline{s}'$, we have $\pi_{ab}(S) \in \overline{S}'$; hence $\overline{S}' \in \overline{\mathcal{W}}$. ■

Proof of Proposition 3.4

It is obvious that δ is reflexive and antisymmetric. Now let $\overline{s}$, $\overline{r}$ and $\overline{v} \in \Lambda(\mathcal{W})$ such that: $\overline{s}\ \delta\ \overline{r}$ and $\overline{r}\ \delta\ \overline{v}$. It follows that, $\sigma^s_{i,j} \geq \sigma^r_{i,j}\ \forall i,j$ and $\sigma^r_{i,j} \geq \sigma^v_{i,j}\ \forall i,j$, hence $\sigma^s_{i,j} \geq \sigma^v_{i,j}\ \forall i,j$. Thus $\overline{s}\ \delta\ \overline{v}$ and δ is an ordering on $\Lambda(\mathcal{W})$.

In Example 3.8 the tripartitions $(13,4,2)$ and $(23,14,\emptyset)$ which are represented by the models: $\overline{s} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\overline{v} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ are not δ-comparable since $\sigma(\overline{s}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix}$ and $\sigma(\overline{v}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}$. Hence δ is a partial order. $\blacksquare$

Proof of Proposition 3.5

($\Rightarrow$) Let us suppose that for any two models $\overline{s}$ and $\overline{r}$ of $\Lambda(\mathcal{W})$, $\sigma(\overline{s}) \succcurlyeq \sigma(\overline{r})$. We shall prove that:

$$\begin{cases} \text{either } \overline{s}_1 = \overline{r}_1 \text{ and } \overline{s}_2\ \delta'\ \overline{r}_2 \\ \text{or } \overline{s}_1 \neq \overline{r}_1 \text{ and } (\overline{s}_1\ \delta'\ \overline{r}_1 \text{ and } (\overline{s}_1 + \overline{s}_2)\ \delta'\ (\overline{r}_1 + \overline{r}_2)) \end{cases}$$

$\sigma(\overline{s}) \succcurlyeq \sigma(\overline{r})$ means that $\sigma^s_{i,j} \geq \sigma^r_{i,j}\ \forall i,j$.

Since $\sigma^s_{i,1} \geq \sigma^r_{i,1}\ \forall i = 1,2,\ldots,t$, we have $\sum_{i' \leq i} s_{i',1} \geq \sum_{i' \leq i} r_{i',1}\ \forall i = 1,2,\ldots,t$. It is easy to remark that $\sum_{i' \leq i} s_{i',1} = \Sigma_i(\overline{s}_1)$. So, $\Sigma_i(\overline{s}_1) \geq \Sigma_i(\overline{r}_1)$ and hence $\overline{s}_1\ \delta'\ \overline{r}_1$.

- If there exists a row i such that $\Sigma_i(\overline{s}_1) > \Sigma_i(\overline{r}_1)$ then $\overline{s}_1 \neq \overline{r}_1$. Since $\sigma^s_{i,2} \geq \sigma^r_{i,2}\ \forall i = 1,2,\ldots,t$, we have $\sum_{i' \leq i, j' \leq 2} s_{i',j'} \geq \sum_{i' \leq i, j' \leq 2} r_{i',j'}\ \forall i = 1,2,\ldots,t\ (\star)$. We remark that $\sum_{i' \leq i, j' \leq 2} s_{i',j'} = \Sigma_i(\overline{s}_1 + \overline{s}_2)$ hence, $\Sigma_i(\overline{s}_1 + \overline{s}_2) = \Sigma_i(\overline{r}_1 + \overline{r}_2)\ \forall i = 1,2,\ldots,t$ and thus $(\overline{s}_1 + \overline{s}_2)\delta'(\overline{r}_1 + \overline{r}_2)$.
- If not, then $\overline{s}_1 = \overline{r}_1$ and using $(\star)$ we conclude that $\Sigma_i(\overline{s}_1 + \overline{s}_2) \geq \Sigma_i(\overline{r}_1 + \overline{r}_2)\ \forall i = 1,2,\ldots,t$. Since $\Sigma_i(\overline{s}_1 + \overline{s}_2) = \Sigma_i(\overline{s}_1) + \Sigma_i(\overline{s}_2)$ we conclude that $\Sigma_i(\overline{s}_2) \geq \Sigma_i(\overline{r}_2)\ \forall i = 1,2,\ldots,t$, that is, $\overline{s}_2\ \delta'\ \overline{r}_2$.

$\Leftarrow$) Let us now suppose that for any two models $\overline{s}$ and $\overline{r}$ of $\Lambda(\mathcal{W})$ we have:

$$\begin{cases} \text{either } \overline{s}_1 = \overline{r}_1 \text{ and } \overline{s}_2\ \delta'\ \overline{r}_2 \\ \text{or } \overline{s}_1 \neq \overline{r}_1 \text{ and } (\overline{s}_1\ \delta'\ \overline{r}_1 \text{ and } (\overline{s}_1 + \overline{s}_2)\ \delta'\ (\overline{r}_1 + \overline{r}_2)) \end{cases}$$

We need to prove that $\sigma(\overline{s}) \succcurlyeq \sigma(\overline{r})$.

- If $\overline{s}_1\ \delta'\ \overline{r}_1$ and $(\overline{s}_1 + \overline{s}_2)\ \delta'\ (\overline{r}_1 + \overline{r}_2)$ then we have, $\sigma^s_{i,1} \geq \sigma^r_{i,1}$ and $\sigma^s_{i,2} \geq \sigma^r_{i,2}\ \forall i = 1,2,\ldots,t$. Thanks to the equalities $\sigma^s_{i,1} = \Sigma_i(\overline{s}_1)$ and $\sigma^s_{i,2} = \Sigma_i(\overline{s}_1 + \overline{s}_2)\ \forall i = 1,2,\ldots,t$, we obtain $\sigma^s_{i,3} = \Sigma_i(\overline{s}_1 + \overline{s}_2 + \overline{s}_3) = n_1 + n_2 + \cdots + n_i = \Sigma_i(\overline{r}_1 + \overline{r}_2 + \overline{r}_3) = \sigma^r_{i,3}\ \forall i = 1,2,\ldots,t$. Hence $\sigma^s_{i,j} \geq \sigma^r_{i,j}\ \forall i = 1,2,\ldots,t$ and $j = 1,2,3$.

- If $\overline{s}_1 = \overline{r}_1$ and $\overline{s}_2 \; \delta' \; \overline{r}_2$ it can easily be checked as well that $\sigma^s_{i,j} \geq \sigma^r_{i,j}$ $\forall i = 1, 2, \ldots, t$ and $j = 1, 2, 3$.

  Finally, we conclude that $\sigma(\overline{s}) \succcurlyeq \sigma(\overline{r})$. ■

Proof of Lemma 3.6

Let $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$.

1. Let $1 \leq i \leq t$ and $j = 1, 2, 3$, $\sigma^u_{i,j} = M'_{i,j} \leq M_{i,j} = \max(\sigma^r_{i,j}; \sigma^s_{i,j}) \geq \sigma^r_{i,j}$, hence $\sigma(\overline{u}) \succcurlyeq \sigma(\overline{r})$ and thus $\overline{u} \; \delta \; \overline{r}$. Likewise we have $\overline{u} \; \delta \; \overline{s}$.

2. Next, consider $\overline{v} \in \Lambda(\mathcal{W})$ such that $\overline{v} \in Maj(\overline{s}, \overline{r})$. We shall prove that if $\overline{u} \; \delta \; \overline{v}$ then $\overline{v} = \overline{u}$. It is then useful to prove that $\overline{v} \; \delta \; \overline{u}$. It follows from the definition of M' that $M' \succcurlyeq M$ and for this purpose we will distinguish two cases.

   - **Case1** : $M = M'$
     Let $1 \leq i \leq t$ and $j = 1, 2, 3$ then $\sigma^u_{i,j} = M'_{i,j} = M_{i,j} = \max(\sigma^s_{i,j}; \sigma^r_{i,j}) \leq \sigma^v_{i,j}$ since $\overline{v} \in Maj(\overline{s}, \overline{r})$, hence $\overline{v} \; \delta \; \overline{u}$ and thus $\overline{v} = \overline{u}$.
   - **Case2** : $M' \succcurlyeq M$ and $M' \neq M$

     It follows from $\overline{v} \; \delta \; \overline{r}$ and $\overline{v} \; \delta \; \overline{s}$ that $\sigma(\overline{v}) \succcurlyeq M$.

     It follows from the definition of M' that there exists l, $(1 < l \leq t)$ such that $M'_{l,2} > M_{l,2}$ and for all i, $(1 \leq i \leq t)$ such that $M'_{i,j} \leq M_{i,j}$, we have $M'_{i,j} = M_{i,j}$ for $j = 1, 2, 3$. For such i, we have $\sigma^v_{i,j} \leq \sigma^u_{i,j} = M'_{i,j} = M_{i,j} \leq \sigma^v_{i,j}$ hence $\sigma^v_{i,j} = M_{i,j} = \sigma^u_{i,j}$.

     We also have $\sigma^u_{l,2} \leq \sigma^v_{l,2}$. Indeed if $\sigma^u_{l,2} > \sigma^v_{l,2}$ then we would have:
     $v_{l,2} = \sigma^v_{l,2} + \sigma^v_{l-1,1} - \sigma^v_{l,1} - \sigma^v_{l-1,2} < M'_{l,2} + M_{l-1,1} - M_{l,1} - M_{l-1,2} = 0$, since $\sigma^v_{l,2} < \sigma^u_{l,2} = M'_{l,2}$.
     Hence $v_{l,2} < 0$ which is impossible since $\overline{v} \in \Lambda(\mathcal{W})$. Therefore, $\sigma(\overline{v}) \succcurlyeq \sigma(\overline{u})$ and thus $\overline{v} \; \delta \; \overline{u}$. ■

Proof of Lemma 3.7

Let $\overline{r}, \overline{s} \in \Lambda(\mathcal{W})$.

1. First let us show that $\overline{s} \; \delta \; \overline{d}$ and $\overline{r} \; \delta \; \overline{d}$.

   Let $1 \leq i \leq t$ and $j = 1, 2, 3$ then $\sigma^d_{i,j} = m'_{i,j} \leq m_{i,j} = \min(\sigma^s_{i,j}; \sigma^r_{i,j}) \leq \sigma^s_{i,j}$, hence $\sigma(\overline{s}) \succcurlyeq \sigma(\overline{d})$ and thus $\overline{s} \; \delta \; \overline{d}$. Likewise we prove that $\overline{r} \; \delta \; \overline{d}$.

2. Next, consider $\overline{v} \in \Lambda(\mathcal{W})$ such that $\overline{v} \in Min(\overline{s}, \overline{r})$. We shall prove that if $\overline{v} \; \delta \; \overline{d}$ then $\overline{v} = \overline{d}$. It's then useful to prove that $\overline{d} \; \delta \; \overline{v}$. It follows from the definition of m' that $m \succcurlyeq m'$ and for this purpose we will distinguish two cases.

   - **Case1** : $m = m'$
     Let $1 \leq i \leq t$ and $j = 1, 2, 3$ then $\sigma^d_{i,j} = m'_{i,j} = m_{i,j} = \min(\sigma^s_{i,j}; \sigma^r_{i,j}) \geq \sigma^v_{i,j}$ since $\overline{v} \in Min(\overline{s}, \overline{r})$, hence $\overline{d} \; \delta \; \overline{v}$ and thus $\overline{v} = \overline{d}$.

- **Case2** : $m \succcurlyeq m'$ and $m \neq m'$

  It follows from $\overline{r}\ \delta\ \overline{t}$ and $\overline{s}\ \delta\ \overline{t}$ that $m \succcurlyeq \sigma(\overline{v})$.

  It follows from the definition of m' that there exists l, $(1 \leq l < t)$ such that $m'_{l,2} < m_{l,2}$ and for all i, $(1 \leq i \leq t)$ such that $m'_{i,j} \geq m_{i,j}$, we have $m'_{i,j} = m_{i,j}$ for $j = 1, 2, 3$. For such i, we have $\sigma^v_{i,j} \geq \sigma^d_{i,j} = m'_{i,j} = m_{i,j} \geq \sigma^v_{i,j}$ hence $\sigma^v_{i,j} = m_{i,j} = \sigma^d_{i,j}$.

  We also have $\sigma^d_{l,2} \geq \sigma^v_{l,2}$. Indeed if $\sigma^v_{l,2} > \sigma^d_{l,2}$ then we would have:
  $v_{l+1,2} = \sigma^v_{l+1,2} + \sigma^v_{l,1} - \sigma^v_{l+1,1} - \sigma^v_{l,2} < m_{l+1,2} + m_{l,1} - m_{l+1,1} - m'_{l,2} = 0$, since $\sigma^v_{l,2} > \sigma^d_{l,2} = m'_{l,2}$. Hence $v_{l+1,2} < 0$ which is impossible since $\overline{v} \in \Lambda(\mathcal{W})$. Therefore, $\sigma(\overline{d}) \succcurlyeq \sigma(\overline{v})$ and thus $\overline{d}\ \delta\ \overline{v}$. ■

Proof of Lemma 3.10

It suffices to prove that for any $S, R \in 3^N$ if $R \subseteq^3 S$ or there exists $u, v \in N$ with $u \in S_m,\ v \in S_l$ with $l \leq m$ and $u \geq_I v$, such that $\pi_{uv}(R) = S$ then $\overline{s}\ \delta\ \overline{r}$.

If $R \subseteq^3 S$ then $R_1 \subseteq S_1$ and $R_1 \cup R_2 \subseteq S_1 \cup S_2$. It then follows that $r_{i,1} \leq s_{i,1}$ and $r_{i,1} + r_{i,2} \leq s_{i,1} + s_{i,2}$ $\forall i = 1, \ldots, t$. This later inequality implies $\Sigma_i(\overline{r}_1) \leq \Sigma_i(\overline{s}_1)$ (i) and $(\overline{s}_1 + \overline{s}_2)\ \delta'\ (\overline{r}_1 + \overline{r}_2)$. (ii)

- If $\overline{s}_1 = \overline{r}_1$ then $r_{i,2} \leq s_{i,2}$ $\forall i = 1, \ldots, t$, hence $\Sigma_i(\overline{r}_2) \leq \Sigma_i(\overline{s}_2)$ $\forall i = 1, \ldots, t$ and consequently $\overline{s}_2\ \delta'\ \overline{r}_2$; thus, $\overline{s}\ \delta\ \overline{r}$.
- If $\overline{s}_1 \neq \overline{r}_1$ then $\overline{s}_1\ \delta'\ \overline{r}_1$ from (i) and with (ii) we have $\overline{s}\ \delta\ \overline{r}$.

If $\pi_{uv}(R) = S$ with $u \geq_I v$, $u \in S_m,\ v \in S_l$ and $l \leq m$, let $u \in N_p$ and $v \in N_q$ then $p \leq q$ since $u \geq_I v$.

- If $l = m$ then $\overline{s} = \overline{r}$ and $\overline{s}\ \delta\ \overline{r}$.
- If $l < m$: on one hand, if $p = q$ then $\overline{s} = \overline{r}$ and thus $\overline{s}\ \delta\ \overline{r}$. On the other hand, if $p < q$ then $s_{p,l} = r_{p,l} + 1,\ s_{q,l} = r_{q,l} - 1,\ s_{p,m} = r_{p,m} - 1,\ s_{q,m} = r_{q,m} + 1,\ s_{i,j} = r_{i,j}$ for all $j \neq l, m$ and all $i \neq p, q$. This means that $\overline{s}$ is an elementary positive shift of $\overline{r}$ and therefore, $\overline{s}\ \delta\ \overline{r}$. ■

Proof of Theorem 3.11

We need to prove that $\begin{array}{cccc} \Phi : & (3^N/\sim_I, \succeq_I) & \longrightarrow & (\Lambda(\mathcal{W}), \delta) \\ & \overline{S} & \longmapsto & \overline{s} = (s_{i,j})_{\substack{i=1,\ldots,t \\ j=1,2,3}} \end{array}$ is an isomorphism of ordered sets.

It follows from proposition 2.11 that Φ is well defined and bijective. Now, let $S, R \in 3^N$: set $\overline{s} = \Phi(\overline{S})$ and $\overline{r} = \Phi(\overline{R})$.

The implication $\overline{S} \succeq_I \overline{R} \Rightarrow \overline{s}\ \delta\ \overline{r}$ follows directly from the lemma above.

Now assume that $\overline{s}\ \delta\ \overline{r}$: let us prove that $\overline{S} \succeq_I \overline{R}$. In order to achieve this, we will construct a matrix $\overline{U}$ such that $\overline{S} \succeq_I \overline{U} \succeq_I \overline{R}$, that is, $S \succsim_I U \succsim_I R$.

As $\overline{s}\ \delta\ \overline{r}$, we have:

$$\begin{cases} \text{either } \overline{s}_1 = \overline{r}_1 \text{ and } \overline{s}_2 \ \delta' \ \overline{r}_2 \\ \text{or } \overline{s} \neq \overline{r} \text{ and } (\overline{s}_1 \ \delta \ \overline{r}_1 \text{ and } \Sigma_i(\overline{s}_1 + \overline{s}_2) \geq \Sigma_i(\overline{r}_1 + \overline{r}_2) \ \forall i = 1, 2 \dots t) \end{cases}$$

Case 1: If $\overline{s}_1 = \overline{r}_1$ and $\overline{s}_2 \ \delta' \ \overline{r}_2$, let $l < t$ be the smallest index such that : $\Sigma_l(\overline{s}_2) \geq \Sigma_t(\overline{r}_2)$. Consider the matrix $\overline{u} = (u_{i,j})$ defined as follow:

$$\begin{cases} u_{i,1} = s_{i,1} = r_{i,1} \ \forall i = 1, ..., t \\ u_{i,2} = s_{i,2} \ \forall i < l, \ u_{l,2} = \Sigma_t(\overline{r}_2) - \Sigma_{l-1}(\overline{s}_2), \ u_{i,2} = 0 \ \forall i > l \\ u_{i,3} = s_{i,3} \ \forall i < l, \ u_{l,3} = n_l - u_{l,1} - u_{l,2}, \ u_{i,3} = n_i - u_{i,1} \ \forall i > l \end{cases}$$

By construction, $\overline{u} \in \Lambda(\mathcal{W})$. Let $\overline{U} \in 3^N / \sim_I$ such that $\overline{u} = \Phi^{-1}(\overline{U})$. By definition of $\overline{u}$ we have : $\overline{s}_2 \ \delta' \ \overline{u}_2$ (1); $\Sigma_t(\overline{u}_2) = \Sigma_t(\overline{r}_2)$ (2); $\overline{s} \ \delta \ \overline{u}$ (3) and $\overline{u} \ \delta \ \overline{r}$ (4).

Since $\overline{s}_1 = \overline{u}_1 = \overline{r}_1$, there exist two maps $f : N \to N$ and $g : N \to N$ product of transpositions of indifferent players such that $f(R_1) = U_1$ and $g(U_1) = S_1$. Denote $f(R) = (U_1, R'_2, R'_3)$ and $g(U) = (S_1, U'_2, U'_3)$: then $f(R) \sim_I R$ and $g(U) \sim_I U$.

- In order to prove that $U \succsim_I R$, let us proceed by induction on $m = \displaystyle\sum_{u_{i,2} \geq r_{i,2}} (u_{i,2} - r_{i,2})$.

- If $m = 0$ then $\overline{u}_2 = \overline{r}_2$ and since $\overline{u}_1 = \overline{r}_1$, it then follow that $\overline{u} = \overline{r}$ and hence $U \succsim_I R$.

- If $m > 0$ then, let $k < t$ be the smallest index such that $u_{k,2} > r_{k,2}$. From (4), we have $u_{1,2} = r_{1,2}$, ..., $u_{k-1,2} = r_{k-1,2}$. Thanks to (2) there exist an index $h > k$ such that $u_{h,2} < r_{h,2}$. Consider the smallest such index h : and let $a \in (U_2 \cap N_k) \setminus R'_2$ and $b \in (R'_2 \cap N_h) \setminus U_2$. Since $k < h$, we have $a >_I b$ and $b \not\geq_I a$. Let $\pi^{(1)} = \pi_{ab}$ and $R''_2 = \pi_{ab}(R'_2)$. Then R''_2 verifies (1) and (2) and $\displaystyle\sum_{u_{i,2} \geq r'_{i,2}} (u_{i,2} - r'_{i,2}) = m - 1$. We obtain by induction a sequence $(\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \dots, \pi^{(m)})$ of transpositions of players of different classes such that $\pi^{(m)} \circ \pi^{(m-1)} \circ \dots \circ \pi^{(1)}(R'_2) = U_2$. Let $\Gamma = \pi^{(m)} \circ \pi^{(m-1)} \circ \dots \circ \pi^{(1)}$. Then $\Gamma(R'_2) = U_2$. Since $\Gamma(f(R)) = \Gamma(U_1, R'_2, R'_3) = (U_1, U_2, \Gamma(R'_3))$, it follows that $\Gamma(f(R)) = U$ and hence $U \succsim_I f(R)$. Since $f(R) \sim_I R$ we deduce that $U \succsim_I R$.

- In order to prove that $S \succsim_I U$, we will proceed by induction on $m = |U'_2 \setminus S_2|$.

Subscase 1 : If $m = 0$ then $U'_2 \subseteq S_2$ and $g(U) \subseteq^3 S$, thus, $S \succsim_I g(U)$.

Subscase 2 : If $m > 0$, then there exists $a \in N : a \in U'_2 \setminus S_2$. With no loss of the generality, let us assume that $a \in (U'_2 \setminus S_2) \cap N_i$. Since $s_{i,2} \geq u_{i,2}$, there exists $b \in (S_2 \setminus U'_2) \cap N_i$. Thus, $a \equiv_I b$, $U''_2 = \pi_{ab}(U'_2)$ satisfies (1) and $|U''_2 \setminus S_2| = m - 1$. We obtain by induction a sequence $(\pi_{ab} = \pi^{(1)}, \pi^{(2)}, \dots, \pi^{(m)})$ of transpositions of equivalent players such that $\pi^{(m)} \circ \pi^{(m-1)} \circ \dots \circ \pi^{(1)}(U'_2) \subseteq S_2$. By letting $\Gamma' = \pi^{(m)} \circ \pi^{(m-1)} \circ \dots \circ \pi^{(1)}$, we have $\Gamma'(U''_2) \subseteq S_2$. It is straightforward that $\Gamma'(g(U)) \subseteq^3 S$; hence $S \succsim_I g(U)$. Since $g(U) \sim_I U$ we deduce that $S \succsim_I U$.

The case 1 is now complete, thanks to $U \succsim_I R$ and $S \succsim_I U$, it follows that $S \succsim_I R$, which means that $\overline{S} \succeq_I \overline{R}$.

Case 2 : If $\overline{s}_1 \ \delta' \ \overline{r}_1$, $\overline{s}_1 \neq \overline{r}_1$ and $(\overline{s}_1 + \overline{s}_2) \ \delta' \ (\overline{r}_1 + \overline{r}_2)$. Again, let $l < t$ be the smallest index such that $\Sigma_l(\overline{s}_1) \geq \Sigma_t(\overline{r}_1)$. Consider the following matrix $\overline{u} = (u_{i,j})$:

$$\begin{cases} u_{i,1} = s_{i,1} \ \forall i < l, \ u_{l,1} = \Sigma_t(\overline{r}_1) - \Sigma_{l-1}(\overline{s}_1) \ , u_{i,1} = 0 \ \forall i > l \\ u_{i,2} = n_i - u_{i,1} - u_{i,3}, \ \forall i = 1, ..., t \\ u_{i,3} = r_{i,3} \ \forall i = 1, ..., t \end{cases}$$

Once more, as $\overline{u} \in \Lambda(\mathcal{W})$, let $\overline{U} \in 3^N / \sim_I$ such that $\overline{u} = \Phi^{-1}(\overline{U})$. We deduce from the definition of $\overline{u}$ that : $s_{i,1} \geq u_{i,1} \ \forall i = 1, 2, \ldots, t$ (1) ; $\Sigma_t(\overline{u}_1) = \Sigma_t(\overline{r}_1)$ (2); $\Sigma_i(\overline{s}_1 + \overline{s}_2) \geq \Sigma_i(\overline{u}_1 + \overline{u}_2) \ \forall i = 1, 2, \ldots, t$ (3); $\overline{s} \ \delta \ \overline{u}$ (4) $\overline{u} \ \delta \ \overline{r}$ (5). Recall that we want to prove that $\overline{S} \succeq_I \overline{U} \succeq_I \overline{R}$, that is, $S \succsim_I U \succsim_I R$.
Since $\overline{u}_3 = \overline{r}_3$ there exists a mapping $h : N \to N$, product of transpositions of equivalent players such that: $h(R_3) = U_3$. Let $h(R) = (R'_1, R'_2, U_3)$: then $h(R) \sim_I R$.

- Let us show that $U \succsim_I R$: let $m = \sum_{u_{i,1} \geq r_{i,1}} (u_{i,1} - r_{i,1})$.

  By using a similar reasoning as in Case 1, and proceeding by induction on m, we obtain $U \succsim_I h(R)$. Now thanks to the fact that $h(R) \sim_I R$, it follows that $U \succsim_I R$.

- Let us show that $S \succsim_I U$. For this purpose, let $m = |U_1 \setminus S_1|$.

Subcase 1 : If $m = 0$, then $U_1 \subseteq S_1$ because $s_{i,1} \geq u_{i,1} \ \forall i = 1, \ldots, t$.

Let $U' = (S_1, U'_2, U'_3)$ where $U'_2 = U_2 \setminus A$, $A = (S_1 \setminus U_1) \cap U_2$, $U'_3 = U_3 \setminus B$ with $B = (S_1 \setminus U_1) \cap U_3$. We then have $U \subseteq^3 U'$ because $U_1 \subseteq S_1$ and $U_1 \cup U_2 = S_1 \cup U'_2$. So, $U' \succsim_I U$ $(\star)$. Now, it is enough to show that $S \succsim_I U'$. Since $\overline{s}_1 = \overline{u}'_1$ and $\Sigma_t(\overline{u}'_2) \leq \Sigma_t(\overline{u}_2)$, U' verifies (4) and hence $\Sigma_t(\overline{s}_2) \geq \Sigma_t(\overline{u}'_2)$.

If $\Sigma_t(\overline{s}_2) = \Sigma_t(\overline{u}'_2)$ then by induction on $m' = \sum_{s_{i,2} \geq u'_{i,2}} (s_{i,2} - u'_{i,2})$, we prove as above that $S \succsim_I U'$. In addition, if $\Sigma_t(\overline{s}_2) > \Sigma_t(\overline{u}'_2)$, then there exists a mapping $K_1 : N \to N$ product of transpositions of equivalent players such that $K_1(U') = (S_1, U''_2, U''_3)$ where U''_2 satisfies (1). In addition, $U' \sim_I K_1(U')$ $(\star\star)$.
By induction on $m'' = |U''_2 \setminus S_2|$, we prove identically that $S \succsim_I K_1(U')$ $(\star\star\star)$.
Now, thanks to $(\star)$, $(\star\star)$ and $(\star\star\star)$, we deduce that $S \succsim_I U' \succsim_I U$ hence $S \succsim_I U$.

Subcase 2 : If $m > 0$, then by proceeding as we did in Subcase 2 of case 1, we prove the existence of a mapping $K_2 : N \to N$ product of transpositions of equivalent players such that $K_2(U_1) \subseteq S_1$. Let $K_2(U) = (K_2(U_1), U'_2, U'_3)$: this meets the subcase just done above ($m = 0$) by merely replacing U with $K_2(U)$. We can therefore use the same reasoning to conclude that $S \succsim_I U' \sim_I U$ and hence $S \succsim_I U$. Finally, we obtain $S \succsim_I U \succsim_I R$. ■

Proof of Proposition 4.1

From the inclusions $\mathcal{W}^{sm} \subseteq \mathcal{W}^m \subseteq \mathcal{W}$, it follows that $\overline{\mathcal{W}^{sm}} \subseteq \overline{\mathcal{W}^m} \subseteq \overline{\mathcal{W}}$. Hence $\overline{\mathcal{W}}^m \subseteq \overline{\mathcal{W}^m} \subseteq \overline{\mathcal{W}}$ since $\overline{\mathcal{W}^{sm}} = \overline{\mathcal{W}}^m$. ■

Proof of Theorem 4.2

We consider $\mathcal{M}$ as defined in the text.

1. This point comes from the fact that any $\overline{m}^p$ belongs to $\Lambda(\mathcal{W})$.

2. If $r > 1$, let $p \neq q$. Then $\overline{m}^p$ and $\overline{m}^q$ are not δ-comparable : indeed, if (for example) $\overline{m}^p \; \delta \; \overline{m}^q$, then, $\overline{S}_q \notin \overline{\mathcal{W}}^m$ with $\overline{m}^q = \Phi(\overline{S}_q)$ which is a contradiction.

3. (i) Assume that : $t = r = 1$. If the vector $\overline{m}^1 = (0, 0, n)$ then $(\emptyset, \emptyset, N)$ would be a winning tripartition, which is impossible.

   (ii) Now assume that $t > 1$. If the condition [there exists some p such that $(m^p_{i,1} > 0$ and $m^p_{i+1,1} < n_{i+1})$ or $(m^p_{i,3} < n_i$ and $m^p_{i+1,3} > 0)$] is not true for some $i < t$, then any component of the vector $\mathcal{M}$ should be one of the following four forms :

Form 1: $\begin{pmatrix} m^p_{1,1} & m^p_{1,2} & m^p_{1,3} \\ m^p_{2,1} & m^p_{2,2} & m^p_{2,3} \\ \cdots & \cdots & \cdots \\ 0 & 0 & n_i \\ \cdots & \cdots & \cdots \\ m^p_{t,1} & m^p_{t,2} & m^p_{t,3} \end{pmatrix}$, Form 2: $\begin{pmatrix} m^p_{1,1} & m^p_{1,2} & m^p_{1,3} \\ \cdots & \cdots & \cdots \\ 0 & m^p_{i,2} & m^p_{i,3} \\ m^p_{(i+1),1} & m^p_{(i+1),2} & 0 \\ \cdots & \cdots & \cdots \\ m^p_{t,1} & m^p_{t,2} & m^p_{t,3} \end{pmatrix}$, Form 3: $\begin{pmatrix} m^p_{1,1} & m^p_{1,2} & m^p_{1,3} \\ \cdots & \cdots & \cdots \\ 0 & 0 & n_i \\ n_{i+1} & 0 & 0 \\ \cdots & \cdots & \cdots \\ m^p_{1t} & m^p_{2t} & m^p_{3t} \end{pmatrix}$,

Form 4: $\begin{pmatrix} m^p_{1,1} & m^p_{1,2} & m^p_{1,3} \\ m^p_{2,1} & m^p_{2,2} & m^p_{2,3} \\ \cdots & \cdots & \cdots \\ n_{i+1} & 0 & 0 \\ \cdots & \cdots & \cdots \\ m^p_{t,1} & m^p_{t,2} & m^p_{t,3} \end{pmatrix}$.

   (a) • In the sequel, we will prove that in either form, for all $a \in N_i$ and $b \in N_{i+1}$ we have $b \geq_I a$.

Let $a \in N_i$ and $b \in N_{i+1}$. We will prove that $b \geq_{D^+} a$, $b \geq_{D^-} a$ and $b \geq_{D^\pm} a$.

The proof of $b \geq_{D^+} a$

Let $S \in 3^N$ such that $a, b \in S_2$ and $(S_1 \cup a, S_2 \setminus a, S_3) \in \mathcal{W}$. We need to prove that $(S_1 \cup b, S_2 \setminus b, S_3) \in \mathcal{W}$, that is, there exists p such that $\overline{s}' = \Phi\overline{(S_1 \cup b, S_2 \setminus b, S_3)} \; \delta \; \overline{m}^p$. Let $\overline{s} = \Phi\overline{(S_1 \cup a, S_2 \setminus a, S_3)}$: since $(S_1 \cup a, S_2 \setminus a, S_3) \in \mathcal{W}$, we have $\overline{s} \; \delta \; \overline{m}^p$ for some p. In the sequel we will prove that $\overline{s}' \; \delta \; \overline{m}^p$.

- If $\overline{m}^p$ is of Form 1, then $\overline{s}_1 \; \delta' \; \overline{m}^p_1$ and $\overline{s}_1 \neq \overline{m}^p_1$ as $s_{i,1} > 0$, $\Sigma_h(\overline{s}_1) \geq \Sigma_h(\overline{m}^p_1)$ $\forall h \neq i$ and $\Sigma_i(\overline{s}_1) > \Sigma_i(\overline{m}^p_1)$.

We claim that $\overline{s}'_1 \; \delta' \; \overline{m}^p_1$: indeed, $\Sigma_h(\overline{s}_1) = \Sigma_h(\overline{s}'_1)$ for all $h < i$, $\Sigma_i(\overline{s}'_1) = \Sigma_i(\overline{s}_1) - 1$, and $\Sigma_h(\overline{s}'_1) = \Sigma_h(\overline{s}_1)$ for all $h \geq i + 1$. We also claim that $\Sigma_h(\overline{s}'_1 + \overline{s}'_2) \geq \Sigma_h(\overline{m}^p_1 + \overline{m}^p_2)$ for all $h = 1, ..., t$. Indeed, $\Sigma_h(\overline{s}'_1 + \overline{s}'_2) = \Sigma_h(\overline{s}_1 + \overline{s}_2) \geq \Sigma_h(\overline{m}^p_1 + \overline{m}^p_2)$ for all $h \neq i$ and $\Sigma_i(\overline{s}'_1 + \overline{s}'_2) = \Sigma_i(\overline{s}_1 + \overline{s}_2) - 1 \geq \Sigma_i(\overline{m}^p_1 + \overline{m}^p_2)$.

So, if $\overline{s}'_1 \neq \overline{m}^p_1$, then it follows that $\overline{s}' \; \delta \; \overline{m}^p$.

However, if it happens that $\overline{s}'_1 = \overline{m}^p_1$, in order to conclude that $\overline{s}' \; \delta \; \overline{m}^p$, we will show that $\overline{s}'_2 \; \delta' \; \overline{m}^p_2$.

We proved above that for all $h = 1, ..., t$, $\Sigma_h(\overline{s}'_1 + \overline{s}'_2) \geq \Sigma_h(\overline{m}^p_1 + \overline{m}^p_2)$ and $\overline{s}'_1 = \overline{m}^p_1$ implies $\Sigma_h(\overline{s}'_1) = \Sigma_h(\overline{m}^p_1)$, thus $\Sigma_h(\overline{s}'_2) \geq \Sigma_h(\overline{m}^p_2)$ for all h, hence, $\overline{s}'_2 \; \delta' \; \overline{m}^p_2$.

- If $\overline{m}^p$ is of Form 2 or 3, the proof is quite similar to that of the case where $\overline{m}^p$ is of Form 1.

- If $\overline{m}^p$ is of Form 4, then $\overline{s}_1\ \delta'\ \overline{m}_1^p$ and $\overline{s}_1 \neq \overline{m}_1^p$ because $s_{(i+1),1} < n_{i+1}$. Since $s_{(i+1),1} < n_{i+1}$ and $\Sigma_{i+1}(\overline{s}_1) \geq \Sigma_{i+1}(\overline{m}_1^p)$, it follows that $\Sigma_i(\overline{s}_1) > \Sigma_i(\overline{m}_1^p)$. Since $s_{(i+1),1} < n_{i+1}$ and $\Sigma_{i+1}(\overline{s}_1) \geq \Sigma_{i+1}(\overline{m}_1^p)$, it follows that $\Sigma_i(\overline{s}_1) > \Sigma_i(\overline{m}_1^p)$. We can now proceed exactly as in the first form to get $\overline{s}'\ \delta\ \overline{m}^p$.

The proof of $b \geq_{D^{\pm}} a$: similar to the proof of $b \geq_{D^+} a$

The proof of $b \geq_{D^-} a$

Let $S \in 3^N$ such that $a, b \in S_2$ and $(S_1, S_2 \cup a, S_3 \setminus a) \in \mathcal{W}$. We need to prove that $(S_1, S_2 \cup b, S_3 \setminus b) \in \mathcal{W}$, that is, there exists p such that $\overline{s}' = \Phi\overline{(S_1, S_2 \cup b, S_3 \setminus b)}\ \delta\ \overline{m}^p$. Let $\overline{s} = \Phi\overline{(S_1, S_2 \cup a, S_3 \setminus a)}$: since $(S_1, S_2 \cup a, S_3 \setminus a) \in \mathcal{W}$, we have $\overline{s}\ \delta\ \overline{m}^p$ for some p. In the sequel we will prove that $\overline{s}'\ \delta\ \overline{m}^p$.

Since $\overline{s}\ \delta\ \overline{m}^p$, either $[\overline{s}_1\ \delta'\ \overline{m}_1^p,\ \overline{s}_1 \neq \overline{m}_1^p$ and $\Sigma_i(\overline{s}_1 + \overline{s}_2) \geq \Sigma_i(\overline{m}_1^p + \overline{m}_2^p)\ \forall i,]$ or $[\overline{s}_1 = \overline{m}_1^p$ and $\overline{s}_2\ \delta'\ \overline{m}_2^p]$.

- If $\overline{m}^p$ is of Form 1, then $\Sigma_h(\overline{s}_1') = \Sigma_h(\overline{s}_1)\ \forall h$, $\Sigma_h(\overline{s}_2) = \Sigma_h(\overline{s}_2')\ \forall h < i$, $\Sigma_i(\overline{s}_2') = \Sigma_i(\overline{s}_2) - 1$ and $\Sigma_h(\overline{s}_2') = \Sigma_h(\overline{s}_2)$ for all $h > i$.

- If $[\overline{s}_1\ \delta'\ \overline{m}_1^p,\ \overline{s}_1 \neq \overline{m}_1^p$ and $\Sigma_i(\overline{s}_1 + \overline{s}_2) \geq \Sigma_i(\overline{m}_1^p + \overline{m}_2^p)\ \forall i]$, then $\overline{s}_1'\ \delta'\ \overline{m}_1^p$ and $\overline{s}_1' \neq \overline{m}_1^p$ because $\overline{s}_1' = \overline{s}_1$. In addition, $\Sigma_h(\overline{s}_1' + \overline{s}_2') = \Sigma_h(\overline{s}_1 + \overline{s}_2) \geq \Sigma_h(\overline{m}_1^p + \overline{m}_2^p)$ for all $h \neq i$ and $\Sigma_i(\overline{s}_1' + \overline{s}_2') = \Sigma_i(\overline{s}_1 + \overline{s}_2) - 1 \geq \Sigma_i(\overline{m}_1^p + \overline{m}_2^p)$, hence $\overline{s}'\ \delta\ \overline{m}^p$.

- If $[\overline{s}_1 = \overline{m}_1^p$ and $\overline{s}_2\ \delta'\ \overline{m}_2^p]$, then $\overline{s}_1' = \overline{m}_1^p$. We claim that $\overline{s}_2'\ \delta'\ \overline{m}_2^p$: Indeed, $\Sigma_h(\overline{s}_2') = \Sigma_h(\overline{s}_2) \geq \Sigma_h(\overline{m}_2^p)\ \forall h < i$, $\Sigma_i(\overline{s}_2') = \Sigma_i(\overline{s}_2) - 1 \geq \Sigma_i(\overline{m}_2^p)$, and $\Sigma_h(\overline{s}_2') = \Sigma_h(\overline{s}_2) \geq \Sigma_h(\overline{m}_2^p)$ $\forall h > i$. It then follows that $\overline{s}'\ \delta\ \overline{m}^p$.

- If $\overline{m}^p$ is of Form 2

- If $[\overline{s}_1\ \delta'\ \overline{m}_1^p,\ \overline{s}_1 \neq \overline{m}_1^p$ and $\Sigma_i(\overline{s}_1 + \overline{s}_2) \geq \Sigma_i(\overline{m}_1^p + \overline{m}_2^p)\ \forall i]$

By proceeding as in the form 1, we get $\overline{s}'\ \delta\ \overline{m}^p$.

- Assume that $[\overline{s}_1 = \overline{m}_1^p$ and $\overline{s}_2\ \delta'\ \overline{m}_2^p]$

By construction, we have $\overline{s}_1' = \overline{s}_1$, thus, $\overline{s}_1' = \overline{m}_1^p$. It remains to show that $\overline{s}_2'\ \delta'\ \overline{m}_2^p$.

If $\overline{s}_2 = \overline{m}_2^p$ then $\overline{s}_3 = \overline{m}_3^p$ and hence $s_{(i+1),3} = 0$ which is a contradiction since $b \in S_3 \setminus a$. We then have $s_{(i+1),2} < m^p_{(i+1),2}$. Indeed, $m^p_{(i+1),2} = n_{i+1} - m^p_{(i+1),1} = n_{i+1} - s_{(i+1),1} = s_{(i+1),2} + s_{(i+1),3} > s_{(i+1),2}$ since $s_{(i+1),3} > 0$. It follows that From $\overline{s}_2\ \delta'\ \overline{m}_2^p$ and $s_{(i+1),2} < m^p_{(i+1),2}$, we have $\Sigma_i(\overline{s}_2) > \Sigma_i(\overline{m}_2^p)$.

In summary, $\Sigma_h(\overline{s}_2') = \Sigma_h(\overline{s}_2) \geq \Sigma_h(\overline{m}_2^p)\ \forall h < i$, $\Sigma_i(\overline{s}_2') = \Sigma_i(\overline{s}_2) - 1 \geq \Sigma_i(\overline{m}_2^p)$ and $\Sigma_h(\overline{s}_2') = \Sigma_h(\overline{s}_2) \geq \Sigma_h(\overline{m}_2^p)$ for all $h > i$. We conclude that $\overline{s}_2'\ \delta'\ \overline{m}_2^p$.

- If $\overline{m}^p$ is of Form 3 or 4, the proof is quite similar to that of the case where $\overline{m}^p$ is of Form1. ■

Proof of Theorem 4.3

Proof of a)

$\Rightarrow$) Let $f : (N, \mathcal{W}) \to (N', \mathcal{W}')$ be an isomorphism. Then the inverse map f^{-1} is also an isomorphism and that $\pi_{f(a)f(b)} = f \circ \pi_{ab} \circ f^{-1}$ for all $a, b \in N$. It then follows that $a \geq_I b \Leftrightarrow f(a) \geq_{I'} f(b)$; that is f preserves the influence relation and hence it preserves individual and coalitional indifference and coalitional dominance. Moreover, f induces a multilattice isomorphism $\overline{f} : (\overline{3^N}; \succeq_1) \to (\overline{3^{N'}}; \succeq_1)$ such that $\overline{f}(\overline{\mathcal{W}}) = \overline{\mathcal{W}'}$. The map $\phi : (\Lambda(\mathcal{W}); \delta) \to (\Lambda(\mathcal{W}'); \delta')$ is an isomorphism because $\phi = \Phi' \circ \overline{f} \circ \Phi^{-1}$ is a product of isomorphisms. Both games have, therefore, a common set of models of shift-minimal winning tripartitions. Since they are lexicographically ordered by rows, we conclude that $\mathcal{M} = \mathcal{M}'$.

$\Leftarrow$) Assume that $\mathcal{M} = \mathcal{M}'$: Let us prove that there exists an isomorphism $f : (N, \mathcal{W}) \to (N', \mathcal{W}')$. Let $N_1 = \{k_1, \ldots, k_{n_1}\}$, $N'_1 = \{l_1, \ldots, l_{n_1}\}$ and $N_i = \{k_{n_1+\cdots+n_{i-1}+1}, \ldots, k_{n_1+\cdots+n_i}\}$, $N'_i = \{l_{n_1+\cdots+n_{i-1}+1}, \ldots, l_{n_1+\cdots+n_i}\}$ for all $i = 2, \ldots, t$. Let us consider the mapping :

$$\begin{array}{llll} f: & N & \longrightarrow & N' \\ & k_p & \longmapsto & l_p \end{array}$$

It is obvious that f is bijective and $f(N_i) = N'_i \ \forall i = 1, \ldots, t$.
Assume that $S \in \mathcal{W}$. If $\overline{s} = \Phi(\overline{S})$, then $\overline{s}\ \delta\ \overline{m}^p$ for some p. Let $S' = f(S)$ and $\overline{s}' = \Phi'(\overline{S}')$. Given that $\mathcal{M} = \mathcal{M}'$, $\overline{m}^p$ is an element of $\mathcal{M}'$. Thanks to the equality $\overline{s}' = \overline{s}$ that comes from the definition of f, it follows that $\overline{s}'\ \delta\ \overline{m}^p$ and $S' \in \mathcal{W}'$. Applying the same argument to f^{-1} it follows that the implication $f(S) \in \mathcal{W}' \Rightarrow S \in \mathcal{W}$ holds, thus, f is an isomorphism.

Proof of b)

Let $\mathcal{M}$ satisfying conditions of Theorem 4.2. We need to construct an I-complete $(3, 2)$ game the characteristic invariant of which is $\mathcal{M}$. Let $n = \Sigma_t(\overline{n}) = n_1 + n_2 + \cdots + n_t$, $N = \{1, 2, \ldots, n\}$ and $N_1, N_2, \ldots, N_t$ be the subsets of N formed, respectively, by $n_1, n_2, \ldots, n_t$, elements (which may be chosen following the natural ordering). By theorem (4.2), none of these subsets is empty. For each $S \in 3^N$ we define $\overline{s} = (s_{i,j})_{\substack{j=1,2,3 \\ i=1,\ldots,t}}$ where $s_{i,j} = |S_j \cap N_i| \ \forall j = 1, 2, 3$ and $i = 1, \ldots, t$. Let $\mathcal{W} = \{S \in 3^N :\ \overline{s}\ \delta\ \overline{m}^p\ for\ some\ p\}$. We will prove that $(N, \mathcal{W})$ is a $(3, 2)$ I-complete simple game whose characteristic invariant is $\mathcal{M}$.

1. It is straightforward that $(N, \mathcal{W})$ is a $(3, 2)$ simple game.

2. We shall now prove that $N_1, N_2, \ldots, N_t$ are equivalence classes according to the relation $\equiv_I$ and they are linearly ordered ($N_1 > N_2 > \cdots > N_t$).

- Let $i \in \{1, 2, ..., t\}$, $a, b \in N_i$ and $S \in 3^N$.

- **Proof of** $b \geq_{D^+} a$**.** If $a, b \in S_2$ such that $(S_1 \cup a, S_2 \setminus a, S_3) \in \mathcal{W}$ then by considering $\overline{s} = \Phi(\overline{(S_1 \cup a, S_2 \setminus a, S_3)})$ and $\overline{s}' = \Phi(\overline{(S_1 \cup b, S_2 \setminus b, S_3)})$, we have $\overline{s} = \overline{s}'$. Since $(S_1 \cup a, S_2 \setminus a, S_3) \in \mathcal{W}$ we have, $\overline{s}\ \delta\ \overline{m}^p$ for some p so that $\overline{s}'\ \delta\ \overline{m}^p$. Hence $(S_1 \cup b, S_2 \setminus b, S_3) \in \mathcal{W}$ and $b \geq_{D^+} a$.

- We prove in the same way that $b \geq_{D^-} a$ and $b \geq_{D^\pm} a$, thus $b \geq_I a$. By the same arguments, we obtain $a \geq_I b$ consequently $a \equiv_I b$.

- Now let $i \in \{1, 2, ..., t\}$, $a \in N_i,\ b \in N_{i+1}$ and $S \in 3^N$. We will prove that $a \geq_I b$ and $b \not\geq a$.

If $a, b \in S_2$ such that $(S_1 \cup b, S_2 \setminus b, S_3) \in \mathcal{W}$ then by considering $\overline{s} = \Phi\overline{(S_1 \cup a, S_2 \setminus a, S_3)}$ and $\overline{s}' = \Phi\overline{(S_1 \cup b, S_2 \setminus b, S_3)}$, we have $\overline{s}\ \delta\ \overline{s}'$ since $\overline{s}$ is an elementary positive shift of $\overline{s}'$. With $(S_1 \cup b, S_2 \setminus b, S_3) \in \mathcal{W}$ we have, $\overline{s}'\ \delta\ \overline{m}^p$ for some p. So, $\overline{s}\ \delta\ \overline{m}^p$. Hence $(S_1 \cup a, S_2 \setminus a, S_3) \in \mathcal{W}$ and $a \geq_{D^+} b$. Likewise, it can easily be checked that $a \geq_{D^-} b$ and $a \geq_{D^\pm} b$ which leads to $a \geq_I b$.

- To prove that $b \not\geq_I a$, we use Theorem 4.2 (3(ii)) and take a tripartition $S \in 3^N$ whose model is some matrix $\overline{m}^p$ such that: $(m^p_{i,1} > 0$ and $(m^p_{i+1,2} > 0$ or $m^p_{i+1,3} > 0))$ or $(m^p_{i+1,3} > 0$ and $(m^p_{i,1} > 0$ or $m^p_{i,2} > 0))$. This condition leads to the following three possible cases. $(m^p_{i,1} > 0$ and $m^p_{i+1,2} > 0)$ or $(m^p_{i,1} > 0$ and $m^p_{i+1,3} > 0)$ or $(m^p_{i,2} > 0$ and $m^p_{i+1,3} > 0)$.

Case 1 : If $m^p_{i,1} > 0$ and $m^p_{i+1,2} > 0$ then we can assume that $a \in S_1$ and $b \in S_2$. If $\overline{r}$ is the model of $\pi_{ab}(S)$ then we have $\overline{m}^p\ \delta\ \overline{r}$ and $\overline{m}^p \neq \overline{r}$, since that $\overline{m}^p$ is an elementary positive shift of $\overline{r}$ with $j' = 1,\ j'' = 2,\ i' = i$ and $i'' = i + 1$. Hence, $b \not\geq_{D^+} a$ and thus $b \not\geq_I a$.

Case 2 : If $m^p_{i,1} > 0$ and $m^p_{i+1,3} > 0$ then we can assume that $a \in S_1$ and $b \in S_3$. If $\overline{r}$ is the model of $\pi_{ab}(S)$ then we have $\overline{m}^p\ \delta\ \overline{r}$ and $\overline{m}^p \neq \overline{r}$, since that $\overline{m}^p$ is an elementary positive shift of $\overline{r}$ with $j' = 1,\ j'' = 3,\ i' = i$ and $i'' = i + 1$. Hence $b \not\geq_{D^\pm} a$ and thus $b \not\geq_I a$.

Case 3 : If $m^p_{i,2} > 0$ and $m^p_{i+1,3} > 0$ then we can assume that $a \in S_2$ and $b \in S_3$. If $\overline{r}$ is the model of $\pi_{ab}(S)$ then we have $\overline{m}^p\ \delta\ \overline{r}$ and $\overline{m}^p \neq \overline{r}$, since that $\overline{m}^p$ is an elementary positive shift of $\overline{r}$ with $j' = 2,\ j'' = 3,\ i' = i$ and $i'' = i + 1$. Hence, $b \not\geq_{D^-} a$ and thus $b \not\geq_I a$.

- Finally, Theorem 4.2.(3) and the definition of $\mathcal{W}$ guarantee that $\overline{\mathcal{W}}^m$ contains exactly r models that are $\overline{m}^1, \overline{m}^2, \ldots, \overline{m}^r$. The proof is complete. ■

Proof of Proposition 4.6

Let $(N, \mathcal{W})$ be a I-complete $(3,2)$ game with $\overline{m}^1, \ldots, \overline{m}^r$ being the shift-minimal winning models and $\overline{\alpha}^1, \ldots, \overline{\alpha}^s$ the shift-maximal losing ones.

1) Suppose that $(N, \mathcal{W})$ is a strongly weighted $(3,2)$ game. Then it is weighted and there exists a vector $w = (w_1, w_2, w_3)$ where $w_i : N \to \mathbb{R}$ for each i together with a real number quota q such that for every $S \in 3^N,\ S \in \mathcal{W} \Leftrightarrow w(S) \geq q$ where $w(S) = \sum\limits_{i=1}^{3} \sum\limits_{a \in S_i} w_i(a)$ and $w_1(a) \geq w_2(a) \geq w_3(a)$ for each $a \in N$. Given that the game is I-complete, let us suppose that players are ranked into t $(t \geq 1)$ classes.

Now let $S \in \mathcal{W}^{sm}$, $T \in \mathcal{L}^{\delta M}$ (the set of all shift-maximal losing tripartitions) and let $\overline{m}^p$ and $\overline{\alpha}^q$ with $p \in \{1, 2, \ldots, r\}$, $q \in \{1, 2, \ldots, s\}$ be the respective models of S and T. We have $w(S) > w(T)$, that is, $\sum\limits_{j=1}^{3} \sum\limits_{a \in S_j} w_j(a) > \sum\limits_{j=1}^{3} \sum\limits_{a \in T_j} w_j(a)$. It is easy to check that $\sum\limits_{j=1}^{3} \sum\limits_{a \in S_j} w_j(a) = \sum\limits_{i=1}^{t} \overline{m}^p_i \cdot w(i)$ and $\sum\limits_{j=1}^{3} \sum\limits_{a \in T_j} w_j(a) = \sum\limits_{i=1}^{t} \overline{\alpha}^q_i \cdot w(i)$. This yields $\sum\limits_{i=1}^{t} \overline{m}^p_i \cdot w(i) > \sum\limits_{i=1}^{t} \overline{\alpha}^q_i \cdot w(i)$, and thus $\sum\limits_{i=1}^{t} (\overline{m}^p_i - \overline{\alpha}^q_i) \cdot w(i) > 0$.

2) Conversely, let us suppose that there exists a vector $w = (w(1), w(2), \ldots, w(t))$ such that, $w(i) = (w_1(i), 0, w_3(i))$ with $w_1(i) \geq 0 \geq w_3(i)$ $(i = 1, \ldots, t)$ which satisfies the system of inequalities: $\sum\limits_{i=1}^{t} (\overline{m}^p_i - \overline{\alpha}^q_i) \cdot w(i) > 0 \quad \forall p = 1, 2, \ldots, r$ and $\forall q = 1, 2, \ldots, s$.

Denote by $q = \min\limits_{p=1,\ldots,r} \sum\limits_{i=1}^{t} \overline{m}_i^p \cdot w(i)$. Let $S \in \mathcal{W}$ represented by the model $\overline{m}$.

Since $\sum\limits_{i=1}^{t} \overline{m}_i \cdot w(i) \geq \sum\limits_{i=1}^{t} \overline{m}_i^p \cdot w(i) \quad \forall p = 1, 2, \ldots, r$, we then have $w(S) = \sum\limits_{i=1}^{3} \sum\limits_{a \in S_i} w_i(a) = \sum\limits_{i=1}^{t} \overline{m}_i \cdot w(i) \geq q$ and hence $(N, \mathcal{W})$ is a weighted $(3, 2)$ game. Thus, $(N, \mathcal{W})$ is strongly weighted since it is well known from [9] that any weighted (3,2) game which is I-complete is strongly weighted. ■

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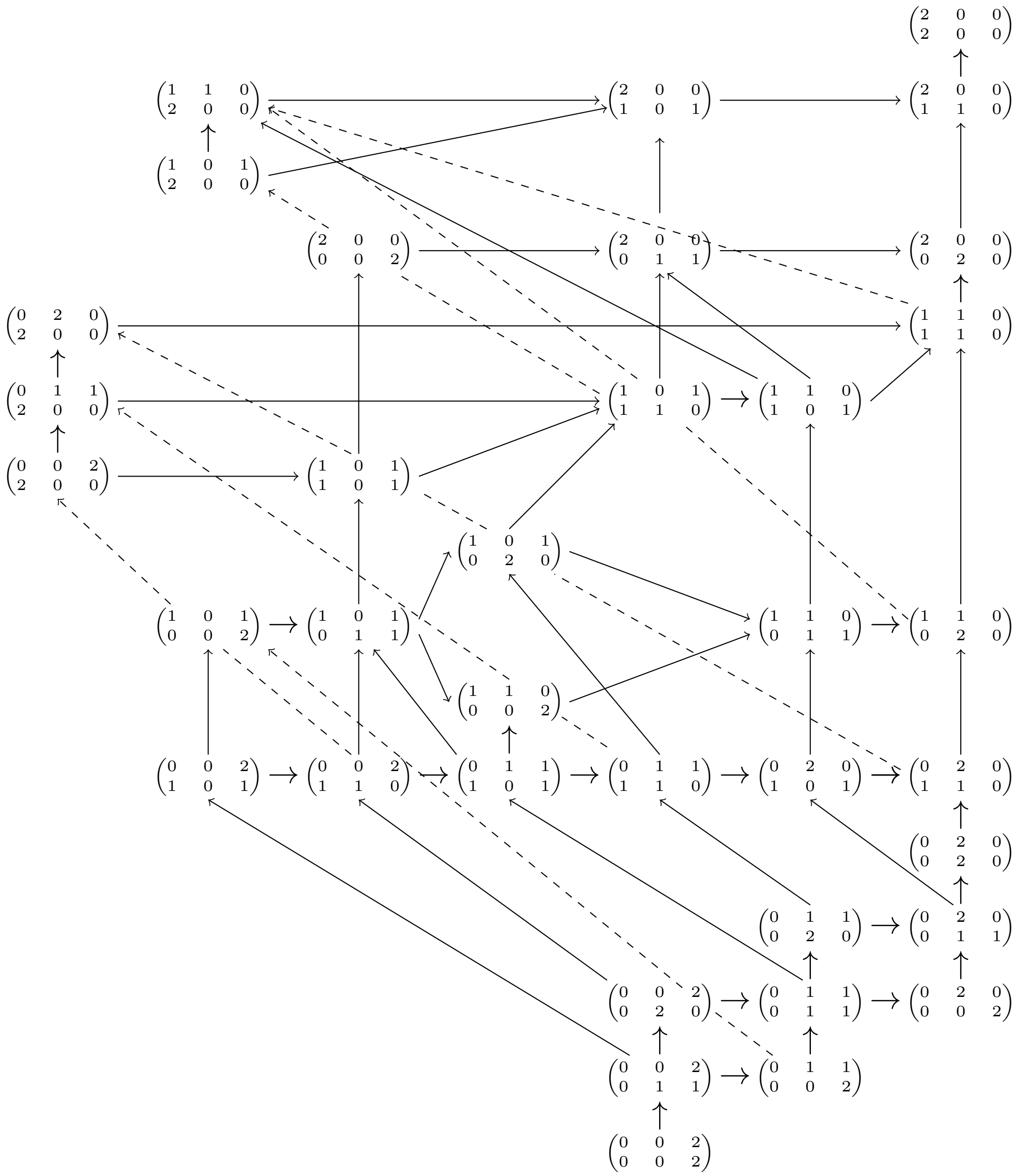

Figure 1: Multilattice associated with Example 2.7.

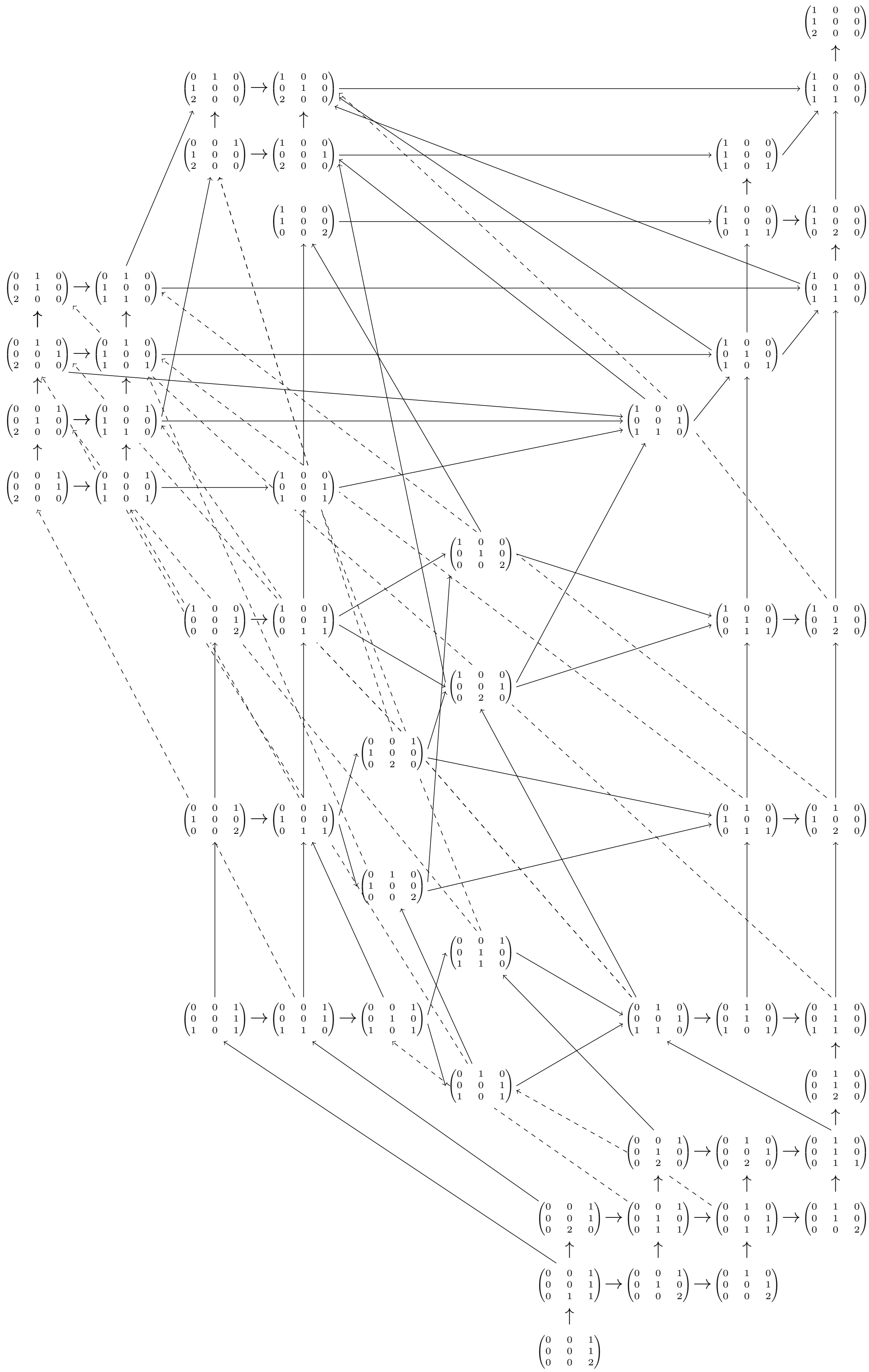

Figure 2: Multilattice associated with Example 3.9.