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Rainbow connectivity of Moore cages of girth 6

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ABSTRACT

Let G be an edge-colored graph. A path P of G is said to be *rainbow* if no two edges of P have the same color. An edge-coloring of G is a *rainbow t-coloring* if for any two distinct vertices u and v of G there are at least t internally vertex-disjoint rainbow (u, v)-paths. The *rainbow t-connectivity* $rc_t(G)$ of a graph G is the minimum integer j such that there exists a rainbow t-coloring using j colors. A $(k; g)$-*cage* is a k-regular graph of girth g and minimum number of vertices denoted $n(k; g)$. In this paper we focus on $g = 6$. It is known that $n(k; 6) \geq 2(k^2 - k + 1)$ and when $n(k; 6) = 2(k^2 - k + 1)$ the $(k; 6)$-cage is called a Moore cage. In this paper we prove that the rainbow k-connectivity of a Moore $(k; 6)$-cage G satisfies that $k \leq rc_k(G) \leq k^2 - k + 1$. It is also proved that the rainbow 3-connectivity of the Heawood graph is 6 or 7.

1. Introduction

All graphs considered in this work are finite, simple and undirected. We follow the book of Bondy and Murty [1] for terminology and notations not defined here. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path. For each vertex $v \in V(G)$ we use $N_G(v)$ and $d_G(v)$ to denote the set of neighbors and the degree of v in G. A graph G is called *k-regular* if each of its vertices has degree k. The *girth* $g(G)$ of G is the length of a shortest cycle in G.

An *edge-coloring* of a graph G is a function $\rho : E(G) \longrightarrow R$, where R is a set of distinct colors. Throughout this paper we only consider edge-colorings. Let G be an edge-colored graph. A path P in G is called *rainbow* if no two edges of P are colored the same. Chartrand, Johns, McKeon and Zhang [3] defined the rainbow connecting colorings. An edge-colored graph G is said to be *rainbow connected* if there exists a rainbow path between every two distinct vertices of G. Clearly, every connected graph G has an edge-coloring that makes it rainbow connected (simply color the edges of G with distinct colors). The *rainbow connection number* $rc(G)$ of a connected graph G is the minimum number of colors that are needed to make G rainbow connected.

Menger [14] proved that a graph G is t-connected if and only if there are at least t internally vertex-disjoint (u, v)-paths for every two distinct vertices u and v. Schiermeyer studied rainbow t-connected graphs with a minimum number of edges [15], and very recently the rainbow connectivity of certain products of graphs has been studied in [12]. Similar to rainbow connecting colorings, an edge-coloring is called a *rainbow t-coloring* if for every pair of distinct vertices u and v there are at least t internally disjoint rainbow (u, v)-paths. Clearly, coloring the edges of a t-connected graph G with as many colors as edges, every two vertices of G are connected by t internally vertex-disjoint rainbow paths. Thus, the rainbow t-connectivity $rc_t(G)$ (defined by Chartrand et al. [4]) of a graph G can be defined as the minimum integer j such that there

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exists a rainbow t-coloring using j colors. Moreover, $rc(G) = rc_1(G)$ and $rc_{t_1}(G) \leq rc_{t_2}(G)$ for $1 \leq t_1 \leq t_2$. The complexity of computing the $rc(G)$ has been studied in [7]. For 2-connected graphs it has been proved that $rc(G) \leq \lceil |V(G)|/2 \rceil$, see [8]. Also for t-connected graphs with $t \geq 5$ and girth $g(G) \geq 5$ it has been proved that $rc(G) < |V(G)|/t + 19$, see [8]. Some $rc_k(G)$ has been computed when G is a complete graph or a complete bipartite graph in [4]. For more references on rainbow connectivity and rainbow k-connectivity see [9] and the book by Li and Sun [11] or the survey by Li, Shi, and Sun [10].

Given two integers $k \geq 2$ and $g \geq 3$ a $(k; g)$-*cage* is a k-regular graph of girth g and minimum number of vertices, that is denoted by $n(k; g)$. For more information on cages see the survey on cages [5]. In this paper we focus on the case $g = 6$. It is known that $n(k; 6) \geq 2(k^2 - k + 1)$ and concerning the connectivity of any $(k; 6)$-cage, it has been proved that they are k-connected [13]. When $n(k; 6) = 2(k^2 - k + 1)$ the $(k; 6)$-cage is called a Moore $(k; 6)$-cage. It is known that the incidence graph of a projective plane of order $k - 1$ is a Moore $(k; 6)$-cage [5,6].

Definition 1.1. A *projective plane* $(\mathcal{P}, \mathcal{L})$ is a non-empty set $\mathcal{P}$ of points together with a set $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called lines, satisfying the following axioms:

GP1. For any two distinct points p and p', there exists a unique line ℓ connecting them.
GP2. For any two distinct lines ℓ and ℓ', there exists a unique point p in their intersection.
GP3. There exist at least four points such that no three of them are collinear.

From this definition it follows that each point $p \in \mathcal{P}$ belongs to $n+1$ lines and each $\ell \in \mathcal{L}$ line contains $n+1$ points yielding that $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$. Thus, the number n is said to be the *order* of the projective plane $(\mathcal{P}, \mathcal{L})$ which must be $n \geq 2$.

The *incidence graph* of a projective plane $(\mathcal{P}, \mathcal{L})$ of order n is a bipartite graph G with vertex set $\mathcal{P} \cup \mathcal{L}$. A vertex $p \in \mathcal{P}$ is adjacent to a vertex $\ell \in \mathcal{L}$ if and only if p is incident with ℓ in $(\mathcal{P}, \mathcal{L})$. Note that G is a Moore $(n + 1; 6)$-cage, because it is a regular graph of degree $n + 1$ with $2(n^2 + n + 1)$ vertices and girth 6. Moreover, the diameter of G is three. A Moore $(n + 1; 6)$-cage has been constructed for $n = q$ where q is a prime power. In Fig. 2 is depicted the $(3; 6)$-cage (Heawood graph), which is the incidence graph of the Fano plane.

Chartrand, Johns, McKeon and Zhang [2] showed that the rainbow 3-connectivity of the Petersen graph is 5, and the rainbow 3-connectivity of the Heawood graph is between 5 and 7 inclusive. In this paper we prove that if G is a Moore $(k; 6)$-cage, then $k \leq rc_k(G) \leq k^2 - k + 1$. It is also proved that the rainbow 3-connectivity of the Heawood graph is 6 or 7.

2. Bounds on the rainbow connectivity of cages

In this section we give a lower bound and an upper bound for the rainbow k-connectivity of a $(k; 6)$-Moore cage.

Theorem 2.1. *Let G be the incidence graph of a projective plane of order $n \geq 3$ and let $\rho : E(G) \to R$ be a coloring of G. If every path of G of length at most 3 is rainbow, then ρ is a rainbow $(n + 1)$-coloring.*

Proof. Let G be the incidence graph of a projective plane $(\mathcal{P}, \mathcal{L})$. Since the diameter of G is three, we distinguish three different cases according to the distance between two vertices in G.

Case 1. Let $a \in \mathcal{P}$ and $L \in \mathcal{L}$ be such that $d_G(a, L) = 3$. Then there is a geodesic (a, L_{ab}, b, L) in G which is rainbow by hypothesis. Let $L_a^{(i)}$, $i = 1, \ldots, n$, be the n lines adjacent to point a different from L_{ab}. Observe that $|N_G(L_a^{(i)}) \cap N_G(L)| = 1$ because G is the incidence graph of a projective plane and let $\{p^{(i)}\} = N_G(L_a^{(i)}) \cap N_G(L)$ for $i = 1, \ldots, n$. Note that $p^{(i)} \neq a, b$ and $p^{(i)} \neq p^{(j)}$ for $i \neq j$ because G has girth 6. The paths $\{(a, L_a^{(i)}, p^{(i)}, L) : 1 \leq i \leq n\}$ are n internally vertex-disjoint paths between a and L, and they are rainbow by hypothesis.

Case 2. Let $a, b \in \mathcal{P}$ be such that $d_G(a, b) = 2$ and let (a, L_{ab}, b) be the geodesic between a and b which is unique because the girth is 6, that is, $N_G(a) \cap N_G(b) = \{L_{ab}\}$. This geodesic is rainbow by hypothesis. Let $L_a^{(i)}$, $i = 1, \ldots, n$, be the n lines adjacent to a different from L_{ab}, and let $L_b^{(i)}$, $i = 1, \ldots, n$, be the n lines adjacent to b different from L_{ab}. Let $\{p^{(i)}\} = N_G(L_a^{(i)}) \cap N_G(L_b^{(i)})$, for $i = 1, \ldots, n$, and observe that $p^{(i)} \neq p^{(j)}$ for $i \neq j$ because G has girth 6. Denote the color of the edge $aL_a^{(i)}$ by $r_i = \rho(aL_a^{(i)})$, $i = 1, \ldots, n$, and note that $r_i \neq r_j$ for $i \neq j$ because by hypothesis paths of length 2 are rainbow. Analogously, denote by $r'_t = \rho(bL_b^{(t)})$, $t = 1, \ldots, n$, and observe that $r'_t \neq r'_h$ for $t \neq h$ by hypothesis. If there is no color in common among these sets of colors $\{r_i\}$, $\{r'_t\}$, then the n paths $\{(a, L_a^{(i)}, p^{(i)}, L) : 1 \leq i \leq n\}$ are n internally vertex-disjoint rainbow paths between a and b. If there are k colors in common, without loss of generality we may assume that $r_i = r'_i$ for $i = 1, \ldots, k$, with $k \leq n$, and $r_j \neq r'_t$ for $j, t = k + 1, \ldots, n$. Then the n paths $\{(a, L_a^{(i)}, u^{(i)}, L_b^{(i+1)}, b) : 1 \leq i \leq n\}$, where $\{u^{(i)}\} = N_G(L_a^{(i)}) \cap N_G(L_b^{(i+1)})$, $i = 1, \ldots, n$, and the sum of superindex is taken modulo n, are internally vertex-disjoint rainbow paths between a and b by hypothesis and because the girth of G is 6. The case when $L, L' \in \mathcal{L}$ such that $d_G(L, L') = 2$ is solved analogously by duality.

Case 3. Let $a \in \mathcal{P}$ and $A \in \mathcal{L}$ be such that $d_G(a, A) = 1$. Let $\{L^{(1)}, \ldots, L^{(n)}\} = N_G(a) - A$ and $\{a^{(1)}, \ldots, a^{(n)}\} = N_G(A) - a$. Moreover, let $\{M_1^{(i)}, \ldots, M_n^{(i)}\} = N_G(a^{(i)}) - A$ and let $\{b_1^{(i)}, \ldots, b_n^{(i)}\} = N_G(L^{(i)}) - a$ for $i = 1, 2, \ldots, n$. Since there exists a perfect matching between the sets $N_G(a^{(i)}) - A$ and $N_G(L_a^{(j)}) - a$, for all i, j, we may assume without loss of generality that $b_j^{(i)} M_j^{(i)} \in E(G)$. Let $r_1 = \rho(aL^{(1)})$ and $s_1 = \rho(Aa^{(1)})$.

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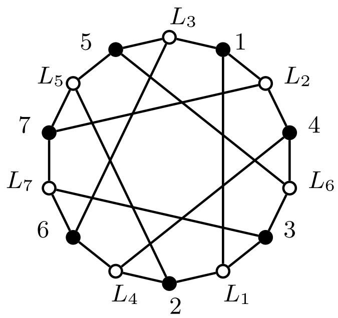

edge	color	edge	color
$1L_1$	3	$6L_4$	4
$2L_1$	1	$2L_5$	7
$3L_1$	2	$5L_5$	2
$1L_2$	4	$7L_5$	5
$4L_2$	7	$3L_6$	5
$7L_2$	1	$4L_6$	3
$1L_3$	6	$5L_6$	4
$5L_3$	1	$3L_7$	6
$6L_3$	5	$6L_7$	7
$2L_4$	6	$7L_7$	3
$4L_4$	2		

Fig. 1. Heawood graph with a σ-coloring.

First, suppose that $\rho(L^{(1)}b_1^{(1)}) = s_2 \neq s_1$. If $\rho(M_1^{(1)}a^{(1)}) = r_2 \neq r_1$, then $\rho(b_1^{(1)}M_1^{(1)}) \notin \{r_1, r_2, s_1, s_2\}$, because by hypothesis paths of length 3 are rainbow. Therefore the path

$$(a, L^{(1)}, b_1^{(1)}, M_1^{(1)}, a^{(1)}, A)$$

is rainbow. Then we have to suppose that $\rho(M_1^{(1)}a^{(1)}) = r_1$, which implies that $\rho(M_j^{(1)}a^{(1)}) = r_j \neq r_1$ for all $j \geq 2$ since paths of length 2 are rainbow by hypothesis. Since $n \geq 3$, we can take $j \in \{2, \ldots, n\}$ such that $\rho(L^{(1)}b_j^{(1)}) = s'_j \neq s_1$. Then $\rho(b_j^{(1)}M_j^{(1)}) \notin \{r_1, r_j, s_1, s'_j\}$, since paths of length 3 are rainbow by hypothesis, which implies that the path

$$(a, L^{(1)}, b_j^{(1)}, M_j^{(1)}, a^{(1)}, A)$$

is rainbow. Second, suppose that $\rho(L^{(1)}b_1^{(1)}) = s_1$. Then $\rho(L^{(1)}b_j^{(1)}) = s_j \neq s_1$ for all $j = 2, \ldots, n$. Since $n \geq 3$, we can take $j \in \{2, \ldots, n\}$ such that $\rho(M_j^{(1)}a^{(1)}) = r'_j \neq r_1$. Then $\rho(b_j^{(1)}M_j^{(1)}) \notin \{r_1, r'_j, s_1, s_j\}$, since paths of length 3 are rainbow by hypothesis, yielding that the path

$$(a, L^{(1)}, b_j^{(1)}, M_j^{(1)}, a^{(1)}, A)$$

is rainbow In either case we can find a rainbow path of length 5 between a and A through vertices $L^{(1)}$, $a^{(1)}$ and vertices in $N_G(L^{(1)}) - a$ and through vertices in $N_G(a^{(1)}) - A$. Repeating this process for each $i = 2, \ldots, n$, we find n internally vertex-disjoint rainbow paths between a and A which along with the edge aA give us $n + 1$ vertex-disjoint (a, A)-paths. ■

Definition 2.1. Let $(\mathcal{P}, \mathcal{L})$ be a projective plane and G the corresponding incidence graph. For all $L \in \mathcal{L}$ let $\sigma_L : L \to L$ be a permutation such that $\sigma_L(a) \neq a$ for every $a \in L$. For each edge aL of G, with $a \in \mathcal{P}$ and $L \in \mathcal{L}$, we color aL with the color $\sigma_L(a)$. This coloring over the edges of G is said to be a σ-coloring.

As an example of Definition 2.1, let us consider the following permutations of lines of Heawood graph defining a σ-coloring shown in Fig. 1.

$$\sigma_{L_1} = (132);\ \sigma_{L_2} = (147);\ \sigma_{L_3} = (165);\ \sigma_{L_4} = (264);\ \sigma_{L_5} = (273);\ \sigma_{L_6} = (354);\ \sigma_{L_7} = (367).$$

Lemma 2.1. *Let G be the incidence graph of a projective plane of order $n \geq 2$ with a σ-coloring. Then every path of length at most three of G is rainbow.*

Proof. If a path has length one, clearly it is rainbow. Let (a, L, b) be a path of length two of G. Since σ_L is a permutation of the points of L and $a, b \in L$ with $a \neq b$, then $\sigma_L(a) \neq \sigma_L(b)$. Let (L, a, L') be a path of length two of G. In this case $\{a\} = L \cap L'$, and $\sigma_L, \sigma_{L'}$ are permutations of the points of L and L', respectively. If $\sigma_L(a) = \sigma_{L'}(a) = \{p\}$, then $p \in L \cap L'$, that is $p = a$, which is a contradiction because $\sigma_L(a) \neq a$ and $\sigma_{L'}(a) \neq a$ according to Definition 2.1.

Let (a, L_{ab}, b, L_b) be a path of length three of G. Then $\sigma_{L_{ab}}(a) \neq \sigma_{L_{ab}}(b) \neq \sigma_{L_b}(b)$. If $\sigma_{L_{ab}}(a) = \sigma_{L_b}(b) = p$, then $p \in L_{ab} \cap L_b = \{b\}$, yielding that $p = b$, which is a contradiction because $\sigma_{L_b}(b) \neq b$ by Definition 2.1. ■

As an immediate consequence of Theorem 2.1 and Lemma 2.1 we can write the following result.

Theorem 2.2. *Let G be the incidence graph of a projective plane of order $n \geq 3$ with a σ-coloring. Then G is rainbow $(n + 1)$-connected and $rc_{(n+1)}(G) \leq n^2 + n + 1$.*

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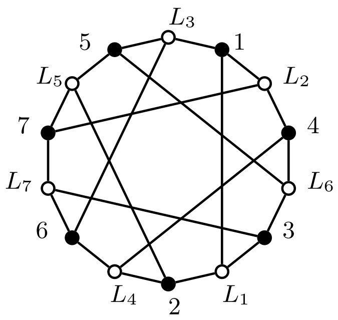

edge	color	edge	color
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$2L_1$	3	$2L_5$	7
$3L_1$	1	$5L_5$	2
$1L_2$	4	$7L_5$	5
$4L_2$	7	$3L_6$	4
$7L_2$	1	$4L_6$	5
$1L_3$	6	$5L_6$	3
$5L_3$	1	$3L_7$	6
$6L_3$	5	$6L_7$	7
$2L_4$	6	$7L_7$	3
$4L_4$	2		

Fig. 2. Heawood graph with a σ-coloring which is not 3-rainbow.

Remark 2.1. In Theorem 2.2, the hypothesis $n \geq 3$ is necessary as shown for the σ-coloring depicted in Fig. 2 of Heawood graph. We can check that this σ-coloring satisfies the hypothesis of Lemma 2.1, but between 1 and L_1 there are no 3 internally rainbow vertex-disjoint paths.

However, the σ-coloring of Heawood graph shown in Fig. 1 does work.

3. Rainbow 3-connectivity of Heawood graph

In the previous section we have described a rainbow 3-coloring of the Heawood graph of 7 colors. We prove that the rainbow 3-connectivity of Heawood graph is at least 6.

Lemma 3.1. *Let G be a k-regular and k-connected graph, and let ρ be a rainbow k-coloring of G. If e_1 and e_2 are two incident edges, then $\rho(e_1) \neq \rho(e_2)$.*

Proof. Suppose by contradiction that there are two incident edges $e_1 = uv$, $e_2 = vw$ of G such that $\rho(e_1) = \rho(e_2)$. Since G is rainbow k-connected there are k vertex disjoint rainbow paths between vertices u and w. Since $d(u) = d(w) = k$, it follows that among these k vertex disjoint rainbow paths there is one containing e_1 and that path cannot contain e_2, and there must be another path containing e_2 and this path cannot contain e_1. A contradiction, because these two paths are not vertex-disjoint. ■

Let ρ be a coloring of a graph G. A *chromatic class* $[r]$ is the set of edges of G with color r. By Lemma 3.1, the following corollary is immediate.

Corollary 3.1. *Let G be the incidence graph of a projective plane of order n and let ρ be a rainbow $(n+1)$-coloring of G. Then every chromatic class is independent.*

It is well known that the Heawood graph can be described as a bipartite graph with $V(G) = \mathbb{Z}_{14}$ and $E(G) = \{\{2i, 2i+1\}, \{2i, 2i-1\}, \{2i+1, 2i+6\} : i = 0, \ldots, 6\}$, see Fig. 3. In the rest of the paper we use this notation for the Heawood graph.

Lemma 3.2. *Let H be the Heawood graph, let $\rho : E(H) \to R$ be a rainbow 3-coloring of H with $|R| = 5$, and let $[r]$ be a chromatic class. The following assertions hold for $i \in \{0, \ldots, 6\}$:*

(i) *If $\{2i-1, 2i\}, \{2i-3, 2i-4\} \in [r]$, then $\{2i+1, 2i+2\}, \{2i+4, 2i+5\} \notin [r]$.*
(ii) *If $\{2i-1, 2i\}, \{2i-7, 2i-6\} \in [r]$, then $\{2i+1, 2i-8\}, \{2i+3, 2i+4\} \notin [r]$.*
(iii) *If $\{2i-1, 2i\}, \{2i+7, 2i+6\} \in [r]$, then $\{2i+5, 2i+4\}, \{2i-5, 2i-6\} \notin [r]$.*
(iv) *If $\{2i-1, 2i\}, \{2i+3, 2i-6\} \in [r]$, then $\{2i+1, 2i+2\}, \{2i-2, 2i+7\} \notin [r]$.*
(v) *If $\{2i-1, 2i\}, \{2i+3, 2i+2\} \in [r]$, then $\{2i-3, 2i-2\}, \{2i-6, 2i-5\} \notin [r]$.*

Proof. Note that if $d_H(a, b) = 2$ for $a, b \in V(H)$, then the shortest (a, b)-path is unique because the girth of H is 6. Let $N(a) = \{c, a', a''\}$ and $N(b) = \{c, b', b''\}$. Then a, c, b is the shortest path between a and b. Since ρ is a rainbow 3-coloring, it follows that between a and b there are another two vertex disjoint rainbow paths which must have even length at least 4 because H is bipartite. Moreover, since $|R| = 5$ these paths must have length exactly 4. If $aa', bb' \in [r]$, then there must be

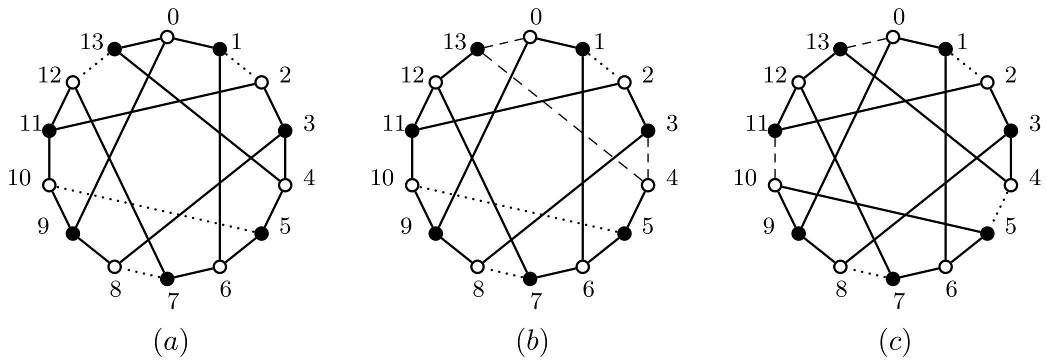

Fig. 3. The dotted edges belong to the class $[r]$, the black edges do not belong to $[r]$ and the dashed edges may belong to $[r]$.

unique paths of length 2 joining a' with b'' and b' with a'' without edges in $[r]$. To prove the lemma we use this fact and we only indicate the shortest path (a, c, b) in most of the cases.

(i) Suppose that $\{2i-1, 2i\}, \{2i-3, 2i-4\} \in [r]$. Let us consider the path of length two $(2i-4, 2i-5, 2i)$. One vertex disjoint rainbow path between $2i-4$ and $2i$ must join $2i-3$ with $2i+1 \in N(2i) \setminus \{2i-5, 2i-1\}$ since $\{2i-1, 2i\} \in [r]$, and having no edges in $[r]$. This path is $(2i-3, 2i+2, 2i+1)$ and $\{2i+1, 2i+2\} \notin [r]$. And the other vertex disjoint rainbow path must join $2i-1$ with $2i+5 \in N(2i-4) \setminus \{2i-5, 2i-3\}$ since $\{2i-4, 2i-3\} \in [r]$, and having no edges in $[r]$. This path is $(2i-1, 2i+4, 2i+5)$ and $\{2i+5, 2i+4\} \notin [r]$.

(ii) Suppose that $\{2i-1, 2i\}, \{2i-7, 2i-6\} \in [r]$. The result follows by considering the path $(2i-1, 2i-2, 2i-7)$.

(iii) Suppose that $\{2i-1, 2i\}, \{2i+7, 2i+6\} \in [r]$. The result follows by considering the path $(2i-1, 2i-2, 2i+7)$.

(iv) Suppose that $\{2i-1, 2i\}, \{2i+3, 2i+8\} \in [r]$. The result follows by considering the path $(2i-1, 2i+4, 2i+3)$.

(v) Suppose that $\{2i-1, 2i\}, \{2i+3, 2i+2\} \in [r]$. The result follows by considering the path $(2i, 2i+1, 2i+2)$. ■

Theorem 3.1. *Let H be the Heawood graph. Then $6 \le rc_3(H) \le 7$.*

Proof. Let $\rho : E(H) \to R$ be a rainbow 3-coloring on the edges of H. We reason by contradiction assuming that $rc_3(H) = |R| = 5$, which implies that there is a chromatic class $[r]$ with $|[r]| \ge 5$ because $|E(H)| = 21 = \sum |[r]|$. Let $[r]$ be such a chromatic class. Observe that a matching of at least 5 edges in Heawood graph always contains two edges at distance 2. Without loss of generality suppose that $\{1, 2\} \in [r]$. At distance two of $\{1, 2\}$ there are 8 edges which induce a cycle of length 8: $C = (7, 12, 13, 4, 5, 10, 9, 8, 7)$. Assume that $\{7, 8\} \in [r]$. Since $\{7, 8\}, \{1, 2\} \in [r]$, by item *(ii)* of Lemma 3.2 (taking $i = 4$), it follows that

$$\{9, 0\}, \{11, 12\} \notin [r]. \quad (1)$$

We consider the following cases according to the edges in $E(C) \cap [r]$.

Suppose that there are four edges in C with color r. In this case, the class $[r]$ must contain the edges $\{1, 2\}, \{7, 8\}, \{9, 10\}, \{5, 4\}$ and $\{13, 12\}$. Since $\{7, 8\}, \{5, 4\} \in [r]$, by item *(i)* of Lemma 3.2 (taking $i = 4$), it follows that $\{9, 10\}, \{13, 12\} \notin [r]$, a contradiction. Hence C contains at most 3 edges in $[r]$ including $\{7, 8\}$.

Suppose that $\{7, 8\}, \{13, 12\} \in E(C) \cap [r]$. Since $\{1, 2\}, \{13, 12\} \in [r]$ it follows that $\{3, 4\} \notin [r]$ by item *(i)* of Lemma 3.2 (taking $i = 1$). If $\{5, 10\} \in [r]$, by Lemma 3.1 and (1) there is no other edge belonging to $[r]$, see Fig. 3(a), and so $|[r]| = 4$, which is a contradiction. Hence, $\{5, 10\} \notin [r]$. If $\{5, 4\} \in [r]$, then taking into account that $\{7, 8\} \in [r]$, it follows by item *(i)* of Lemma 3.2 (taking $i = 4$) that $\{12, 13\} \notin [r]$, a contradiction. Thus, $\{5, 4\} \notin [r]$. If $\{9, 10\} \in [r]$, using that $\{13, 12\} \in [r]$, item *(v)* of Lemma 3.2 (taking $i = 5$) implies that $\{7, 8\} \notin [r]$ which is a contradiction; then $\{9, 10\} \notin [r]$. Furthermore, if $\{5, 6\} \in [r]$ using that $\{12, 13\} \in [r]$, item *(iii)* of Lemma 3.2 (taking $i = 3$) implies that $\{11, 10\} \notin [r]$ yielding that $|[r]| = 4$ which is a contradiction. Hence, if $\{7, 8\} \in [r]$, then $\{12, 13\} \notin [r]$. By symmetry, if $\{7, 8\} \in [r]$, then $\{9, 10\} \notin [r]$. Thus, if $[r]$ contains two edges of C these two edges must be at distance at least 2 in C.

Suppose that $\{7, 8\}, \{5, 10\} \in [r] \cap E(C)$. Observe that the only other edges that can be in $[r]$ are $\{3, 4\}, \{13, 0\}, \{13, 4\}$ (see Fig. 3(b)). By item *(iv)* of Lemma 3.2 (taking $i = 1$), $\{1, 2\}, \{5, 10\} \in [r]$ implies that $\{3, 4\} \notin [r]$, yielding that $|[r]| \le 4$, a contradiction. Thus, $\{5, 10\} \notin [r]$. By symmetry $\{4, 13\} \notin [r]$.

Suppose that $\{7, 8\}, \{5, 4\} \in [r] \cap E(C)$. At this point the only edges that can be in $[r]$ are $\{13, 0\}, \{10, 11\}$ (see Fig. 3(c)). By item *(v)* of Lemma 3.2 (taking $i = 1$), $\{1, 2\}, \{5, 4\} \in [r]$ implies that $\{10, 11\}, \{13, 0\} \notin [r]$, yielding that $|[r]| = 4$ which is a contradiction, Thus, we conclude that $[r] \cap E(C) = \{7, 8\}$.

Therefore, we have all the edges incident with $\{1, 2\}, \{7, 8\}$ (by Lemma 3.1) together with the edges of C minus $\{7, 8\}$, and $\{11, 12\}, \{9, 0\}$ (by (1)) do not belong to $[r]$. Hence, the edges that can be in $[r]$ are $\{3, 4\}, \{5, 6\}, \{10, 11\}$ and $\{13, 0\}$. Suppose $\{13, 0\} \in [r]$. Then $\{7, 8\}, \{13, 0\} \in [r]$ implies that $\{3, 4\} \notin [r]$ by item *(ii)* of Lemma 3.2, and $\{10, 11\}, \{13, 0\} \in [r]$

implies that $\{1, 2\} \notin [r]$ by item *(i)* of Lemma 3.2 which is a contradiction. Therefore, if $\{13, 0\} \in [r]$, $|[r]| = 4$ which is a contradiction. Hence, $\{13, 0\} \notin [r]$. By symmetry $\{10, 11\} \notin [r]$, yielding that $|[r]| \leq 4$ which is a contradiction.

Since in every case we obtain a contradiction we conclude that for each chromatic class $|[r]| \leq 4$ which implies that $|R| \geq 6$. ■

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