Generalized voterlike model on activity-driven networks with attractiveness

Antoine Moinet,[1,2] Alain Barrat,[1,3] and Romualdo Pastor-Satorras[2]
[1]*Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France*
[2]*Departament de Física, Universitat Politècnica de Catalunya, Campus Nord B4, 08034 Barcelona, Spain*
[3]*Data Science Laboratory, ISI Foundation, Torino, Italy*

(Received 23 April 2018; published 6 August 2018)

We study the behavior of a generalized consensus dynamics on a temporal network of interactions, the activity-driven network with attractiveness. In this temporal network model, agents are endowed with an intrinsic activity a, ruling the rate at which they generate connections, and an intrinsic attractiveness b, modulating the rate at which they receive connections. The consensus dynamics considered is a mixed voter and Moran dynamics. Each agent, either in state 0 or 1, modifies his or her state when connecting with a peer. Thus, an active agent copies his or her state from the peer (with probability p) or imposes his or her state to him or her (with the complementary probability $1-p$). Applying a heterogeneous mean-field approach, we derive a differential equation for the average density of voters with activity a and attractiveness b in state 1, which we use to evaluate the average time to reach consensus and the exit probability, defined as the probability that a single agent with activity a and attractiveness b eventually imposes his or her state to a pool of initially unanimous population in the opposite state. We study a number of particular cases, finding an excellent agreement with numerical simulations of the model. Interestingly, we observe a symmetry between voter and Moran dynamics in pure activity-driven networks and their static integrated counterparts that exemplifies the strong differences that a time-varying network can impose on dynamical processes.

DOI: 10.1103/PhysRevE.98.022303

I. INTRODUCTION

A wide variety of complex physical systems are concerned with the problem of an initially disordered configuration that is able to achieve an ordered state by means of local pairwise dynamical interactions. Examples of such systems range from the formation of an opinion consensus in social systems [1–3] to the loss of genetic diversity in evolutionary dynamics [4]. These situations, implying a competition between different alternative states diffusing among the agents, have been modeled with stochastic copying or invasion processes. In these models, each individual is endowed with a state variable and copies or imposes his or her state from or to neighboring sites until one single state finally dominates the whole system. Among those different frameworks, the voter model [5] was introduced to schematically model the opinion spreading in human populations and has become emblematic for its simplicity and analytical tractability. In this model, the agents possess one of two discrete opinions, and at each time step an individual is chosen and adopts the opinion of a randomly chosen neighbor. On the other hand, in the context of evolutionary dynamics, the Moran process [6] considers a population of individuals belonging to different species, that reproduce generating an offspring that replaces a randomly chosen nearest neighbor. The voter and Moran models differ thus in the direction in which the state is transferred between pairs of interacting agents.

Recently, it has been acknowledged that the topology in which such ordering dynamics takes place in real systems is often far from homogeneous, and better represented in terms of a complex network [7,8], in which agents are characterized by a number of neighbors (degree) k that is broadly distributed, with a heterogeneous probability distribution $P(k)$ (degree distribution) often schematically described by a power-law-like form $P(k) \sim k^{-\gamma}$ [9]. This observation has led to an intense research activity in order to unveil the different properties of ordering dynamics in heterogeneous topologies [7,10–15], yielding a good understanding of the problem both at the numerical and analytical levels.

These studies have mainly focused on the case of static networks, in which nodes representing agents are connected by a set of edges, standing for pairwise possible interactions, that are fixed in time and never change. However, many networks, and in particular social ones, are dynamic in nature, given by a pattern of connections that evolves in time. Such temporal networks [16,17] have been the subject of an intense research activity, considering in particular their possible impact on the behavior of dynamical processes running on top of them [17–22]. Despite their relevance, however, consensus dynamics have seldom been studied in detail in temporal topologies [23–25].

Here we contribute to fill this gap by presenting a detailed study of the voter and Moran processes on temporal networks, focusing on a generalization of the recently introduced activity-driven temporal network model [26]. In this model [27] agents are assigned an activity parameter a that determines their propensity to establish social interactions with other individuals, and another variable b defining their attractiveness, which in turn determines the probability that they are chosen by an active agent to interact. We provide a full analysis of basic ordering dynamics through a heterogeneous mean-field approach that allows us to describe the dynamics of the process in the limit of a large system size, and in particular to compute the average time to reach consensus starting from a random configuration. Another quantity of interest when studying consensus or invasion processes is the so-called exit probability [28], defined as the probability that a single agent having a discrepant opinion among an unanimous population manages to spread his or her opinion to the whole population. In our work this quantity plays a significant role as it highlights an interesting symmetry between the voter model and the Moran process when comparing the unfolding of these processes either on static heterogeneous networks or on and the activity-driven network. In particular, we show how, depending on the type of dynamics, the effect of a node's characteristics on the dynamics can be similar or drastically different when the dynamics runs on a temporal network or on the corresponding static aggregated network.

The paper is organized as follows. In Sec. II we define the variation of the voter model we consider, and the class of activity-driven network with attractiveness we use as a substrate for the dynamics. In Sec. III we provide a full description of the dynamics and in particular we exhibit analytical expressions for the exit probability and the average consensus time. In Sec. IV we give more insights for a few particular forms of the joint distribution of the activity and attractiveness $\eta(a,b)$. Section V presents numerical simulations showing agreement with our analytical expressions, which are further studied in some asymptotic limit of the activity distribution in Sec. VI. Finally, we conclude our work discussing our results and exploring perspectives in Sec. VII.

II. CONSENSUS DYNAMICS IN GENERALIZED ACTIVITY-DRIVEN NETWORKS

A. Activity-driven networks with attractiveness

We focus on the class of activity-driven temporal networks [26,27,29], which are based in the key ingredient that the formation of social interactions is driven by some innate activity of individuals, which determines their tendency to establish interactions and is empirically observed to be heterogeneously distributed [26]. In this paradigmatic model, a fixed population of N agents is considered, each endowed with an activity a, representing the rate (probability per unit time) at which she or he becomes active and draws edges (interactions) towards other agents. In the original formulation of the model [26], the chosen agents are selected uniformly at random among all peers. In order to make the model more realistic, one can assign to each agent a parameter b, called attractiveness, such that his or her probability of being selected by an active peer is proportional to b [27]. The activity a and attractiveness b are extracted at random for each agent from a given joint distribution $\eta(a,b)$.

B. Consensus dynamics

Coupling a dynamical process with a temporal network always entails the problem of how to deal with the different time scales inherent in the process and in the evolution of the network. Here we consider the simplest case of a single time scale, imposed by the network evolution. In this way, the state of an agent can only change when she or he interacts with another agent, and is constant in the latency times between interactions. The consensus dynamics are thus defined as follows:

(i) We start from an initial configuration of states $s_i \in \{0,1\}$, assigned to each agent.

(ii) In an interval of time δt, an agent i, in state s_i, becomes active with probability $a_i \delta t$, and chooses as peer another agent j (in state s_j) with probability $\frac{b_j}{\sum_\ell b_\ell}$.

(iii) The states s_i and s_j are updated according to the chosen consensus dynamics.

(iv) Time is updated $t \to t + \delta t$.

We consider three different variations of social dynamics, based on the update dynamics of the state variables. Assuming that, at given time t, agent i becomes active, and chooses agent j to start an interaction, we consider the three different updates:

(i) voter dynamics: $s_i := s_j$ (i.e., i adopts j's state);

(ii) Moran dynamics: $s_j := s_i$;

(iii) mixed dynamics: with probability p, $s_i := s_j$; with the complementary probability $1-p$, $s_j := s_i$.

In what follows, we consider the mixed update rule, as the voter model and the Moran process are particular cases of the latter obtained by setting $p=1$ or $p=0$, respectively.

III. HETEROGENEOUS MEAN-FIELD ANALYSIS

When agent activation is ruled by a Poisson process, it is possible to tackle the behavior of voterlike dynamics by extending the heterogeneous mean-field [7,30] approach developed in Refs. [10,31] to study voter dynamics on static networks. This method is based in a coarse graining of the network, considering that the state of an agent with activity a and attractiveness b depends exclusively on those two quantities. In this way, one considers a fundamental description in terms of the fraction $\rho_{a,b}(t)$ of agents with activity strength a and attractiveness b in the state 1 at time t; in other words, $\rho_{a,b}(t)$ is the probability that a randomly chosen agent with activity a and attractiveness b is in state 1 at time t. The corresponding fraction of agents in state 0 is given by the complementary probability $1-\rho_{a,b}(t)$. The total fraction of agents in state 1, $\rho(t)$, is given by

$$\rho(t) = \sum_{a,b} \eta(a,b)\rho_{a,b}(t). \quad (1)$$

To alleviate notation, we denote the pair (a,b) by the symbol h, writing thus $\rho_{a,b}(t) \equiv \rho_h(t)$.

The relevant functions defining the dynamics are the transition probabilities R_h and L_h for, respectively, increasing and decreasing the number of voters in state 1, among the pool of agents with activity strength a and attractiveness b, in a time interval δt. From these transition probabilities, a differential equation ruling the evolution of $\rho_h(t)$ can be derived, as well as information about the exit probability and the average ordering time. In the dynamical rules described in the previous section, agents activate independently so that *a priori* multiple activations are possible during a single time step. However, the use of the transition probabilities R_h and L_h relies on the implicit hypothesis that only one flip attempt may occur during a single time step, thus in order to ensure the validity of our analysis, we impose that $\langle a\rangle N\delta t \ll 1$ so that the probability of counting more than one activation during δt is almost zero. This is of course always possible as the time step δt is arbitrary. Let us now derive the time-evolution equation of the fraction $\rho_h(t)$ of agents with activity strength a and attractiveness b in state 1 at time t.

A. Evolution equation

In a single time step, the number of agents with activity strength a and attractiveness b in state 1 may either increase by one unit with probability R_h, decrease by one unit with probability L_h, or stay unchanged with probability $1 - R_h - L_h$. Thus on average the variation $\delta\rho_h$ reads

$$\delta\rho_h = (+1) \times \frac{R_h}{N_h} + (-1) \times \frac{L_h}{N_h} + 0 \times \frac{1 - R_h - L_h}{N_h}, \tag{2}$$

where N_h is the number of agents in the state (a, b). In the continuous time limit (for $\delta t \ll 1$) we may write

$$\frac{\partial\rho_h(t)}{\partial t} = \frac{R_h - L_h}{N_h\,\delta t}. \tag{3}$$

We consider the mixed process in which every agent, when activated, might either copy the state of his or her peer with probability p, or impose his or her own state to him with probability $1 - p$. The transition probabilities are thus given by

$$R_h = p\,R_h^V + (1-p)R_h^M \tag{4}$$

$$L_h = p\,L_h^V + (1-p)L_h^V, \tag{5}$$

where the rates L_h^X and R_h^X refer to the voter ($X = V$) and Moran ($X = M$) dynamics, respectively.

In the case of the voter dynamics, these transition probabilities take the form

$$R_h^V = N_h\,a\,\delta t\,(1-\rho_h)\frac{\langle b\,\rho_h\rangle}{\langle b\rangle}, \tag{6}$$

$$L_h^V = N_h\,a\,\delta t\,\rho_h\left(1 - \frac{\langle b\,\rho_h\rangle}{\langle b\rangle}\right). \tag{7}$$

The origin of these expressions is easy to see. For example, in Eq. (6), the probability that the number of agents in state 1, activity a and attractiveness b increases by one unit is proportional to the number of agents in this class in state 0, $N_h[1-\rho_h(t)]$, times the probability that any one of them becomes active in a time interval δt ($a\delta t$), times the probability that an active agent generates a link to an agent in state 1, thus copying the state of this last agent. The latter is the sum over all the agents i of the probability that i is chosen and is in state 1, i.e., $\sum_i \frac{b_i}{\langle b\rangle N} s_i = \frac{\langle b\,\rho_h\rangle}{\langle b\rangle}$ The transition probability L_h^V can be obtained by an analogous reasoning.

In the case of the Moran process, instead, the probability that in a time step the state of node i is flipped from 1 to 0 is

$$P_i(1 \to 0) = \sum_j \frac{s_i\,b_i(1-s_j)}{\langle b\rangle(N-1)} a_j\delta t = s_i\,\frac{b_i}{\langle b\rangle}\delta t(\langle a\rangle - \langle a\rho_h\rangle). \tag{8}$$

Indeed, the probability that the agent i is flipped from 1 to 0 while interacting with j is equal to the probability $a_j\delta t(1-s_j)$ that j becomes active and is in state 0, times the probability $\frac{s_i\,b_i}{\langle b\rangle(N-1)}$ that i is chosen among all the other agents and is in state 1. We then sum over all the agents j to obtain the total probability. Then, summing over all nodes i with activity a and attractiveness b we get

$$L_h^M = N_h\,\delta t\,\rho_h\,(\langle a\rangle - \langle a\,\rho_h\rangle)\frac{b}{\langle b\rangle}\,. \tag{9}$$

We obtain in a similar fashion

$$R_h^M = N_h\,\delta t\,\langle a\,\rho_h\rangle\,(1-\rho_h)\frac{b}{\langle b\rangle}. \tag{10}$$

From these two particular cases we deduce the time-evolution equation of the fraction of nodes with activity a and attractiveness b in state 1 in the general mixed case, which is given by

$$\frac{\partial\rho_h(t)}{\partial t} = pa\left(\frac{\langle b\,\rho_h\rangle}{\langle b\rangle} - \rho_h\right) + (1-p)\langle a\rangle\frac{b}{\langle b\rangle}\left(\frac{\langle a\,\rho_h\rangle}{\langle a\rangle} - \rho_h\right). \tag{11}$$

B. Conservation law

In the case of the voter model on a complete static graph, the total fraction ρ of voters in state 1 is conserved by the dynamics. In our model, it is clear from the previous equation that this in not in general true. Nevertheless, we may look for a conserved quantity of the form

$$\Omega = \sum_h \lambda_h\,\rho_h, \tag{12}$$

where the weights λ_h are normalized as $\sum_h \lambda_h = 1$. Using Eq. (11), we can check that the condition $\partial\Omega/\partial t = 0$ is fulfilled if the functions λ_h satisfy the self-consistent equation

$$\lambda_h = \eta(h)\,\frac{p\,b\left[\sum_{h'} a'\lambda_{h'}\right] + (1-p)\,a\left[\sum_{h'} b'\lambda_{h'}\right]}{pa\langle b\rangle + (1-p)\langle a\rangle b}, \tag{13}$$

where $\sum_h a\lambda_h$ and $\sum_h b\lambda_h$ are determined by the normalization of the weights λ_h (see details in Appendix):

$$\sum_h a\lambda_h = \frac{1}{Q_p}\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle, \tag{14}$$

$$\sum_h b\lambda_h = \frac{\langle a\rangle}{Q_p\,\langle b\rangle}\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle, \tag{15}$$

where we have defined

$$\Delta_{h,p} = pa\langle b\rangle + (1-p)\langle a\rangle b \tag{16}$$

and

$$Q_p = p\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle\left\langle\frac{b}{\Delta_{h,p}}\right\rangle + (1-p)\frac{\langle a\rangle}{\langle b\rangle}\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\left\langle\frac{a}{\Delta_{h,p}}\right\rangle. \tag{17}$$

Notice that this last quantity depends only on p.

C. Exit probability

As in the case of the standard voter model [32], the presence of a conservation law allows us to estimate directly the exit probability E for a single agent with state 1 in a population of agents with state 0, i.e., the probability that all agents finally adopt the state 1. Indeed, the final state with all voters in state 1, corresponding to $\Omega = 1$, takes place with probability E (by definition), while the final state with all voters in state 0, with $\Omega = 0$, happens with probability $1 - E$. The conservation of Ω implies that $\Omega(t = 0) = E \times 1 + (1 - E) \times 0$, from where we immediately obtain

$$E = \sum_h \lambda_h \, \rho_h(0), \tag{18}$$

which depends exclusively on the initial state in which the system is prepared. For the particular initial conditions consisting of a single voter with variables $h = (a, b)$, i.e., activity a and attractiveness b, in state 1 in a background of voters in state 0, we have that $\rho_{h'}(0) = \delta_{h',h} \, N_h^{-1}$, which leads to an exit probability

$$E_{a,b} = \frac{\lambda(a, b)}{N\eta(a, b)}, \tag{19}$$

which, using Eq. (13) can be more explicitly expressed as

$$E_{a,b} = \frac{1}{NQ_p} \frac{pb\left\langle \frac{a^2}{\Delta_{h,p}}\right\rangle + (1-p)a\frac{\langle a\rangle}{\langle b\rangle}\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle}{pa\langle b\rangle + (1-p)b\langle a\rangle}. \tag{20}$$

Interestingly, this exit probability is a function of the ratio $\frac{a}{b}$ only.

D. Average consensus time

In order to compute the consensus time we can follow [10,31] and apply a one-step calculation to write down the recursion relation for the time $T[\{\rho_h\}]$ to reach consensus starting from a configuration $\{\rho_h\}$:

$$\begin{aligned} T[\{\rho_h\}] &= \delta t + \left(1 - \sum_h (R_h + L_h)\right) T[\{\rho_h\}] \\ &\quad + \sum_h (R_h T[\{\rho_{h'}, \rho_h + 1/N_h\}] \\ &\quad + L_h T[\{\rho_{h'}, \rho_h - 1/N_h\}]), \end{aligned} \tag{21}$$

where the notation $\{\rho_{h'}, \rho_h \pm 1/N_h\}$ denotes a modification of the configuration $\{\rho_h\}$ by the flip of one agent of variables h (either from state 0 to the state 1, for the +case, or vice versa for the − case).

This equation essentially amounts to consider that the consensus time for a given configuration is equal to the consensus time at the configuration obtained after a transition taking place in a time δt, weighted by the corresponding transition probabilities, plus δt. Expanding Eq. (21) at second order in $1/N_h$ we obtain the backward Kolmogorov equation [33]

$$\sum_h v_h \frac{\partial T}{\partial \rho_h} + \sum_h D_h \frac{\partial^2 T}{\partial \rho_h^2} = -1, \tag{22}$$

where

$$v_h = pa\left(\frac{\langle b\,\rho_h\rangle}{\langle b\rangle} - \rho_h\right) + (1-p)\langle a\rangle \frac{b}{\langle b\rangle}\left(\frac{\langle a\,\rho_h\rangle}{\langle a\rangle} - \rho_h\right) \tag{23}$$

and

$$\begin{aligned} D_h &= \frac{pa}{2N_h}\left(\frac{\langle b\,\rho_h\rangle}{\langle b\rangle} + \rho_h - 2\,\frac{\langle b\,\rho_h\rangle}{\langle b\rangle}\rho_h\right) \\ &\quad + \frac{(1-p)\langle a\rangle b}{2\langle b\rangle N_h}\left(\frac{\langle a\,\rho_h\rangle}{\langle a\rangle} + \rho_h - 2\,\frac{\langle a\,\rho_h\rangle}{\langle a\rangle}\rho_h\right) \end{aligned} \tag{24}$$

are the drift and diffusion coefficients, respectively [33]. After a transient time depending on the distribution $\eta(a, b)$, the system reaches a steady state where $\rho_h = \Omega$, $\forall h$. Then we may drop the drift term in Eq. (22), and, considering Eq. (12), change variable from ρ_h to Ω [10,31]

$$\frac{\partial T}{\partial \rho_h} = \lambda_h \frac{\partial T}{\partial \Omega}. \tag{25}$$

Substituting into Eq. (22) and simplifying (see details in the Appendix), we finally obtain

$$\Omega(1 - \Omega)\frac{\partial^2 T}{\partial \Omega^2} = \frac{-N\langle b\rangle}{\left(\sum_h a\lambda_h\right)\left(\sum_h b\lambda_h\right)}. \tag{26}$$

This last equation can be directly integrated, yielding the consensus time

$$T = \tau\frac{N}{\langle a\rangle}\left((1-\Omega)\ln\frac{1}{1-\Omega} + \Omega\ln\frac{1}{\Omega}\right), \tag{27}$$

where we defined the characteristic adimensional consensus time

$$\tau = \frac{\langle a\rangle\langle b\rangle}{\left(\sum_h a\lambda_h\right)\left(\sum_h b\lambda_h\right)}. \tag{28}$$

The model is then entirely solved in terms of the previous expressions for the consensus time and the exit probability. These expressions are, however, quite intricate and it is quite insightful to study particular cases of interest, for given forms of the distribution of the activity and attractiveness $\eta(a, b)$ and particular values of the mixing probability p. We present this analysis in the following section.

IV. PARTICULAR CASES

A. $p = 1/2$

From the definition of $\Delta_{h,p}$ in Eq. (16), one obtains, by multiplying this equation, respectively, by $a/\Delta_{h,p}$ and $b/\Delta_{h,p}$ and averaging,

$$\langle b\rangle = p\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle\langle b\rangle + (1-p)\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\langle a\rangle, \tag{29}$$

$$\langle a\rangle = p\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle\langle b\rangle + (1-p)\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle\langle a\rangle. \tag{30}$$

Thus for $p = 1/2$, one obtains, eliminating $\langle ab/\Delta_{h,p}\rangle$ between these two equations,

$$\frac{\langle a\rangle}{\langle b\rangle}\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle = \frac{\langle b\rangle}{\langle a\rangle}\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle, \tag{31}$$

and Eq. (17) becomes

$$\begin{aligned} Q_{1/2} &= \frac{1}{2}\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\frac{\langle a\rangle}{\langle b\rangle}\left(\left\langle\frac{a}{\Delta_{h,p}}\right\rangle + \left\langle\frac{b}{\Delta_{h,p}}\right\rangle\frac{\langle a\rangle}{\langle b\rangle}\right) \\ &= \left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\frac{\langle a\rangle}{\langle b\rangle^2} = \frac{\langle 1\rangle}{\langle a\rangle}\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle. \end{aligned} \tag{32}$$

From here, it follows that $\sum_h a\,\lambda_h = \langle a \rangle$ and $\sum_h b\,\lambda_h = \langle b \rangle$, which finally implies that $\lambda_h = \eta_h$ and $\tau = 1$.

In this case, the dynamics becomes identical to the standard link update dynamics of the voter model [34], and it is totally independent on a and b (in terms of the number of flip attempts) because the probability that the total number of voters in state 1 is increased during an update attempt is exactly compensated by the probability that this same number is decreased.

B. Pure voter model

The voter model, corresponding to $p = 1$, leads in Eq. (13) to

$$\lambda(a, b) = \eta(a, b) \frac{b\,a^{-1}}{\langle b\,a^{-1} \rangle}\ . \tag{33}$$

We also obtain $\Delta_{h,p} = a\langle b \rangle$ and $Q_1 = \langle a \rangle \langle b a^{-1} \rangle / \langle b \rangle^2$, leading for the consensus time in Eq. (28) to the simple form:

$$\tau = \langle a \rangle \frac{\langle b\,a^{-1} \rangle^2}{\langle b^2 a^{-1} \rangle}. \tag{34}$$

The exit probability is also straightforward to derive from Eq. (19):

$$E_{a,b} = \frac{b\,a^{-1}}{N \langle b\,a^{-1} \rangle}. \tag{35}$$

C. Moran process

The Moran process corresponds to $p = 0$, then Eq. (13) reduces to

$$\lambda(a, b) = \eta(a, b) \frac{a\,b^{-1}}{\langle a\,b^{-1} \rangle}, \tag{36}$$

leading, with $\Delta_{h,p} = b\langle a \rangle$ and $Q_0 = \langle a b^{-1} \rangle / \langle a \rangle$, from Eq. (28) to

$$\tau = \langle b \rangle \frac{\langle a\,b^{-1} \rangle^2}{\langle a^2 b^{-1} \rangle}. \tag{37}$$

The exit probability reads in this case

$$E_{a,b} = \frac{a\,b^{-1}}{N \langle a\,b^{-1} \rangle}. \tag{38}$$

It is noteworthy that the results for the Moran process are obtained from the ones of the voter model by simply exchanging a and b. In fact, we see from Eq. (13) that the dynamics of the mixed process is the same as the dynamics of the symmetrical process (i.e., with $p \leftarrow 1 - p$) upon exchanging a and b. This is intuitively clear if we examine the process from a stochastic point of view: at each update attempt, the node i is chosen at random with probability $\frac{a_i}{\langle a \rangle}$ and the node j with probability $\frac{b_j}{\langle b \rangle}$. Besides, changing p into $1 - p$ is equivalent to reversing the roles of i and j, which has no effect if a and b are exchanged. This is, however, valid only when the time is counted as the number of update attempts, the physical time being multiplied by $\frac{\langle a \rangle}{\langle b \rangle}$ when swapping a and b.

D. Pure activity-driven networks

The original activity driven network model [26] does not consider a heterogeneous attractiveness, and this corresponds to a joint distribution $\eta(a, b) = F(a)\delta_{b,b_0}$, where $F(a)$ is the activity distribution and $b = b_0$, constant. In this case we have $Q_p = \frac{\langle a \rangle}{b_0} \langle [pa + (1 - p)\langle a \rangle]^{-1} \rangle$, and the characteristic consensus time reads

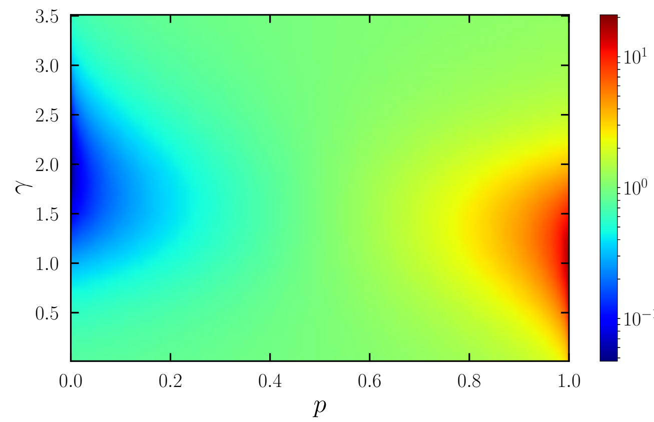

FIG. 1. Characteristic consensus time τ for the dynamics on a pure activity-driven network, i.e., fixed attractivity $b = b_0$, and a distribution of activities $F(a)$ given by Eq. (41), as a function of p and of the exponent γ of the distribution F. $\epsilon = 10^{-3}$.

$$\tau = \frac{\langle a \rangle^2 \langle [pa + (1 - p)\langle a \rangle]^{-1} \rangle}{\langle a^2 [pa + (1 - p)\langle a \rangle]^{-1} \rangle}, \tag{39}$$

while the exit probability is given by

$$E_a = \frac{1}{N} \frac{p\frac{\langle a \rangle}{\tau} + (1 - p)a}{pa + (1 - p)\langle a \rangle}. \tag{40}$$

In order to study the behavior of the consensus time, in Fig. 1 we plot the analytical evaluation of τ, Eq. (39), for a normalized activity distribution with a power-law form, as empirically observed in Ref. [26],

$$F(a) = \frac{1 - \gamma}{1 - \epsilon^{1-\gamma}} a^{-\gamma}, \quad a \in [\epsilon, 1]. \tag{41}$$

where ϵ is the minimum activity in the system, imposed in order to avoid divergences in the normalization and moments of $F(a)$. From Fig. 1 we see that the consensus time has a minimum around $\gamma = 2$ for the Moran process ($p = 0$) and a maximum around $\gamma = 1$ for the voter model ($p = 1$). Note that by virtue of the symmetry property discussed above, the dynamics of the pure attractiveness model (setting $a_i = a_0, \forall i$), taking the same distribution F for b and imposing $a_0 = b_0$, is the same upon exchanging p by $1 - p$. In particular, the consensus time is obtained by reversing the p axis in Fig. 1.

E. Independent activity and attractiveness

In the case where a and b are drawn independently from the same distribution F, we have $\eta(a, b) = F(a)F(b)$. In Fig. 2 we plot the characteristic consensus time τ as a function of p and γ for F given by Eq. (41). For this particular form of the distribution $\eta(a, b)$ [and in general for any symmetric joint distribution such that $\eta(a, b) = \eta(b, a)$], the dynamics remains the same when changing p into $1 - p$ because exchanging

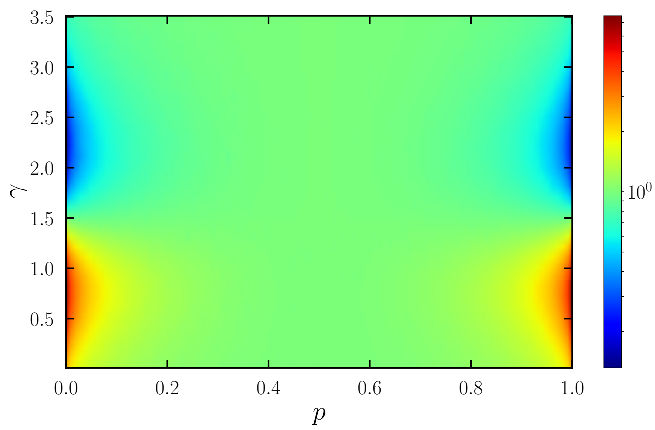

FIG. 2. Characteristic consensus time τ as a function of γ and p in the case $\eta(a,b) = F(a)F(b)$, with F given by Eq. (41) and $\epsilon = 10^{-3}$.

a and b has no effect. This is clearly observed in Fig. 2. Additionally, we see that both the voter model and the Moran process have a minimum consensus time for $\gamma \simeq 2.25$ and a maximum consensus time for $\gamma = 0.75$, respectively.

F. Strongly correlated activity and attractiveness

As we previously mentioned, the weight function λ_h is the product of η_h and a function of the ratio $\frac{a}{b}$. This fact straightforwardly implies that, in the maximally correlated case $\eta(a,b) = F(a)\delta_{a,b}$, where $a = b$ for every agent, the dynamics is the same as in a fully connected static network, i.e., the average density $\rho = \langle \rho_h \rangle$ of voters in state 1 is conserved, the reduced consensus time τ is equal to 1, and the exit probability is homogeneous and equals $1/N$.

G. Discussion

The results obtained above relate the average consensus time and the exit probability with the moments of the joint distribution $\eta(a,b)$ and the value of the activity a and the attractiveness b of the initial invading voter. Remarkably, when we compare these results with the ones obtained in the case of static networks with a given degree distribution $P(k)$ [10,31] we observe interesting symmetries between voter and Moran dynamics.

This symmetry is of particular interest when we consider the pure activity-driven network (setting $b = 1$). Let us consider, for instance, the invasion exit probability, Eqs. (35) and (38). In the case of the voter model, this exit probability is inversely proportional to the activity of the node with initial state 1 ($E_a^{\text{voter}} \propto 1/a$), while for the Moran process, it is proportional to the activity ($E_a^{\text{Moran}} \propto a$). This can be understood by the fact that an active node will often change state in the voter model, by contacting other nodes, while in the Moran process it will often spread his or her state towards the other nodes contacted.

These results have to be compared with the result for the voter and Moran processes in static networks, in which the exit probability for a single node of degree k with state 1 is $E_k^{\text{voter}} \sim k$ for the voter and $E_k^{\text{Moran}} \sim k^{-1}$ for the Moran process [10,31]. Intuitively indeed, in the case of static networks, in the voter model high degree nodes are chosen to be copied with high probability [35], implying that they are very efficient spreaders of their own state to the rest of the network. Hence, the larger k, the higher the exit probability. In the case of the Moran process, by applying the same argument, high degree nodes are prone to often change state by adopting the state of a neighbor [10,31], and hence the exit probability decreases with the degree.

Let us now recall that, for a pure activity-driven network, the aggregated degree of a node with activity a takes the value $\bar{k}_a(t) \sim (a + \langle a \rangle)t$ at time t: nodes with high activity tend to have large integrated degree [26,29]. Putting this in relation with the behavior of the exit probability as a function of activity in temporal networks and of degree in static networks, we thus obtain that the dynamics on the temporal activity-driven network yields a completely different and opposite result when compared with the dynamics on the static, integrated network counterpart: High-activity nodes are more prone to spread under Moran dynamics, while low-activity nodes are more prone under voter dynamics.

This symmetry voter-Moran between pure activity-driven networks and their integrated counterpart occurs as well at the level of the average consensus time when we measure it as a function of the update attempts. Considering that a randomly chosen node becomes active with average probability $\langle a \rangle$, we have that, as a function of updated attempts, the convergence time is $\bar{T}_N \equiv \langle a \rangle T_N$. We have thus, for homogeneous initial conditions ($\Omega = 1/2$),

$$\bar{T}_N^{\text{voter}} = N\langle a \rangle \langle a^{-1} \rangle \ln 2, \quad \bar{T}_N^{\text{Moran}} = N\frac{\langle a \rangle^2}{\langle a^2 \rangle} \ln 2. \tag{42}$$

Comparing with the results for static networks [10,31],

$$\bar{T}_N^{\text{Moran}} = N\langle k \rangle \langle k^{-1} \rangle \ln 2, \quad \bar{T}_N^{\text{voter}} = N\frac{\langle k \rangle^2}{\langle k^2 \rangle} \ln 2, \tag{43}$$

we observe that the formulas for voter and Moran dynamics are indeed mirror images, with the activity distribution a in the temporal representation substituted by the degree distribution in the integrated representation.

Let us now consider instead the pure attractiveness temporal network model (setting $a = 1$). In that case, the exit probability is proportional to the attractiveness for the voter model, $E_b^{\text{voter}} \propto b$, while for the Moran process, it is inversely proportional to the attractiveness, $E_b^{\text{Moran}} \propto 1/b$. Moreover, the integrated degree of a node with attractiveness b is $\bar{k}_b(t) \sim \frac{b}{\langle b \rangle} t$. Here therefore, we have the same kind of behavior on the temporal and corresponding integrated static network when making an equivalence between attractiveness in the temporal network and degree in the static network. This equivalence between a static network with a degree distribution $P(k)$ and a pure attractiveness temporal network with the same distribution $P(b)$ is also obtained by looking at the consensus time measured as the number of update attempts.

V. NUMERICAL RESULTS

In order to check the analytical predictions made above, we have performed simulations of the mixed process defined earlier on activity-driven networks with attractiveness, choosing a marginal activity distribution following a power law, Eq. (41), similar to the distribution observed empirically in some real

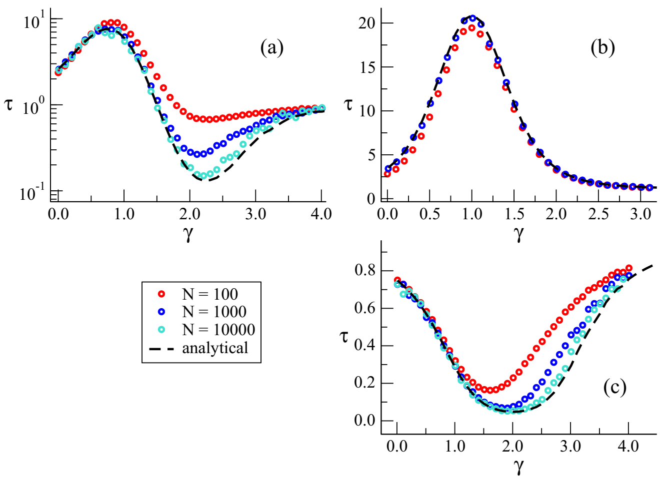

FIG. 3. Voterlike dynamics in temporal activity driven networks with attractiveness. Reduced consensus time as a function of γ for different values of the network size N, for an activity distribution $F(a)$ given by Eq. (41) with $\epsilon = 10^{-3}$. (a) Voter and Moran processes (equivalent) for $\eta(a,b) = F(a)F(b)$. (b) Voter model on pure activity-driven network ($b = 1$). (c) Moran process on pure activity-driven network ($b = 1$). In each case, the dashed line corresponds to the analytical expression given by Eq. (28).

networks [26,36]. We have performed simulations for network sizes $N = 10^2$, 10^3, and 10^4, averaging over 10^3 realizations.

In Fig. 3 we plot the reduced consensus time τ as a function of γ for three different values of the network size and three different dynamics: voter and Moran processes on a pure activity-driven network, and voter model on an activity-driven network with attractiveness with a and b independently and equally distributed $\eta(a,b) = F(a)F(b)$. The curves are compared to the theoretical value given in Eq. (28). We see that for $N = 10^4$ the dynamics already matches well the expected behavior in the infinite size limit. We deduce that our heterogeneous mean-field analysis captures efficiently the opinion dynamics on the activity-driven network with attractiveness.

VI. ASYMPTOTIC BEHAVIOUR

In Figs. 1–3, we see that the consensus time presents minima and maxima when γ varies. To investigate this point in more details, we analyze the asymptotic behavior of the moments of the distribution $F(a)$ when ϵ tends to zero. For an activity distributed with Eq. (41), the moments of a take the form

$$\langle a^n \rangle = \frac{1-\gamma}{n+1-\gamma}\,\frac{1-\epsilon^{1+n-\gamma}}{1-\epsilon^{1-\gamma}}. \quad (44)$$

The dynamics of the voter and Moran processes on pure activity-driven network and on activity-driven network with equally and independently distributed a and b depends on the moments $\langle a^{-1}\rangle$, $\langle a\rangle$ and $\langle a^2\rangle$ only. The asymptotic behavior of these three quantities for $\epsilon \to 0$ are summarized in Tables I and II, along with the resulting behavior of the consensus time in the case with a and b independently and equally distributed and $p = 1$ (or equivalently $p = 0$), for which

$$\tau = \frac{\langle a\rangle^3 \langle a^-1\rangle}{\langle a^2\rangle}. \quad (45)$$

TABLE I. Asymptotic behavior of the moments of the distribution $F(a)$ defined in Eq. (41), and resulting asymptotic behavior of the reduced consensus time τ given by Eq. (45) when ϵ tends to zero, as a function of the exponent γ. For $\gamma = 0, 1, 2, 3$, logarithmic corrections are present, given in Table II.

γ	]0,1[	]1,2[	]2,3[	>3
$\langle a^{-1}\rangle$	$\mathcal{O}(\epsilon^{-\gamma})$	$\mathcal{O}(\epsilon^{-1})$	$\mathcal{O}(\epsilon^{-1})$	$\mathcal{O}(\epsilon^{-1})$
$\langle a\rangle$	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon^{\gamma-1})$	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon)$
$\langle a^2\rangle$	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon^{\gamma-1})$	$\mathcal{O}(\epsilon^{\gamma-1})$	$\mathcal{O}(\epsilon^2)$
τ	$\mathcal{O}(\epsilon^{-\gamma})$	$\mathcal{O}(\epsilon^{2\gamma-3})$	$\mathcal{O}(\epsilon^{3-\gamma})$	$\mathcal{O}(1)$

For all values of γ, the physical ordering time $T \sim \tau/\langle a\rangle$ diverges when ϵ tends to zero. This is not surprising if we notice that the consensus of the voter dynamics is strongly limited by the agent with the smallest activity ϵ, which activates and copies the opinion of a peer for the first time in a time $1/\epsilon$ on average. However, we observe that for $0 < \gamma < 1.5$ the consensus time τ measured as the number of update attempts, in Eq. (45), diverges when ϵ goes to zero, and tends instead to zero for $1.5 < \gamma < 3$. The number of update attempts required to reach consensus may either be infinite or null depending on the exponent γ, with a sharp transition between both regimes at $\gamma = 1.5$. This nontrivial behavior reveals the critical importance of the temporal network's topology in this case, and is not easily predictable with qualitative arguments. We also recover the fact that the fastest consensus is reached for $\gamma = 2$ and the slowest for $\gamma = 1$ and that for $0 < \gamma < 3$ the consensus time exhibits a symmetry with respect to the axis $\gamma = 1.5$: $\tau(\gamma) = \frac{1}{\tau(3-\gamma)}$. Finally, for $\gamma \gg 3$, the heterogeneity of the distribution of a is no longer significant, so that the dynamics is that of a fully connected static network. In Fig. 4 we plot the reduced consensus time for the voter dynamics on a network with independent activity and attractiveness as obtained by direct numerical simulations of a voter model on a temporal network, compared with the analytical predictions of Eq. (45), for various values of ϵ and $N = 10^4$. The simulations confirm the predicted asymptotic behavior of τ when ϵ tends to zero. We also observe stronger finite-size effects when epsilon tends to zero due to a poorer sampling of the activity distribution given by Eq. (41).

VII. CONCLUSIONS

In this paper, we have studied in detail the properties of consensus processes mixing the voter and Moran models

TABLE II. Asymptotic behavior of the moments of the distribution $F(a)$ defined in Eq. (41), and resulting asymptotic behavior of the reduced consensus time τ given by Eq. (45) when ϵ tends to zero, for the specific cases $\gamma = 0, 1, 2, 3$.

γ	0	1	2	3
$\langle a^{-1}\rangle$	$-\ln\epsilon$	$-(\epsilon\ln\epsilon)^{-1}$	$(2\epsilon)^{-1}$	$\frac{2}{3}\epsilon^{-1}$
$\langle a\rangle$	$1/2$	$-(\ln\epsilon)^{-1}$	$-\epsilon\ln\epsilon$	2ϵ
$\langle a^2\rangle$	$1/3$	$-(2\ln\epsilon)^{-1}$	ϵ	$-2\epsilon^2\ln\epsilon$
τ	$-\frac{3}{8}\ln\epsilon$	$-2\epsilon^{-1}(\ln\epsilon)^{-3}$	$-\frac{1}{2}\epsilon(\ln\epsilon)^3$	$-\frac{8}{3}(\ln\epsilon)^{-1}$

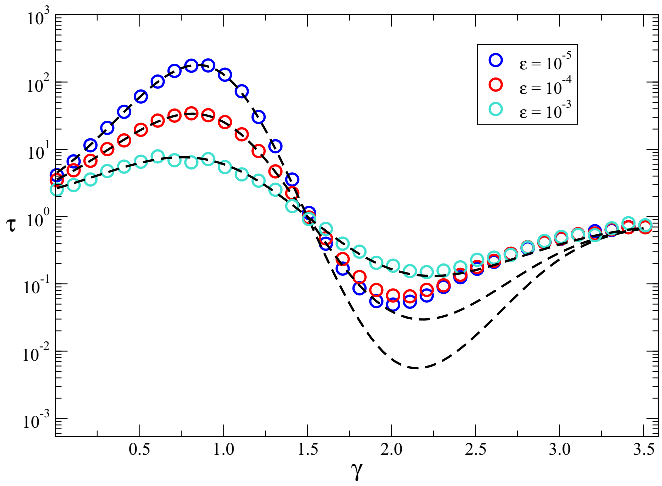

FIG. 4. Voter dynamics in temporal networks with independent and equally distributed activity and attractiveness. Consensus time τ as a function of γ for activity and attractiveness distributed according to Eq. (41) and different values of ϵ. The analytical expression given by Eq. (45) is shown in dashed lines. The numerical simulations are performed with network size $N = 10^4$.

update rules, on temporal network models based on the activity driven paradigm. Through a heterogeneous mean-field approach, we have derived the evolution equation of the average density of voters in state 1 with activity a and attractiveness b. This has allowed us to identify a conserved quantity of the dynamics, and subsequently to compute the average time to reach consensus and the probability that a single agent with a discrepant opinion among an otherwise unanimous population spreads his or her opinion to the whole network, called the exit probability. Surprising results arise from the study of particular cases of the distribution of the parameters a and b. When the attractiveness is taken to be proportional to the activity a, the dynamics is the same as if the copying process were running on a static complete graph. The average activity $\langle a\rangle$ determines the time scale of the dynamics, but otherwise the precise distribution of activity among the agents is no longer relevant. This holds for all values of the probability p determining the state update rule, and in particular for the voter model ($p = 1$) and the Moran process ($p = 0$). The same behavior happens when $p = 1/2$, regardless of the distribution $\eta(a, b)$ of activity and attractiveness: surprisingly, the exit probability is equal to $1/N$ and does not depend of the parameters of the initial invading node.

Interestingly, when the activity and the attractiveness are independent and equally distributed, the dynamics is unchanged when replacing p by $1 - p$. In fact, it appears that by construction, when the time is counted as the number of update attempts, exchanging a and b on all the nodes is equivalent to replace p by $1 - p$ in the update rule. One of our main results lies in the observation that the voter model and the Moran process on a pure activity-driven network (setting $b = 1$ for all nodes) are in some sense mirror images of their static network counterparts. Indeed, the dynamics of the voter model on an activity-driven network with a distribution $F(a)$ is the same as the Moran dynamics running on top of a static network with a degree distribution $P(k) = F(k)$. The same holds for the Moran process on the activity-driven network and the voter model on the static network. This implies that the apparently appealing operation consisting in considering an activity-driven network and its integrated counterpart as similar substrates for this kind of opinion dynamics process would be misleading, despite the fact that the degree distribution of the integrated network is practically equal to the activity distribution of the temporal network [26]. On the contrary, a pure attractiveness temporal network (setting $a = 1$ for all nodes) and its integrated counterpart are equivalent substrates for the voter and Moran processes. It would be very interesting to check whether similar conclusions hold for other consensus formation processes with more complex update rules such as the majority rule process. Our results will hopefully motivate further research in this direction.

ACKNOWLEDGMENTS

We acknowledge financial support from the Spanish MINECO, under Project No. FIS2016-76830-C2-1-P. R.P.-S. acknowledges additional financial support from ICREA Academia, funded by the Generalitat de Catalunya.

APPENDIX: DETAILS OF SOME COMPUTATIONS

From the expression of the weights λ_h in Eq. (13) we obtain, by multiplying by a and summing over h (in the next equations we write $\sum a\lambda$ and $\sum b\lambda$ as shorthand for $\sum_h a\lambda_h$ and $\sum_h b\lambda_h$, respectively):

$$\sum_h a\lambda_h = p\left\langle \frac{ab}{\Delta_{h,p}}\right\rangle\Big[\sum a\lambda\Big] + (1-p)\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle\Big[\sum b\lambda\Big], \tag{A1}$$

which gives a relation between $\sum a\lambda$ and $\sum b\lambda$

$$\sum b\lambda = \frac{1 - p\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle}{(1-p)\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle}\sum a\lambda. \tag{A2}$$

The normalization of the weights gives

$$p\left\langle\frac{b}{\Delta_{h,p}}\right\rangle\Big[\sum a\lambda\Big] + (1-p)\left\langle\frac{a}{\Delta_{h,p}}\right\rangle\Big[\sum b\lambda\Big] = 1. \tag{A3}$$

Combining Eqs. (A2) and (A3) leads to

$$\sum a\lambda = \frac{\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle}{p\left\langle\frac{b}{\Delta_{h,p}}\right\rangle\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle + \left(1 - p\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle\right)\left\langle\frac{a}{\Delta_{h,p}}\right\rangle}. \tag{A4}$$

Besides, by definition of $\Delta_{h,p}$ we wrote Eq. (29), which, after dividing by $\langle b\rangle$ gives

$$1 - p\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle = (1-p)\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\frac{\langle a\rangle}{\langle b\rangle}. \tag{A5}$$

Inserting this into Eqs. (A4) and (A2) one recovers the correct expressions given in Eqs. (14) and (15).

To derive the expression of the average consensus time, we write, combining Eqs. (22), (24), and (25)

$$\Omega(1-\Omega)\frac{\partial^2 T}{\partial\Omega^2}\sum_h \eta_h\,\Delta_{h,p}\left(\frac{\lambda_h}{\eta_h}\right)^2 = -N\langle b\rangle. \tag{A6}$$

We have

$$\begin{aligned}
\sum_h \eta_h\,\Delta_{h,p}\left(\frac{\lambda_h}{\eta_h}\right)^2 &= \sum_h \eta_h\,\frac{\left(pb\left[\sum a\lambda\right]+(1-p)a\left[\sum b\lambda\right]\right)^2}{\Delta}\\
&= \left[\sum a\lambda\right]\left[\sum b\lambda\right]\left(p^2\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\frac{\sum a\lambda}{\sum b\lambda}+(1-p)^2\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle\frac{\sum b\lambda}{\sum a\lambda}+2p(1-p)\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle\right)\\
&= \frac{\left[\sum a\lambda\right]\left[\sum b\lambda\right]}{\langle a\rangle\langle b\rangle}\left(p^2\left\langle\frac{a^2}{\Delta_{h,p}}\right\rangle\langle b\rangle^2+(1-p)^2\left\langle\frac{b^2}{\Delta_{h,p}}\right\rangle\langle a\rangle^2+2p(1-p)\left\langle\frac{ab}{\Delta_{h,p}}\right\rangle\langle a\rangle\langle b\rangle\right)\\
&= \frac{\left[\sum a\lambda\right]\left[\sum b\lambda\right]}{\langle a\rangle\langle b\rangle}\left\langle\frac{\left(pa\langle b\rangle+(1-p)\langle a\rangle b\right)^2}{\Delta_{h,p}}\right\rangle\\
&= \left[\sum a\lambda\right]\left[\sum b\lambda\right],
\end{aligned} \tag{A7}$$

which, inserted in Eq. (A6), finally yields Eq. (26).

[1] S. Galam, Global Econ. Mgmt. Rev. **18**, 2 (2013).
[2] S. Qian, Y. Liu, and S. Galam, Physica A **432**, 187 (2015).
[3] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).
[4] B. Drossel, Adv. Phys. **50**, 209 (2001).
[5] T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[6] P. Moran, *The Statistical Processes of Evolutionary Theory* (Clarendon Press, Oxford, 1962).
[7] A. Barrat, M. Barthélemy, and A. Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, Cambridge, 2008).
[8] M. E. J. Newman, *Networks: An introduction* (Oxford University Press, Oxford, 2010).
[9] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[10] V. Sood, T. Antal, and S. Redner, Phys. Rev. E **77**, 041121 (2008).
[11] N. Masuda, N. Gibert, and S. Redner, Phys. Rev. E **82**, 010103 (2010).
[12] F. Vazquez and V. M. Eguíluz, New J. Phys. **10**, 063011 (2008).
[13] M. Á. Serrano, K. Klemm, F. Vazquez, V. M. Eguíluz, and M. S. Miguel, J. Stat. Mech.: Theor. Exp. (2009) P10024.
[14] A. Baronchelli, C. Castellano, and R. Pastor-Satorras, Phys. Rev. E **83**, 066117 (2011).
[15] P. Moretti, S. Y. Liu, A. Baronchelli, and R. Pastor-Satorras, Eur. Phys. J. B **85**, 88 (2012).
[16] P. Holme and J. Saramäki, Phys. Rep. **519**, 97 (2012).
[17] P. Holme, Eur. Phys. J. B **88**, 234 (2015).
[18] R. Lambiotte, L. Tabourier, and J.-C. Delvenne, Eur. Phys. J. B **86**, 320 (2013).
[19] L. Isella, J. Stehlé, A. Barrat, C. Cattuto, J.-F. Pinton, and W. V. den Broeck, J. Theor. Biol **271**, 166 (2011).
[20] M. Karsai, M. Kivelä, R. K. Pan, K. Kaski, J. Kertész, A.-L. Barabási, and J. Saramäki, Phys. Rev. E **83**, 025102 (2011).
[21] B. Min, K.-I. Goh, and A. Vazquez, Phys. Rev. E **83**, 036102 (2011).
[22] M. Starnini, A. Baronchelli, A. Barrat, and R. Pastor-Satorras, Phys. Rev. E **85**, 056115 (2012).
[23] J. Fernández-Gracia, V. M. Eguíluz, and M. San Miguel, Phys. Rev. E **84**, 015103 (2011).
[24] F. Karimi and P. Holme, Physica A **392**, 3476 (2013).
[25] M. Li and H. Dankowicz, arXiv:1804.03525.
[26] N. Perra, B. Gonçalves, R. Pastor-Satorras, and A. Vespignani, Sci. Rep. **2**, 00469 (2012).
[27] L. Alessandretti, K. Sun, A. Baronchelli, and N. Perra, Phys. Rev. E **95**, 052318 (2017).
[28] S. Galam and A. C. R. Martins, Europhys. Lett. **95**, 48005 (2011).
[29] M. Starnini and R. Pastor-Satorras, Phys. Rev. E **87**, 062807 (2013).
[30] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[31] V. Sood and S. Redner, Phys. Rev. Lett. **94**, 178701 (2005).
[32] P. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).
[33] C. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences*, 4th ed. (Springer-Verlag, Berlin, 2010).
[34] K. Suchecki, V. M. Eguíluz, and M. S. Miguel, Europhys. Lett. **69**, 228 (2005).
[35] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
[36] T. Kobayashi, T. Takaguchi, and A. Barrat, arXiv:1804.08828 [physics.soc-ph].