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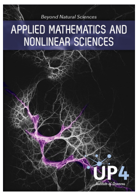

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Regularizing algorithm for mixed matrix pencils

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Abstract

P. Van Dooren (1979) constructed an algorithm for computing all singular summands of Kronecker's canonical form of a matrix pencil. His algorithm uses only unitary transformations, which improves its numerical stability. We extend Van Dooren's algorithm to square complex matrices with respect to consimilarity transformations $A \mapsto SA\bar{S}^{-1}$ and to pairs of $m \times n$ complex matrices with respect to transformations $(A,B) \mapsto (SAR, SB\bar{R})$, in which S and R are nonsingular matrices.

Keywords: Regularizing algorithm; Matrix pencils; Consimilarity; Unitary transformations; Canonical forms.
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1 Introduction

Van Dooren [7] gave an algorithm that for each pair (A,B) of complex matrices of the same size constructs its *regularizing decomposition*; that is, it constructs a matrix pair that is simultaneously equivalent to (A,B) and has the form

$$(A_1,B_1) \oplus \cdots \oplus (A_t,B_t) \oplus (\underline{A},\underline{B})$$

in which $(\underline{A},\underline{B})$ is a pair of nonsingular matrices and each other summand has one of the forms:

$$(F_n,G_n), \quad (F_n^T,G_n^T), \quad (I_n,J_n(0)), \quad (J_n(0),I_n),$$

where $J_n(0)$ is the singular Jordan block and

$$F_n := \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}, \qquad G_n := \begin{bmatrix} 1 & & 0 \\ 0 & \ddots & \\ & \ddots & 1 \\ 0 & & 0 \end{bmatrix}$$

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are $n\times(n-1)$ matrices; $n\geq 1$. Note that $(F_1,G_1)=(0_{10},0_{10})$; we denote by 0_{mn} the zero matrix of size $m\times n$, where $m,n\in\{0,1,2,\dots\}$. The algorithm uses only unitary transformations, which improves its computational stability.

We extend Van Dooren's algorithm to square complex matrices up to consimilarity transformations $A\mapsto SA\bar{S}^{-1}$ and to pairs of $m\times n$ matrices up to transformations $(A,B)\mapsto(SAR,SB\bar{R})$, in which S and R are nonsingular matrices.

A regularizing algorithm for matrices of undirected cycles of linear mappings was constructed by Sergeichuk [6] and, independently, by Varga [8]. A regularizing algorithm for matrices under congruence was constructed by Horn and Sergeichuk [5].

All matrices that we consider are complex matrices.

2 Regularizing unitary algorithm for matrices under consimilarity

Two matrices A and B are *consimilar* if there exists a nonsingular matrix S such that $SA\bar{S}^{-1}=B$. Two matrices are consimilar if and only if they represent the same semilinear operator, but in different bases. Recall that a mapping $\mathscr{A}:U\to V$ between complex vector spaces is *semilinear* if

$$\mathscr{A}(au_1+bu_2)=\bar{a}\mathscr{A}u_1+\bar{b}\mathscr{A}u_2$$

for all $a,b\in\mathbb{C}$ and $u_1,u_2\in U$.

The canonical form of a matrix under consimilarity is the following (see [3] or [4]):
Each square complex matrix is consimilar to a direct sum, uniquely determined up to permutation of direct summands, of matrices of the following types:

- *a Jordan block $J_k(\lambda)$ with $\lambda\geq 0$, and*
- $\begin{bmatrix}0&1\\ \mu&0\end{bmatrix}$ *with $\mu\notin\mathbb{R}$ or $\mu<0$.*

Thus, each square matrix A is consimilar to a direct sum

$$J_{n_1}(0)\oplus\cdots\oplus J_{n_k}(0)\oplus\underline{A},$$

in which $\underline{A}$ is nonsingular and is determined up to consimilarity; the other summands are uniquely determined up to permutation. This sum is called a *regularizing decomposition* of A. The following algorithm admits to construct a regularizing decomposition using only unitary transformations.

Algorithm 1. *Let A be a singular $n\times n$ matrix. By unitary transformations of rows, we reduce it to the form*

$$S_1A=\begin{bmatrix}0_{r_1n}\\ A'\end{bmatrix},\qquad S_1 \text{ is unitary,}$$

in which the rows of A' are linearly independent. Then we make the coninverse transformations of columns and obtain

$$S_1A\bar{S}_1^{-1}=\begin{bmatrix}0_{r_1}&0\\ \star&A_1\end{bmatrix}$$

We apply the same procedure to A_1 and obtain

$$S_2A_1\bar{S}_2^{-1}=\begin{bmatrix}0_{r_2}&0\\ \star&A_2\end{bmatrix},\qquad S_2 \text{ is unitary,}$$

in which the rows of $[\star\ A_2]$ are linearly independent.

We repeat this procedure until we obtain

$$S_t A_{t-1} \bar{S}_t^{-1} = \begin{bmatrix} 0_{r_t} & 0 \\ \star & A_t \end{bmatrix}, \qquad S_t \text{ is unitary,}$$

in which A_t is nonsingular. The result of the algorithm is the sequence $r_1, r_2, \dots, r_t, A_t$.

For a matrix A and a nonnegative integer n, we write

$$A^{(n)} := \begin{cases} 0_{00}, & \text{if } n = 0; \\ A \oplus \cdots \oplus A \ (n \text{ summands}), & \text{if } n \geq 1. \end{cases}$$

Theorem 1. *Let $r_1, r_2, \dots, r_t, A_t$ be the sequence obtained by applying Algorithm 1 to a square complex matrix A. Then*

$$r_1 \geq r_2 \geq \cdots \geq r_t$$

and A is consimilar to

$$J_1{}^{(r_1-r_2)} \oplus J_2{}^{(r_2-r_3)} \oplus \cdots \oplus J_{t-1}^{(r_{t-1}-r_t)} \oplus J_t^{(r_t)} \oplus A_t \tag{1}$$

in which $J_k := J_k(0)$ and A_t is determined by A up to consimilarity and the other summands are uniquely determined.

Proof. Let $\mathscr{A} : V \to V$ be a semilinear operator whose matrix in some basis is A. Let $W := \mathscr{A}V$ be the image of $\mathscr{A}$. Then the matrix of the restriction $\mathscr{A}_1 : W \to W$ of $\mathscr{A}$ on W is A_1. Applying Algorithm 1 to A_1, we get the sequence $r_2, \dots, r_t, A_t$. Reasoning by induction on the length t of the algorithm, we suppose that $r_2 \geq r_3 \geq \cdots \geq r_t$ and that A_1 is consimilar to

$$J_1{}^{(r_2-r_3)} \oplus \cdots \oplus J_{t-2}^{(r_{t-1}-r_t)} \oplus J_{t-1}^{(r_t)} \oplus A_t. \tag{2}$$

Thus, $\mathscr{A}_1 : W \to W$ is given by the matrix (2) in some basis of W.

The direct sum (2) defines the decomposition of W into the direct sum of invariant subspaces

$$W = (W_{21} \oplus \cdots \oplus W_{2,r_2-r_3}) \oplus \cdots \oplus (W_{t1} \oplus \cdots \oplus W_{tr_t}) \oplus W'.$$

Each W_{pq} is generated by some basis vectors $e_{pq2}, e_{pq3}, \dots, e_{pqp}$ such that

$$\mathscr{A} : \ e_{pq2} \mapsto e_{pq3} \mapsto \cdots \mapsto e_{pqp} \mapsto 0.$$

For each W_{pq}, we choose $e_{pq1} \in V$ such that $\mathscr{A} e_{pq1} = e_{pq2}$. The set

$$\{e_{pqp} \,|\, 2 \leq p \leq t, \ 1 \leq q \leq r_p - r_{p+1}\} \quad (r_{t+1} := 0)$$

consists of r_2 basis vectors belonging to the kernel of $\mathscr{A}$; we supplement this set to a basis of the kernel of $\mathscr{A}$ by some vectors $e_{111}, \dots, e_{1,r_1-r_2,1}$.

The set of vectors e_{pqs} supplemented by the vectors of some basis of W' is a basis of V. The matrix of $\mathscr{A}$ in this basis has the form (1) because

$$\mathscr{A} : \ e_{pq1} \mapsto e_{pq2} \mapsto e_{pq3} \mapsto \cdots \mapsto e_{pqp} \mapsto 0$$

for all $p = 1, \dots, t$ and $q = 1, \dots, r_p - r_{p+1}$. This completes the proof of Theorem 1.

Example 1. *Let a square matrix A define a semilinear operator $\mathscr{A} : V \to V$ and let the singular part of its regularizing decomposition be $J_2 \oplus J_3 \oplus J_4$. This means that V possesses a set of linear independent vectors forming the Jordan chains*

$$\begin{aligned} \mathscr{A} : \quad & e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto 0 \\ & f_1 \mapsto f_2 \mapsto f_3 \mapsto 0 \\ & g_1 \mapsto g_2 \mapsto 0 \end{aligned} \tag{3}$$

Applying the first step of Algorithm 1, we get A_1 whose singular part corresponds to the chains

$$\begin{array}{rl}\mathscr{A}: & e_2 \mapsto e_3 \mapsto e_4 \mapsto 0 \\ & f_2 \mapsto f_3 \mapsto 0 \\ & g_2 \mapsto 0\end{array}$$

On the second step, we delete e_2, f_2, g_2 and so on. Thus, r_i is the number of vectors in the ith column of (3)*: $r_1 = 3,\ r_2 = 3,\ r_3 = 2, r_4 = 1$. We get the singular part of regularizing decomposition of A:*

$$J_1{}^{(r_1-r_2)} \oplus \cdots \oplus J_{t-1}^{(r_{t-1}-r_t)} \oplus J_t^{(r_t)} = J_1{}^{(3-3)} \oplus J_2{}^{(3-2)} \oplus J_3^{(2-1)} \oplus J_4^{(1)} = J_2 \oplus J_3 \oplus J_4.$$

In particular, if

$$A = \left[\begin{array}{cccc|ccc|cc} 0&0&0&0&&&&& \\ 1&0&0&0&&&&& \\ 0&1&0&0&&&&& \\ 0&0&1&0&&&&& \\ \hline &&&&0&0&0&& \\ &&&&1&0&0&& \\ &&&&0&1&0&& \\ \hline &&&&&&&0&0 \\ &&&&&&&1&0 \end{array}\right] \begin{array}{l} e_1\\ e_2\\ e_3\\ e_4\\ f_1\\ f_2\\ f_3\\ g_1\\ g_2 \end{array}, \qquad (4)$$
$$\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & f_1 & f_2 & f_3 & g_1 & g_2 \end{array}$$

then we can apply Algorithm 1 using only transformations of permutational similarity and obtain

$$\left[\begin{array}{ccc|ccc|cc|c} 0&0&0&&&&&& \\ 0&0&0&&&&&& \\ 0&0&0&&&&&& \\ \hline 1&0&0&0&0&0&&& \\ 0&1&0&0&0&0&&& \\ 0&0&1&0&0&0&&& \\ \hline &&&1&0&0&0&0& \\ &&&0&1&0&0&0& \\ \hline &&&&&&1&0&0 \end{array}\right] \begin{array}{l} e_1\\ f_1\\ g_1\\ e_2\\ f_2\\ g_2\\ e_3\\ f_3\\ e_4 \end{array}$$
$$\begin{array}{ccccccccc} e_1 & f_1 & g_1 & e_2 & f_2 & g_2 & e_3 & f_3 & e_4 \end{array}$$

(all unspecified blocks are zero), which is the Weyr canonical form of (4)*, see* [4].

3 Regularizing unitary algorithm for matrix pairs under mixed equivalence

We say that pairs of $m \times n$ matrices (A,B) and (A',B') are *mixed equivalent* if there exist nonsingular S and R such that

$$(SAR, SB\bar{R}) = (A',B').$$

The *direct sum* of matrix pairs (A,B) and (C,D) is defined as follows:

$$(A,B) \oplus (C,D) = \left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} \right).$$

The canonical form of a matrix pair under mixed equivalence was obtained by Djoković [2] (his result was extended to undirected cycles of linear and semilinear mappings in [1]):

Each pair (A,B) of matrices of the same size is mixed equivalent to a direct sum, determined uniquely up to permutation of summands, of pairs of the following types:

$$(I_n, J_n(\lambda)),\ (I_n, \left(\begin{smallmatrix}0 & 1\\ \mu & 0\end{smallmatrix}\right)),\ (J_n(0), I_n),\ (F_n, G_n),\ (F_n^T, G_n^T),$$

in which $\lambda \geq 0$ and $\mu \notin \mathbb{R}$ or $\mu < 0$.

Thus, (A,B) is mixed equivalent to a direct sum of a pair $(\underline{A}, \underline{B})$ of nonsingular matrices and summands of the types:

$$(I_n, J_n(0)),\ (J_n(0), I_n),\ (F_n, G_n),\ (F_n^T, G_n^T),$$

in which $(\underline{A}, \underline{B})$ is determined up to mixed equivalence and the other summands are uniquely determined up to permutation. This sum is called a *regularizing decomposition* of (A,B). The following algorithm admits to construct a regularizing decomposition using only unitary transformations.

Algorithm 2. *Let (A,B) be a pair of matrices of the same size in which the rows of A are linearly dependent. By unitary transformations of rows, we reduce A to the form*

$$S_1 A = \begin{bmatrix} 0 \\ A' \end{bmatrix}, \qquad S_1 \text{ is unitary,}$$

in which the rows of A' are linearly independent. These transformations change B:

$$S_1 B = \begin{bmatrix} B' \\ B'' \end{bmatrix}.$$

By unitary transformations of columns, we reduce B' to the form $[B_1'\ 0]$ in which the columns of B_1' are linearly independent, and obtain

$$BR_1 = \begin{bmatrix} B_1' & 0 \\ \star & B_1 \end{bmatrix}, \qquad R_1 \text{ is unitary.}$$

These transformations change A:

$$S_1 A \bar{R}_1 = \begin{bmatrix} 0_{k_1 l_1} & 0 \\ \star & A_1 \end{bmatrix}.$$

We apply the same procedure to (A_1, B_1) and obtain

$$(S_2 A_1 \bar{R}_2, S_2 B_1 R_2) = \left(\begin{bmatrix} 0_{k_2 l_2} & 0 \\ \star & A_2 \end{bmatrix}, \begin{bmatrix} B_2' & 0 \\ \star & B_2 \end{bmatrix} \right),$$

in which the rows of $[\star\ A_2]$ are linearly independent, S_2 and R_2 are unitary, and the columns of B_2' are linearly independent.

We repeat this procedure until we obtain

$$(S_t A_{t-1} \bar{R}_t, S_t B_{t-1} R_t) = \left(\begin{bmatrix} 0_{k_t l_t} & 0 \\ \star & A_t \end{bmatrix}, \begin{bmatrix} B_t' & 0 \\ \star & B_t \end{bmatrix} \right),$$

in which the rows of A_t are linearly independent. The result of the algorithm is the sequence

$$(k_1, l_1),\ (k_2, l_2),\ \dots,\ (k_t, l_t),\ (A_t, B_t).$$

For a matrix pair (A,B) and a nonnegative integer n, we write

$$(A,B)^{(n)} := \begin{cases} (0_{00}, 0_{00}), & \text{if } n = 0; \\ (A,B) \oplus \cdots \oplus (A,B)\ (n \text{ summands}), & \text{if } n \geq 1. \end{cases}$$

Theorem 2. *Let (A,B) be a pair of complex matrices of the same size. Let us apply Algorithm 2 to (A,B) and obtain*

$$(k_1,l_1),\ (k_2,l_2),\ \dots,\ (k_t,l_t),\ (A_t,B_t).$$

Let us apply Algorithm 2 to $(\underline{A},\underline{B}) := (B_t^T, A_t^T)$ and obtain

$$(\underline{k}_1,\underline{l}_1),\ (\underline{k}_2,\underline{l}_2),\ \dots,\ (\underline{k}_{\underline{t}},\underline{l}_{\underline{t}}),\ (\underline{A}_{\underline{t}},\underline{B}_{\underline{t}}).$$

Then (A,B) is mixed equivalent to

$$\begin{aligned}
&(F_1,G_1)^{(k_1-l_1)} \oplus \cdots \oplus (F_{t-1},G_{t-1})^{(k_{t-1}-l_{t-1})} \oplus (F_t,G_t)^{(k_t-l_t)}\\
&\oplus (J_1,I_1)^{(l_1-k_2)} \oplus \cdots \oplus (J_{t-1},I_{t-1})^{(l_{t-1}-k_t)} \oplus (J_t,I_t)^{(l_t)}\\
&\oplus (F_1^T,G_1^T)^{(\underline{k}_1-\underline{l}_1)} \oplus \cdots \oplus (F_{\underline{t}-1}^T,G_{\underline{t}-1}^T)^{(\underline{k}_{\underline{t}-1}-\underline{l}_{\underline{t}-1})} \oplus (F_{\underline{t}}^T,G_{\underline{t}}^T)^{(\underline{k}_{\underline{t}}-\underline{l}_{\underline{t}})}\\
&\oplus (I_1,J_1)^{(\underline{l}_1-\underline{k}_2)} \oplus \cdots \oplus (I_{\underline{t}-1},J_{\underline{t}-1})^{(\underline{l}_{\underline{t}-1}-\underline{k}_{\underline{t}})} \oplus (I_{\underline{t}},J_{\underline{t}})^{(\underline{l}_{\underline{t}})}\\
&\oplus (\underline{B}_{\underline{t}}^T,\underline{A}_{\underline{t}}^T)
\end{aligned}$$

(all exponents in parentheses are nonnegative). The pair $(\underline{B}_{\underline{t}}^T,\underline{A}_{\underline{t}}^T)$ consists of nonsingular matrices; it is determined up to mixed equivalence. The other summands are uniquely determined by (A,B).

The rows of A_t in Theorem 2 are linearly independent, and so the columns of $\underline{B} := A_t^T$ are linearly independent. As follows from Algorithm 2, the columns of $\underline{B}_{\underline{t}}$ are linearly independent too. Since the rows of $\underline{A}_{\underline{t}}$ are linearly independent and the columns of $\underline{B}_{\underline{t}}$ are linearly independent, we have that the matrices in $(\underline{A}_{\underline{t}},\underline{B}_{\underline{t}})$ have the same size, these matrices are square, and so they are nonsingular. The pairs (I_n,J_n^T) and (G_n^T,F_n^T) are permutationally equivalent to (I_n,J_n) and (F_n^T,G_n^T). Therefore, the following lemma implies Theorem 2.

Lemma 1. *Let (A,B) be a pair of complex matrices of the same size. Let us apply Algorithm 2 to (A,B) and obtain*

$$(k_1,l_1),\ (k_2,l_2),\ \dots,\ (k_t,l_t),\ (A_t,B_t).$$

Then (A,B) is mixed equivalent to

$$\begin{aligned}
&(F_1,G_1)^{(k_1-l_1)} \oplus \cdots \oplus (F_{t-1},G_{t-1})^{(k_{t-1}-l_{t-1})} \oplus (F_t,G_t)^{(k_t-l_t)}\\
&\oplus (J_1,I_1)^{(l_1-k_2)} \oplus \cdots \oplus (J_{t-1},I_{t-1})^{(l_{t-1}-k_t)}\\
&\oplus (J_t,I_t)^{(l_t)} \oplus (A_t,B_t)
\end{aligned} \tag{5}$$

(all exponents in parentheses are nonnegative). The rows of A_t are linearly independent. The pair (A_t,B_t) is determined up to mixed equivalence. The other summands are uniquely determined by (A,B).

Proof. We write

$$(A,B) \Longrightarrow (k_1,l_1,(A_1,B_1))$$

if $k_1,l_1,(A_1,B_1)$ are obtained from (A,B) in the first step of Algorithm 2.

First we prove two statements.

Statement 1: If

$$\begin{aligned}
(A,B) &\Longrightarrow (k_1,l_1,(A_1,B_1)),\\
(\widetilde{A},\widetilde{B}) &\Longrightarrow (\tilde{k}_1,\tilde{l}_1,(\widetilde{A}_1,\widetilde{B}_1)),
\end{aligned} \tag{6}$$

and (A,B) is mixed equivalent to $(\widetilde{A},\widetilde{B})$, then $k_1=\tilde{k}_1$, $l_1=\tilde{l}_1$, and (A_1,B_1) is mixed equivalent to $(\widetilde{A}_1,\widetilde{B}_1)$.

Let m be the number of rows in A. Then

$$k_1 = m - \operatorname{rank} A = m - \operatorname{rank} \widetilde{A} = \tilde{k}_1.$$

Since (A,B) and $(\widetilde{A},\widetilde{B})$ are mixed equivalent and they are reduced by mixed equivalence transformations to

$$\left(\begin{bmatrix}0_{k_1l_1} & 0\\ X & A_1\end{bmatrix},\begin{bmatrix}B_1' & 0\\ Y & B_1\end{bmatrix}\right),\quad \left(\begin{bmatrix}0_{\tilde{k}_1\tilde{l}_1} & 0\\ \widetilde{X} & \widetilde{A}_1\end{bmatrix},\begin{bmatrix}\widetilde{B}_1' & 0\\ \widetilde{Y} & \widetilde{B}_1\end{bmatrix}\right), \tag{7}$$

there exist nonsingular S and R such that

$$\left(S\begin{bmatrix}0_{k_1l_1} & 0\\ X & A_1\end{bmatrix}, S\begin{bmatrix}B_1' & 0\\ Y & B_1\end{bmatrix}\right) = \left(\begin{bmatrix}0_{\tilde{k}_1\tilde{l}_1} & 0\\ \widetilde{X} & \widetilde{A}_1\end{bmatrix}R,\ \begin{bmatrix}\widetilde{B}_1' & 0\\ \widetilde{Y} & \widetilde{B}_1\end{bmatrix}\bar{R}\right). \tag{8}$$

Equating the first matrices of these pairs, we find that S has the form

$$S = \begin{bmatrix}S_{11} & 0\\ S_{21} & S_{22}\end{bmatrix},\qquad S_{11}\text{ is }k_1\times k_1.$$

Equating the second matrices of the pairs (8), we find that

$$S_{11}[B_1'\ 0] = [\widetilde{B}_1'\ 0]\bar{R}, \tag{9}$$

and so

$$l_1 = \operatorname{rank}[B_1'\ 0] = \operatorname{rank}[\widetilde{B}_1'\ 0] = \tilde{l}_1.$$

Since B_1' and $\widetilde{B}_1'$ are $k_1\times l_1$ and have linearly independent columns, (9) implies that R is of the form

$$R = \begin{bmatrix}R_{11} & 0\\ R_{21} & R_{22}\end{bmatrix},\qquad R_{11}\text{ is }l_1\times l_1.$$

Equating the (2,2) entries in the matrices (8), we get

$$S_{22}A_1 = \widetilde{A}_1R_{22},\qquad S_{22}B_1 = \widetilde{B}_1\bar{R}_{22},$$

hence (A_1,B_1) and $(\widetilde{A}_1,\widetilde{B}_1)$ are mixed equivalent, which completes the proof of Statement 1.

Statement 2: If (6), *then*

$$(A,B)\oplus(\widetilde{A},\widetilde{B}) \Longrightarrow (k_1+\tilde{k}_1, l_1+\tilde{l}_1, (A_1\oplus\widetilde{A}_1, B_1\oplus\widetilde{B}_1)).$$

Indeed, if (A,B) and $(\widetilde{A},\widetilde{B})$ are reduced to (7), then $(A,B)\oplus(\widetilde{A},\widetilde{B})$ is reduced to

$$\left(\begin{bmatrix}0_{k_1l_1}\oplus 0_{\tilde{k}_1\tilde{l}_1} & 0\oplus 0\\ X\oplus\widetilde{X} & A_1\oplus\widetilde{A}_1\end{bmatrix},\begin{bmatrix}B_1'\oplus\widetilde{B}_1' & 0\oplus 0\\ Y\oplus\widetilde{Y} & B_1\oplus\widetilde{B}_1\end{bmatrix}\right),$$

which is permutationally equivalent to

$$\left(\begin{bmatrix}0_{k_1l_1} & 0\\ X & A_1\end{bmatrix},\begin{bmatrix}B_1' & 0\\ Y & B_1\end{bmatrix}\right)\oplus\left(\begin{bmatrix}0_{\tilde{k}_1\tilde{l}_1} & 0\\ \widetilde{X} & \widetilde{A}_1\end{bmatrix},\begin{bmatrix}\widetilde{B_1'} & 0\\ \widetilde{Y} & \widetilde{B}_1\end{bmatrix}\right).$$

We are ready to prove Lemma 1 for any pair (A,B). Due to Statement 1, we can replace (A,B) by any mixed equivalent pair. In particular, we can take

$$(A,B) = (F_1,G_1)^{(r_1)}\oplus\cdots\oplus(F_t,G_t)^{(r_t)}\oplus(J_1,I_1)^{(s_1)}\oplus\cdots\oplus(J_t,I_t)^{(s_t)}\oplus(C,D) \tag{10}$$

for some nonnegative $t, r_1,\dots,r_t, s_1,\dots,r_t$ and some pair (C,D) in which C has linearly independent rows.

Clearly,

$$(J_i, I_i) \Longrightarrow \begin{cases} (1,1,(J_{i-1}, I_{i-1})), & \text{if } i \neq 1; \\ (1,1,(0_{00}, 0_{00})), & \text{if } i = 1, \end{cases}$$

and

$$(F_i, G_i) \Longrightarrow \begin{cases} (1,1,(F_{i-1}, G_{i-1})), & \text{if } i \neq 1; \\ (1,0,(0_{00}, 0_{00})), & \text{if } i = 1. \end{cases}$$

Due to Statement 2,

- $k_1 = m - \operatorname{rank} A$ is the number of all summands of the types (J_i, I_i) and (F_i, G_i),
- l_1 is the number of all summands of the types (J_i, I_i) and (F_i, G_i), except for (F_1, G_1),
- and

$$(A_1, B_1) = (F_1, G_1)^{(r_2)} \oplus \cdots \oplus (F_{t-1}, G_{t-1})^{(r_t)} \oplus (J_1, I_1)^{(s_2)} \oplus \cdots \oplus (J_{t-1}, I_{t-1})^{(s_t)} \oplus (C, D). \tag{11}$$

We find that $k_1 - l_1$ is the number of summands of the type (F_1, G_1).

Applying the same reasoning to (11) instead of (10) we get that

- k_2 is the number of all summands of the types (J_i, I_i) and (F_i, G_i) with $i \geq 2$,
- l_1 is the number of all summands of the types (J_i, I_i) with $i \geq 2$ and (F_i, G_i) with $i \geq 3$,
- $(A_2, B_2) = (F_1, G_1)^{(r_3)} \oplus \cdots \oplus (F_{t-2}, G_{t-2})^{(r_t)} \oplus (J_1, I_1)^{(s_3)} \oplus \cdots \oplus (J_{t-2}, I_{t-2})^{(s_t)} \oplus (C, D)$.

We find that $k_2 - l_2$ is the number of summands of the type (F_2, G_2), and that $l_1 - k_2$ is the number of summands of the type (J_1, I_1), and so on, until we obtain (5).

The fact that the pair (A_t, B_t) in (5) is determined up to mixed equivalence and the other summands are uniquely determined by (A, B) follows from Statement 1 (or from the canonical form of a matrix pair up to mixed equivalence). This concludes the proof of Lemma 1 and Theorem 1.

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