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ON THE STABILITY IN PHASE-LAG HEAT CONDUCTION WITH TWO TEMPERATURES

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Abstract: We investigate the well-posedness and the stability of the solutions for several Taylor approximations of the phase-lag two-temperatures equations. We give conditions on the parameters which guarantee the existence and uniqueness of solutions as well as the stability and the instability of the solutions for each approximation.

Keywords: phase-lag heat conduction, well-posedness, stability, instability, energy methods, spectral analysis.

1. Introduction

The Fourier formulation to describe heat conduction is widely used by mathematicians, physicists and engineers. In this model, the heat flux is proportional to the temperature's gradient. However, this formulation jointly with the usual energy equation (see (3)) leads to the instantaneous propagation of heat, a drawback of the model because this fact is incompatible with real observations. In order to overcome this difficulty, alternative proposals have been stated. For example, Cattaneo [4] proposed an hyperbolic equation with dissipation. At the end of the 60's, Chen and Gurtin [5] introduced the two-temperatures theory which was discussed later by Chen *et al.* [6, 7, 32]. Other theories that involved the temperature's history were stated by Gurtin [14]. Two more proposals arose in the middle of the 90's. On the one hand, we can mention the theory of Green and Naghdi [12, 13] that considers three different cases, one of which coincides with the Fourier formulation. On the other hand, Tzou proposed a theory in which the heat flux and the gradient of the temperature have a delay in the constitutive equations [31]. When this consideration is taken into account, it is usual to speak of phase-lag theories. In that case, the constitutive equations are given by:

$$q_i(\mathbf{x}, t+\tau_1) = -k\theta_{,i}(\mathbf{x}, t+\tau_2), \qquad k>0. \tag{1}$$

Here $q=(q_i)$ is the heat flux vector, θ is the temperature and τ_1 and τ_2 are the delay parameters which are assumed to be positive. As usual, the notation $\theta_{,i}$ means the derivative of θ with respect to the variable x_i, and repeated subscripts means summation. The derivative with respect to the time will be denoted using a dot over the function.

This equation suggests that the temperature gradient established across a material volume at position $\mathbf{x}$ and time $t+\tau_2$ results in a heat flux to flow at a different time $t+\tau_1$. These delays can be understood in terms of the microstructure of the medium.

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In 2007, Choudhuri [8] proposed an extension of Tzou's theory in which the heat flux is described using the following constitutive equations:

$$q_i(\mathbf{x}, t+\tau_1) = -k_1\alpha_{,i}(\mathbf{x}, t+\tau_3) - k_2\theta_{,i}(\mathbf{x}, t+\tau_2), \tag{2}$$

where $\dot{\alpha} = \theta$. Here, α is the *thermal displacement* introduced by Green and Naghdi and τ_3 is another delay parameter.

These two aforementioned theories have several derivations when the heat flux and the gradients of the temperature and the thermal displacement are replaced by Taylor approximations. In fact, one can think that Choudhuri's proposal aims to recover Green and Naghdi theories when different Taylor approximations are considered. This new approach gives rise to different equations (depending on the selected Taylor approximation) to describe heat conduction that have been analyzed by many authors (see, for example, [1, 3, 15, 18, 21, 24, 25, 26, 27, 28, 29, 33]).

It is worth noting that both proposals, those of Tzou and Choudhuri, lead to *ill-posed* problems in the sense of Hadamard. To be more precise, it has been shown in a recent paper that, combining either equation ((1) or (2), respectively) with the energy equation

$$c\dot{\theta} + \text{ div } q = 0, \quad (c > 0) \tag{3}$$

there exists a sequence of elements in the point spectrum such that its real part tends to infinity [9].

In order to obtain a heat conduction theory with delays but without such an explosive behavior, Quintanilla [22, 23] combined the delay parameters of Tzou and Choudhuri with the two-temperatures theory proposed by Chen and Gurtin. The basic constitutive equations read

$$q_i(\mathbf{x}, t+\tau_1) = -k_1\beta_{,i}(\mathbf{x}, t+\tau_3) - k_2T_{,i}(\mathbf{x}, t+\tau_2), \tag{4}$$

where $\alpha = \beta - a\Delta\beta$, $\theta = T - a\Delta T$ and a is a positive constant.

In these cases, there are uniqueness of the solutions and continuous dependence with respect to the parameters of the system of equations. Therefore, it seems that these new proposals are on the right way to better describe the heat conduction phenomenon.

In this work we investigate the well-posedness and the stability of the solutions for several approximations (in the Taylor sense) for the equations proposed by Quintanilla. It is worth noting that the phase-lag with two-temperatures theories for heat conduction have been extensively considered in the literature (see [2, 10, 11, 16, 19, 20, 30] among others).

We study three different cases. In the first one we take $k_1 = 0$, a second-order approximation of q_i and a first-order approximation of T. In the second case, we assume that k_1 and k_2 are both positive and we approximate q_i, T and β by their first-order Taylor approximations. Finally, in the third case, the heat flux vector is approximated by its second-order Taylor approximation while first-order is taken to approximate β and T. We want to recall here that similar studies for the Tzou and Choudhuri theories have been developed [3, 21, 25, 26, 27].

For each case we need to set initial and boundary conditions. The boundary conditions can be the same for the three of them. However, the initial conditions will be given individually for each case.

Let B be a three-dimensional bounded domain with boundary smooth enough to apply the divergence theorem. From the definition of θ, it is clear that

$$\int_B \theta^2 dV = \int_B \Big(T^2 + 2a|\nabla T|^2 + a^2|\Delta T|^2\Big)dV \tag{5}$$

when we assume null Dirichlet boundary conditions. Therefore, taking into account the Poincaré inequality, we have

$$\int_B \theta^2 dV \approx \int_B (|\nabla T|^2 + |\Delta T|^2) dV. \tag{6}$$

In this paper we assume that the delay parameters τ_1, τ_2 and τ_3 are nonnegative and, in each section, we will impose several conditions on them to guarantee the stability or instability of the solutions. A similar assumption is made on k_1 and k_2.

2. First case

In this first case we assume that $k_1 = 0$ and $k_2 > 0$ in the basic constitutive equations and, consequently, we do not care about function β. We take a second-order Taylor approximation for the heat flux q_i and a first-order Taylor approximation for the inductive temperature T. Hence, we obtain the following approximate equations for our model:

$$\begin{aligned} q_i(\mathbf{x}, t+\tau_1) &\approx q_i(\mathbf{x}) + \tau_1 \dot{q}_i(\mathbf{x}) + \tfrac{\tau_1^2}{2} \ddot{q}_i(\mathbf{x}), \\ T(\mathbf{x}, t+\tau_2) &\approx T(\mathbf{x}) + \tau_2 \dot{T}(\mathbf{x}). \end{aligned} \tag{7}$$

Replacing the above expressions into the constitutive equations, we obtain the partial differential equation

$$c\left(\dot{\theta} + \tau_1 \ddot{\theta} + \frac{\tau_1^2}{2} \dddot{\theta}\right) = k_2(\Delta T + \tau_2 \Delta \dot{T}) \tag{8}$$

whose solutions we want to study.

To have a well-posed problem we need to impose initial and boundary conditions. We assume null Dirichlet boundary conditions, that is,

$$T(\mathbf{x}, t) = 0 \text{ for } \mathbf{x} \in \partial B \text{ and } t > 0. \tag{9}$$

As far as the initial conditions are concerned, we assume that

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}), \dot{T}(\mathbf{x}, 0) = T_1(\mathbf{x}), \ddot{T}(\mathbf{x}, 0) = T_2(\mathbf{x}) \text{ for } \mathbf{x} \in B. \tag{10}$$

2.1. Well-posedness.

We now show that the problem determined by equation (8), the boundary conditions (9) and the initial conditions (10) is well-posed in the sense of Hadamard. It is worth pointing out that we do not impose any condition on the coefficients τ_1, τ_2 and neither on k_2 apart from the one proposed in the introduction.

We will transform the given problem into an abstract problem involving a convenient Hilbert space. First, we note that $Id - a\Delta : T \to T - a\Delta T = \theta$ is an isomorphism on $W^{2,2}(B) \cap W_0^{1,2}(B)$ and takes values in $L^2(B)$, where $W^{2,2}(B), W_0^{1,2}(B)$ and $L^2(B)$ are the usual Hilbert spaces. We shall denote by $\Phi(\theta) = T$ the inverse operator.

We will work in the Hilbert space

$$\mathcal{H} = L^2(B) \times L^2(B) \times L^2(B). \tag{11}$$

We need to introduce a suitable notation to work in $\mathcal{H}$. Let us denote by $(\theta, \theta^{\{1\}}, \theta^{\{2\}})$ the elements in $\mathcal{H}$. We consider the usual inner product in the Hilbert space $\mathcal{H}$.

To propose a synthetic expression to the above problem, we define the operators:

$$A^*(\theta) = \frac{2k_2}{c\tau_1^2}\Delta\Phi(\theta), \;\; B^*(\theta^{\{1\}}) = \frac{2k_2\tau_2}{c\tau_1^2}\Delta\Phi(\theta^{\{1\}}) - \frac{2}{\tau_1^2}\theta^{\{1\}}, \;\; C(\theta^{\{2\}}) = -\frac{2}{\tau_1^2}\theta^{\{2\}}. \tag{12}$$

Therefore, our problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \;\; U(0) = (\theta_0, \theta_0^{\{1\}}, \theta_0^{\{2\}}), \tag{13}$$

where $\theta_0 = T_0 - a\Delta T_0$, $\theta_0^{\{1\}} = T_1 - a\Delta T_1$, $\theta_0^{\{2\}} = T_2 - a\Delta T_2$ and

$$\mathcal{A} = \begin{pmatrix} 0 & Id & 0 \\ 0 & 0 & Id \\ A^* & B^* & C \end{pmatrix}. \tag{14}$$

We will prove that $\mathcal{A}$ generates a quasi-contractive semigroup. We first note that the domain, $\mathcal{D}$, of $\mathcal{A}$ agrees with the whole Hilbert space $\mathcal{H}$ and then it is dense.

Lemma 2.1. *There exists a positive constant K such that*

$$\langle \mathcal{A}U, U\rangle \leq K||U||^2, \tag{15}$$

for every $U \in \mathcal{D}$.

Proof. Notice that from relation (5) or (6) it is easy to see that the operators A^*, B^* and C are bounded in $L^2(B)$. The proof of the lemma is a direct consequence of this fact. □

Lemma 2.2. *There exists a positive constant δ large enough such that $\delta Id - \mathcal{A}$ is exhaustive.*

Proof. We consider $(f_1, f_2, f_3) \in \mathcal{H}$. We must prove that the system

$$\left.\begin{aligned} \delta\theta - \theta^{\{1\}} &= f_1 \\ \delta\theta^{\{1\}} - \theta^{\{2\}} &= f_2 \\ \delta\theta^{\{2\}} - A^*(\theta) - B^*(\theta^{\{1\}}) - C(\theta^{\{2\}}) &= f_3 \end{aligned}\right\}$$

has a solution in $\mathcal{H}$. After substitution we obtain the equation:

$$\delta^3\theta - A^*(\theta) - \delta B^*(\theta) - \delta^2 C(\theta) = \delta^2 f_1 + \delta f_2 + f_3 - B^*(f_1) - \delta C(f_1) - C(f_2). \tag{16}$$

We first note that the right-hand side of (16) is in $L^2(B)$. As the operators A^*, B^* and C are bounded, we see that the bilinear form

$$M(\theta_1, \theta_2) = \langle \delta^3\theta_1 - A^*(\theta_1) - \delta B^*(\theta_1) - \delta^2 C(\theta_1), \theta_2\rangle \tag{17}$$

is bounded. Moreover, it is coercive for δ large enough. Therefore, equation (16) has a solution. Then, we can also obtain the values for $\theta^{\{1\}}$ and $\theta^{\{2\}}$ and the lemma is proved. □

In view of the Lumer-Phillips corollary to the Hille-Yosida theorem we can state the following result (see [17], page 136).

Theorem 2.3. *The operator $\mathcal{A}$ generates a quasi-contractive semigroup in $\mathcal{H}$.*

As a consequence, we obtain the existence and uniqueness of solutions to our problem.

Theorem 2.4. *For any $U(0) \in \mathcal{H}$, there exists a unique solution to the problem determined by (8)–(10) such that $U(t) \in C^1([0, t_1], \mathcal{H})$.*

Remark 2.5. The continuous dependence of solutions on initial data and supply terms (in case they were assumed) can also be obtained.

These facts prove that the problem is well-posed in the sense of Hadamard.

2.2. **On the stability.** The spectral analysis of the equation can give some information on the behavior of the solutions with respect to time. In fact, we have the following result.

Theorem 2.6. (1) *Let us assume that* $2ac + k_2(2\tau_2 - \tau_1) > 0$. *Then the point spectrum of the problem determined by Equation (8), the boundary conditions (9) and the initial conditions (10) lies on the left-hand side of the line* $\Re z = -\epsilon$, *where* ϵ *is a positive real number.*
(2) *Let us assume that* $2ac+k_2(2\tau_2-\tau_1) < 0$. *Then the solutions to the above stated problem are unstable.*

Proof. Let us assume that there are solutions of Equation (8) with the boundary conditions (9) of the form $T_n(\mathbf{x}, t) = e^{\omega t}\phi_n(\mathbf{x})$, with $\Delta\phi_n(\mathbf{x}) + \lambda_n\phi_n(\mathbf{x}) = 0$ and $\lambda_n > 0$. Replacing $T_n(\mathbf{x}, t)$ in Equation (8) and making the corresponding calculations we get the following equation:

$$e^{t\omega}\phi_n(\mathbf{x})\left(c\tau_1^2(a\lambda_n + 1)\omega^3 + 2c\tau_1(a\lambda_n + 1)\omega^2 + (2c(a\lambda_n + 1) + 2k_2\lambda_n\tau_2)\omega + 2k_2\lambda_n\right) = 0.$$

Denote by $p(\omega)$ the third-order polynomial inside brackets in the left-hand side of the above equation. We are going to prove that for ϵ small enough the elements of the point spectrum are located at the left side of the vertical line $\Re(z) = -\epsilon$. To do so, we replace ω by $\omega - \epsilon$ in $p(\omega)$ and study its roots.

We recall the Routh–Hurwitz criterium. It assesses that all roots of the polynomial

$$a_0x^3 + a_1x^2 + a_2x + a_3$$

have negative real part if and only if $a_i > 0$, for $i = 0, 1, 2, 3$, and $d = a_1a_2 - a_0a_3 > 0$.

In our case, the explicit expressions of the coefficients of $p(\omega - \epsilon)$ are as follows:

$$\begin{aligned}
a_0 &= c\tau_1^2(a\lambda_n + 1),\\
a_1 &= 2c\tau_1(a\lambda_n + 1) - 3c\tau_1^2(a\lambda_n + 1)\epsilon,\\
a_2 &= 2c(a\lambda_n + 1) + 2k_2\lambda_n\tau_2 - 4c\tau_1(a\lambda_n + 1)\epsilon + 3c\tau_1^2(a\lambda_n + 1)\epsilon^2,\\
a_3 &= 2k_2\lambda_n - (2c(a\lambda_n + 1) + 2k_2\lambda_n\tau_2)\,\epsilon + 2c\tau_1(a\lambda_n + 1)\epsilon^2 - c\tau_1^2(a\lambda_n + 1)\epsilon^3.
\end{aligned}$$

In view of the assumptions on the constitutive coefficients and being $\lambda_n > 0$, it is clear that $a_i > 0$ for all i and for ϵ small enough. Nevertheless, the sign of d depends on the relations among the coefficients. In fact,

$$d = 2c(a\lambda_n + 1)\tau_1\Big(\big(2ac + k_2(2\tau_2 - \tau_1)\big)\lambda_n + 2c\Big) + \epsilon(\lambda_n^2S_1 + \lambda_nS_2 + S_3),$$

where S_i, $i = 1, 2, 3$, are polynomial functions involving the coefficients of the system and ϵ.

Notice that, for ϵ small enough, whenever $2ac+k_2(2\tau_2-\tau_1) > 0$, we have $d > 0$ and, consequently, all the roots of $p(\omega)$ have negative real parts. This proves the first part of the theorem.

Moreover, if $2ac + k_2(2\tau_2 - \tau_1) < 0$, then instability of the solutions can be shown. In that case we take $\epsilon = 0$ and λ_n large enough to guarantee that there are solutions of the polynomial $p(\omega)$ with positive real parts. This proves the second part of the theorem. □

From the previous theorem, one suspects that whenever the condition $2ac + k_2(2\tau_2 - \tau_1) > 0$ holds, exponential stability of solutions would be obtained. We will use the energy method to prove the exponential stability of the solutions assuming that $2\tau_2 - \tau_1 > 0$. Notice that, as k_2, a and c are assumed to be positive, this is a stronger condition than $2ac + k_2(2\tau_2 - \tau_1) > 0$.

Theorem 2.7. *Let us assume that* $2\tau_2 - \tau_1 > 0$. *Then the solutions to the problem determined by Equation (8), the boundary conditions (9) and the initial conditions (10) are exponentially stable.*

Proof. Let us denote $\tilde{f} = f + \tau_1 \dot{f} + \frac{\tau_1^2}{2}\ddot{f}$. We define the following function:

$$E(t) = \frac{1}{2}\int_B \Big(c\tilde{\theta}^2 + k_2(\tau_1+\tau_2)\big(|\nabla T|^2 + a(\Delta T)^2\big) + k_2\tau_1^2\big(\nabla T \nabla \dot{T} + a\Delta T \Delta \dot{T}\big) + \frac{k_2\tau_1^2\tau_2}{2}\big(|\nabla \dot{T}|^2 + a(\Delta \dot{T})^2\big)\Big)\, dV. \tag{18}$$

Notice that $E(t)$ is positive, since the quadratic form with matrix

$$A = k_2 \begin{pmatrix} \tau_1 + \tau_2 & \frac{\tau_1^2}{2} \\ \frac{\tau_1^2}{2} & \frac{\tau_1^2\tau_2}{2} \end{pmatrix}$$

is positive definite: its determinant is $\frac{k_2^2\tau_1^2}{4}\big(\tau_1(2\tau_2 - \tau_1) + 2\tau_2^2\big)$, which is positive because of the assumption $2\tau_2 - \tau_1 > 0$. Moreover, this energy function is equivalent to

$$\int_B \big(\theta^2 + \dot{\theta}^2 + \ddot{\theta}^2\big) dV.$$

Our next aim is to compute the time derivative of $E(t)$. Direct calculations give

$$E'(t) = -k_2 \int_B \big(|\nabla T|^2 + a(\Delta T)^2\big) dV - \frac{k_2\tau_1}{2}(2\tau_2 - \tau_1)\int_B \big(|\nabla \dot{T}|^2 + a(\Delta \dot{T})^2\big) dV.$$

There is still one part of $E(t)$ that needs to be controlled. To do so, we consider

$$G_1(t) = \int_B \left(\frac{c}{\tau_1^2}|\dot{\theta}|^2 + \frac{c}{2}|\ddot{\theta}|^2 + \frac{2k_2}{\tau_1^2}(\nabla \dot{T} \nabla T + a\Delta \dot{T} \Delta T) + \frac{k_2\tau_2}{\tau_1^2}(|\nabla \dot{T}|^2 + a|\Delta \dot{T}|^2)\right) dV.$$

The derivative of $G_1(t)$ is given by

$$G_1'(t) = \int_B \left(-\frac{2c}{\tau_1}\ddot{\theta}^2 + \frac{2k_2}{\tau_1^2}\big(|\nabla \dot{T}|^2 + a(\Delta \dot{T})^2\big)\right) dV.$$

Therefore, for ϵ small enough, we take $E + \epsilon G_1$, which is equivalent to E. That is, there exist two positive constants M_1 and M_2 such that

$$M_1\, E(t) \leq E(t) + \epsilon G_1(t) \leq M_2\, E(t).$$

From the previous calculations we also have

$$E'(t) + \epsilon G_1'(t) \leq -\delta^* E(t),$$

for an appropriate positive constant δ^*. Therefore, a usual argument involving several equivalent energy functions proves the exponential stability of the solutions. □

3. Second case

In this section we assume that $k_1 > 0$ and $k_2 > 0$. We take a first-order Taylor approximation for q_i, for β and for T. Therefore, we assume that

$$\begin{aligned} q_i(\mathbf{x}, t + \tau_1) &\approx q_i(\mathbf{x}) + \tau_1 \dot{q}_i(\mathbf{x}), \\ \beta(\mathbf{x}, t + \tau_3) &\approx \beta(\mathbf{x}) + \tau_3 \dot{\beta}(\mathbf{x}), \\ T(\mathbf{x}, t + \tau_2) &\approx T(\mathbf{x}) + \tau_2 \dot{T}(\mathbf{x}). \end{aligned} \tag{19}$$

Substituting these expressions into the constitutive equations, we obtain the following partial differential equation:

$$c\left(\ddot{\theta}+\tau_1\dddot{\theta}\right)=k_1\Delta T+\tau_4\Delta\dot{T}+k_2\tau_2\Delta\ddot{T}, \tag{20}$$

where $\tau_4=\tau_3k_1+k_2$.

We assume the same boundary and initial conditions as in the previous section.

3.1. **Well-posedness.** The arguments proposed in subsection 2.1 can be adapted to this case to obtain again a result on the existence, uniqueness and continuous dependence of solutions. We do not repeat all the procedure because it is very similar to the one done before. Nevertheless, we state the definition of the new operators A^*, B^* and C for this situation (we abuse a little bit the notation and denote the operators as in subsection 2.1):

$$A^*(\theta)=\frac{k_1}{c\tau_1}\Delta\Phi(\theta),\;\; B^*(\theta^{\{1\}})=\frac{\tau_4}{c\tau_1}\Delta\Phi(\theta^{\{1\}}),\;\; C(\theta^{\{2\}})=-\frac{k_2\tau_2}{c\tau_1}\Delta\Phi(\theta^{\{2\}})-\frac{1}{\tau_1}\theta^{\{2\}}. \tag{21}$$

Therefore, we can prove:

Theorem 3.1. *The operator $\mathcal{A}$, defined as in (14), generates a quasi-contractive semigroup in $\mathcal{H}$.*

As a consequence, we obtain the following result:

Theorem 3.2. *For any $U(0)\in\mathcal{H}$, there exists a unique solution to the problem defined by Equation (20) with boundary conditions (9) and initial conditions (10) such that $U(t)\in C^1([0,t_1],\mathcal{H})$.*

This fact proves the well-posedness of the problem.

Let us highlight the fact that no other condition on the parameters τ_1, k_1, τ_2, k_2 and τ_4 has been assumed apart from the one proposed in the introduction.

3.2. **On the stability.** As above, a spectral analysis gives some clues on the conditions on the coefficients to get the exponential stability.

Theorem 3.3.
1. *Let us assume that $\tau_4-k_1\tau_1>0$. Then the solutions to the problem determined by Equation (20), the boundary conditions (9) and the initial conditions (10) are exponentially stable.*
2. *Let us assume that $\tau_4-k_1\tau_1<0$. Then there exist domains B such that the solutions are unstable.*
3. *Les us assume that $k_2\tau_2\tau_4+ac(\tau_4-k_1\tau_1)<0$. Then the solutions are unstable.*

Proof. We first prove the first claim.

We proceed as in Section 2, defining an appropriate energy function for the equation. We abuse a little bit the notation and denote it by $E(t)$ again:

$$\begin{aligned}E(t)=\frac{1}{2}\int_B&\Big(c(\dot{\theta}+\tau_1\ddot{\theta})^2+k_1\big(|\nabla T+\tau_1\nabla\dot{T}|^2\big)+a(\Delta T+\tau_1(\Delta\dot{T})^2)\\&+\big(\tau_1(\tau_4-k_1\tau_1)+k_2\tau_2\big)\big(|\nabla\dot{T}|^2+a(\Delta\dot{T})^2\big)\Big)\,dV,\end{aligned} \tag{22}$$

which, taking into account that $\tau_4-k_1\tau_1>0$, defines a function that is equivalent to

$$\int_B\big(\theta^2+\dot{\theta}^2+\ddot{\theta}^2\big)dV.$$

Direct calculations give

$$E'(t) = -(\tau_4 - k_1\tau_1)\int_B \left(|\nabla \dot{T}|^2 + a(\Delta \dot{T})^2\right) dV - k_2\tau_1\tau_2 \int_B \left(|\nabla \ddot{T}|^2 + a(\Delta \ddot{T})^2\right) dV.$$

As in the previous case, we need an auxiliary function. We define

$$G_2(t) = \int_B \left(c\theta(\dot{\theta} + \tau_1\ddot{\theta}) + \frac{\tau_4}{2}\left(|\nabla T|^2 + a(\Delta T)^2\right) + k_2\tau_2\left(\nabla T \nabla \dot{T} + a\Delta T \Delta \dot{T}\right)\right) dV.$$

We now compute the time derivative of $G_2(t)$:

$$G_2'(t) = \int_B \left(-k_1\left(|\nabla T|^2 + a(\Delta T)^2\right) + c\dot{\theta}(\dot{\theta} + \tau_1\ddot{\theta}) + k_2\tau_2\left(|\nabla \dot{T}|^2 + a(\Delta \dot{T})^2\right)\right) dV.$$

Therefore, as in the previous section, we consider for $\epsilon > 0$ small enough the function $E(t) + \epsilon G_2(t)$, which is equivalent to $E(t)$. As above, we have the inequality $E'(t) + \epsilon G_2'(t) \leq -\delta^* E(t)$, which proves the exponential stability of the solutions.

We now concentrate on the second claim of the theorem. Assuming that there are solutions of the form $T_n(\mathbf{x}, t) = e^{\omega t}\phi_n(\mathbf{x})$ with $\Delta\phi_n(\mathbf{x}) + \lambda_n\phi_n(\mathbf{x}) = 0$ and $\lambda_n > 0$, replacing $T_n(\mathbf{x}, t)$ in Equation (20) and making the corresponding calculations, we obtain again a third-order polynomial,

$$b_0x^3 + b_1x^2 + b_2x + b_3$$

with positive coefficients,

$$\begin{aligned} b_0 &= c\tau_1(a\lambda_n + 1), \\ b_1 &= c + \lambda_n(ac + k_2\tau_2), \\ b_2 &= \tau_4\lambda_n, \\ b_3 &= k_1\lambda_n. \end{aligned}$$

To apply the Routh–Hurwitz criterium we also compute $d' = b_1b_2 - b_0b_3$:

$$d' = \lambda_n^2\left(k_2\tau_2\tau_4 + ac(\tau_4 - k_1\tau_1)\right) + \lambda_n c(\tau_4 - k_1\tau_1).$$

If $\tau_4 - k_1\tau_1 < 0$, instability is obtained, because d' can be negative for λ_n small enough.

The third assertion of the theorem is straightforward from the expression of d'. □

Remark 3.4. When $\tau_4 = k_1\tau_1$ it is possible to also prove the exponential stability. We consider another auxiliary function:

$$H_1(t) = -\int_B \left(\nabla T \nabla \dot{T} + a\Delta T \Delta \dot{T}\right) dV.$$

Therefore, for ϵ small enough we take $E(t) + \epsilon G_2(t) + \epsilon^{3/4}H_1(t)$ which is equivalent to E. Notice that

$$H_1'(t) = -\int_B \left(\nabla \dot{T} \nabla \dot{T} + a\Delta \dot{T} \Delta \dot{T} + \nabla T \nabla \ddot{T} + a\Delta T \Delta \ddot{T}\right) dV.$$

Hence, it can be shown that $E'(t) + \epsilon G_2'(t) + \epsilon^{3/4}H_1'(t) \leq -\delta' E(t)$ using usual arguments again.

4. Third case

We assume again that $k_1 > 0$ and $k_2 > 0$, but we now take a second-order Taylor approximation for q_i and a first-order Taylor approximations for β and T:

$$
\begin{aligned}
&q_i(\mathbf{x}, t+\tau_1) \approx q_i(\mathbf{x}) + \tau_1 \dot{q}_i(\mathbf{x}) + \tfrac{\tau_1^2}{2}\ddot{q}_i(\mathbf{x}),\\
&\beta(\mathbf{x}, t+\tau_3) \approx \beta(\mathbf{x}) + \tau_3 \dot{\beta}(\mathbf{x}),\\
&T(\mathbf{x}, t+\tau_2) \approx T(\mathbf{x}) + \tau_2 \dot{T}(\mathbf{x}).
\end{aligned} \tag{23}
$$

Therefore, replacing these expressions into the constitutive equations, we obtain the following partial differential equation:

$$
c\left(\ddot{\theta} + \tau_1 \dddot{\theta} + \frac{\tau_1^2}{2}\ddddot{\theta}\right) = k_1 \Delta T + \tau_4 \Delta \dot{T} + k_2 \tau_2 \Delta \ddot{T}, \tag{24}
$$

where (as in the previous section) $\tau_4 = \tau_3 k_1 + k_2$.

We assume null Dirichlet boundary conditions (9). As initial conditions, we impose (10), but also

$$
\dddot{T}(\mathbf{x}, 0) = T_3(\mathbf{x}) \text{ for } \mathbf{x} \in B. \tag{25}
$$

4.1. **Well-posedness.** The arguments proposed in subsection 2.1 can be adapted again to this new setting. However, there are some differences which will be pointed out.

The first thing to take into account is that the problem is of fourth-order with respect to time. Therefore, we will need to work in the following Hilbert space (to simplify the notation we denote it again by $\mathcal{H}$):

$$
\mathcal{H} = L^2(B) \times L^2(B) \times L^2(B) \times L^2(B). \tag{26}
$$

The elements in this space will be denoted by $(\theta, \theta^{\{1\}}, \theta^{\{2\}}, \theta^{\{3\}})$. We will work with the operators (we abuse the notation again)

$$
\begin{aligned}
&A^*(\theta) = \tfrac{2k_1}{c\tau_1^2}\Delta\Phi(\theta), && B^*(\theta^{\{1\}}) = \tfrac{2\tau_4}{c\tau_1^2}\Delta\Phi(\theta^{\{1\}}),\\
&C(\theta^{\{2\}}) = -\tfrac{2k_2\tau_2}{c\tau_1^2}\Delta\Phi(\theta^{\{2\}}) - \tfrac{2}{\tau_1^2}\theta^{\{2\}}, && D(\theta^{\{3\}}) = -\tfrac{2}{\tau_1}\theta^{\{3\}}.
\end{aligned} \tag{27}
$$

We can state the problem in the form

$$
\frac{dU}{dt} = \mathcal{A}U, \;\; U(0) = (\theta_0, \theta_0^{\{1\}}, \theta_0^{\{2\}}, \theta_0^{\{3\}}), \tag{28}
$$

where $\theta_0, \theta_0^{\{1\}}, \theta_0^{\{2\}}$ are defined as in subsection 2.1, $\theta_0^{\{3\}} = T_3 - a\Delta T_3$ and

$$
\mathcal{A} = \begin{pmatrix} 0 & Id & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & Id \\ A^* & B^* & C & D \end{pmatrix}
$$

The domain of this operator is the whole Hilbert space $\mathcal{H}$ and being the operators A^*, B^*, C, and D bounded in $L^2(B)$, we can obtain an inequality similar to that given in (15).

To prove that the operator $\mathcal{A}$ generates a quasi-contractive semigroup and the existence and uniqueness theorem we need to show that $\delta Id - \mathcal{A}$ is exhaustive (as in Lemma 2.2). Hence, we consider an element $(f_1,f_2,f_3,f_4)\in\mathcal{H}$ and we have to prove that the system

$$\left.\begin{aligned}\delta\theta-\theta^{\{1\}}&=f_1\\ \delta\theta^{\{1\}}-\theta^{\{2\}}&=f_2\\ \delta\theta^{\{2\}}-\theta^{\{3\}}&=f_3\\ \delta\theta^{\{3\}}-A^*(\theta)-B^*(\theta^{\{1\}})-C(\theta^{\{2\}})-D(\theta^{\{3\}})&=f_4\end{aligned}\right\}$$

has a solution in $\mathcal{H}$.

After substitution we obtain that

$$\begin{aligned}\delta^4\theta-A^*(\theta)-\delta B^*(\theta)-\delta^2C(\theta)-\delta^3D(\theta)=\\ \delta^3f_1+\delta^2f_2+\delta f_3+f_4-B^*(f_1)-\delta C(f_1)-\delta^2D(f_1)-C(f_2)-\delta D(f_2)-D(f_3).\end{aligned}\tag{29}$$

An argument similar to the one proposed to solve equation (16) can be used to prove the existence of a solution θ to this equation in $L^2(B)$. Hence we can obtain the solutions for $\theta^{\{1\}},\theta^{\{2\}},\theta^{\{3\}}$. Therefore, we have obtained the following result.

Theorem 4.1. *The operator $\mathcal{A}$ generates a quasi-contractive semigroup.*

As a consequence, we obtain the existence and uniqueness of solutions.

Theorem 4.2. *For any $U(0)\in\mathcal{H}$, there exists a unique solution to the problem determined by Equation (24), boundary conditions (9) and initial conditions (10) and (25) such that $U(t)\in C^1([0,t_1],\mathcal{H})$.*

Summarizing: we see that the problem considered in this section is well-posed in the sense of Hadamard without imposing any other condition on the sign of τ_1,τ_2,k_1,k_2 and τ_4 apart from the ones proposed in the introduction.

4.2. **On the stability.** We now study the time behavior of the solutions to this problem.

Theorem 4.3.
(1) *Let us assume that $2(ac+k_2\tau_2)\tau_4>2ack_1\tau_1+\tau_1\tau_4^2$ and $\tau_4>k_1\tau_1$. Then the point spectrum of the problem determined by Equation (24), the boundary conditions (9) and the initial conditions (10) and (25) lies on the left-hand side of the line $\Re z=-\epsilon$, where ϵ is a positive real number.*
(2) *Let us assume that $\tau_4<k_1\tau_1$. Then there exist domains B such that the solutions to the above stated problem are unstable.*
(3) *Les us assume that $2(ac+k_2\tau_2)\tau_4<2ack_1\tau_1+\tau_1\tau_4^2$. Then the solutions to the above stated problem are unstable.*

Proof. As in the previous sections, we first make a spectral analysis to get some possible conditions on the coefficients for the exponential stability. We now obtain a fourth-order polynomial

$$c_0x^4+c_1x^3+c_2x^2+c_3x+c_4$$

whose coefficients are

$c_0=c\tau_1^2(1+a\lambda_n)$,
$c_1=2c\tau_1(1+a\lambda_n)(1-2\epsilon\tau_1)$,
$c_2=2c(1+a\lambda_n)+2k_2\tau_2\lambda_n-6c(1+a\lambda_n)\tau_1\epsilon+6c(1+a\lambda_n)\tau_1^2\epsilon^2$,
$c_3=2\lambda_n\tau_4-2\epsilon\left(2c\left(a\lambda_n+1\right)+2k_2\lambda_n\tau_2\right)+6c\tau_1\epsilon^2\left(a\lambda_n+1\right)-4c{\tau_1}^2\epsilon^3\left(a\lambda_n+1\right)$,
$c_4=2k_1\lambda_n-2\tau_4\lambda_n\epsilon+\left(2c\left(a\lambda_n+1\right)+2k_2\lambda_n\tau_2\right)\epsilon^2-2c\tau_1\epsilon^3\left(a\lambda_n+1\right)+c{\tau_1}^2\epsilon^4\left(a\lambda_n+1\right)$.

We note that $c_i>0$ for all i when ϵ is small enough.

We now apply the Routh–Hurwitz criterium to locate the elements on the point spectrum. We have to study the sign of the determinants of four different matrices

$$D_1 = (c_1/c_0), D_2 = \begin{pmatrix} c_1 & c_0 \\ c_3 & c_2 \end{pmatrix}, D_3 = \begin{pmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & c_4 & c_3 \end{pmatrix} \text{ and } D_4 = \begin{pmatrix} c_1 & c_0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 \\ 0 & c_4 & c_3 & c_2 \\ 0 & 0 & 0 & c_4 \end{pmatrix}.$$

For ϵ small enough, it is clear that the determinant of D_1 is positive: $\det D_1 = 2/\tau_1 - 4\epsilon$.

The determinant of D_2 is given by

$\det D_2 = \big(2ac\tau_1(2ac+2k_2\tau_2-\tau_1\tau_4+q_1(\epsilon)\big)\lambda_n^2+\big(2c\tau_1(4ac+2k_2\tau_2-\tau_1\tau_4+q_2(\epsilon)\big)\lambda_n+4c^2\tau_1+q_3(\epsilon),$

where the $q_i(\epsilon)$ are polynomials of ϵ depending on the parameters and such that $q_i(0) = 0$.

In view of the first assumption on the parameters, it is clear that $\det D_2 > 0$ for ϵ small enough.

The determinant of D_3 is a little bit more complicated. It is a third-order polynomial of λ_n given by

$$\begin{aligned} \det D_3 = {}& \big(4ac\tau_1(2ac\tau_4 + 2k_2\tau_2\tau_4 - 2ack_1\tau_1 - \tau_1\tau_4^2) + r_1(\epsilon)\big)\lambda_n^3 \\ & +\big(4c\tau_1(4ac\tau_4 + 2k_2\tau_2\tau_4 - 4ack_1\tau_1 - \tau_1\tau_4^2) + r_2(\epsilon)\big)\lambda_n^2 \\ & +\big(8c^2\tau_1(\tau_4 - k_1\tau_1) + r_3(\epsilon)\big)\lambda_n + r_4(\epsilon), \end{aligned}$$

where the $r_i(\epsilon)$ are polynomials of ϵ depending on the parameters and such that $r_i(0) = 0$.

Again, in view of the hypotheses on the coefficients, $\det D_3 > 0$ for ϵ small enough. The sign of $\det D_4$ coincides with the sign of $\det D_3$ when ϵ is small enough and, therefore, we do not need to compute it. This proves the first claim of the theorem.

To prove the second claim we can consider $\det D_3$ with $\epsilon = 0$. When we assume that $\tau_4 < k_1\tau_1$, this determinant is negative for λ_n small enough.

Finally, the third claim also follows by considering $\det D_3$ with $\epsilon = 0$. As we can take λ_n as large as necessary, this determinant will be negative. □

As in the previous sections, we will prove the exponential stability using energy methods under the assumptions proposed above. As the conditions on the coefficients are fairly complicated, we will consider stronger but easier ones: $\tau_4 > k_1\tau_1$ and $2k_2\tau_2 > \tau_1\tau_4$.

Theorem 4.4. *Let us assume that $\tau_4 > k_1\tau_1$ and $2k_2\tau_2 > \tau_1\tau_4$. Then the solutions to the problem determined by Equation (24), the boundary conditions (9) and the initial conditions (10) and (25) are exponentially stable.*

Proof. We define the energy function (that we again denote by $E(t)$)

$$\begin{aligned} E(t) = \frac{1}{2}\int_B \Big(& c(\dot{\tilde{\theta}})^2 + k_1\Big(|\nabla(T + \tau_1\dot{T} + \frac{\tau_1^2}{2}\ddot{T})|^2 + a|\Delta(T + \tau_1\dot{T} + \frac{\tau_1^2}{2}\ddot{T})|^2\Big) \\ & +\tau_1(\tau_4 - k_1\tau_1)\Big(|\nabla(\dot{T} + \frac{\tau_1^2}{2}\ddot{T})|^2 + a|\Delta(\dot{T} + \frac{\tau_1^2}{2}\ddot{T})|^2\Big)+ \\ & +(k_2\tau_2 - k_1\frac{\tau_1^2}{2})\Big(|\nabla\dot{T}|^2 + a|\Delta\dot{T}|^2\Big) + \frac{\tau_1^2}{2}(k_2\tau_2 - \frac{\tau_1\tau_4}{2})\Big(|\nabla\ddot{T}|^2 + a|\Delta\ddot{T}|^2\Big)\Big)\, dV. \end{aligned} \tag{30}$$

Direct calculations give

$$E'(t) = -(\tau_4 - k_1\tau_1)\int_B \Big(|\nabla\dot{T}|^2 + a|\Delta\dot{T}|^2\Big)dV - \tau_1\big(k_2\tau_2 - \frac{\tau_1\tau_4}{2}\big)\int_B \Big(|\nabla\ddot{T}|^2 + a|\Delta\ddot{T}|^2\Big)dV,$$

which, taking into account the assumptions on the coefficients, is negative.

We also consider

$$G_3(t) = \frac{1}{2}\int_B \Bigg(\frac{\tau_1^2}{2}c(\dddot{\theta})^2 + c(\ddot{\theta})^2 + k_2\tau_2\Big(|\nabla\ddot{T}|^2 + a|\Delta\ddot{T}|^2\Big) + 2\tau_4\Big(\nabla\dot{T}\nabla\ddot{T} + a\Delta\dot{T}\Delta\ddot{T}\Big) + 2k_1\Big(\nabla T\nabla\ddot{T} + a\Delta T\Delta\ddot{T}\Big)\Bigg)\,dV \tag{31}$$

and

$$G_4(t) = \frac{\tau_4}{2}\int_B \Big(\Big(|\nabla T|^2 + a|\Delta T|^2\Big) + k_2\tau_2\Big(\nabla T\nabla\dot{T} + a\Delta T\Delta\dot{T}\Big) + c\theta\dot{\ddot{\theta}}\Big)\,dV. \tag{32}$$

Computing the time derivatives of both functions we obtain

$$G_3'(t) = -\tau_1 c\int_B |\dddot{\theta}|^2 dV + \int_B \Big(\tau_4\big(|\nabla\ddot{T}|^2 + a|\Delta\ddot{T}|^2\big) + k_1\big(\nabla\dot{T}\nabla\ddot{T} + a\Delta\dot{T}\Delta\ddot{T}\big)\Big)\,dV$$

and

$$G_4'(t) = -k_1\int_B \big(|\nabla T|^2 + a|\Delta T|^2\big)dV + k_2\tau_2\int_B \Bigg(\big(|\nabla\dot{T}|^2 + a|\Delta\dot{T}|^2\big) + c|\dot{\theta}|^2 + c\tau_1\dot{\theta}\ddot{\theta} + c\frac{\tau_1^2}{2}\dot{\theta}\dddot{\theta}\Bigg)\,dV.$$

Hence, if we consider $\epsilon_2 G_4(t) + G_3(t)$, we can see that

$$\epsilon_2 G_4'(t) + G_3'(t) \le -\epsilon_2 k_1\int_B \big(|\nabla T|^2 + a|\Delta T|^2\big)dV - \frac{c\tau_1}{2}\int_B |\dddot{\theta}|^2 dV + C^*\int_B \Big(|\nabla\dot{T}|^2 + a|\Delta\dot{T}|^2 + |\nabla\ddot{T}|^2 + a|\Delta\ddot{T}|^2\Big)\,dV,$$

where C^* is a computable positive constant.

Finally, if we consider $E + \epsilon\big(G_3 + \epsilon_2 G_4\big)$, which is equivalent to E, we can obtain the desired inequality. □

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