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EXPONENTIAL STABILITY TO LOCALIZED TYPE III THERMOELASTICITY

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ABSTRACT. This paper investigates the system proposed by a wave equation and a heat equation of type III in one part of the domain; a wave equation and a heat equation of type II in another part of the domain, coupled in a certain pattern. In this paper we prove the exponential stability of the solutions under suitable conditions for the thermal conductivity and coupling term.

1. INTRODUCTION

The model for heat conduction based on the Fourier's law and combined with the energy equation

$$c\dot{\theta} = \nabla \cdot q$$

allows the *infinite speed of propagation phenomena*. This is not well accepted from the physical point of view, because it violates the causality principle. Several modifications in the constitutive relations have been proposed along the years to overcome this drawback. The most known is the one proposed by Maxwell and Cattaneo [23]. This theory modifies Fourier's law and introduces a relaxation time to obtain the constitutive equation

$$\chi_0 q_t + q = b \nabla \theta,$$

where q is the heat flux, χ_0 is the relaxation time, b is the thermal conductivity and θ is the temperature. The combination of this relation with the previous energy equation brings to a heat conduction equation of hyperbolic type with dissipation. There are two thermoelastic theories having this heat conduction equation. Green and Lindsay [1] introduced one and the second was proposed by Lord and Shulman [16]. In both cases, the thermoelasticity is described by a system of hyperbolic equations. The hyperbolic heat equation has not been the only alternative proposition to the parabolic equation of heat. We can also cite (among others) the two temperatures model suggested by Chen, Gurtin and Williams [24–26] or the time reversal thermoelasticity [28] to overcome the difficulties remarked before.

In the decade of the 90's Green and Naghdi proposed three thermoelastic theories [2–4] which were based on an entropy balance law rather than the usual entropy inequality¹. They were proposed from a rational point of view and were based in the axioms of thermomechanics. The main difference between them came from the choice of the independent variables. If the independent variables were the gradient of deformation, the temperature and the gradient of temperature one obtains the theory that Green and Naghdi called of Type I. The system obtained when we

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¹This represents an alternative way to obtain the basic constitutive equations in thermomechanics

consider the linear version coincides with the classical system of the thermoelasticity based on the Fourier law and it proposes an hyperbolic/parabolic system. The Type II theory corresponds to assume that the independent variables are the gradient of deformation, the temperature and the gradient of the thermal displacement, where the thermal displacement is defined by

$$\tau(\cdot, t) = \int_0^t \theta(\cdot, s) ds + \tau(\cdot, 0).$$

This theory is also known as *thermoelasticity without energy dissipation*, because in this case the energy is not dissipated. It also proposes an hyperbolic/hyperbolic system, which is very different from the ones proposed by Lord and Shulman or Green and Lindsay. The most general system is the one called of Type III, because in this case the independent variables are the gradient of deformation, the temperature, the gradient of the thermal displacement and the gradient of the temperature. It is worth noting that Type I and Type II are two limiting cases of it. Type III proposes an equation, which is similar to the one proposed for the Kelvin-Voigt viscoelasticity, for the heat conduction. Therefore we also obtain a hyperbolic/parabolic system which is very different from the one proposed for the Type I. It is worth noting the big difference between the time behaviour for these two last theories. Meanwhile the Type II theory is conservative, we have stability and we do not have asymptotic stability, the Type III thermoelasticity proposes generically the asymptotic stability [21] and in the one-dimensional case we always have exponential stability (see [19, 21]).

In order to clarify the applicability of these new theories it is needed a mathematical and physical study. In fact, they are under a deep mathematical interest. It is suitable to recall the works of Hetnarski and Ignaczak [5, 6] as well as the books Ignaczak and Ostoja-Starzewski [7] and Straughan [20]. As we are going to focus our attention to the new theories of Green and Naghdi it is suitable to say that they have attracted much interest in the last years (see [22, 31–36, 38–41, 43], among others) .

In the absence of source terms the field equations that govern the one-dimensional version of the linear Type III thermoelastic theory are

$$\rho(x) u_{tt} - (a(x)u_x - \beta(x)\theta)_x = 0, \quad (1.1)$$

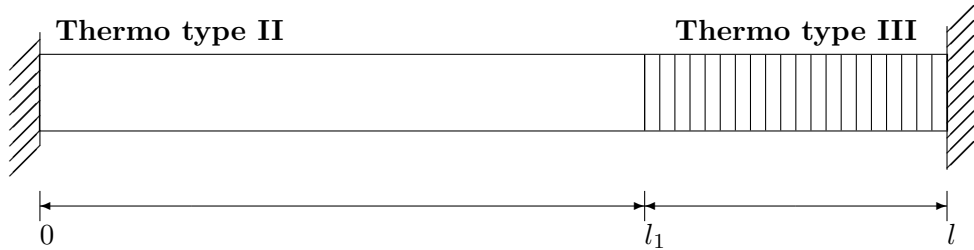
$$c(x) \tau_{tt} + \beta(x) u_{tx} - (k(x)\tau_x + b(x)\theta_x)_x = 0. \quad (1.2)$$

Here $\rho(x)$ is the mass density, $a(x)$ is the elasticity, $\beta(x)$ is the coupling term, $c(x)$ is the thermal capacity, $k(x)$ is the conductivity rate which is a constitutive function typical in the Green and Naghdi theories and $b(x)$ is the thermal conductivity which must be non-negative by virtue of the basic axioms of the thermomechanics. This system of equations defines the one-dimensional Type II thermoelasticity theory when $b(x) = 0$.

An interesting question to clarify in continuous mechanics is the decay of the perturbations when the dissipation of the system is localized [27, 29, 30, 37, 42, 44, 45]. That is when the dissipation only holds in a proper part of the body. Looking for such kind of situation, we can consider the corresponding problem for the Type III thermoelasticity when the thermal

conductivity is strictly positive only in a proper sub-domain of the domain. In that case we have that the energy dissipates only in a proper part meanwhile in the remains of the body we have the Type II system. To be precise we will consider the oscillations of a string configured over the interval $]0, \ell[$ with thermoelasticity of type III. This is a typical problem called *transmission problem*. We have a dissipative structure in a part of the domain and a conservative structure in the remains. The natural question is to know how the behaviour of the dissipative part transmits to the whole domain. We here assume that the dissipative effect of the difference of temperature is effective only over an interval $]\ell_1, \ell[$.

Model



In this paper we consider the problem determined by the system

$$u_{tt} - (au_x - \beta\tau_t)_x = 0, \quad]0, \ell[\times]0, \infty[\tag{1.3}$$

$$\tau_{tt} - (k\tau_x + b\tau_{xt})_x + \beta u_{xt} = 0, \quad]0, \ell[\times]0, \infty[\tag{1.4}$$

where a, k are positive constants, but b is a function of the material point as well as β . The meaning of the positivity of the constants a and k is related with the elastic stability and the non-negativity of $b(x)$ is a consequence of the basic axioms of thermomechanics as we are pointed out before.

To get a well-posed initial-boundary value problem we need to prescribe the initial data

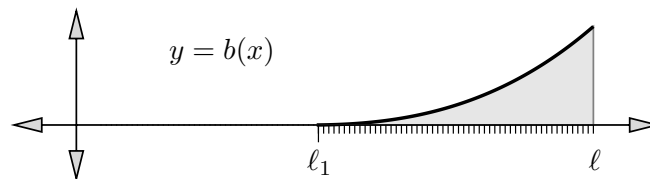
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in }]0, \ell[, \tag{1.5}$$

$$\tau(x, 0) = \tau_0(x), \quad \tau_t(x, 0) = \tau_1(x), \quad \text{in }]0, \ell[, \tag{1.6}$$

and the boundary conditions

$$u(0, t) = u(\ell, t) = \tau(0, t) = \tau(\ell, t) = 0, \quad \forall t > 0. \tag{1.7}$$

Here we consider $b \in C^1(0, \ell)$ as a non-negative function such that $\text{sup}(b) \subset]0, \ell[$, and whose graphic is given by



That is, we have a system that is of Type II in a certain sub-domain and of the Type III in the remains. We want to know the time decay behaviour of this system. Liu and Quintanilla [12] obtained an answer to this question in the case when that the coupling function is given by a constant and under suitable conditions of regularity for the thermal conductivity. The authors proved that the solutions decay in a polynomial way. This means that the decay of solutions is upper bounded by a polynomial function. In this paper, we have proposed how to improve this result and to know when an exponential negative can control the decay, which is a faster rate of decay. However to obtain this new kind of decay we need to change a little bit the assumptions (see (2.1)). We need to impose again a certain regularity condition on the thermal conductivity and that the coupling function must be controlled by the thermal conductivity in the neighbourhood of the point where the thermal conductivity vanishes. Certainly this condition is very different from the one proposed in [12] and it implies that the coupling function cannot be a constant; however the thermomechanical meaning of this condition does not seem transparent.

From the mathematical point of view we are working with a model with localized dissipation acting on the temperature distribution. We are dealing with a coupled model where the motion equation is of the hyperbolic type, but the equation for the temperature difference does not have a well defined classification, it is certainly hyperbolic on one component of the material, but on the component associated with type III thermoelasticity it no longer is. As in the wave equation with localized dissipation, the main tool to show uniform stability is the observability inequalities. However, due to the non-hyperbolic nature of the system, the observability is not immediate and a more complex procedure is necessary to obtain the corresponding inequalities.

In the literature we also have models to thermoelastic beams, which are described by equations of fourth order for the movement equation, these models have a different nature to the ones studied here, for example when the Fourier law is used for the heat flow the corresponding system defines an analytical semigroup as shown by Lasiecka and Triggiani [48] and also Renardy and Liu [49]. To non linear models see [46, 47]

In the next section we recall the semigroup approach to the proposed problem and in the last section we obtain the exponential decay of solutions under suitable conditions for the functions $b(x)$ and $\beta(x)$.

2. THE SEMIGROUP APPROACH

In this section we recall the basic properties of the solutions of our problem.

Here we impose the basic hypotheses on the functions $b(x)$ and $\beta(x)$. We assume that there exists a positive constant c such that the following conditions

$$|b'(x)|^2 \leq c|b(x)|, \quad |\beta| + |\beta'|^2 \leq c|b(x)|, \quad \forall x \in]\ell_1, \ell_1 + \delta[. \quad (2.1)$$

hold, with δ small, but positive. It is worth noting that we assume that the only point where the function β vanishes inside the interval $[\ell_1, \ell]$ is the point ℓ_1 .

We define the phase space where we are going to work

$$\mathcal{H} = H_0^1(0, \ell) \times L^2(0, \ell) \times H_0^1(0, \ell) \times L^2(0, \ell)$$

with the norm

$$\|z\|_{\mathcal{H}}^2 = \int_0^\ell |v(s)|^2 + a|u_x(s)|^2 + |\theta(s)|^2 + k|\tau_x(s)|^2 dx$$

where $z = (u, v, \tau, \theta)$. The operator \mathcal{A} is given by

$$\mathcal{A}z = \begin{pmatrix} u_t \\ (au_x + \beta\theta)_x \\ \tau_t \\ (k\tau_x + b\theta_x)_x - \beta v_x \end{pmatrix}.$$

We have that

$$D(\mathcal{A}) = \{z \in \mathcal{H}; v, \theta \in H_0^1(0, \ell), u_x, (k\tau_x + b\theta_x) \in H^1(0, \ell)\}.$$

So we have the following result:

Theorem 2.1. *Under the above conditions the operator \mathcal{A} is the infinitesimal generator of a contraction semigroup over the space \mathcal{H} .*

PROOF.- The proof follows by standard procedures.

Note that this damping produces a class of invariants semigroups. In fact, let us consider the space \mathbb{V} defined by

$$\mathbb{V} = \left\{ w \in L^2(0, \ell); \int_0^\ell b(x)|w_x|^2 dx < \infty \right\}$$

It is not difficult to prove that the space

$$\mathcal{H}_b = H_0^1(0, \ell) \times L^2(0, \ell) \times H_0^1(0, \ell) \times \mathbb{V}$$

is an invariant subspace of the semigroup $S(t) = e^{\mathcal{A}t}$. We also note that this is a Hilbert space with the norm

$$\|z\|_{\mathcal{H}_b}^2 = \int_0^\ell |v(s)|^2 + a|u_x(s)|^2 + |\theta(s)|^2 + b|\theta_x(s)|^2 + k|\tau_x(s)|^2 dx.$$

3. EXPONENTIAL STABILITY

In this section we prove the exponential stability of the solutions to the proposed problem whenever the conditions on the functions $b(x)$ and $\beta(x)$ hold. The main tool we use is the characterizations of the exponential stability establish by Huang and Pruess [8, 18], that is

Theorem 3.1. *A contraction semigroup $S(t) = e^{\mathcal{A}t}$ over a Hilbert space \mathcal{H} is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall \lambda \in \mathbb{R}$$

Lemma 3.2. *Under the above conditions, $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. The proof can be obtained in a similar way to the one proposed in [12]. □

Given $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}_b$ and $\lambda \in \mathbb{R}$, where $|\lambda|$ large enough, we consider the resolvent equation. Let $z = (u, v, \tau, \theta)$ be the unique solution of the resolvent equation. In terms of its components we can write

$$i\lambda u - v = f_1, \quad]0, \ell[\quad (3.1)$$

$$i\lambda v - (au_x + \beta\theta)_x = f_2, \quad]0, \ell[\quad (3.2)$$

$$i\lambda\tau - \theta = f_3, \quad]0, \ell[\quad (3.3)$$

$$i\lambda\theta - (k\tau_x + b\theta_x)_x + \beta v_x = f_4, \quad]0, \ell[. \quad (3.4)$$

We see that

$$\int_0^\ell b|\theta_x|^2 dx \leq C\|z\|\|F\| \quad (3.5)$$

where C is a calculable positive constant.

Let us introduce the following notations. For the interval where the local continuous viscosity is effective we denote as

$$I_u(s) = |v(s)|^2 + a|u_x(s)|^2, \quad E_u(l_1, l_2) = \int_{l_1}^{l_2} I_u(s) ds$$

$$I_\tau(s) = |\theta(s)|^2 + |k\tau_x(s) + b\theta_x(s)|^2, \quad E_\tau(l_1, l_2) = \int_{l_1}^{l_2} I_b(s) ds$$

Finally, let us denote as

$$P_\ell = au_x(\ell)\beta[k\bar{\tau}_x(\ell) + b(\ell)\bar{\theta}_x(\ell)].$$

Note that

$$|P_\ell| \leq C^* I_\tau(\ell)^{1/2} I_u(\ell)^{1/2}$$

where C^* is another calculable positive constant.

Lemma 3.3. *Let us suppose that conditions (2.1) hold. Then for any $\epsilon > 0$ there exists a positive constant C_ϵ such that*

$$\int_0^\ell b|\lambda\theta|^2 dx \leq C_\epsilon\|z\|\|F\| + C_\epsilon\|F\|^2 + \epsilon\|z\|^2 + C|P_\ell| \quad (3.6)$$

whenever $|\lambda|$ is large enough.

Proof. We multiply the relation (3.4) by $\overline{i\lambda b\theta}$, after an integration it follows that

$$\begin{aligned}
\int_0^\ell b|\lambda\theta|^2 dx &= \operatorname{Re} \int_0^\ell (k\tau_x + b\theta_x)_x \overline{i\lambda b\theta} dx - \operatorname{Re} \int_0^\ell \beta v_x \overline{i\lambda b\theta} dx + \operatorname{Re} \int_0^\ell f_4 \overline{i\lambda b\theta} dx \\
&= -\operatorname{Re} \int_0^\ell (k\tau_x + b\theta_x) \overline{i\lambda b'\theta} dx - \operatorname{Re} \int_0^\ell (k\tau_x + b\theta_x) \overline{i\lambda b\theta_x} dx - \operatorname{Re} \int_0^\ell \beta v_x \overline{i\lambda b\theta} dx + R \\
&= -\operatorname{Re} \int_0^\ell (k\tau_x \overline{i\lambda b'\theta} + b\theta_x \overline{i\lambda b'\theta}) dx - \operatorname{Re} \int_0^\ell k\tau_x \overline{i\lambda b\theta_x} dx - \operatorname{Re} \int_0^\ell \beta v_x \overline{i\lambda b\theta} dx + R \\
&= -\operatorname{Re} \int_0^\ell (kb'\theta_x \bar{\theta} + bb'\theta_x \overline{i\lambda\theta}) dx + \operatorname{Re} \int_0^\ell kb|\theta_x|^2 dx - \operatorname{Re} \int_0^\ell \beta v_x \overline{i\lambda b\theta} dx + R \\
&\leq C_\epsilon \int_0^\ell b|\theta_x|^2 dx + \epsilon \int_0^\ell |\theta|^2 dx + \epsilon \int_0^\ell b|\lambda\theta|^2 dx \\
&+ \operatorname{Re} \int_0^\ell kb|\theta_x|^2 dx - \operatorname{Re} \int_0^\ell \beta v_x \overline{i\lambda b\theta} dx + R \\
&\leq C_\epsilon \int_0^\ell b|\theta_x|^2 dx + \epsilon \|z\|^2 + \epsilon \int_0^\ell b|\lambda\theta|^2 dx - \underbrace{\operatorname{Re} \int_0^\ell \beta v_x \overline{i\lambda b\theta} dx}_{:=I} + R. \tag{3.7}
\end{aligned}$$

Here R is such that

$$|R| \leq \epsilon \int_0^\ell b|\lambda\theta|^2 dx + c_\epsilon \|F\|^2.$$

So we have that

$$\begin{aligned}
I &= - \int_{\ell_1}^\ell i\lambda v [\beta \bar{b}\bar{\theta}]_x dx \\
&= - \int_{\ell_1}^\ell i\lambda v [\beta b]'\bar{\theta} dx - \int_{\ell_1}^\ell i\lambda v \beta b \bar{\theta}_x dx \\
&= - \int_{\ell_1}^\ell i\lambda v [\beta b]'\bar{\theta} dx - \int_{\ell_1}^\ell i\lambda v \beta [k\bar{\tau}_x + b\bar{\theta}_x] dx + \int_{\ell_1}^\ell i\lambda v \beta k \bar{\tau}_x dx.
\end{aligned}$$

Using equation (3.2) we get

$$\begin{aligned}
I &= - \int_{\ell_1}^\ell [au_{xx} + (\beta\theta)_x][\beta b]'\bar{\theta} dx - \int_{\ell_1}^\ell [au_{xx} + (\beta\theta)_x]\beta[k\bar{\tau}_x + b\bar{\theta}_x] dx - \int_{\ell_1}^\ell v\beta k \bar{\theta}_x dx + R_1 \\
&= - \int_{\ell_1}^\ell [au_{xx}][\beta b]'\bar{\theta} dx - \int_{\ell_1}^\ell [au_{xx}]\beta[k\bar{\tau}_x + b\bar{\theta}_x] dx + Q_0 + R_1
\end{aligned}$$

where

$$|R_1| \leq C \|z\| \|F\|$$

and

$$Q_0 = - \int_{\ell_1}^\ell [(\beta\theta)_x][\beta b]'\bar{\theta} dx - \int_{\ell_1}^\ell [(\beta\theta)_x]\beta[k\bar{\tau}_x + b\bar{\theta}_x] dx - \int_{\ell_1}^\ell v\beta k \bar{\theta}_x dx.$$

We note that there are constant c and c_ϵ

$$|Q_0| \leq \epsilon \|z\|^2 + \frac{c}{|\lambda|^2} \int_{\ell_1}^\ell b|\lambda\theta|^2 dx + c_\epsilon \int_{\ell_1}^\ell b|\theta_x|^2 dx.$$

From now on we will denote by Q_i , $i \geq 1$ any function satisfying the above inequality with suitable positive constants.

$$\begin{aligned}
I &= \int_{\ell_1}^{\ell} au_x ([\beta b]' \bar{\theta})_x dx + \underbrace{au_x(\ell)\beta[k\bar{\tau}_x(\ell) + b(\ell)\bar{\theta}_x(\ell)]}_{:=P_\ell} - \int_{\ell_1}^{\ell} au_x \beta [k\bar{\tau}_x + b\bar{\theta}_x]_x dx \\
&\quad - \underbrace{\int_{\ell_1}^{\ell} au_x \beta' [k\bar{\tau}_x + b\bar{\theta}_x] dx}_{:=Q_1} + Q_0 + R_1 \\
&= - \int_{\ell_1}^{\ell} au_x \beta [i\lambda\bar{\theta} + \beta\bar{v}_x] dx + P_\ell + \underbrace{\int_{\ell_1}^{\ell} au_x [\beta b]'' \bar{\theta} dx + \int_{\ell_1}^{\ell} au_x [\beta b]' \bar{\theta}_x dx}_{:=Q_2} + Q_1 + R_1.
\end{aligned}$$

Taking the real part we get

$$\begin{aligned}
\operatorname{Re} I &= -\operatorname{Re} \int_{\ell_1}^{\ell} au_x \beta [i\lambda\bar{\theta} + \beta\bar{v}_x] dx + \operatorname{Re} P_\ell + \operatorname{Re} Q_2 + \operatorname{Re} R_1 \\
&= -\operatorname{Re} \int_{\ell_1}^{\ell} au_x \beta [i\lambda\bar{\theta}] dx + \operatorname{Re} P_\ell + \operatorname{Re} Q_2 + \operatorname{Re} R_1 \\
&= \operatorname{Re} P_\ell + \operatorname{Re} \int_{\ell_1}^{\ell} au \beta' [i\lambda\bar{\theta}] dx + \underbrace{\operatorname{Re} \int_{\ell_1}^{\ell} au \beta [i\lambda\bar{\theta}_x] dx}_{:=Q_3} + \operatorname{Re} Q_2 + \operatorname{Re} R_1.
\end{aligned}$$

We also note that

$$\operatorname{Re} \int_{\ell_1}^{\ell} au \beta' [i\lambda\bar{\theta}] dx \leq \epsilon \int_{\ell_1}^{\ell} b |\lambda\theta|^2 dx + \frac{C_\epsilon}{\lambda^2} \int_{\ell_1}^{\ell} (|v|^2 + |F|^2) dx.$$

After substitution into (3.7) our conclusion follows. \square

Lemma 3.4. *For any $\ell_2, \ell_3 \in [\ell_1, \ell]$ and q where q will be one of the following options: $q = x - \ell_2$, $q = x - \ell_3$ or $q = x - (\ell_2 + \ell_3)/2$, and for all λ large enough, we have that*

$$E_u(\ell_2, \ell_3) \leq q(\ell) I_u(\ell) |_{\ell_2}^{\ell_3} + C \|z\| \|F\| + C_\epsilon \|F\|^2 + \epsilon \|z\|^2 + \frac{C_\epsilon}{|\lambda|^2} |P_\ell|$$

and reciprocally

$$(\ell_3 - \ell_2) [I_u(\ell_2) + I_u(\ell_3)] \leq E_u(\ell_2, \ell_3) + C \|z\| \|F\| + C_\epsilon \|F\|^2 + \epsilon \|z\|^2 + \frac{C_\epsilon}{|\lambda|^2} |P_\ell|.$$

Proof. Let us multiply the equation (3.2) by $q(x)\bar{u}_x$, where q is any of the above polynomials, to get

$$\int_{\ell_2}^{\ell_3} q(x) i \lambda v \bar{u}_x dx - \int_{\ell_2}^{\ell_3} q(x) (au_x + \beta\theta)_x \bar{u}_x dx = \int_{\ell_2}^{\ell_3} q(x) f_2 \bar{u}_x dx.$$

We have that

$$- \int_{\ell_2}^{\ell_3} q(x) v i \bar{\lambda} \bar{u}_x \, dx - \int_{\ell_2}^{\ell_3} q(x) a u_{xx} \bar{u}_x \, dx - \int_{\ell_2}^{\ell_3} q(x) (\beta \theta)_x \bar{u}_x \, dx = \int_{\ell_2}^{\ell_3} q(x) f_2 \bar{u}_x \, dx$$

and

$$\underbrace{- \int_{\ell_2}^{\ell_3} q(x) v \bar{v}_x \, dx - \int_{\ell_2}^{\ell_3} q(x) a u_{xx} \bar{u}_x \, dx - \int_{\ell_2}^{\ell_3} q(x) (\beta \theta)_x \bar{u}_x \, dx}_{= -\frac{1}{2} q(\ell) I_u(\ell) \Big|_{\ell_2}^{\ell_3} + \frac{1}{2} E_u(\ell_2, \ell_3)} = \int_{\ell_2}^{\ell_3} q(x) f_2 \bar{u}_x \, dx + \int_{\ell_2}^{\ell_3} q(x) v \bar{f}_{1,x} \, dx.$$

It follows that

$$- q(\ell) I_u(\ell) \Big|_{\ell_2}^{\ell_3} + E_u(\ell_2, \ell_3) = -2 \int_{\ell_2}^{\ell_3} q(x) (\beta \theta)_x \bar{u}_x \, dx + 2 \int_{\ell_2}^{\ell_3} q(x) (f_2 \bar{u}_x + v \bar{f}_1) \, dx.$$

Note that

$$\begin{aligned} \left| \int_{\ell_2}^{\ell_3} q(x) (\beta \theta)_x \bar{u}_x \, dx \right| &\leq \left| \int_{\ell_2}^{\ell_3} q(x) \beta' \theta \bar{u}_x \, dx \right| + \left| \int_{\ell_2}^{\ell_3} q(x) \beta \theta_x \bar{u}_x \, dx \right| \\ &\leq C_\epsilon \int_{\ell_2}^{\ell_3} |\beta'|^2 |\theta|^2 \, dx + \epsilon \|z\|^2 + c_\epsilon \int_{\ell_2}^{\ell_3} \beta |\theta_x|^2 \, dx \\ &\leq \frac{C_\epsilon}{|\lambda|^2} \int_{\ell_2}^{\ell_3} |\beta'|^2 |\lambda \theta|^2 \, dx + \epsilon \|z\|^2 + c_\epsilon \int_{\ell_2}^{\ell_3} \beta |\theta_x|^2 \, dx. \end{aligned}$$

Using Lemma 3.3, the assumption that $(|\beta(x)| + |\beta'(x)|) \leq c|b(x)|$ and the estimate (3.5) we get

$$\left| \int_{\ell_2}^{\ell_3} q(x) (\beta \theta)_x \bar{u}_x \, dx \right| \leq C_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2 + \epsilon \|z\|^2 + \frac{C_\epsilon}{|\lambda|^2} |P_\ell|.$$

The second assertion of the lemma can be obtained in a similar way by taking $q = x - (\ell_2 + \ell_3)/2$. \square

Lemma 3.5. *Under the above notations we have that for any $\epsilon > 0$ there exists $c_\epsilon > 0$ such that*

$$I_u(\ell) + \int_{\ell_1}^{\ell} \beta^2 (|v|^2 + a|u_x|^2) \, dx \leq \epsilon \|z\|^2 + c_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2 + \frac{c}{|\lambda|^2} I_\tau(\ell).$$

Proof. Multiplying equation (3.4) by $\beta \bar{u}_x$ and after an integration we obtain

$$\int_{\ell_1}^{\ell} i \lambda \theta \beta \bar{u}_x \, dx - \int_{\ell_1}^{\ell} (k \tau_x + b \theta_x)_x \beta \bar{u}_x \, dx + i \lambda \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx = R.$$

From where we get

$$\int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx = - \int_{\ell_1}^{\ell} \theta \beta \bar{u}_x \, dx + \frac{1}{i \lambda} \int_{\ell_1}^{\ell} (k \tau_x + b \theta_x)_x \beta \bar{u}_x \, dx + \frac{1}{i \lambda} R.$$

Therefore we have that

$$\begin{aligned} \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx &= - \int_{\ell_1}^{\ell} \theta \beta \bar{u}_x \, dx + \frac{1}{i \lambda} \bar{P}_\ell - \frac{1}{i \lambda} \int_{\ell_1}^{\ell} (k \tau_x + b \theta_x) \beta' \bar{u}_x \, dx \\ &\quad - \frac{1}{i \lambda} \int_{\ell_1}^{\ell} (k \tau_x + b \theta_x) \beta \bar{u}_{xx} \, dx + \frac{1}{i \lambda} R. \end{aligned} \tag{3.8}$$

We know that

$$\left| - \int_{\ell_1}^{\ell} \theta \beta \overline{u_x} dx \right| \leq \epsilon \|z\|^2 + \frac{c_\epsilon}{|\lambda|^2} \int_{\ell_1}^{\ell} \beta^2 |\lambda \theta|^2 dx \quad (3.9)$$

and

$$\begin{aligned} -\frac{1}{i\lambda} \int_{\ell_1}^{\ell} (k\tau_x + b\theta_x) \beta \overline{u_{xx}} dx &\leq -\frac{1}{ia\lambda} \int_{\ell_1}^{\ell} \beta (k\tau_x + b\theta_x) [\overline{i\lambda v + (\beta\theta)_x}] dx + C\|F\|^2 + R \\ &\leq \underbrace{\frac{1}{a} \int_{\ell_1}^{\ell} \beta (k\tau_x + b\theta_x) \overline{v} dx}_{:=J_1} - \underbrace{\frac{1}{ia\lambda} \int_{\ell_1}^{\ell} \beta (k\tau_x + b\theta_x) [\overline{(\beta\theta)_x}] dx}_{:=J_2} + C\|F\|^2 + R. \end{aligned}$$

Note that

$$\begin{aligned} |J_1| &= \left| \frac{1}{ai\lambda} \int_{\ell_1}^{\ell} \beta k [i\lambda \tau_x] \overline{v} dx + \frac{1}{a} \int_{\ell_1}^{\ell} \beta b \theta_x \overline{v} dx \right| \\ &\leq \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} |v|^2 dx + \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} \beta^2 |\theta_x|^2 dx + c_\epsilon \int_{\ell_1}^{\ell} \beta^2 |\theta_x|^2 dx + \epsilon \|z\|^2 + R. \end{aligned}$$

Therefore for λ large enough we have

$$|J_1| \leq \epsilon \|z\|^2 + c_\epsilon \|z\| \|F\|.$$

Similarly for J_2 we have

$$\begin{aligned} |J_2| &= \left| \frac{1}{ia\lambda} \int_{\ell_1}^{\ell} \beta (k\tau_x + b\theta_x) [\overline{(\beta\theta)_x}] dx \right| \\ &\leq \frac{c}{|\lambda|^2} \int_{\ell_1}^{\ell} \beta |\theta_x| |\beta' \theta| dx + \frac{c}{|\lambda|^2} \int_{\ell_1}^{\ell} \beta |\theta_x| |\beta \theta_x| dx + \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} \beta |b \theta_x| |(\beta\theta)_x| dx + R \\ &\leq \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} |\theta|^2 dx + \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} b^2 |\theta_x|^2 dx + R \\ &\leq \epsilon \|z\|^2 + c \|z\| \|F\|. \end{aligned}$$

So we see

$$\left| -\frac{1}{i\lambda} \int_{\ell_1}^{\ell} (k\tau_x + b\theta_x) \beta \overline{u_{xx}} dx \right| \leq \epsilon \|z\|^2 + c \|z\| \|F\|. \quad (3.10)$$

If we substitute the inequalities (3.9), (3.10), into (3.8) we get

$$\begin{aligned} \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 dx &\leq \epsilon \|z\|^2 + \frac{c_\epsilon}{|\lambda|^2} \int_{\ell_1}^{\ell} \beta^2 |\lambda \theta|^2 dx + \frac{1}{|\lambda|} |P_\ell| \\ &\quad \left| -\frac{1}{i\lambda} \int_{\ell_1}^{\ell} (k\tau_x + b\theta_x) \beta' \overline{u_x} dx \right| + \epsilon \|z\|^2 + c \|z\| \|F\|. \end{aligned}$$

In view of the Lemma 3.3 we find

$$\int_{\ell_1}^{\ell} \beta^2 |u_x|^2 dx \leq \epsilon \|z\|^2 + c_\epsilon \|z\| \|F\| + \frac{1}{|\lambda|} |P_\ell| + c \|F\|^2 + \left| \frac{1}{i\lambda} \int_{\ell_1}^{\ell} (k\tau_x + b\theta_x) \beta' \overline{u_x} dx \right|.$$

We also note that

$$\begin{aligned} \left| \frac{1}{i\lambda} \int_{\ell_1}^{\ell} (k\tau_x + b\theta_x) \beta' \bar{u}_x \, dx \right| &\leq \frac{c}{|\lambda|} \|z\|^2 + \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} b\beta' |\theta_x| |u_x| \, dx \\ &\leq \frac{c}{|\lambda|} \|z\|^2 + \frac{c}{|\lambda|} \int_{\ell_1}^{\ell} b^2 |\theta_x|^2 \, dx. \end{aligned}$$

From the above inequalities and from the Lemma 3.4 we get, for $|\lambda|$ large enough,

$$\int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx \leq \epsilon \|z\|^2 + c_{\epsilon} \|z\| \|F\| + \frac{1}{|\lambda|} |P_{\ell}| + c \|F\|^2. \quad (3.11)$$

Multiplying equation (3.2) by $\beta^2 \bar{u}$ we get

$$\begin{aligned} \int_{\ell_1}^{\ell} \beta^2 |v|^2 \, dx &= a \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx + a \int_{\ell_1}^{\ell} 2\beta\beta' u_x \bar{u} \, dx - \int_{\ell_1}^{\ell} (\beta\theta)_x \beta^2 \bar{u} \, dx + R \\ &= a \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx - \frac{a}{i\lambda} \int_{\ell_1}^{\ell} 2\beta\beta' u_x \bar{v} \, dx + \frac{1}{i\lambda} \int_{\ell_1}^{\ell} (\beta\theta)_x \beta^2 \bar{v} \, dx + R. \end{aligned}$$

We also note that

$$\left| -\frac{a}{i\lambda} \int_{\ell_1}^{\ell} 2\beta\beta' u_x \bar{v} \, dx \right| \leq \frac{a}{2} \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx + \underbrace{\frac{c}{|\lambda|^2} \int_{\ell_1}^{\ell} |v|^2 \, dx}_{\leq \epsilon \|z\|^2}$$

for λ large enough and

$$\left| \frac{1}{i\lambda} \int_{\ell_1}^{\ell} (\beta\theta)_x \beta^2 \bar{v} \, dx \right| \leq \frac{1}{2} \int_{\ell_1}^{\ell} \beta^2 |v|^2 \, dx + \frac{c}{|\lambda|^2} \int_{\ell_1}^{\ell} |\theta|^2 \, dx + \frac{c}{|\lambda|^2} \int_{\ell_1}^{\ell} b^2 |\theta_x|^2 \, dx.$$

From where we get

$$\int_{\ell_1}^{\ell} \beta^2 |v|^2 \, dx \leq \frac{3a}{2} \int_{\ell_1}^{\ell} \beta^2 |u_x|^2 \, dx + \epsilon \|z\|^2 + c_{\epsilon} \|z\| \|F\|$$

for $|\lambda|$ large enough. Using the estimate (3.11) we get

$$\int_{\ell_1}^{\ell} \beta^2 |v|^2 \, dx \leq \epsilon \|z\|^2 + c_{\epsilon} \|z\| \|F\| + \frac{c}{|\lambda|} |P_{\ell}| + c \|F\|^2. \quad (3.12)$$

That is we get that

$$\int_{\ell_1}^{\ell} \beta^2 |v|^2 + \beta^2 |u_x|^2 \, dx \leq \epsilon \|z\|^2 + c_{\epsilon} \|z\| \|F\| + \frac{c}{|\lambda|} |P_{\ell}| + c \|F\|^2.$$

Recalling the definition of E_u and that $\beta(x)^2 > 0$ for $x > \ell_1$ the above inequality implies that for $\ell_2 \in]\ell_1, \ell[$ the estimate

$$E_u(\ell_2, \ell) \leq \epsilon \|z\|^2 + c_{\epsilon} \|z\| \|F\| + \frac{c}{|\lambda|} |P_{\ell}| + c \|F\|^2$$

holds. Taking $\ell_3 = \ell$ in Lemma 3.4 we get that

$$(\ell_3 - \ell_2)[I_u(\ell_2) + I_u(\ell)] \leq C \|z\| \|F\| + C_{\epsilon} \|F\|^2 + \epsilon \|z\|^2 + \frac{c}{|\lambda|} |P_{\ell}|.$$

Since

$$\frac{c}{|\lambda|} |P_\ell| \leq \frac{c}{|\lambda|} I_u^{1/2}(\ell) I_\tau^{1/2}(\ell) \leq \frac{\ell_3 - \ell_2}{2} I_u(\ell) + \frac{c}{|\lambda|^2} I_\tau(\ell)$$

we have that

$$(\ell_3 - \ell_2) I_u(\ell) \leq C \|z\| \|F\| + C_\epsilon \|F\|^2 + \epsilon \|z\|^2 + \frac{c}{|\lambda|^2} I_\tau(\ell).$$

From where our conclusion follows. \square

Lemma 3.6. *Let us take $\ell_2 \in]\ell_1, \ell[$, then for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$I_\tau(\ell) \leq C_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2 + \epsilon \|z\|^2$$

Proof. Using identity (3.4) over $[\ell_2, \ell]$ for $q = x - \ell_2$ we have

$$-q(\ell) I_\tau(\ell) + E_\tau(\ell_2, \ell) + 2 \underbrace{\int_{\ell_2}^{\ell} \beta v_x q [\overline{k\tau_x + b\theta_x}] dx}_{:=J_0} = R_0 - 2 \underbrace{\int_{\ell_2}^{\ell} i\lambda\theta q \overline{b\theta_x} dx}_{:=J_1}. \quad (3.13)$$

Because of the Lemma 3.3 and the dissipative term we have

$$J_1 \leq C \|z\| \|F\| + C \|F\|^2 + \epsilon |P_\ell| + \epsilon \|z\|^2.$$

Note that

$$\begin{aligned} J_0 &= -2 \int_{\ell_2}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_2}^{\ell} q\beta v [\overline{b\theta_x + k\tau_x}]_x dx \\ &= -2 \int_{\ell_2}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_2}^{\ell} q\beta v [i\lambda\theta + \beta v_x] dx + R \\ &= -2 \int_{\ell_2}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_2}^{\ell} q\beta i\lambda v \bar{\theta} dx - 2 \int_{\ell_1}^{\ell} q\beta v [\overline{\beta v_x}] dx + R \\ &= -2 \int_{\ell_2}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_2}^{\ell} q\beta [au_{xx} - (\beta\theta)_x] [\bar{\theta}] dx - \int_{\ell_1}^{\ell} q\beta^2 \frac{d}{dx} |v|^2 dx + R \\ &= Q_5 - 2 \int_{\ell_2}^{\ell} q\beta [au_{xx}] [\bar{\theta}] dx - \int_{\ell_2}^{\ell} q\beta^2 \frac{d}{dx} |v|^2 dx + Q_4 + R \end{aligned}$$

and

$$-2 \int_{\ell_2}^{\ell} q\beta [au_{xx}] \bar{\theta} dx = 2 \int_{\ell_2}^{\ell} [q\beta]' [au_x] \bar{\theta} dx + 2 \int_{\ell_2}^{\ell} q\beta [au_x] \bar{\theta}_x dx + R = Q_6 + R$$

$$J_0 = Q_4 + Q_5 + Q_6 + \int_{\ell_2}^{\ell} (q\beta^2)' |v|^2 dx + R.$$

Recalling the condition satisfying the functions Q we get that

$$|J_0| \leq \epsilon \|z\|^2 + \frac{c}{|\lambda|^2} \int_{\ell_1}^{\ell} b |\lambda\theta|^2 dx + c_\epsilon \int_{\ell_1}^{\ell} b |\theta_x|^2 dx + \int_{\ell_2}^{\ell} (q\beta^2)' |v|^2 dx + R.$$

Using Lemma 3.5 and Lemma 3.4 we get

$$|J_0| \leq \epsilon \|z\|^2 + c_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2 + \frac{c}{|\lambda|^2} I_\tau(\ell).$$

Finally, since

$$\int_{\ell_1}^{\ell} |\theta_x|^2 dx \leq C \|z\| \|F\|$$

we have that

$$\int_{\ell_1}^{\ell} |\theta|^2 + |\tau_x|^2 dx \leq C \|z\| \|F\|.$$

Therefore

$$E_{\tau}(\ell_2, \ell) \leq C \|z\| \|F\| \quad (3.14)$$

Substitution (3.14) into (3.13) and using the Lemma 3.5 we get

$$q(\ell)I_{\tau}(\ell) \leq E_{\tau}(\ell_2, \ell) - 2J_0 + 2J_1 + R_0$$

Therefore we have

$$q(\ell)I_{\tau}(\ell) \leq \epsilon \|z\|^2 + c_{\epsilon} \|z\| \|F\| + C_{\epsilon} \|F\|^2 + \frac{c}{|\lambda|^2} I_{\tau}(\ell).$$

For λ large enough our conclusion follows. □

Let us introduce the notation

$$\mathcal{X} = 2k \int_0^{\ell} q \beta' u_x \bar{\theta} dx.$$

Now we consider the observability condition

Theorem 3.7. *The semigroup is exponentially stable provided (2.1) holds for δ small enough.*

PROOF.- Let us suppose that $|\beta'(x)| \leq M$ over $]0, \ell_1[$. Let us divide the interval $]0, \ell_1[$ into n subintervals of size h where

$$h = \frac{\ell_1}{n} \leq \frac{1}{2M}.$$

Let us denote by $s_i = ih$. Note that $s_n = \ell_1$. Let us take $q(x) = (x - s_{n-1})$.

Let us multiply the equation (3.4) by $q[\overline{k\tau_x + b\theta_x}]$. After integration over $[s_{n-1}, \ell]$ we get

$$\int_{s_{n-1}}^{\ell} i\lambda \theta q[\overline{k\tau_x + b\theta_x}] dx - \int_{s_{n-1}}^{\ell} (k\tau_x + b\theta_x)_x q[\overline{k\tau_x + b\theta_x}] dx + \int_{s_{n-1}}^{\ell} \beta v_x q[\overline{k\tau_x + b\theta_x}] dx = R_0$$

where

$$R_0 = \int_{s_{n-1}}^{\ell} f_4 q[\overline{k\tau_x + b\theta_x}] dx.$$

From where we have that

$$\int_{s_{n-1}}^{\ell} i\lambda \theta q \overline{k\tau_x} dx - \int_{s_{n-1}}^{\ell} (k\tau_x + b\theta_x)_x q[\overline{k\tau_x + b\theta_x}] dx + \int_{s_{n-1}}^{\ell} \beta v_x q[\overline{k\tau_x + b\theta_x}] dx = R_0 - \int_{s_{n-1}}^{\ell} i\lambda \theta q \overline{b\theta_x} dx.$$

Thus, we get

$$-\frac{k}{2} \int_{s_{n-1}}^{\ell} q \frac{d}{dx} |\theta|^2 dx - \frac{1}{2} \int_{s_{n-1}}^{\ell} q \frac{d}{dx} |k\tau_x + b\theta_x|^2 dx + \int_{s_{n-1}}^{\ell} \beta v_x q [\overline{k\tau_x + b\theta_x}] dx = R_0 - \int_{s_{n-1}}^{\ell} i\lambda\theta q \overline{b\theta_x} dx.$$

Performing an integration by parts we also get

$$-q(\ell)I_{\tau}(\ell) + E_{\tau}(s_{n-1}, \ell) + 2 \underbrace{\int_{s_{n-1}}^{\ell} \beta v_x q [\overline{k\tau_x + b\theta_x}] dx}_{:=J_2} = R_0 - 2 \underbrace{\int_{s_{n-1}}^{\ell} i\lambda\theta q \overline{b\theta_x} dx}_{:=J_3}. \quad (3.15)$$

Using Lemma 3.3 and the dissipative inequality we get that

$$\begin{aligned} J_3 &= 2\operatorname{Re} \int_{s_{n-1}}^{\ell} i\lambda\theta q \overline{b\theta_x} dx \\ &\leq \int_{s_{n-1}}^{\ell} \epsilon b |\lambda\theta|^2 + C_{\epsilon} b |\theta_x|^2 dx \\ &\leq \epsilon P_{\ell} + C_{\epsilon} \|z\| \|F\| + C_{\epsilon} \|F\|^2 + \epsilon \|z\|^2. \end{aligned} \quad (3.16)$$

On the other hand

$$\begin{aligned} J_2 &= 2 \underbrace{\int_{s_{n-1}}^{\ell} q \beta v_x [\overline{k\tau_x}] dx}_{:=J_4} + 2 \int_{\ell_1}^{\ell} q \beta v_x [\overline{b\theta_x + k\tau_x}] dx - 2 \underbrace{\int_{\ell_1}^{\ell} q \beta v_x [\overline{k\tau_x}] dx}_{:=Q_7} \\ &= J_4 - 2 \int_{\ell_1}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_1}^{\ell} q \beta v [\overline{b\theta_x + k\tau_x}]_x dx + Q_7 \\ &= J_4 - 2 \int_{\ell_1}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_1}^{\ell} q \beta v [\overline{i\lambda\theta + \beta v_x}] dx + Q_7 \\ &= J_4 - 2 \int_{\ell_1}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_1}^{\ell} q \beta i \lambda v [\overline{\theta}] dx - 2 \int_{\ell_1}^{\ell} q \beta v [\overline{\beta v_x}] dx + Q_7 \\ &= J_4 - 2 \int_{\ell_1}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx - 2 \int_{\ell_1}^{\ell} q \beta [au_{xx} - (\beta\theta)_x] [\overline{\theta}] dx - \int_{\ell_1}^{\ell} q \beta^2 \frac{d}{dx} |v|^2 dx + Q_7 \\ &= J_4 + Q_8 - 2 \int_{\ell_1}^{\ell} q \beta [au_{xx}] [\overline{\theta}] dx - \int_{\ell_1}^{\ell} q \beta^2 \frac{d}{dx} |v|^2 dx + Q_7 \end{aligned}$$

where

$$Q_8 = 2 \int_{\ell_1}^{\ell} (q\beta)' v [\overline{b\theta_x + k\tau_x}] dx + 2 \int_{\ell_1}^{\ell} q \beta [(\beta\theta)_x] [\overline{\theta}] dx.$$

Note that

$$-2 \int_{\ell_1}^{\ell} q \beta [au_{xx}] [\overline{\theta}] dx = 2 \int_{\ell_1}^{\ell} [q\beta]' [au_x] [\overline{\theta}] dx + 2 \int_{\ell_1}^{\ell} q \beta [au_x] [\overline{\theta}_x] dx = Q_9.$$

Therefore

$$J_2 = J_4 + \underbrace{Q_7 + Q_8 + Q_9}_{:=Q_{10}} + \underbrace{\int_{\ell_1}^{\ell} (q\beta^2)' |v|^2 dx}_{:=J_5}. \quad (3.17)$$

Note that

$$\begin{aligned}
 J_4 &= -2k \int_{s_{n-1}}^{\ell} qu_x[(\beta\bar{\theta})_x] dx + 2k \int_{s_{n-1}}^{\ell} qu_x[\beta'\bar{\theta}] dx + R \\
 &= 2k \int_{s_{n-1}}^{\ell} qu_x[i\lambda v - au_{xx} - f_2] dx + 2k \int_{s_{n-1}}^{\ell} qu_x[\beta'\bar{\theta}] dx + R \\
 &= -kq(\ell)I_u(\ell) + kE_u(s_{n-1}, \ell) + 2k \int_0^{\ell} qu_x[\beta'\bar{\theta}] dx + R.
 \end{aligned}$$

Now, we recall that β and q are continuous functions such that

$$\lim_{x \rightarrow \ell_1} (q\beta^2)' = 0.$$

Then, for any $\epsilon > 0$ there exists $\delta > 0$, such that $0 < x - \ell_1 < \delta$ implies that $|(q\beta^2)'| < \epsilon$. Therefore

$$\begin{aligned}
 J_5 &= \int_{\ell_1}^{\ell} (q\beta^2)' |v|^2 dx \\
 &= \int_{\ell_1}^{\ell_1+\delta} \epsilon |v|^2 dx + \int_{\ell_1+\delta}^{\ell} (q\beta^2)' |v|^2 dx.
 \end{aligned}$$

Note that for $x \in [0, \ell]$, we have that $|(q\beta^2)'| \leq C$. Therefore using Lemma 3.5 and Lemma 3.6 we get

$$\begin{aligned}
 \int_{\ell_1+\delta}^{\ell} (q\beta^2)' |v|^2 dx &\leq \epsilon \|z\|^2 + c_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2 + \frac{c}{|\lambda|^2} I_\tau(\ell) \\
 &\quad C_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2 + \epsilon \|z\|^2.
 \end{aligned}$$

We then conclude that

$$|J_5| \leq \epsilon \|z\|^2 + C_\epsilon \|z\| \|F\| + C_\epsilon \|F\|^2.$$

Substitution of J_4 and J_5 in (3.17) we get

$$J_2 = -kq(\ell)I_u(\ell) + kE_u(s_{n-1}, \ell) + R + Q_7 + \underbrace{2k \int_{s_{n-1}}^{\ell} qu_x[\beta'\bar{\theta}] dx}_{:=\mathcal{X}}. \quad (3.18)$$

Finally, the substitution of the above inequality into (3.15) implies that

$$-q(\ell)[I_\tau(\ell) + kI_u(\ell)] + E_\tau(s_{n-1}, \ell) + kE_u(s_{n-1}, \ell) + Q_7 + \mathcal{X} = R_0 - R - J_1.$$

Therefore recalling the definition of Q_i and using Lemma 3.6 we get

$$E_\tau(s_{n-1}, \ell) + kE_u(s_{n-1}, \ell) + \mathcal{X} \leq \epsilon \|z\|^2 + C \|z\| \|F\| + C \|F\|^2$$

for $|\lambda|$ large enough. Since

$$|\mathcal{X}| \leq 2\sqrt{k}MhE_\tau(s_{n-1}, \ell)^{1/2}E_u(s_{n-1}, \ell)^{1/2}$$

and

$$h = \frac{\ell_1}{n} \leq \frac{1}{2M}$$

we conclude that

$$E_\tau(s_{n-1}, \ell) + kE_u(s_{n-1}, \ell) \leq 2\epsilon\|z\|^2 + 2C\|z\|\|F\| + 2C\|F\|^2.$$

Repeating the process for $q(x) = x - s_{n-2}$ we get

$$E_\tau(s_{n-2}, \ell) + kE_u(s_{n-2}, \ell) + \underbrace{2k \int_{s_{n-2}}^{\ell_1} qu_x[\beta'\bar{\theta}]}_{:=\mathcal{X}} \leq \epsilon\|z\|^2 + C\|z\|\|F\| + C\|F\|^2. \quad (3.19)$$

But

$$2k \int_{s_{n-2}}^{\ell_1} qu_x[\beta'\bar{\theta}] dx = 2k \int_{s_{n-2}}^{s_{n-1}} qu_x[\beta'\bar{\theta}] dx + 2k \int_{s_{n-1}}^{\ell_1} qu_x[\beta'\bar{\theta}] dx.$$

So we have

$$\left| 2k \int_{s_{n-2}}^{\ell_1} qu_x[\beta'\bar{\theta}] dx \right| \leq \underbrace{\left| 2k \int_{s_{n-2}}^{s_{n-1}} qu_x[\beta'\bar{\theta}] dx \right|}_{\mathcal{X}_1} + 2\epsilon\|z\|^2 + 2C\|z\|\|F\| + 2C\|F\|^2.$$

But

$$|\mathcal{X}_1| \leq 2\sqrt{k}MhE_\tau(s_{n-2}, s_{n-1})^{1/2}E_u(s_{n-2}, s_{n-1})^{1/2}.$$

Substitution on (3.19) we get

$$E_\tau(s_{n-2}, \ell) + kE_u(s_{n-2}, \ell) \leq 5\epsilon\|z\|^2 + 5C\|z\|\|F\| + 5C\|F\|^2$$

using induction we get

$$E_\tau(0, \ell) + kE_u(0, \ell) \leq (2n+1)\epsilon\|z\|^2 + (2n+1)C\|z\|\|F\| + (2n+1)C\|F\|^2.$$

From where our conclusion follows.

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