Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

On sets of points with few ordinary hyperplanes

Enrique Jimenez Izquierdo

Supervised by Simeon Ball

June, 2018

Abstract

Let S be a set of n points in the projective d-dimensional real space $\mathbb{RP}^d$ such that not all points of S are contained in a single hyperplane and such that any subset of d points of S span a hyperplane. Let an ordinary hyperplane of S be an hyperplane of $\mathbb{RP}^d$ containing exactly d points of S. In this paper we study the minimum number of ordinary hyperplanes spanned by any set S of n points in 4 dimensions, following the work of Ben Green and Terence Tao in the planar version of the problem, as well as the work of Simeon Ball in the 3 dimensional case. We classify the sets of points in 4 dimensions that span few ordinary hyperplanes, showing that if S is a set spanning less than Kn^3 ordinary hyperplanes, for some $K = o(n^{\frac{1}{6}})$, then all but $O(K)$ points of S must be contained in the intersection of 5 linearly independent quadrics.

Keywords

Discrete Geometry, Incidence and Arrangement Problems, Sylvester-Gallai-Type Problems, Computational Geometry, Combinatorial Geometry

Contents

1. Introduction

In 1893, James Joseph Sylvester proposed in Educational Times the following problem:

"Let n given points have the property that the line joining any two of them passes through a third point of the set. Must the n points all lie on one line?"

That problem would be known later as the Sylvester's problem, and a solution was presented 40 years later (1944) by Tibor Gallai, in the form of the Sylvester-Gallai theorem:

Theorem (Sylvester-Gallai Theorem)**.** *Let S be a set of points in the plane not all contained in a line. Then there is at least one ordinary line through S, that is to say, a line that contains exactly two points of S.*

After having the question posed by Sylvester answered, the natural follow-up problem is to try to minimize the number of ordinary lines (lines passing by exactly two points) generated by sets of n points, not all collinear.

There has been work on this generalization of the Sylvester problem since 1968, and the asymptotic solution was proven by Ben Green and Terence Tao in 2013 [11].

In this thesis we focus on a generalization of this problem to higher dimensions. Different generalizations of this problem to higher dimensions can be considered, but we will focus on the one used by Simeon Ball:

"Let S be a set of n points in the projective d-dimensional real space $\mathbb{RP}^d$ such that they are not contained in one hyperplane and that any subset of d points of S spans a hyperplane. Let us call an ordinary hyperplane to an hyperplane that contains exactly d points of the set S. Then, what is the minimum number of ordinary hyperplanes spanned by a set S of n points?"

The extra condition of any subset of d points of S spanning a hyperplane was added to the natural generalization of the problem. This is because trivial examples of sets without ordinary hyperplanes can be found if this condition is missing (allowing 3 or more collinear points, etc...). It is obvious, though, that the restriction of this problem to dimension $d = 2$ is equivalent to the original Sylvester's problem.

In general, we will denote as $e_d(n)$ to the minimum number of ordinary hyperplanes spanned by a set S of n points in a d-dimensional space. Our problem, then, consists on finding a closed expression for that number.

Work on this generalization to higher dimensions has been done in [6], [7] and [9]. The solution of the problem is know for several small values of d and n, and several bounds have been found in some special cases. Simeon Ball managed to prove in [9] the asymptotic result for the case $d = 3$, taking inspiration on the work done by Green and Tao in the two dimensional case. The solution for the general problem, though, is still missing.

In this paper we try to tackle the problem in dimensions greater than 3, since these are the areas where most work is still to be done. The version of Sylvester's problem from dimension 4 upward has a special interest, since it is from here on that we lack the knowledge of significant examples of points spanning few ordinary hyperplanes. This makes the study a bit more challenging, as we have to work without a strong guess on the solution.

Although the aim of this work is the study of the problem in dimensions higher than 3, the main focus of this paper will be the version of the Sylvester's problem on $\mathbb{RP}^4$, as this is the simplest of the not studied cases and it gives us a strong basis for the general problem.

We will begin our thesis with an introduction to the Sylvester's problem, explaining the problem in the planar and 3 dimensional case, showing the important results on these dimensions, aiming to help the understanding of our own results later.

In section 3 we will show our results on the asymptotic solution of Sylvester's problem in the fourth dimension. We will take inspiration on the work done in two and three dimensions, finishing with our own version of the structure theorem, the main result of this thesis.

To end this paper, in section 4 we will present some minor results on the solution of the Sylvester's problem for small values of d and n. We will be exploring the value of $e_d(n)$ for a particular case of values (d, n), giving strong bound on this value and managing to compute it exactly for one of cases.

2. Previous results

As a first section of our thesis we will to take a quick look at the known results on the problem. We want to review the work done on dimensions 2 and 3, as we will need many of those results for our study. Also we are interested in the path used to prove some of these results, as we will need to use similar techniques for our own ones.

We will use the planar case to introduce the basic concepts and tricks that we will be using on our thesis, as it stands as the original Sylvester's problem. We will show on this section the basic examples of configurations of points related to the solution of the problem and we will present some of the basic results made in this area that will be of use to us later on. For the more advanced results on the subject we will mainly review the work by Ben Green and Terence Tao in [11].

For the 3 dimensional case we will follow the work done by Simeon Ball in [9], which is the main reference for this generalization of Sylvester's problem on $\mathbb{RP}^3$.

As this section is just to show the results made previously on our problem and to introduce the various concepts of the work, we won't be showing the proofs of the more complicated theorems or the technical parts of the studies. Instead we will be focusing on just presenting the most important results, just showing the proofs we think relevant to our own study later.

2.1 Planar case

The first step in our study is the planar case, which corresponds to the original Sylvester's problem. The study of the planar case helps us familiarize with the kind of work we are going to be doing, and with the tricks and techniques needed for our proofs.

2.1.1 Melchior's proof

Before studying the stronger results proven by Green and Tao for the asymptotic solution to the problem we want to show briefly a proof of the original problem posed by Sylvester, by proving a simple lower bound for the problem, showing that there cannot be a finite set of points without ordinary lines.

There are plenty of proofs of Sylvester's problem. Here we will be showing the proof given by Melchior (1940), as it is very simple and seems more elegant than the proof presented by Gallai later on. The proof using the dualization of the set of points S and the Euler's formula in the projective space, techniques that we will be using later to prove some of our results.

Theorem 2.1 (Sylvester-Gallai)**.** *Let S be a set of points in the plane not all contained in a line. Then there is at least one ordinary line through S, that is to say, a line that contains exactly two points of S.*

Proof. Let S be a set of n points in the projective real plane $\mathbb{RP}^2$. Let us consider $S^* = \{p^* : p \in S\}$, the dual set of S consisting on n lines in $\mathbb{RP}^2$.

The set of lines S^* determine the drawing of a graph Γ_S in $\mathbb{RP}^2$. The edges of this graph are the segments of the lines $p^* \in S^*$, while the vertices of the graph are the intersection of two or more lines $p^*, q^* ... \in S^*$, which corresponds to the dual of the line joining the points $p, q ... \in S$.

By hypothesis the set S is not entirely contained in a single line, thus we can assume that there is more than one vertex in the graph Γ_S and that any line in S^* meets at least two vertices.

As a graph, we can say that Γ_S partitions the projective plane $\mathbb{RP}^2$ into a set of vertices V, a set of edges E and a set of faces F. If we apply Euler's formula this partition we get:

$$|V| + |F| - |E| = 1 \tag{1}$$

as the Euler's characteristic for the projective plane is equal to 1.

Let $V_i \subseteq V$ be the set of vertices of V which are incident with exactly i lines of S^*, and let $v_i = |V_i|$. A vertex is incident with two edges of Γ_S for each line in S^* that passes through it, so the degree of a vertex $v \in V_i$ is $d(v_i) = 2i$. If we count the number of edges of the graph using this fact we obtain:

$$2|E| = \sum_{v \in V} d(v) = \sum_{i=2}^{n} 2iv_i \tag{2}$$

Let us also denote as $F_i \subseteq F$ to the set of faces of F that have exactly i sides, and let $f_i = |F_i|$. We can count the number of edges of Γ_S again with this:

$$2|E| = \sum_{i=3}^{n} if_i \tag{3}$$

where we have used that each edge is in exactly two faces of F, and that each face of Γ_S has at least 3 sides.

We can combine the three equations (1), (2), (3) to get the following expression:

$$\begin{aligned}
|V| + |F| - |E| = 1 \Rightarrow \\
3|V| + 3|F| - 3|E| = 3 \Rightarrow \\
3\sum_{i=2}^{n} v_i + 3\sum_{i=3}^{n} f_i - \sum_{i=2}^{n} iv_i - \sum_{i=3}^{n} if_i = 3 \Rightarrow \\
\sum_{i=2}^{n}(3-i)v_i + \sum_{i=3}^{n}(3-i)f_i = 3
\end{aligned}$$

$$\Rightarrow v_2 = 3 + \sum_{i=4}^{n}(i-3)v_i + \sum_{i=4}^{n}(i-3)f_i \tag{4}$$

As every summand in the right hand side is positive, we can deduce that the term v_2 is also positive (in fact, we can deduce it is at least 3). By definition, v_2 is the number of vertices in Γ_S that are incident with exactly two lines of S^*, which is equal to the number of lines in $\mathbb{RP}^2$ that goes through exactly two points of S, that is to say, the number of ordinary lines. Proving, then, that v_2 is greater than 0 proves that any finite set of points S in the plane must span ordinary lines.

$\square$

Melchior's proof tells us that any finite set of points in the plane that is not in a line must span ordinary lines, but it does not tell us much about how many (although we can get that there must be at least three ordinary lines).

In the next, we will explore some examples of constructions of sets of points in the plane that span few ordinary lines.

2.1.2 Böröczky examples

After having the question of whether there must be ordinary lines in all set of points in the plane, we want to solve the problem of how many must there be in any set of n points.

For an arbitrary set of points in the plane we would expect an $O(n^2)$ number of ordinary lines spanned by the set. It is obvious, though, that this can be greatly improved.

The trivial example for a set of n points with few ordinary lines is the configuration of points where $n-1$ of the points are in the same line and the last point outside. For this configuration the number of ordinary lines is $n-1$, which is linear in n instead of quadratic.

This trivial example can be generalized to higher dimensions easily. Indeed, if we are working in the problem on the $\mathbb{RP}^d$ space, one can easily construct the trivial configuration where $n-1$ points are contained in one hyperplane, and the last points is outside. For these configurations we get $\binom{n-1}{d-1}$ ordinary hyperplanes, which would be the starting upper bound for these problems.

In the planar case, though, we can do better than the trivial example. Now we are going to introduce the **Böröczky examples**, which are an improvement on this bound. These sets were first described by Böröczky (cited in [2]):

Definition 2.2 (Böröczky examples)**.** Let m be an integer greater than 2. We will denote as X_{2m} to the set of $2m$ points in $\mathbb{RP}^2$ described as:

$$X_{2m} = \left\{ \left[cos\frac{2\pi j}{m}, sin\frac{2\pi j}{m}, 1 \right] : 0 \leq j < m \right\} \\ \cup \left\{ \left[-sin\frac{\pi j}{m}, cos\frac{\pi j}{m}, 0 \right] : 0 \leq j < m \right\} \tag{5}$$

which consists on the m points of an m-gon in the unit circle and m points on the line at infinity aligned with the lines generated by the m-gon. We will call this set of points the Böröczky set for $2m$.

As we have already said, the Böröczky examples span few ordinary lines. This is due to how they are constructed. The idea is that we take a regular polygon of m sides and we take every pair of points on the polygon and put a point in the line at infinity in the direction of the line formed by those two points. In that way, those points won't be forming an ordinary line anymore and since we are beginning with a regular polygon at the start, only m additional points are needed.

In the figure 1 we show one of the Böröczky examples, in particular (X_12). We can see that from the 12 points shown in the image (6 of them in the line at infinity), only 6 ordinary lines are spanned by them.

Let us check the exact number of ordinary lines generated by these examples.

Proposition 2.3. *The Böröczky set X_{2m} spans exactly m ordinary lines.*

Proof. This is a rather simple check. We just have to check which pairs of points can span a ordinary line in the Böröczky example.

First of all, for $m > 2$ there will be 3 or more points in the line at infinity, thus, no two points at infinity will span a ordinary line. Secondly, by construction any two points of the m-gon in the unit circle will be aligned with one point in the line at infinity, so no two points in the m-gon will span an ordinary line.

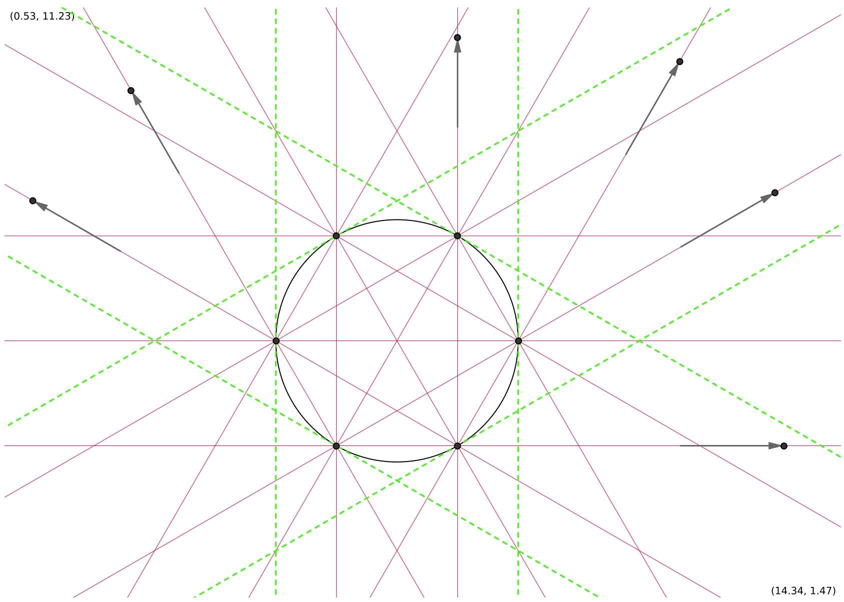


Figure 1: Böröczky configuration X_{12}. The points pointed by arrows are points in the line at infinity. Green lines mark the ordinary lines of the example.

The only case where an ordinary line is generated is joining a point of the circle with a point of the line at infinity. But it turns out that for any point in the m-gon there is only one point in the infinity which is not aligned with any other point in the circle. So that gives us one line for each point in the m-gon, and we end up with just m ordinary lines. $\square$

The Börčzky examples are really good configurations of points, spanning only one ordinary line for every two points, but we can only build a proper Böröczky set for an even number of points.

For n odd we would have to use the **near-Böröczky examples**, which can be constructed from a Böröczky set by either adding or removing a point.

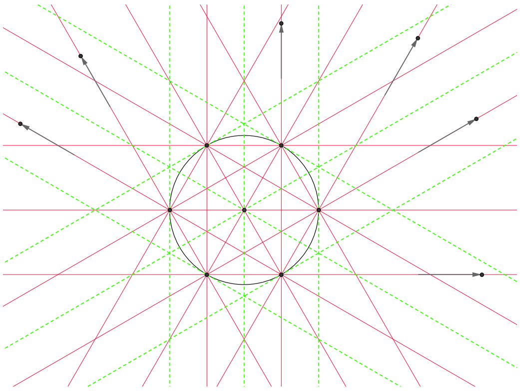
Proposition 2.4 (Near-Börčzky examples)**.** *Let $m \geq 2$ be an integer. Then:*

- *The set X_{4m} plus the point at the origin $[0, 0, 1]$ is a set containing $4m + 1$ points and that spans exactly $3m$ ordinary lines.*
- *The set X_{4m} minus any point on the line at infinity contains $4m - 1$ points and spans precisely $3m - 3$ ordinary lines.*
- *The set X_{4m+2} minus any point on the line at infinity contains $4m + 1$ points and span $3m$ ordinary lines.*

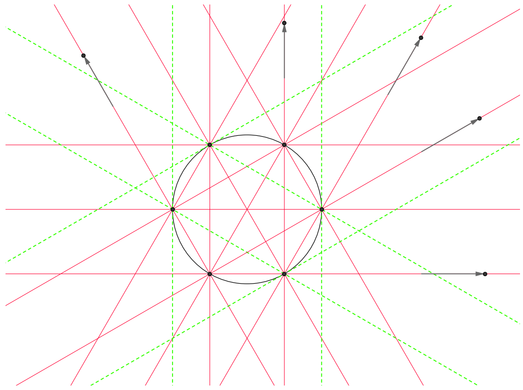
In figures 2a and 2b we show two of these near-Böröczky examples.

Ben Green and Terence Tao proved in [11], via their structural theorem, that the Böröczky and near Böröczky examples are optimal for larger values of n. This gives us the asymptotic solution of the Sylvester's problem (the value of $e_2(n)$ for $n \geq n_0$).

It is important to note here that, even though their work serves to prove the asymptotic bound of the problem and that the Böröczky examples are optimal there, the same is not proven for small values of n.



(a) Near Böröczky configuration of the set X_{12} plus the point in the center. The set spans 9 ordinary lines, marked in green.



(b) Near Böröczky configuration of the set X_{12} minus one of the points on the line at infiniry. The points pointed by arrows are points in the line at infinity. The set span 6 ordinary lines, marked in green.

In fact, there are known examples for n small that best the bound reached with the Böröczky and near Böröczky configurations. The very existence of these examples is an interesting add to the study of this problem.

2.1.3 Structural theorems

Now we want to mention the work of Green and Tao. As we have already said, their work aims to prove that the optimal solution for the Sylvester's problem are the Böröczky examples, and thus:

$$e_2(n) = \begin{cases} \frac{n}{2} & if \quad n \equiv 0 mod(2) \\ 3\lfloor \frac{n}{4} \rfloor & if \quad n \equiv 1 mod(2) \end{cases} \tag{6}$$

We are not going to immerse ourselves into the whole proof, which is too large for this thesis, but we will mention the scheme and some of the ideas of the proof. Later for our own work we will be trying to prove similar results in dimension 4 and some of these ideas will prove key to our study.

The proof of Green and Tao works in the dualization of the set of points S, looking at the graph defined by that dualization, just as we saw in Melchior's proof. The idea behind the whole approach is that, intuitively, a set S of points with few ordinary lines should be rich in lines passing through exactly 3 points. Then they prove that the dual of these kind of sets of points must contain some structures (triangular grids), which allows them to categorize these sets of points.

The whole work of Green and Tao ends with their full structure theorem. The purpose of the theorem is to classify the set of points of the projective plane that span few ordinary lines. With this intent, they define the concept of sets spanning few ordinary lines as sets of points which span less than Kn of these lines for some K, and then state which sets of points can satisfy this.

Theorem 2.5 (Full structure theorem)**.** *Let S be set of n points in the projective real plane $\mathbb{RP}^2$ and suppose that S spans at most Kn ordinary lines for $n \geq exp\ exp(CK^C)$ for some constant C. Then the set S differs by at most $O(K)$ points from some of the following:*

(i) $n - O(K)$ *points in a line.*

(ii) *The Böröczky set* X_{2m}*, for some* $m = \frac{n}{2} + O(K)$*.*

(iii) *A coset* $H \oplus g$*,* $3g \in H$*, of a finite subgroup* H *of the real points on an elliptic curve* $\{[x, y, z] : y^2 z = x^3 + axz^2 + bz^3\}$ *in Weierstrass normal form, with* H *having cardinality* $n + O(K)$*.*

It is easy to spot that the first case of the theorem corresponds to the trivial example mentioned earlier, or sets close to the near example, having almost all the points in a single line. The second case on the other hand, corresponds to the Böröczky and near Börözcky examples described earlier. The third case is a bit harder to explain, but it still keeps a close relation with the other two cases.

We will expand later on some of the arguments that Green and Tao used to arrive at this theorem and how these seemingly different sets of points come up next to each other in the same theorem. But the main idea is that they prove that for a set of points to span few ordinary lines big part of it should lie on a finite subgroup of the group defined on a cubic curve (see explanation on [11]).

These three cases come from the different nature of the cubic curve in question. Case 1 would come from a line, case 2 from a conic curve plus a line, and the case 3 from an irreducible cubic curve.

One can check that, even though all three kind of sets span few ordinary lines (less than Kn), the Böröczky examples are the ones that give the best bound, and thus the optimal examples for the Sylvester's problem.

Indeed, checking for the first set one would expect $O(Kn) + O(K^2)$ ordinary lines generated by this set. This number can be a bit optimized by aligning some of the $O(K)$ points outside of the line. But, if we call x to the number of points not in the line ($x \in O(K)$), we see that any of those points would span at least $n - 2x$ ordinary lines. Which means that the whole set span $n - O(K)$ ordinary lines.

It can also be proven that the third case examples span as well a number $n - O(K)$ of ordinary lines, and in the end we conclude that the Böröczky bound is the best possible outcome, giving us the result in (6).

2.2 3d case

We will now comment on the results known for the Sylvester's problem in dimension 3.

Our generalization of Sylvester's problem restricted to 3 dimensions consists on finding the minimum number of ordinary planes spanned by a set of points S in the 3 dimensional space such that not all S is in a line and such that there are not three collinear points in S.

The condition of non-collinearity is necessary for our generalization in order to not have sets without ordinary planes and to have a proper generalization of the Sylvester's problem. Indeed, if this condition were missing, for $n \geq 6$, the example with the n points distributed between two skew lines, each of them having at least three points does not span any ordinary plane.

Also, this non-collinearity condition in the 3 dimensional case is the motivation behind the condition added to the general d-dimensional problem of having every subset of d points in S span a hyperplane.

The main reference we will use for this will be the work of Simeon Ball in [9]. In his article Simeon Ball tries to prove a similar result as that of Green and Tao in [11], giving a structural theorem for sets of points in dimension 3 spanning few ordinary planes.

As in the planar case, we do not want to immerse ourselves too deep explaining in detail the work done by Simeon Ball, but we think interesting to show the examples with few ordinary planes for 3 dimensions, as well as the most important results.

Also, we feel important to show these results as this work is closer to the study of the problem in 4 dimensions, which we will be facing in the later section of this thesis.

First we will show which are the examples in 3 dimensions of sets that span few ordinary planes. Then we will show the structural theorem for the 3 dimensions.

2.2.1 Prism and antiprism examples

In this section we want to present briefly the examples of collections of points on $\mathbb{RP}^3$ that span few ordinary planes.

The expected number of ordinary planes in a random configuration is of the order of $O(n^3)$. The trivial example for $\mathbb{RP}^3$, which is all but one of the n points contained in the same plane, is the first example to improve this bound, producing $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ ordinary planes.

To improve this bound further, we are going to present the key examples in three dimensions: the prism and the antiprism.

Definition 2.6 (Prism)**.** Let m be an integer greater than 2, and let π_1 and π_2 be two planes of $\mathbb{RP}^3$. Let us consider two sets R_1 and R_2 of m points each, distributed respectively in π_1 and π_2 such that both R_1 and R_2 are projectively equivalent to a regular m-gon. Now we say that the collection of $2m$ points $R = R_1 \cup R_2$ is a prism if there is a point p_{inf} such that for any point $p \in R_1$, the line joining p and p_{inf} intersects π_2 into a point of R_2.

We refer to the configuration described above as prism, and we will denote it as P_{2m}.

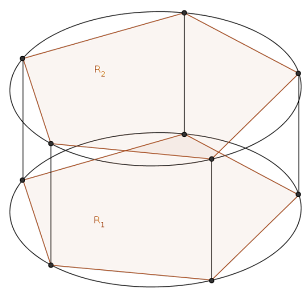


Figure 3: The prism configuration P_{10}.

In the image 3 we show a prism in the more geometrical sense. Notice that the definition above is describes a prism in a more general way, but it can simply be interpreted as the sets which can be transformed into a prism (classically speaking) by a projective transformation. Because of this, in the following we will be thinking of the prism examples as if they were in the form presented in 3.

We have said that the prism example is a configuration of points spanning few ordinary planes in $\mathbb{RP}^3$. In the following we show the exact number of ordinary planes spanned by one prism P_{2m}.

Proposition 2.7. *Let m be an integer greater than 3, then the prism set P_{2m} spans $2m\lfloor\frac{m-1}{2}\rfloor$ ordinary planes.*

Proof. Let us count the number of ordinary planes by checking the planes generated by each set of 3 points.

The prism P_{2m} has its $2m$ points divided between the two planes π_1 and π_2, in the form of a regular polygon on each plane.

Let us choose 3 points out of the $2m$ of the prism. If the three points are contained in the same plane π_i, then the plane spanned by the three points will be the plane π_i, and since each π_i has m points and $m > 3$, it is not an ordinary plane.

So any ordinary plane must be spanned by two points on π_1 and one point on π_2 or the other way around.

Let us fix x the only point in π_1. Now let us consider two points y and z in the plane π_2. Let us draw a line through x parallel to the line $\langle y, z\rangle$. If this line passes through another point of π_1, this would mean that this other point will be contained in the plane generated by x, y and z.

Let us consider the lines generated by a pair of points in π_2. These lines go on m different directions. If we draw a line on each of these m directions, only one of the lines will not contain any other point of π_1, which is the line tangent to the circumscribed circle of the m-gon on π_1. Let us call this line l.

The only pairs of points y, z on π_2 that will span an ordinary plane with x are the ones such that the line $\langle y, z\rangle$ is parallel to the tangent l. There are $\lfloor\frac{m-1}{2}\rfloor$ of these pairs. This is portrayed in the figure 4.

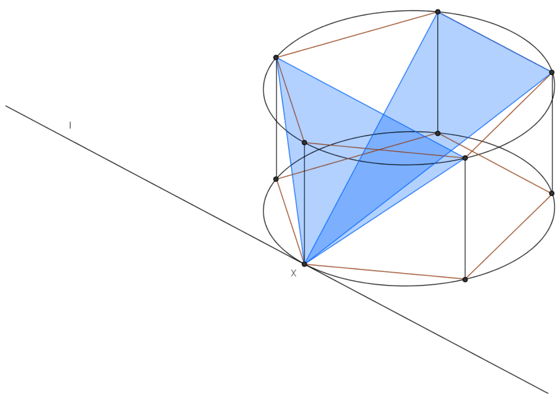


Figure 4: Prism configuration P_10 with the ordinary planes (in blue) generated from a point X.

By symmetry, we obtain the same number of ordinary planes for each selection of the point x. So we get a total of $2m \times \lfloor\frac{m-1}{2}\rfloor$ ordinary planes spanned by P_{2m}. □

There is another example of structure with few ordinary planes in $\mathbb{RP}^3$, which is the anti-prism, as referred to in [6] and [9]. An anti-prism can be thought as a prism where the top face has been given a little twist, as depicted in 5. We will denote the anti-prism set with $2m$ vertices as A_{2m}.

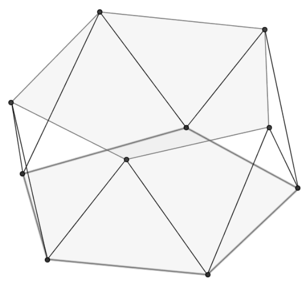


Figure 5: Anti-Prism configuration of 10 points spanning 20 ordinary planes.

Because of the similarities between the two structures, it is easy to see that the anti-prism span, as the prism examples, few ordinary planes. Doing similar arguments as before one can prove the following:

Proposition 2.8. *Let m be an integer greater than 3. Then, the anti-prism set A_{2m} spans $2m\lfloor \frac{m}{2} \rfloor$.*

This coincides with the number of ordinary planes for m odd, and is worse otherwise. In fact, one can prove that for m odd, one can do a projective transformation to transform the anti-prism (image 5) into a prism (image 3).

This is more easily seen if we notice that the anti-prism for m odd fits in the definition 2.6 (by taking p_{inf} to be the center point of the anti-prism).

Now, as in the two dimensional case, these presented examples are good for the cases when n is even. For the odd cases, we would need to content ourselves with the near examples, that is to say, a prism with a point removed or an anti-prism with a point removed.

We will not include here the analysis of these cases to compute the exact number of ordinary planes, but the whole proof of those results can be found on [6].

Before continuing, we think it is worth to mention the relation between the sets described in this section (the prism and anti-prism examples) and the key examples in the 2 dimensional version of the Sylvester's problem (the Böröczky examples).

To see this relation, we have to make the following observation: If we have a set of points S in $\mathbb{RP}^3$ and project this set from one of its points $p \in S$ to $\mathbb{RP}^2$, the number of ordinary lines in the projection is equal to the number of ordinary planes of S that contains p. This is because every ordinary plane containing p projects into an ordinary line in $\mathbb{RP}^2$.

It is only natural, then, that if we want to build sets of points in $\mathbb{RP}^3$ spanning few ordinary planes we should be looking at configurations of points that project into sets in $\mathbb{RP}^2$ with few ordinary lines.

The prism and anti-prism examples are perfect in this sense, since they project from any of their points into Böröczky sets. This assures that there are few ordinary planes passing through each point, and is the reason of the good quality of these examples for the Sylvester's problem.

2.2.2 Structural theorem in dimension 3

We have mentioned that the prism examples are the optimal configurations for the 3 dimensional version of the Sylvester's problem. Joining the results mentioned in the last section we are able to get the value of the optimal number of spanned ordinary planes by a set of n points.

$$e_3(n) = \begin{cases} n\lfloor \frac{n-1}{4} \rfloor & \textit{if} \quad n \equiv 0 mod(2) \\ \frac{3}{8}n^2 - n + \frac{5}{8} & \textit{if} \quad n \equiv 1 mod(4) \\ \frac{3}{8}n^2 - \frac{3}{2}n + \frac{17}{8} & \textit{if} \quad n \equiv 3 mod(4) \end{cases} \tag{7}$$

This holds true for large values of n, having some small counterexamples when n is small, which can be seen as simple corner cases.

This is proven in [9] by Simeon Ball when he proofs the optimality of the prism and anti-prism example. This he does by creating his own version of the structure theorem of Ben Green and Terence Tao.

In his structure theorem, with inspiration on its version on 2 dimensions, Simeon classifies the nature of the sets of points in $\mathbb{RP}^3$ which span few ordinary plane (at most Kn^2).

Theorem 2.9. *Let S be a set of n points in $\mathbb{RP}^3$, no three collinear and not all coplanar, spanning less than Kn^2 ordinary planes, for some $K = o(n^{\frac{1}{7}})$. Then one of the following holds:*

(i) There are two distinct quadrics such that all but at most $O(K)$ points of S are contained in the intersection of the quadrics. Furthermore, all but at most $O(K)$ points of S are incident with at least $\frac{3}{2}n - O(K)$ ordinary planes.

(ii) There are two planar conic sections of a quadric which contain at least $\frac{1}{2}n - O(K)$ points of S.

(iii) All but at most $2K$ points of S are contained in a plane.

There are parallelisms between the structure theorem by Green and Tao (2.5) and the one here. One can see that the third case corresponds to the trivial example and the second case corresponds to the prism and anti-prism examples, which we have seen have relation with the Böröczky examples in $\mathbb{RP}^2$. The first case can also be related with the remaining case of the structural theorem in 2 dimensions.

In fact, Simeon Ball uses a few times in his proof of this theorem the projection of the set S from a point to $\mathbb{RP}^2$, with the argument that the projections of the set from a point with few ordinary planes give sets in 2 dimensions with few ordinary lines. This allows him to use the work of Green and Tao, and in proving the characteristics of the projection of his sets, he is later able to nature of the sets themselves.

An important part of the proof of Simeon Ball is the relationship between the problem and the intersection of two quadrics. As Green and Tao proved in their paper that sets with few ordinary lines should mostly be contained in a cubic curve, Simeon managed to prove that for any set in $\mathbb{RP}^3$ spanning few ordinary planes, all but a small part of it must be contained in the intersection of two quadrics.

We will expand later in the relation between the cubic curve appearing in the Green and Tao's work and the intersection of two quadrics.

Parting from the structure theorem 2.9, Simeon Ball reaches the following theorem as an immediate consequence:

Theorem 2.10. *Let S be a set of n points in $\mathbb{RP}^3$, no three collinear and not all coplanar. There is a constant c such that if S spans less than $\frac{1}{2}n^2 - cn$ ordinary planes then, for n sufficiently large, S is either a prism, an anti-prism, a prism with a point removed or an anti-prism with a point removed.*

Which proves that the prism and anti-prism examples are the optimal configurations of points when n is large enough.

This follows naturally from the structure theorem 2.9, as one can easily prove that the first and third cases of the theorem yield sets with more than $\frac{1}{2}n^2 - cn$ for a large enough constant c. That leaves us only with the prism and anti-prism examples.

3. Sylvester's problem in $\mathbb{RP}^4$

In this section we are going to introduce our own study of the problem in 4 dimensions.

After the work of Ben Green and Terence Tao in one hand and Simeon Ball in the other, we are able to classify the sets of points with few ordinary hyperplanes in $\mathbb{RP}^2$ and $\mathbb{RP}^3$, obtaining then the asymptotic solution of the Sylvester's problem for those dimensions.

For us the next natural step is the study of the problem in 4 dimensions. After the work in the lower dimensions, one would think a similar result could be achieved in dimension 4 by the same means used in the previous studies. Our objective will be exactly that.

This work, though, presents some complications not present in the equivalent studies in the previous two dimensions. For one thing, just the fact of working in 4 dimensions is in itself a complication with respect to the other studies, as the visualization of the kind of objects we will be working with in 4 dimensions is somewhat more challenging.

On the other hand, and most problematic of all, unlike in the cases of dimensions 2 and 3, we start our study in 4 dimensions without known examples of structures of points spanning few ordinary hyperplanes (aside from the trivial example). The existence of the Böröczky examples for the planar case and the prism and anti-prism examples for the 3-dimensional case help the study of those cases, giving a clear direction to the research, as one can suspect from the start these examples to be optimal.

In our case, though, we are helpless in this aspect. The restriction that any subset of 4 points must span a hyperplane implies that we cannot have in our sets of points any 3 points collinear or 4 points coplanar. This condition is very strong and has prevented us from finding any simple generic examples (similar to those in 2 and 3 dimensions) of structures with few ordinary hyperplanes, aside from the trivial one. The lack of this kind of examples forces us to work from scratch, having to guess if either no such structure existed in the first place or simply we could not find it.

This alone makes us suspect that the trivial example is the optimal configuration for dimension 4. But even with this suspicion we need to do a structural study similar to the one in lower dimensions to prove this.

We will follow the same steps as Simeon or Green and Tao did for their respective results.

First we will describe the equivalent concept of triangular grids in 4 dimensions, as to classify the sets with few ordinary hyperplanes we will have our eyes set on configurations of points rich in $(d-1)$-hyperplanes. We will prove some algebraic results over triangular grids in 4 dimensions, in the end arriving at the characterization of the sets of points whose dual form a triangular grid: points whose dual defines a triangular grid in 4 dimensions must be contained in the intersection of 5 quadrics.

After these results, we will try to give our own version of the structure theorem for 4 dimensions, arriving, then, to the solution of the Sylvester's problem for this case.

3.1 Triangular grids in 4 dimensions

One of the key ingredients in the work of Green and Tao are the structures they define as triangular grids. Their definition of triangular grids can be roughly presented as:

Definition 3.1. Let I, J, K be three discrete intervals in $\mathbb{Z}$. A triangular grid of dimensions I, J, K is a collection of lines $(p_i^*)_{i\in I}$, $(q_j^*)_{j\in J}$, $(r_k^*)_{k\in K}$ in $\mathbb{RP}^2$ such that p_i^*, q_j^* and r_k^* intersect into a point if and only if $i+j+k=0$ and no other line in the grid passes through the point of intersection.

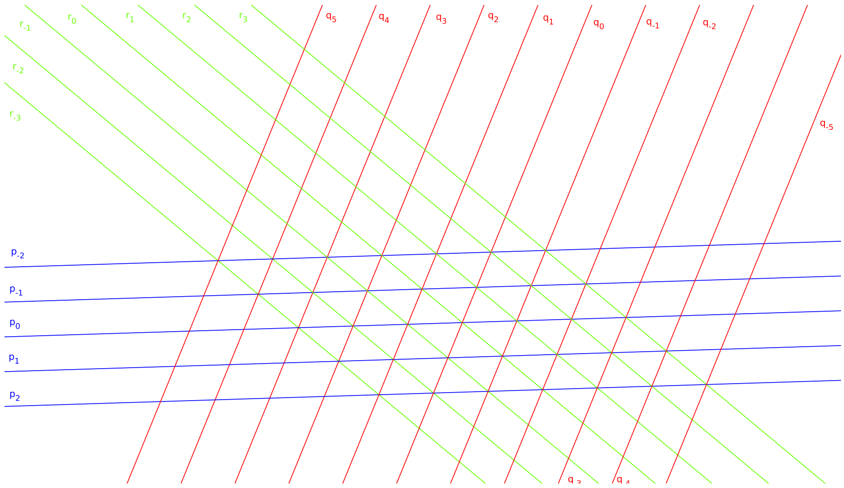


Figure 6: Triangular grid of dimensions $[-2, 2]$, $[-3, 3]$ and $[-5, 5]$.

In their study, Green and Tao prove two things about triangular grids that serve to prove their structural theorem.

- First they prove a set of points in $\mathbb{RP}^2$ whose dual forms a triangular grid is contained in a single cubic curve.
- Secondly they prove that the dual of any set of points in $\mathbb{RP}^2$ that spans few ordinary lines must be mostly by triangular grids.

Joining these together they manage to prove that any set of points spanning few ordinary lines is contained in the union of a handful of cubic curves, which then they manage to reduce to a single cubic curve. The structural theorem in $\mathbb{RP}^2$ comes just by studying different cases on the nature of the cubic curve.

The interest in triangular grids in the study of Sylvester's problem comes from the intuitive notion that sets with few ordinary lines should be rich in lines with exactly 3 points. This connects with the triangular grid naturally, as any point of intersection in the triangular grid dualizes into a line with exactly 3 points.

Starting with this intuitive notion, one can think of the results just mentioned as proving that any set spanning few ordinary lines must be rich in 3-lines, and that sets of points rich in 3-lines are almost contained in a cubic curve.

For the study in three dimensions, Simeon Ball had to make use of an equivalent structure in $\mathbb{RP}^3$. The definition of what Simeon calls a 'tetra-grid' is the natural generalization of the definition by Green and Tao to the 3 dimensions.

Similar statements as the ones proved by Green and Tao can be reached by using the concepts derived from the tetra-grid. Simeon does that in his work. First proving that a set of points that dualizes into a tetra-grid is contained in the intersection of two linearly independent quadrics, using this to prove in the end that any set of points in $\mathbb{RP}^3$ spanning few ordinary planes must be contained in two linearly independent quadrics.

As in the two dimensional case, the idea behind the tetra-grid is the characterization of sets of points rich in planes containing exactly 4 points.

To face the problem in 4 dimensions we will need to do the same kind of study.

In this section we are going to use the natural generalization of the definition of triangular grids to higher dimensions. Our goal will be to find which algebraic variety plays the same role as the cubic curve in two dimensions or the intersection of two quadrics in three.

After the exposition on Green and Tao's work on the triangular grid, the generalization to the fourth dimension of this structure should be obvious. We will call this structure a 5-cell grid:

Definition 3.2. Let I, J, K, L, M be discrete intervals in $\mathbb{Z}$. A 5-cell grid of dimensions I, J, K, L, M is a collection of hyperplanes $(p_i^*)_{i\in I}$, $(q_j^*)_{j\in J}$, $(r_k^*)_{k\in K}$, $(s_l^*)_{l\in L}$, $(t_m^*)_{m\in M}$ in $\mathbb{RP}^4$ such that p_i^*, q_j^*, r_k^*, s_l^*, t_m^* intersect into a point if and only if $i+j+k+l+m=0$ and no other hyperplane of the grid passes through the point of intersection.

Now, along with the notion of 5-cell grid, we want to introduce the notion of good/bad edges to help us in our study. In their paper, Green and Tao defined good and bad edges in order to characterize edges of the graph Γ according to whether they contributed to the triangular grid structure of the graph.

According to Green and Tao, a good edge in the graph in $\mathbb{RP}^2$ was an edge that was the side of two triangular faces and whose end vertices were each of degree 6. A bad edge was simply any edge that was not good.

For $\mathbb{RP}^4$ we will call a good edge to the following:

Definition 3.3 (Good and bad edges)**.** A good edge of the graph Γ in $\mathbb{RP}^4$ is any edge e in Γ such that any face that contains e is a triangle and such that the end vertices of e are each incident with exactly 5 hyperplanes of S^*. A bad edge is any edge of Γ that is not good.

To state this in the same way as Green and Tao we could say that the end vertices of a good edge have degree 20.

Let us also define the concept of rather good edges as follows:

Definition 3.4 (Rather good edges)**.** We say an edge of the graph Γ is a rather good edge if it is a good edge and every edge coming from its end vertices is also a good edge. We will call an edge of Γ slightly bad if it is not a rather good edge.

These two concepts are important, as their appearance comes naturally inside the 5-cell structure. The other implication is also true, as it can be proven that the structure of Γ around a rather good edge is that of a 5-cell grid. This fact will be more useful to us, and will help us later.

Although it seems a rather obvious statement, to really prove it we need to perform a little check first:

Proposition 3.5. *Let e be a rather good edge of the graph Γ lying on the line $p^*\cap q^*\cap r^*$. If there is a triangle in the plane $p^*\cap q^*$ with side e that has its other sides cut out by s^* and r^*, then there must be triangles in the planes $p^*\cap r^*$ and $q^*\cap r^*$ with side e whose sides are also cut by r^* and s^* respectively.*

From this the earlier mention result follows:

Corollary 3.6. *The structure of* Γ *around a rather good edge is that of a* 5*-cell grid.*

Now, we have mentioned the relationship between good and rather good edges and 5-cell grids. Also, as we have mention earlier, the objective of introducing the grids in the first place was the idea we have that sets of points spanning few ordinary hyperplanes should be rich in 5-hyperplanes, and thus, should contain 5-cell grids.

One of the early results we can show indicating this has to do with the notions we have just introduced of good edges and rather good edges. The key thing to notice here is that, when we are dealing with sets of points spanning few ordinary hyperplanes, the number of bad/slightly bad edges in the graph Γ is low.

We state this as the following two propositions:

Lemma 3.7. *Let* S *be a set of* n *points in* $\mathbb{RP}^4$ *not all in an hyperplane and such that any subset of* 4 *points span a hyperplane. Let us assume that* S *spans less than* Kn^3 *ordinary hyperplanes. Then in the graph* $\Gamma = \Gamma_S$ *there are at most* $48Kn^3$ *bad edges.*

Proof. First of all, let us call V and E respectively to the set of vertices and edges on the graph Γ.

Let π and π' be two elements of S^*. As these are hyperplanes of $\mathbb{RP}^4$, the intersection $\pi \cap \pi'$ forms a plane.

Let us call $V_{\pi,\pi'}$ to the set of vertices in Γ which are in both π and π', and let us define in the same way the set $E_{\pi,\pi'}$ of edges in the intersection. Let us also consider the set of faces $F_{\pi,\pi'}$ in the plane $\pi \cap \pi'$ defined by these vertices and edges. As $\pi \cap \pi'$ is a projective plane, Euler's formula gives us:

$$|V_{\pi,\pi'}| - |E_{\pi,\pi'}| + |F_{\pi,\pi'}| = 1 \tag{8}$$

Let us call F to the set of faces in the graph Γ, which we will take as the union of the faces defined in the planes $\pi \cap \pi'$ for all $\pi, \pi' \in S^*$:

$$F = \bigcup_{\pi,\pi' \in S^*} F_{\pi,\pi'}$$

Now, taking the Euler's formula (8) into account, if we sum for all the choices of $\pi \cap \pi'$ in S^* we end up with:

$$\sum_{i=4}^{n} \frac{i(i-1)}{2} v_i - 3|E| + |F| = \frac{n(n-1)}{2} \tag{9}$$

where the numbers v_i represent the number of vertices in Γ which are incident with exactly i hyperplanes of S^*. This formula comes naturally for the following:

- Any face is exactly in one plane $\pi \cap \pi'$.
- Any line in $\mathbb{RP}^4$ is the intersection of 3 hyperplanes. Thus, any edge of E is in exactly 3 planes $\pi \cap \pi'$.
- A vertex which is incident with exactly i hyperplanes of S^* will be contained in $\frac{i(i-1)}{2}$ planes $\pi \cap \pi'$.

Now we will use some counting arguments to derive some relations between the number of faces, edges and vertices of the graph.

First, since each edge is contained in exactly three planes $\pi \cap \pi'$ and since in each of these planes, it is the bound between two faces, we can obtain the following equation by counting pairs (e, f) of edges and faces in two different ways:

$$\#(e, f) = 6|E| = \sum_{j \geq 3} j f_j \tag{10}$$

where f_i denotes the number of faces in Γ with j edges.

Secondly let us count pairs (v, e) of edges and vertices in two different ways. For that we will note that for a vertex incident with exactly i hyperplanes there are $\frac{i(i-1)(i-2)}{6}$ lines (which are intersection of three hyperplanes) incident with it. Then:

$$\begin{gathered} \#(v, e) = 2|E| = \sum_{i \geq 4} 2 \times \frac{i(i-1)(i-2)}{6} v_i \\ \Downarrow \\ 6|E| = \sum_{i \geq 4} i(i-1)(i-2) v_i \end{gathered} \tag{11}$$

Now we will use these equations and the fact that $v_4 \leq Kn^3$ (since v_4 represents the number of ordinary hyperplanes), to obtain the bound for the number of bad edges in Γ.

We can combine the Euler's formula (9) with the equations (10) and (11) to obtain the following:

$$\begin{aligned} |F| - 3|E| + \sum_{i \geq 4} \frac{i(i-1)}{2} v_i &= \frac{n(n-1)}{2} \\ 6|F| - 18|E| + 3 \sum_{i \geq 4} i(i-1) v_i &= 3n(n-1) \\ 6|F| - \left[2 \sum_{j \geq 3} j f_j \right] - \left[\sum_{i \geq 4} i(i-1)(i-2) v_i \right] + 3 \sum_{i \geq 4} i(i-1) v_i &= 3n(n-1) \\ -2 \sum_{j \geq 3} (j-3) f_j - \sum_{i \geq 4} i(i-1)(i-5) v_i &= 3n(n-1) \end{aligned}$$

$$12 v_4 = 3n(n-1) + 2 \sum_{j \geq 4} (j-3) f_j + \sum_{i \geq 6} i(i-1)(i-5) v_i \tag{12}$$

We can use this last equation (12) to obtain some bounds, since every summand on the right hand side is non-negative.

First we can deduce for the faces that:

$$
\begin{aligned}
\sum_{j\geq 4}(j-3)f_j \leq 6v_4 & \leq 6Kn^3 \\
\sum_{j\geq 4} f_j \leq \sum_{j\geq 4}(j-3)f_j & \leq 6Kn^3 \\
\sum_{j\geq 4} jf_j \leq 6Kn^3 + 18Kn^3 & = 24Kn^3
\end{aligned}
$$

This gives us a bound for the number of edges incident with a face which is not a triangle. Now, if we look at the vertices:

$$
\begin{aligned}
\sum_{i\geq 6} i(i-1)(i-5)v_i \leq 12v_4 & \leq 12Kn^3 \\
\sum_{i\geq 6} i(i_1)v_i \leq \sum_{i\geq 6} i(i-1)(i-5)v_i & \leq 12Kn^3 \\
\sum_{i\geq 6} i(i-1)(i-2)v_i \leq 12Kn^3 + 36Kn^3 & = 48Kn^3
\end{aligned}
$$

From this last equation we can find a bound for the number of edges incident to a vertex which is not incident to exactly 5 hyperplanes:

$$
\sum_{i\neq 5} \frac{i(i-1)(i-2)}{3} v_i = 8v_4 + \sum_{i\geq 6} \frac{i(i-1)(i-2)}{3} v_i \leq 8Kn^3 + \frac{48Kn^3}{3} = 24Kn^3 \tag{13}
$$

Now, an edge can be bad if either it is contained in some non-triangular face or if one of its end vertices is not incident with exactly 5 hyperplanes (i.e. the vertex has degree different than 20. Because of that, if we join the two equations (3.1) and (13) we obtain the following bound:

$$
\#\text{Bad Edges} \leq 24Kn^3 + 24Kn^3 = 48Kn^3 \tag{14}
$$

□

Lemma 3.8. *The number of slightly bad edges is at most* $1872Kn^3$.

Proof. This is quickly proven by counting. A slightly bad edge is either a bad edge itself or a good edge which is incident with a bad edge. We know from lemma 3.7 that the number of bad edges in Γ is at most $48Kn^3$. In the other hand, we know that the end vertices of a good edge have degree 20 (since they are incident to 5 hyperplanes of S^*. Thus, we can bound the number of slightly bad edges by the following rough estimation:

$$
\#\text{Slightly Bad Edges} \leq 48Kn^3 + 2 \times (20-1) \times 48Kn^3 = 1872Kn^3 \tag{15}
$$

□

3.2 Intersection of 5 quadrics

We have defined in the previous section the concepts of 5-cell grid and good/bad edges and how they relate to our problem. To keep following the steps of the work of Green and Tao now we want to study if sets of points which dualize into 5-cell grids are contained in a certain variety and, in that case, what is the nature of that variety.

The answer to that question is that, indeed, sets of points which dualizes in $\mathbb{RP}^4$ into 5-cell grids are contained the intersection of 5 quadrics.

In this section we will prove this, along with some other results that we will need later about these varieties. Also, we will discuss briefly the nature of this kind of variety and how it relates with its counterparts in 2 and 3 dimensions.

The first thing we want to do, before studying the characteristics of the variety, is prove that the sets of points forming the 5-cell are contained in the intersection of 5 quadrics. For this we will need two lemmas, the first of which is the following:

Lemma 3.9. *Let $\{p_0, p_1, q_0, q_1, r_{-1}, r_0, s_{-1}, s_0, t_{-1}, t_0, t_1\}$ be eleven points of $\mathbb{RP}^4$ such that p_i, q_j, r_k, s_x, t_y are contained in a hyperplane if and only if $i + j + k + x + y = 0$. Then there are 5 linearly independent quadrics that contains the 11 points.*

Proof. With this set of points we have defined the following 14 hyperplanes:

$$
\begin{aligned}
H_{11} &:= \{p_0, q_0, r_0, s_0, t_0\} & H_{12} &:= \{p_1, q_1, r_{-1}, s_{-1}, t_0\} \\
H_{21} &:= \{p_1, q_0, r_{-1}, s_0, t_0\} & H_{22} &:= \{p_0, q_1, r_0, s_{-1}, t_0\} \\
H_{31} &:= \{p_1, q_0, r_0, s_{-1}, t_0\} & H_{32} &:= \{p_0, q_1, r_{-1}, s_0, t_0\} \\
H_{41} &:= \{p_0, q_1, r_0, s_0, t_{-1}\} & H_{42} &:= \{p_1, q_0, r_{-1}, s_{-1}, t_1\} \\
H_{51} &:= \{p_1, q_1, r_{-1}, s_0, t_{-1}\} & H_{52} &:= \{p_0, q_0, r_0, s_{-1}, t_1\} \\
H_{61} &:= \{p_1, q_1, r_0, s_{-1}, t_{-1}\} & H_{62} &:= \{p_0, q_0, r_{-1}, s_0, t_1\} \\
H_{71} &:= \{p_1, q_0, r_0, s_0, t_{-1}\} & H_{72} &:= \{p_0, q_1, r_{-1}, s_{-1}, t_1\}
\end{aligned}
$$

Any two hyperplanes H_{j1} and H_{j2} form an hyperplane pair quadric. Let us denote by $Q_j = H_{j1} \cup H_{j2}$.

Let us take the four quadrics Q_j for $j = 4, 5, 6, 7$. These quadrics are four linearly independent quadrics that contains the 10 points $\{p_0, p_1, q_0, q_1, r_{-1}, r_0, s_{-1}, s_0, t_{-1}, t_1\}$ (the eleven points without t_0).

Now let us consider two quadrics $W_k = Q_1 + \lambda_k Q_k$ for $k = 2, 3$. These quadrics contain the 9 points $\{p_*, q_*, r_*, s_*, t_0\}$. We choose λ_k so that W_k passes also through the point t_1.

Let us consider a map γ such that it maps the points $p_0 \leftrightarrow p_1$, $q_0 \leftrightarrow q_1$, $r_{-1} \leftrightarrow r_0$, $s_{-1} \leftrightarrow s_0$ and $t_{-1} \leftrightarrow t_1$. The map γ satisfies for $j = 1..7$ that:

$$
\begin{aligned}
\gamma(H_{j1}) &= H_{j2} \\
\gamma(H_{j2}) &= H_{j1}
\end{aligned}
$$

This implies that the quadrics Q_j remain the same through γ, and therefor also the quadrics W_k. On the other side, because γ maps the points $t_{-1} \leftrightarrow t_1$, and because the quadrics W_k contain the point t_1

and are unaltered by γ we can deduce that the point t_{-1} is also contained in the quadrics W_k.

With this we get a total of 6 linearly independent quadrics passing through the 10 points $\{p_*, q_*, r_*, s_*, t_{-1}, t_1\}$, which are $\{W_1, W_2, Q_4, Q_5, Q_6, Q_7\}$. These quadrics form a subspace of dimension 6 on the space of quadrics on $\mathbb{RP}^4$.

As we have a subspace of dimension 6 of the space of quadrics passing through those 10 points, adding an extra restriction to the space will give us a subspace of dimension 5. So, adding the point t_0 we get that there are 5 linearly independent quadrics containing the original 11 points. $\square$

Notice that the space of quadrics in $\mathbb{RP}^4$ has dimension $\binom{6}{2} = 15$, and that, in a generic situation, each point we force our quadrics to contain should impose a linearly independent condition on the space of quadrics. Thus, in that general situation we would expect the space of quadrics passing through 11 points to be of dimension 4 and the space of quadrics passing through 10 points to be of dimension 5.

In this last lemma we have managed to prove, though, that there are 6 linearly independent quadrics passing through the 10 points used in the proof, and 5 quadrics through the whole set of 11 points. This constitute one more dimension than what we would expect in a general situation.

From this first lemma we obtain the variety that we were looking for, the intersection of 5 linearly independent quadrics. In order to extend this result to the entire 5-cell grid, we will need a further result, which is closely related with the proof we did on the previous one:

Lemma 3.10. *Let $\{p_0, p_1, q_0, q_1, r_{-1}, r_0, s_{-1}, s_0, t_{-1}, t_1\}$ be 10 points on $\mathbb{RP}^4$ such that p_i, q_j, r_k, s_x, t_y are contained in a hyperplane if and only if $i + j + k + x + y = 0$. Then any quadric that contains 9 of the 10 points must also contain the tenth one.*

Proof. As we have said, the expected dimension of the space of quadrics passing through the 10 points should be 5, instead of having the 6 linearly independent quadrics we proved in the 3.9 lemma.

Now we want to prove that the space of quadrics through any subset of 9 points of these 10 is of dimension 6, as one would expect. This will imply that the 6 linearly independent quadrics passing through any 9 points are the same as the ones going through the 10 points, and thus, any quadric going through 9 points should contain the tenth point of this set.

For this we will try to prove that any subset of 9 points of these impose 9 linearly independent conditions in the space of quadrics.

Without loss of generality, let us suppose that the subset of 9 points we choose is the one without the point t_1. We can apply a projective transformation so the points have the following coordinates:

$$
\begin{aligned}
p_0 &= [1, 0, 0, 0, 0] & p_1 &= [a_1, a_2, a_3, a_4, a_5] \\
q_0 &= [0, 1, 0, 0, 0] & q_1 &= [b_1, b_2, b_3, b_4, b_5] \\
r_{-1} &= [0, 0, 1, 0, 0] & r_0 &= [c_1, c_2, c_3, c_4, c_5] \\
s_{-1} &= [0, 0, 0, 1, 0] & s_0 &= [d_1, d_2, d_3, d_4, d_5] \\
t_{-1} &= [0, 0, 0, 0, 1] & &
\end{aligned}
$$

Now let us consider ϕ a generic quadric in $\mathbb{RP}^4$ with coefficients as follows:

$$\phi(X) = \alpha_{11}X_1^2 + ... + \alpha_{55}X_5^2 + \alpha_{12}X_1X_2 + ... + \alpha_{45}X_4X_5$$

To impose that the quadric ϕ contains a certain point x is the same as to impose the equation $\phi(x) = 0$. Now, if we want to impose that this quadric contains the 9 points mentioned above, we obtain the following system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_1a_2 & a_1a_3 & a_1a_4 & a_1a_5 & a_2a_3 & a_2a_4 & a_2a_5 & a_3a_4 & a_3a_5 & a_4a_5 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 & b_1b_2 & b_1b_3 & b_1b_4 & b_1b_5 & b_2b_3 & b_2b_4 & b_2b_5 & b_3b_4 & b_3b_5 & b_4b_5 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & c_1c_2 & c_1c_3 & c_1c_4 & c_1c_5 & c_2c_3 & c_2c_4 & c_2c_5 & c_3c_4 & c_3c_5 & c_4c_5 \\ d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & d_1a_2 & d_1d_3 & d_1d_4 & d_1d_5 & d_2d_3 & d_2d_4 & d_2d_5 & d_3d_4 & d_3d_5 & d_4d_5 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ \alpha_{44} \\ \alpha_{55} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15} \\ \alpha_{23} \\ \alpha_{24} \\ \alpha_{25} \\ \alpha_{34} \\ \alpha_{35} \\ \alpha_{45} \end{bmatrix} = 0 \tag{16}$$

The fact that we want to prove is that these 9 equations impose 9 linearly independent conditions, giving the space of quadrics only 6 degrees of liberty, and thus, dimension 6.

This is the same as proving that the matrix (16) has rank 9.

The five points corresponding to the basis of $\mathbb{RP}^4$ give us immediately rank 5 on the matrix. We just need to prove that the 4×10 sub-matrix corresponding to the last four rows is of rank 4.

The key ingredient to prove this comes from the nature of the set we are dealing with. We will use the fact that these points form 5-hyperplanes and that the points p_i, q_j, r_k, s_x, t_y are in an hyperplane if and only if $i + j + k + x + y = 0$.

First of all, we can suppose that none of the coefficients a_5, b_5, c_5, d_5 are 0. This is because, if any of them were, that point would be contained in the hyperplane formed by p_0, q_0, r_0, s_0 and by hypothesis they are not.

Secondly, we know that the 5 points p_1, q_1, r_{-1}, s_{-1} and t_{-1} are not contained in an hyperplane, as their indexes sum to -1. This means that the 5 points are linearly independent and that the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}$$

has rank 5.

Now, because we know that the coefficients a_5, b_5, c_5, d_5 are different from 0, we can scale the points to make them 1 and get rid of the X_5 coefficients. After doing that, looking at the sub-matrix given by the columns corresponding to the coefficients of X_1X_5, X_2X_5, X_3X_5 and X_4X_5, we get the following matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

which we know has rank 4.

From this we get that the 9 points we chose impose 9 linearly independent conditions to the space of quadrics, and thus, there should be just 6 linearly independent quadrics through the 9 points.

If the point left out was other than t_1 we will need to use other coordinates and use other points as the basis, but we are able to do the same reasoning.

As we have proven that there are just 6 linearly independent quadrics through each subset of 9 points, we conclude that any quadric passing through said subset of 9 points should also contain the tenth point. □

From these two lemmas 3.9, 3.10 we have proved that some small sets of points on a 5-cell grid are contained in the intersection of 5 linearly independent quadrics. What we want to prove now, using these two results, is that the whole 5-cell grid structure must be contained in the intersection of 5 quadrics, which is the main theorem of this section.

In order to use these lemmas later, we will need to be careful and verify that the points to which we apply them maintain the same structure as the points in the lemma. With this in mind and to avoid complications later, we will present the sets of points in these two lemmas as follows.

For the lemma 3.9:

p_1	q_1			t_1
p_0	q_0	r_0	s_0	t_0
		r_{-1}	s_{-1}	t_{-1}

And for the lemma 3.10:

p_1	q_1			t_1
p_0	q_0	r_0	s_0	
		r_{-1}	s_{-1}	t_{-1}

The exact statement that we will prove is slightly different of what we have mentioned, as we will be proving that a 5-cell grid at the neighborhood of a segment of rather good edges is contained in the intersection of 5 linearly independent quadrics. We present the theorem in this way as it will be more useful for us later if stated like this. Though, one can see that the proof we present here can be easily adapted to take into account the whole grid.

Theorem 3.11. *Let S be a set of points in $\mathbb{RP}^4$. Let $p, q, r \in S$ be points in S such that the line $p^* \cap q^* \cap r^*$ in Γ contains a segment T of m rather good edges. Then there are 5 linearly independent quadrics such that they contain the points p, q and r and all the points s such that s^* intersects $p^* \cap q^* \cap r^*$ in T.*

Proof. First of all, because the segment T is a segment of m rather good edges, we know that the structure of Γ around T is that of a 5-cell grid of dimensions $3 \times 3 \times 3 \times (m+1) \times (m+1)$.

Now, because it has the structure of a 5-cell grid, we can rename the points involved to be as follows:

$$\begin{array}{cccccccc} & & p_{-1} & p_0 & p_1 & & \\ & & q_{-1} & q_0 & q_1 & & \\ & & r_{-1} & r_0 & r_1 & & \\ s_{-m} & \dots & s_{-1} & s_0 & & & \\ & & & t_0 & t_1 & \dots & t_m \end{array}$$

having $p = p_0$, $q = q_0$ and $r = r_0$.

Now is the time to apply the lemmas 3.9 and 3.10 in order to get our 5 quadrics. It will be important when we use this lemmas that the 11 (respectively 10) points we apply the lemmas to hold the same structure as the points used in the lemmas and that one can form a bijection between the sets of points that keeps invariant the spanned hyperplanes of the set.

First of all we can apply the lemma 3.9 to the set:

t_1	r_1			p_1
t_0	r_0	s_0	q_0	p_0
		s_{-1}	q_{-1}	p_{-1}

It is clear that this set of points hold the same structure as that of the set in 3.9. So the lemma applies and we know that there are 5 linearly independent quadrics containing the 11 points.

Let us call $\{Q_1, Q_2, Q_3, Q_4, Q_5\}$ to this set of quadrics.

Now let us consider the set of 10 points:

t_1	r_1			q_1
t_0	r_0	s_0	p_0	
		s_{-1}	p_{-1}	q_{-1}

It holds the same structure as the set of points in lemma 3.10, so the lemma applies and any quadric passing through 9 of these points passes through the tenth. But because of what we saw earlier, we know that the 9 points $\{t_0, t_1, r_0, r_1, s_{-1}, s_0, p_{-1}, p_0, q_{-1}\}$ are contained in the 5 quadrics Q_i. Thus, the tenth point q_1 is also contained in these quadrics.

We can do this same trick to argue that the point r_{-1} is also contained in the quadrics.

Now we want to extend this argument to include all the points s_{-i} and t_i. We will manage this by induction:

First of all, we know already that the points $\{s_0, s_{-1}, t_0, t_1\}$ are contained in the 5 quadrics.

Now, for the induction hypothesis, let us suppose that the set of points $\{s_0, s_{-1} \dots s_{-i}, t_0, t_1 \dots t_i\}$ are all contained in our set of 5 quadrics. We will now prove that both s_{-i-1} and t_{i+1} are also contained in the 5 quadrics.

To begin, let us consider the following set of points:

p_1	q_0			t_{i+1}
p_0	q_{-1}	r_0	s_{-i+1}	
		r_{-1}	s_{-i}	t_{i-1}

Although it a bit more complicated than before, we can also check that this set of points hold the same hyperplane relation as the one in lemma 3.10. Indeed, this set span the following 8 hyperplanes:

$$
\begin{aligned}
H_{11} &= \{p_0, q_0, r_0, s_{-i+1}, t_{i-1}\} & H_{21} &= \{p_0, q_0, r_{-1}, s_{-i}, t_{i+1}\} \\
H_{12} &= \{p_1, q_0, r_0, s_{-i}, t_{i-1}\} & H_{22} &= \{p_0, q_{-1}, r_0, s_{-i}, t_{i+1}\} \\
H_{13} &= \{p_1, q_0, r_{-1}, s_{-i+1}, t_{i-1}\} & H_{23} &= \{p_0, q_{-1}, r_{-1}, s_{-i+1}, t_{i+1}\} \\
H_{14} &= \{p_1, q_{-1}, r_0, s_{-i+1}, t_{i-1}\} & H_{24} &= \{p_1, q_{-1}, r_{-1}, s_{-i}, t_{i+1}\}
\end{aligned}
$$

With this, lemma 3.10 applies. Now, as we know from the induction hypothesis that the 9 points $\{p_0, p_1, q_{-1}, q_0, r_{-1}, r_0, s_{-i+1}, s_{-i}, t_{i-1}\}$ are all contained in the 5 quadrics Q_i, we deduce that the tenth point t_{i+1} is also contained in the quadrics.

On the other hand, let us consider the following points:

p_{-1}	q_0			s_{-i-1}
p_0	q_1	r_0	t_{i-1}	
		r_1	t_i	s_{-i+1}

As in the previous case we can prove the same structure holds, having the set span the same hyperplanes:

$$
\begin{aligned}
H_{11} &= \{p_0, q_0, r_0, t_{i-1}, s_{-i+1}\} & H_{21} &= \{p_0, q_0, r_1, t_i, s_{-i-1}\} \\
H_{12} &= \{p_{-1}, q_0, r_0, t_i, s_{-i+1}\} & H_{22} &= \{p_0, q_1, r_0, t_i, s_{-i-1}\} \\
H_{13} &= \{p_{-1}, q_0, r_1, t_{i-1}, s_{-i+1}\} & H_{23} &= \{p_0, q_1, r_1, t_{i-1}, s_{-i-1}\} \\
H_{14} &= \{p_{-1}, q_1, r_0, t_{i-1}, s_{-i+1}\} & H_{24} &= \{p_{-1}, q_1, r_1, t_i, s_{-i-1}\}
\end{aligned}
$$

So the lemma 3.10 applies, and by the same argument as before, as the induction hypothesis tells us that the 9 points $\{p_0, p_{-1}, q_0, q_1, r_0, r_1, t_{i-1}, t_i, s_{-i+1}\}$ are already contained in our 5 quadrics we get that the tenth point of the set s_{-i-1} is also contained in the quadrics. □

With this theorem we have proven that the dual set of a 5-cell grid (or a segment of rather good edges in Γ) is contained in the intersection of 5 quadrics. Later we will prove using this that any set of points spanning few ordinary hyperplanes must also be mostly contained in the intersection of 5 quadrics.

Thus, it is obvious the importance of this variety for our study. For this reason, before moving on, we want to take some time to comment on the nature of this variety, and some of the properties it present relevant to our work.

First of all we want to mention the work made by Glynn on [5]. In his paper, Glynn talks, between other things, about normal rational curves and the intersection of quadrics. Although his work is made over the finite field $\mathbb{F}_q$, most of his results can be adapted to work over the reals.

We are particularly interested in the theorem 3.1 of his paper:

Theorem 3.12. *Consider a set Q of quadrics on $\mathbb{RP}^d$ generated by a collection of $\binom{d}{2}$ independent quadrics. Let $B = \bigcap Q$ be the intersection of the quadrics in Q. Suppose that B generates the whole $\mathbb{RP}^d$ and that Q does not contain any quadric that is the union of two hyperplanes of $\mathbb{RP}^d$. Then B is an arc of $\mathbb{RP}^d$.*

where he defines an k-arc of $\mathbb{RP}^d$ as a set of k points of $\mathbb{RP}^d$ such that every subset of $d+1$ points span the whole space.

This is very interesting for us, because this means that if instead of 5 linearly independent quadrics we had that our set S is included in the intersection of 6 linearly independent quadrics such that they do not span any hyperplane pair quadric, then the set S would be an arc. By the definition of arc, this would mean that every hyperplane spanned by the set S would be ordinary, and that would make the set S really bad for our problem.

Glynn's theorem can be extended to the following proposition:

Proposition 3.13. *Suppose that we have 5 linearly independent quadrics in $\mathbb{RP}^4$ that does not span a quadric which is an hyperplane pair and such that the intersection of the quadrics does not contain a line. Then the intersection of the 5 quadrics is a set S with the property that any hyperplane of $\mathbb{RP}^4$ intersects S in at most 5 points.*

Proof. Suppose that this is false, and let H be an hyperplane containing at least 6 points of S. Without loss of generality we can assume that H is the hyperplane $X_5 = 0$ and that the six points are:

$$\begin{aligned}
p_1 &= [1, 0, 0, 0, 0]\\
p_2 &= [0, 1, 0, 0, 0]\\
p_3 &= [0, 0, 1, 0, 0]\\
p_4 &= [0, 0, 0, 1, 0]\\
p_5 &= [d_1, d_2, d_3, d_4, 0]\\
p_6 &= [e_1, e_2, e_3, e_4, 0]
\end{aligned}$$

It is easy to check that the two last points impose distinct conditions on the space of quadrics. Indeed, the points p_5 and p_6 will impose linearly independent conditions in the space of quadrics unless the determinant of the matrix

$$\begin{bmatrix} d_1 d_2 & d_1 d_3 & d_1 d_4 & d_2 d_3 & d_2 d_4 & d_3 d_4 \\ e_1 e_2 & e_1 e_3 & e_1 e_4 & e_2 e_3 & e_2 e_4 & e_3 e_4 \end{bmatrix}$$

was zero. This can only happen if the two points are the same or if only one of the coefficients $X_i X_j$ is non-zero, which means that the points p_5 and p_6 are on a line with the two points of the basis. By hypothesis this cannot happen, so p_5 and p_6 impose different conditions on the space of quadrics.

So the 6 points p_i impose 6 linearly independent conditions on the space of quadrics defined in the hyperplane $H = \mathbb{RP}^3$. As the space of quadrics in $\mathbb{RP}^3$ has dimension 10, this means that we have at most 4 linearly independent quadrics on H containing the points $S \cap H$.

As we have 5 linearly independent quadrics which are zero on S, this means that in the subspace of quadrics generated by them, there must be two independent quadrics that agree on their restriction to H.

This implies that there is a quadric in the subspace generated by the 5 quadrics which is zero on H. This quadric is necessarily an hyperplane pair quadric, which is a contradiction with the hypothesis. □

This result implies that the set of points in the intersection of 5 linearly independent quadrics does not span hyperplanes containing more than 5 points. This is also an interesting conclusion, since most of the examples that we have explained until now (trivial, Böröczky or prism examples) stand out for having one or more hyperplanes containing a big number of the points of S.

The next thing we want to talk about is the relation between the intersection of 5 quadrics and the other key varieties in the lower dimensions.

We have already mentioned that the fact that each of these varieties show as in the optimal examples for this problem is no accident, and have hinted at a relation between these. In the next proposition we want to show this relation between an irreducible cubic and the intersection of 5 quadrics.

Proposition 3.14. *The lift of an elliptic curve to $\mathbb{RP}^4$ is a variety equivalent to the intersection of 5 quadrics.*

Proof. In [4] Massimo Giullietti defines a lifting mechanism for elliptic curves by means of the morphism from the points of the elliptic curve on $\mathbb{RP}^2$ to the k dimensional space $\mathbb{RP}^2$.

For the case which is of interest for us:

Let us denote the homogeneous coordinates of $\mathbb{RP}^4$ as $[X_1, X_2, X_3, X_4, X_5]$, and let us denote the non-homogeneous coordinates of $\mathbb{RP}^2$ as $[1, X, Y]$.

Let ε be an elliptic curve in $\mathbb{RP}^2$ with equation in normal Weierstrass form:

$$Y^2 = X^3 + aX + b \tag{17}$$

Let us define the map ϕ from the points of the elliptic curve ε to $\mathbb{RP}^4$:

$$\begin{aligned} \phi\colon \quad \varepsilon \quad &\rightarrow \quad \mathbb{RP}^4 \\ [1, X, Y] \quad &\rightarrow [1, X, Y, X^2, XY] \end{aligned} \tag{18}$$

Now, we want to find 5 linearly independent quadrics such that the points in the image of ϕ are contained in the intersection of the quadrics.

Just by the definition of the map ϕ we can deduce that the points in $\phi(\varepsilon)$ are contained in the following three quadrics:

$$\begin{aligned} Q_1 := \quad & X_2^2 &= X_1X_4 \\ Q_2 := \quad & X_2X_3 &= X_1X_5 \\ Q_3 := \quad & X_2X_5 &= X_3X_4 \end{aligned}$$

For the other two quadrics that we need we will look at the equation of the elliptic curve (17). We can obtain one quadric by using the direct equation. To obtain the last quadric we can multiply the equation of the elliptic curve by X, obtaining then another linearly independent quadric. The results would be:

$$\begin{aligned} Q_4 := \quad & X_3^2 &= X_2X_4 + aX_1X_2 + b \\ Q_5 := \quad & X_3X_5 &= X_4X_4 + aX_2X_2 + bX_2 \end{aligned}$$

It is easy to check that all the quadrics Q_i for $i = 1..5$ contain the image of the elliptic curve by the mapping ϕ. The linearly independence of the quadrics can also be checked quite easily, giving us the proof of the proposition. $\square$

To end this section we want to make a comment on why there are sets of points spanning few hyperplanes in the varieties we have seen in each dimension (cubic curve, intersection of two/five quadrics...).

We have shown in previous sections that sets of points in $\mathbb{RP}^2$ and $\mathbb{RP}^3$ spanning few ordinary lines/-planes must be mostly contained in a cubic curve or the intersection of two quadrics respectively. Also, we will prove in the next section the same result for sets of points in $\mathbb{RP}^4$ and the intersection of five quadrics.

But it should be clear that not every set of points included in these varieties is a good set of points, in the sense that it doesn't span few ordinary hyperplanes.

We have talked about the Böröczky examples for two dimensions and the prism and anti-prism examples at length. But we have not made much effort into proving the existence of examples for the cases of an irreducible cubic or the intersection of two generic quadrics in their respective structure theorems.

The key reason why we can find examples of sets with few ordinary hyperplanes in these varieties is the existence of a group structure within the variety which contains finite subgroups.

Ben Green and Terence Tao make a great exposition of this in their paper [11, pp. 14..17] for the planar case. They talk about the group structure of the set of non-singular points of an elliptic curve, and then they describe the Sylvester examples, which consist on the n points of a subgroup of order n of the non-singular points of the curve.

For our case, to be able to assure the existence of finite examples off sets with few ordinary hyperplanes, we are interested to know if there exist the same kind of group structure in the intersection of 5 quadrics.

It is not obvious the existence of the group structure, but we are able to prove it by looking at the relation between the cubic curve and the intersection of 5 quadrics.

In 3.14 we have proven that the intersection of 5 quadrics is the lift of an elliptic curve to four dimensions, by using an injective map from the curve to $\mathbb{RP}^4$ and proving that the image is contained in the intersection of 5 linearly independent quadrics.

In [1], proposition 3, it is explained how this lift induces a group structure in the image of the map. This proposition also shows the relation between this group structure and our problem, claiming that for a lift from the elliptic curve to $\mathbb{RP}^d$, any $d + 1$ points on the induced group will be contained in the same hyperplane if and only if they sum to 0 (using the group operation).

The existence of this group structure is enough to assure that we can build examples of sets of points in the intersection of 5 quadrics with few ordinary hyperplanes.

3.3 Structural theorem in 4 dimensions

After all the work done in the previous sections, here we are ready to state our own version of the structure theorem in 4 dimensions.

First of all, we are going to present the weak version of the structure theorem. This version of the structure theorem, although it seems just a weaker result of the main structure theorem we will present later, has a merit of its own. This is because for this one we are not making any assumptions with respect to the value of K, in contrast with the main structure theorem, where we need K to be of the order of $o(n^{\frac{1}{6}})$.

We will state our weak structure theorem in the same fashion as Green and Tao. The exact numbers on the theorem are not that much important, but it is important that we can prove that the set S is contained in the union of an $O(K)$ number of varieties, each of which is the intersection of 5 quadrics.

Theorem 3.15 (Weak structure theorem in 4 dimensions)**.** *Let S be a set of n points in $\mathbb{RP}^4$, such that any subset of 4 points span a hyperplane and not all S is contained in a hyperplane. If S spans at most Kn^3 ordinary hyperplanes, then S is contained in the union of at most $33699K$ varieties, each of which is the intersection of 5 quadrics.*

Proof. From lemma 3.8 we get that there are at most $1872Kn^3$ slightly bad edges in the graph Γ. Using the pigeon-hole principle we know that there have to be points $p, q, r \in S$ such that the number of slightly bad edges on the line $l = p^* \cap q^* \cap r^*$ is:

$$\frac{1872Kn^3}{\binom{n}{3}} = 11232K\frac{n^2}{(n-1)(n-2)}$$

We can assume that $n \geq 33699$ since otherwise the theorem is immediately proven by choosing a variety from each point. This alone is enough to bound the number of slightly bad edges of l by $b = 11233K$.

The slightly bad edges will partition the line l into a set of at most b segments of consecutive good edges. By theorem 3.11 we know that for any of these segments T, there are 5 linearly independent quadrics containing the points p, q, r and such that any point $s \in S$ such that s^* intersects the line l into T is contained in the intersection of the 5 quadrics. So we can cover all the points whose dual intersect l in the middle of a segment of good edges with less than $11233K$ of these varieties.

In the other hand, any point whose dual intersects l into a vertex of a bad edge of l has to be treated separately. We have, though, at most $2 \times 11233K$ vertex incident with a bad edge in l, and for each one of these vertices, we can construct a variety (intersection of 5 quadrics) such that all the points whose dual intersect l in that vertex are contained in the variety (this is obvious, since the dual of the vertex itself is a hyperplane that contains all of these points).

From the number of varieties covering the points incident with the segments of good edges and the ones incident with the bad edges, we conclude that all the points of S can be covered with a collection of at most $11233K + 2 \times 11233K = 33699K$ varieties each of which is the intersection of 5 quadrics. $\square$

Now we want to present the full structure theorem, which is the main theorem of this section, as well as the whole thesis. The purpose of the theorem is to classify the sets of points in $\mathbb{RP}^4$ that span few ordinary hyperplanes, as did the structure theorems of Green and Tao and Simeon Ball respectively.

Theorem 3.16 (Full Structure Theorem)**.** *Let S be a set of n points in $\mathbb{RP}^4$ such that any subset of 4 points span a hyperplane and such that not all S is contained in a hyperplane. If S spans less than Kn^3 ordinary hyperplanes, for some $K = o(n^{\frac{1}{6}})$, then one of the following holds:*

(i) All but at most $2K$ points of S are contained in a hyperplane.

(ii) There are 5 linearly independent quadrics such that all but at most $O(K)$ points of S are contained in the intersection of the 5 quadrics.

Proof. We will make the proof of the statement in several steps.

One of the first thing we will be doing to prove the theorem is to look at the projection of our set S to 3 and 2 dimensions, and make use of the respective structure theorems in those dimensions. Looking at the possible structures of the projection of our set will help us to classify and discard the different possibilities we will have.

Next we will need to extract from the structure of the projections the necessary information to argue the existence of the 5 linearly independent quadrics that contain the set.

Let us consider the set S' consisting on the points $p \in S$ such that p is contained in at most dKn^2 ordinary hyperplanes, for some absolute constant d. We can bound the size of S' in the following way:

As we know that the points not contained in S' span more than dKn^2 ordinary hyperplanes, and since every ordinary hyperplane contains exactly 4 points, we get:

$$|S \backslash S'| dKn^2 < 4Kn^3$$

Which gives us:

$$|S'| > \left(1 - \frac{4}{d}\right) n \tag{19}$$

If we project the set S from any point $p \in S'$ we obtain a set in $\mathbb{RP}^3$ spanning less than dKn^2 ordinary planes. The structure theorem in dimension 3 by Simeon Ball 2.9 tell us the different possible sets that span few ordinary planes in $\mathbb{RP}^3$. These possibilities were:

(a) There are two distinct quadrics such that all but at most $O(K)$ points of S are contained in the intersection of the quadrics. And all but at most $O(K)$ points of S are incident with at least $\frac{3}{2}n - O(K)$ ordinary planes.

(b) There are two planar sections of a quadric which contain $\frac{1}{2}n - O(K)$ points of S each.

(c) All but at most $2K$ points of S are contained in a plane.

First of all we want to rule out the case where the projection of the set S is almost contained in a plane (case (c)), since this is a corner case.

It is obvious that if we have a point such that the projection of S from it is of the type (c), then we will be in the case (i).

So in the following we will suppose that no point of S' projects the set S into a set almost contained in a plane.

Now we want to prove that almost none of the points of S' can project the set S into a set of the type (b).

Let us suppose that there are at least 4 points in S' such that the projection of S from these points is of the case (b). Let us call p_1, p_2, p_3 and p_4 to these points. This means that, for each of these points, there are two planar conics containing $\frac{n}{2} - O(K)$ points of the projection each.

The lift of these two planes to $\mathbb{RP}^4$ consist on two hyperplanes H_1 and H_2 containing $\frac{n}{2} - O(K)$ points of S each.

Notice that the lift of the planes produced by each of the points p_i must result in the same two hyperplanes H_1 and H_2. Suppose not, and suppose that the lift of one of the planes produced by some of the p_i produces an hyperplane H_3 different from H_1 and H_2. The intersections $\pi_1 = H_1 \cap H_3$ and $\pi_2 = H_2 \cap H_3$ are both planes on $\mathbb{RP}^4$. Since all H_1, H_2 and H_3 contain $\frac{n}{2} - O(K)$ points of S, it is obvious that one of π_1 or π_2 would have at least $\frac{n}{4} - O(K)$ points of S, which is a contradiction since we cannot have 4 coplanar points in S.

Now, as all p_i project the points in the two hyperplanes H_1 and H_2 into two planar conics, we deduce that the points p_i are all contained in the hyperplanes H_1 and H_2. That implies they are in the intersection of the two hyperplanes which, because the hyperplanes are distinct, is a plane of $\mathbb{RP}^4$. But since we cannot have 4 coplanar points in the set S we reach a contradiction.

Thus, we conclude that there are at most 3 points of S' which project the set S into a set of type (b).

So in the case we are studying we know that most of the points of S' project the set S into a set of the type (a). This correspond to the case in $\mathbb{RP}^2$ of all but at most $O(K)$ points contained in an irreducible cubic.

Now we want to prove that in this scenario, the set S is almost contained in the intersection of 5 quadrics.

For this let us consider a line $p^* \cap q^* \cap r^*$ on the graph Γ and a segment T on that line of m consecutive rather good edges, for an arbitrarily large constant m. As we have mentioned before, we can assure the existence of this segment, as the number of slightly bad edges on Γ is low.

We have proved that for a segment T of rather good edges, there are 5 linearly independent quadrics $\{Q_i \colon i = 1..5\}$ containing p, q, r and all the points s such that s^* intersect the line $p^* \cap q^* \cap r^*$ on the segment T.

But also we know that from all these points, the projection of the set S is contained in the intersection of two quadrics. This means that for each point p_j of these, from the subspace of quadrics generated by the Q_i, there are two linearly independent quadrics that are degenerate on p_j and contain all but at most $O(K)$ points of S.

Now, let us consider two of these points p_1 and p_2 and let us apply a projective transformation so they become the points $[1, 0, 0, 0, 0]$ and $[0, 1, 0, 0, 0]$.

Let us call q_1 and q_1' to the two quadrics degenerate at p_1, and let us also call q_2 and q_2' to the quadrics degenerate at p_2. Then these quadrics will have coefficient 0 at the X_1 and X_2 terms respectively.

$$\begin{aligned}
q_1 &\equiv a_{22}X_2^2 + \cdots + a_{55}X_5^2 + a_{23}X_2X_3 + \ldots a_{45}X_4X_5 \\
q_1' &\equiv b_{22}X_2^2 + \cdots + b_{55}X_5^2 + b_{23}X_2X_3 + \ldots b_{45}X_4X_5 \\
q_2 &\equiv c_{22}X_1^2 + c_{33}X_3^2 + \cdots + c_{55}X_5^2 + c_{13}X_1X_3 + \ldots c_{45}X_4X_5 \\
q_2' &\equiv d_{22}X_1^2 + d_{33}X_3^2 + \cdots + d_{55}X_5^2 + d_{13}X_1X_3 + \ldots d_{45}X_4X_5
\end{aligned}$$

Both pairs $\langle q_1, q_1' \rangle$ and $\langle q_2, q_2' \rangle$ are linearly independent quadrics. This together with the fact that q_2 and q_2' have coefficients on the terms X_1 and not q_1 and q_1' allows us to assure that these four quadrics are linearly independent.

Now let us choose a point p_3 from the points contained in the 5 quadrics such that the one of the two degenerate quadrics at p_3 has non-zero coefficient on the term X_1X_2. We can assure that such a p_3 exists.

Indeed, suppose that the two degenerate quadrics at p_3 have zero coefficient at X_1X_2. Then, if we project these quadrics from p_1 the plane, then the point p_2 will be a singularity of the cubic curve. Since the projection of S from all our points down to the plane is an irreducible cubic, there are few singularities, and we can solve this problem by choosing another point p_2.

We have enough freedom to do this, since we assure the existence of a segment T of rather good edges sufficiently large.

Any of these quadrics will be linearly independent with the four quadrics described above, since none of them have term X_1X_2. All of these quadrics are taken from the 5 linearly independent quadrics containing the segment T, and we can assure that each of them contains all but at most $O(K)$ points of S.

Then, joining the quadrics together we get a set of 5 linearly independent quadrics, such that all but at most $O(K)$ points of S are contained in the intersection of the five quadrics.

□

3.4 Conjecture for the problem on $d > 4$

Although on this thesis we have focused on the version of the Sylvester's problem in $\mathbb{RP}^4$, a great part of our study can be applied to higher dimensions.

We cannot make here the whole study for the problem on higher dimensions, though just the work we have already done give us some clues to the solution.

It is easy to see that some of the concepts we have defined throughout this section, such as the 5-cell grid, or the concepts of good and bad edges, are naturally generalized to higher dimensions. But the key ingredient of our study, the nature of the variety containing almost all the points of the sets spanning few ordinary points, is a bit harder to generalize to higher dimensions.

We have talked briefly of the relation between the key varieties of the examples in 2, 3 and 4 dimensions. Proving in an earlier section that the intersection of 5 quadrics is the lift of a cubic curve to four dimensions. One can imagine that for the problem in $d > 4$ the same relation should hold.

The relation between the varieties of different dimensions is very natural. It comes from the fact that, if S is a set with few ordinary hyperplanes, the projection of S from most of its points is a set with few ordinary hyperplanes in the lower dimension. Thus, it is only natural that the varieties containing these set of points are one the lift of the other.

The study made by Glynn give us a reasonable enough conjecture for the nature of these varieties.

Conjecture 3.17. *Let S be a set of points on $\mathbb{RP}^d$ for $d > 4$ such that any subset of d points of S span a hyperplane of $\mathbb{RP}^d$ and such that not all of S is contained in a single hyperplane. Suppose that S spans at most Kn^{d-1} hyperplanes for some K, then one of the followings hold:*

- *All but at most $O(K)$ points of S are contained in a single hyperplane.*
- *There is a set of $\binom{d+2}{2} - 1$ linearly independent quadrics such that all but at most $O(K)$ points of S are contained in the intersection of the quadrics.*

4. Solution for small d and n

Both the work of Green and Tao, and that of Simeon are focused on the asymptotic solution of Sylvester's problem, proving the optimal configurations for large values of n.

Even though those are much more important results, it is interesting to look at what happens for the problem for small values of d and n. The interest we have in the results for smaller values of n comes from known examples that break the asymptotic solutions, meaning that they does not constitute a general solution to the problem.

As the study of the asymptotic behavior of the problem constitute much stronger results we do not want to spend a lot of time in this section. We will speak briefly about the know counterexamples for the general solution of the problem, as well as the known bounds for the unknown values of $e_d(n)$. Then we will introduce some of our own work in this matter.

4.1 Previous known results

In this section we will be going through some of the known results and bounds for the values of $e_d(n)$ for small n. We won't be proving the results mentioned here, as they will occupy lot of space and are quite simple results. All the proofs can be found in [6].

We are going to show here the results we consider more important. Instead of focusing on the particular computations of some of the numbers, we want to show more some of the bounds for general $e_d(n)$. Some of these results will be of use to us in the next section where we try to compute the value of some of these numbers ourselves. The other results we will simply recollect into the table 1

In the following let S be a set of n points in the projective space $\mathbb{RP}^d$.

Let us call a k-hyperplane to an hyperplane of $\mathbb{RP}^d$ that contains exactly k points of the set S. Then an ordinary hyperplane would be a d-hyperplane. For a set S, let us denote as τ_k the number of k-hyperplanes that it spans. Then, obviously $e_d(n) = \tau_d = \#d$-hyperplanes.

With this definitions we can state the following:

Lemma 4.1. *Let S be a set of n points in $\mathbb{RP}^d$. Then the following holds:*

$$\sum_{k=d}^{n} \binom{k}{d} \tau_k = \binom{n}{d} \tag{20}$$

This is the basic equation from which we start our studies of the small cases. It is easily proven by counting subsets of d points in two different ways. Most of the following bounds make use of this equation.

Next we want to show some lower bounds for the numbers $e_d(n)$. We are more interested in the results for lower bounds, since in the other side, the methods for finding upper bounds are constructive, and require us to find examples of configurations of points with few ordinary hyperplanes. That proves to be really hard for higher dimensions if you deviate from the trivial example, and it has to be done for each particular n and d, making it difficult to find general results for improved upper bounds.

The first lower bound we will show is the following:

Lemma 4.2.

$$e_d(n) \geq \frac{n}{d} e_{d-1}(n-1) \tag{21}$$

As simple a result as it seems, it is interesting, as it gives us a first lower bound to work with for any value of n and d. The only problem with this bound is its recursive nature, which makes it hard to obtain good lower bound for greater values of n and d. We will, though, be using this lemma for our results in the next sections. Also, after the results we obtain, this equation allows us to expand our results to improve the bounds of later values just by their relation.

The following lemma gives us a non-recursive lower bound for $e_d(n)$, which solvents the problem of the earlier bound:

Lemma 4.3.

$$e_d(n) \geq \binom{n}{d} - \frac{d+1}{d+2}\binom{n}{d+1} \tag{22}$$

Apart from these bounds, in [6] we can find different results for particular values of d and n. In the next table 1 we show a compilation of these results, along with the ones derived from the bound we just exposed.

n	3	4	5	6	7	8	9	10	11	12	13
$e_2(n)$	3	3	4	3	3	4	6	5	6	6	6
$e_3(n)$		4	6	8	11	8	16-22	20	19-31	24	26-51
$e_4(n)$			5	10	20	25-35	18-56	30-84	55-120	57-165	78-220
$e_5(n)$				6	15	32	54-70	36-126	66-210	132-330	149-495
$e_6(n)$					7	21	56	90-126			
$e_7(n)$						8	28	80			

Table 1: Value of $e_d(n)$ for small n and d

4.2 New bounds

Now we want to make our small contribution to these results. In particular, in the following we will be giving some results for the case when $n = d+4$. We will first find a stronger lower bound for this particular case, by using some combinatoric results. Later we will use this bound and some other of the mentioned results earlier to prove the values for $e_4(8)$ and $e_5(9)$.

4.2.1 Improved bound for n = d + 4

For our first result, the improved lower bound when $n = d+4$, we will need first to introduce the following combinatoric lemma:

Lemma 4.4. *Let $f(n)$ denote the maximum possible number of edge-disjoint K_3 inside of a K_n, where K_r represents the complete graph of n nodes. Then:*

$$f(n) = \left\{ \begin{array}{ll} \frac{n(n-1)}{6} & \text{if } n \equiv 1,3 \text{ mod } 6 \\ \frac{n(n-2)}{6} & \text{if } n \equiv 0,2 \text{ mod } 6 \\ \frac{n(n-1)-8}{6} & \text{if } n \equiv 5 \text{ mod } 6 \\ \frac{n(n-2)-2}{6} & \text{if } n \equiv 4 \text{ mod } 6 \end{array} \right\} \tag{23}$$

There are multiple ways to state this lemma, as it has a lot of applications in different areas. For example, the function $f(n)$ can be defined in the context of code theory as the maximum cardinality of a code of length n with constant weight 3 and minimum distance 4. In our work we will use it to compute a restriction on the number of $(d+1)$-hyperplanes in any configuration of points.

The proof of this lemma is constructive. The upper bounds for $f(n)$ are easily proven by a simple counting argument, and the only thing left is to prove there is always a construction that attains this upper bound. The full proof is quite tedious, as it involves a lot of lemmas for constructing the sets and is very case by case based.

The complete proof is given in [3].

We will be using this lemma to improve the current lower bound for $e_d(d+4)$. Technically the new bound we will be giving is only an improvement in the previous known bound for some of the cases in the lemma (particularly important for the case d even). Though, this alone will be useful for us later.

First of all, let us take a look to the best bounds known so far in this case. As there is no much work done here, only the general bounds are applicable.

For the upper bound, we use the trivial example (a configuration of n points where $n-1$ of these points are contained in the same hyperplane H, and the last one is outside of H). We can count easily an upper bound for the number of ordinary hyperplanes in this trivial example, as every ordinary hyperplane must contain the point outside of H. So the upper bound given by the trivial example is:

$$e_d(n) \leq \binom{n-1}{d-1} \tag{24}$$

Which applied to our case transforms into:

$$e_d(d+4) \leq \binom{d+3}{d-1} = \frac{(d+3)(d+2)(d+1)d}{24} \tag{25}$$

Now for the lower bound we can use the lemma 4.3, which gives us the following lower bound:

$$\begin{aligned} e_d(d+4) \geq \binom{d+4}{d} - \frac{d+1}{d+2}\binom{d+4}{d+1} = \frac{(d+4)(d+3)(d+2)(d+1)}{24} \\ - \frac{(d+4)(d+3)(d+2)(d+1)}{6(d+2)} = \frac{(d+4)(d+3)(d+1)(d-2)}{24} \end{aligned} \tag{26}$$

This is already quite a good result, since just with this bounds, the gap between lower and upper bound is of order $O(n^2)$. Let us denote with u and l to the values of the upper and lower bound in 25 and 26. Then indeed:

$$u - l = \frac{(d+3)(d+2)(d+1)d}{24} - \frac{(d+4)(d+3)(d+1)(d-2)}{24} = \frac{(d+3)(d+1)}{3}$$

Our job now will be to improve the lower bound. For this we will use some case analysis and the important lemma 4.4.

To begin let us first take a look to the equation derived from the lemma 4.1:

$$\tau_d + (d+1)\tau_{d+1} + \binom{d+2}{d}\tau_{d+2} + \binom{d+3}{d}\tau_{d+3} = \binom{d+4}{d} \tag{27}$$

We will consider the different possible values of τ_i to get rid of lots of cases, and focus on the few important ones.

The first thing we will do is get rid of the term τ_{d+3}. If this term was greater than 0, it would need to be 1, and then we would be in the trivial example. Since we want in this section to improve the lower bound, we are not interested in this. So from now on we can suppose easily $\tau_{d+3} = 0$.

The next thing we will argue is that $\tau_{d+2} \leq 1$. To prove this let us suppose that $\tau_{d+2} \geq 2$. This means that there exists at least two different $(d+2)$-hyperplanes, which we will call H_1 and H_2. Since they are $(d+2)$-hyperplanes, each of them contains exactly $d+2$ points of P. Now, because P has only $d+4$ points, the intersection of H_1 and H_2 must necessarily contain d points of P. As every set of d points of P must span a hyperplane, we can deduce that H_1 and H_2 must be equal to this hyperplane and thus, they must be the same. So by contradiction, we get that we can not have $\tau_{d+2} \geq 2$.

Now we want to study two different cases, one for $\tau_{d+2} = 1$ and one for $\tau_{d+2} = 0$. We will prove different bounds for these two cases and we will join them to get the general bound for $e_d(d+4)$. We will be using the lemma 4.4, so obviously the resulting bounds will depend on the parity of d. We will ignore the slight difference in the cases $n \equiv 4, 5 \bmod 6$, as they don't really provide an asymptotic improvement on the bound and it duplicated the amount of calculations to be done.

- Case $\tau_{d+2} = 1$:

  In this case we have exactly one $(d+2)$-hyperplane. The previous equation now transforms into:

  $$\tau_d + (d+1)\tau_{d+1} + \binom{d+2}{d} = \binom{d+4}{d}$$

  To get a lower bound for the value of τ_d we will try maximizing the value of τ_{d+1} (the number of $(d+1)$-hyperplanes).

  Let us call H to our $(d+2)$-hyperplane, and let us call $\{x, y\}$ to the two points of P outside of H.

  We will start by noting that every $(d+1)$-hyperplane must necessarily contain the two points $\{x, y\}$. Indeed, let us suppose we have one $(d+1)$-hyperplane H' that does not contain $\{x, y\}$. Then the intersection of H and H' contains at least d points of P. Again, as every d points of P spans an hyperplane, we conclude that H and H' must be the same, and we arrive to a contradiction.

  Now, all that hyperplanes contain $\{x, y\}$, any $(d+1)$-hyperplane can be seen as a selection of $d-1$ of the points of $H \cap P$. As H contains $d+2$ points of P this is the same as choosing three points of $H \cap P$ (the points excluded from the hyperplane).

  Now let us consider two different $(d+1)$-hyperplanes H_1 and H_2. These two hyperplanes can share at most $d-1$ points of P (for the same argument as before, if not, the d shared points would span a hyperplane and we would reach a contradiction). This implies that the triplets of points representing each hyperplane (triplet of points outside of the hyperplane), can not share more than one point.

With this preparation, the connection with the lemma 4.4 should become clear.

We want to maximize the number of $(d+1)$-hyperplanes, which can be seen as triplets of points chosen from the $d+2$ points of $H \cap P$ such that no pair of triplets intersects in more than one point. The lemma 4.4 gives us, then, a bound for the maximum number of $(d+1)$-hyperplanes we could get.

With this bound in mind, let us solve the equation to get the bound for τ_d.

– If d is odd:

$$\tau_{d+1} \leq \frac{(d+2)(d+1)}{6}$$

so we get the bound for τ_d

$$\begin{aligned}
\tau_d &= \binom{d+4}{d} - \binom{d+2}{d} - (d+1)\tau_{d+1} \\
&\geq \binom{d+4}{d} - \binom{d+2}{d} - (d+1)\frac{(d+2)(d+1)}{6} \\
&= \frac{(d+2)(d+1)}{24}\left[(d+4)(d+3) - 12 - 4(d+1)\right] \\
&= \frac{(d+2)(d+1)(d^2+3d-4)}{24}
\end{aligned}$$

which is already greater than the previous lower bound l.

– If d is even:

$$\tau_{d+1} \leq \frac{(d+2)d}{6}$$

so we get the bound

$$\begin{aligned}
\tau_d &= \binom{d+4}{d} - \binom{d+2}{d} - (d+1)\tau_{d+1} \\
&\geq \binom{d+4}{d} - \binom{d+2}{d} - (d+1)\frac{(d+2)d}{6} \\
&= \frac{(d+2)(d+1)}{24}\left[(d+4)(d+3) - 12 - 4d\right] \\
&= \frac{(d+3)(d+2)(d+1)d}{24}
\end{aligned}$$

which is equal to the upper bound u. So we can conclude that in the even case, we can not improve the trivial example when $\tau_{d+2} = 1$.

- Case $\tau_{d+2} = 0$:

  In this case we have to care only about the $(d+1)$-hyperplanes, and as in the previous case, we want to maximize the number of these hyperplanes in order to get a lower bound for τ_d.

  As in the previous case we have to notice that $(d+1)$-hyperplanes can be seen as the triplets of points outside of them, and that any pair of $(d+1)$-hyperplanes can not share more than $d-1$ points of P, so their triplets can not share more than 1 point of P.

  With those characteristics we can use the lemma 4.4 in the same way we did in the previous case. So we can get a similar lower bound that depends on the parity of d:

– If d is odd:

$$\tau_{d+1} \leq \frac{(d+4)(d+3)}{6} \tag{28}$$

and we just get the bound:

$$\begin{aligned} \tau_d &= \binom{d+4}{d} - (d+1)\tau_{d+1} \geq \binom{d+4}{d} - (d+1)\frac{(d+4)(d+3)}{6} \\ &= \frac{(d+4)(d+3)(d+1)(d-2)}{24} \end{aligned} \tag{29}$$

This is equal to the lower bound l computed with the lemma 4.3, so in the end we can not improve this bound for the case d odd.

– If d is even:

$$\tau_{d+1} \leq \frac{(d+4)(d+2)}{6} \tag{30}$$

and the bound for τ_d ends up being:

$$\begin{aligned} \tau_d &= \binom{d+4}{d} - (d+1)\tau_{d+1} \geq \binom{d+4}{d} - (d+1)\frac{(d+4)(d+2)}{6} \\ &= \frac{(d+4)(d+2)(d+1)(d-1)}{24} \end{aligned} \tag{31}$$

Seen all the cases we can derive the general bound. In the case d odd we have not been able to improve the lower bound from the previous computed bound l. In the other hand, for d even we can improve the bound to:

$$e_d(d+4) \geq \frac{(d+4)(d+2)(d+1)(d-1)}{24} \tag{32}$$

If we compute the gap between this lower bound and the upper bound, we obtain:

$$\frac{(d+3)(d+2)(d+1)d}{24} - \frac{(d+4)(d+2)(d+1)(d-1)}{24} = \frac{(d+2)(d+1)}{6}$$

which is a significant improvement on the gap given by the previous bound. This means that we have reduced significantly the possible values for $e_d(d+4)$.

4.2.2 Case $d = 4$ and $n = 8$

For the case $d = 4, n = 8$, as we are in the even case, the previous section give us strong bound on the value of $e_4(8)$.

The upper bound is the one given by the trivial example, while the lower bound is the one given by the previous section.

$$30 = \frac{8 \times 6 \times 5 \times 3}{24} \leq e_4(8) \leq \frac{7 \times 6 \times 5 \times 4}{24} = 35$$

In the following we are going to prove that $e_4(8) = 35$. For that we are going to reduce the possibility that $e_4(8) < 35$ to one specific case, and then we are going to prove geometrically that that case is impossible.

First, by the work done in the previous section, we know that in order for τ_4 to be less than the upper bound, then τ_6 should be equal to 0 (since we saw that the lower bound of τ_d when $\tau_{d+2} = 1$ was equal to the upper bound).

Since we have only 5-hyperplanes to care about, the equation 4.1 takes the following form:

$$\tau_4 + 5 \times \tau_5 = 70$$

By this equation we observe that τ_5 is necessarily a multiple of 5, so if we want τ_4 to be less than 35 (the upper bound), it can only be equal to 30, and that implies $\tau_5 = 8$.

$\tau_5 = 8$ is the maximum attainable value according to the lemma 4.4. If we number the points $\{1, 2, 3, 4, 5, 6, 7, 8\}$, the following set of triplets is the unique (up to permutations) way of reaching this number:

$$\begin{array}{cc} (1,2,3) & (4,5,6) \\ (1,4,7) & (1,5,8) \\ (2,5,7) & (2,6,8) \\ (3,6,7) & (3,4,8) \end{array}$$

By the way we have been using this lemma, there is a bijection between this triplets and the 5-hyperplanes corresponding to the set of points outside each hyperplane. So we would have 8 hyperplanes consisting on the points:

$$\begin{array}{ll} H_1 = \{p_4, p_5, p_6, p_7, p_8\} & H_2 = \{p_1, p_2, p_3, p_7, p_8\} \\ H_3 = \{p_2, p_3, p_5, p_6, p_8\} & H_4 = \{p_2, p_3, p_4, p_6, p_7\} \\ H_5 = \{p_1, p_3, p_4, p_6, p_8\} & H_6 = \{p_1, p_3, p_4, p_5, p_7\} \\ H_7 = \{p_1, p_2, p_4, p_5, p_8\} & H_8 = \{p_1, p_2, p_5, p_6, p_7\} \end{array}$$

Our objective is to prove that this combination of 8 hyperplanes is geometrically impossible.

Let us take the two hyperplanes H_1 and H_2. These hyperplanes intersect into a plane π that contains the two points $\{p_7, p_8\}$. Let us suppose that we have these two hyperplanes and we will prove that it is impossible to configure the points in order to produce the other 6 hyperplanes.

The points $\{p_4, p_5, p_6\}$ which are inside of H_1 but outside of π form a plane α_1 that intersects π into a line r_1. In the same way, the points $\{p_1, p_2, p_3\}$ form the plane α_2 that intersects π into the line r_2.

Let us take one of the other 6 hyperplanes (H_3 ... H_8). All of them consist on the following set of points: a selection of two points from the set $\{p_1, p_2, p_3\}$, a selection of two points from the set $\{p_4, p_5, p_6\}$ and one point from $\{p_7, p_8\}$.

As each hyperplane has two points from $\{p_4, p_5, p_6\}$, the intersection of the hyperplane with the plane α_1 will be one of the three lines spanned by these three points. Consequently, it will contain the point of intersection of that line with the line r_1. The same argument apply to the points $\{p_1, p_2, p_3\}$ and the line r_2.

Let us denote these points of intersection:

$$\begin{aligned} x_{11} &= \langle p_4, p_5\rangle \cap r_1, & x_{21} &= \langle p_1, p_2\rangle \cap r_2 \\ x_{12} &= \langle p_4, p_6\rangle \cap r_1, & x_{22} &= \langle p_1, p_3\rangle \cap r_2 \\ x_{13} &= \langle p_5, p_6\rangle \cap r_1, & x_{23} &= \langle p_2, p_3\rangle \cap r_2 \end{aligned}$$

With this notation, each of the 6 hyperplanes must contain exactly one of the points x_{1i}, one of the points x_{2j} and one of $\{p_7, p_8\}$. Now, as we are talking about hyperplanes and all these points are contained in the plane π, this transforms into a condtion of collinearity.

As an example of what we mean by this: The hyperplane H_7 contains the points $\{p_1, p_2\}$ and the points $\{p_4, p_5\}$. This means that it contains the points x_{11} and x_{21}. As H_7 also contains the point p_8 and the intersection of H_7 and π is a line, this means that the points x_{11}, x_{21} and p_8 are aligned.

By the same reasoning we reach the following collinearity relations:

$$\begin{array}{cc} (p_8, x_{11}, x_{21}) & (p_7, x_{11}, x_{22}) \\ (p_8, x_{12}, x_{22}) & (p_7, x_{12}, x_{23}) \\ (p_8, x_{13}, x_{23}) & (p_7, x_{13}, x_{21}) \end{array}$$

Now what we are going to argue is that it is impossible to arrange a set of points in a plane that satisfy these relations. All this is derived from our initial situation with the 5-hyperplanes H_1 to H_8 with no other assumption, so if this is impossible, it follows that it is impossible to obtain the configuration with our 8 hyperplanes.

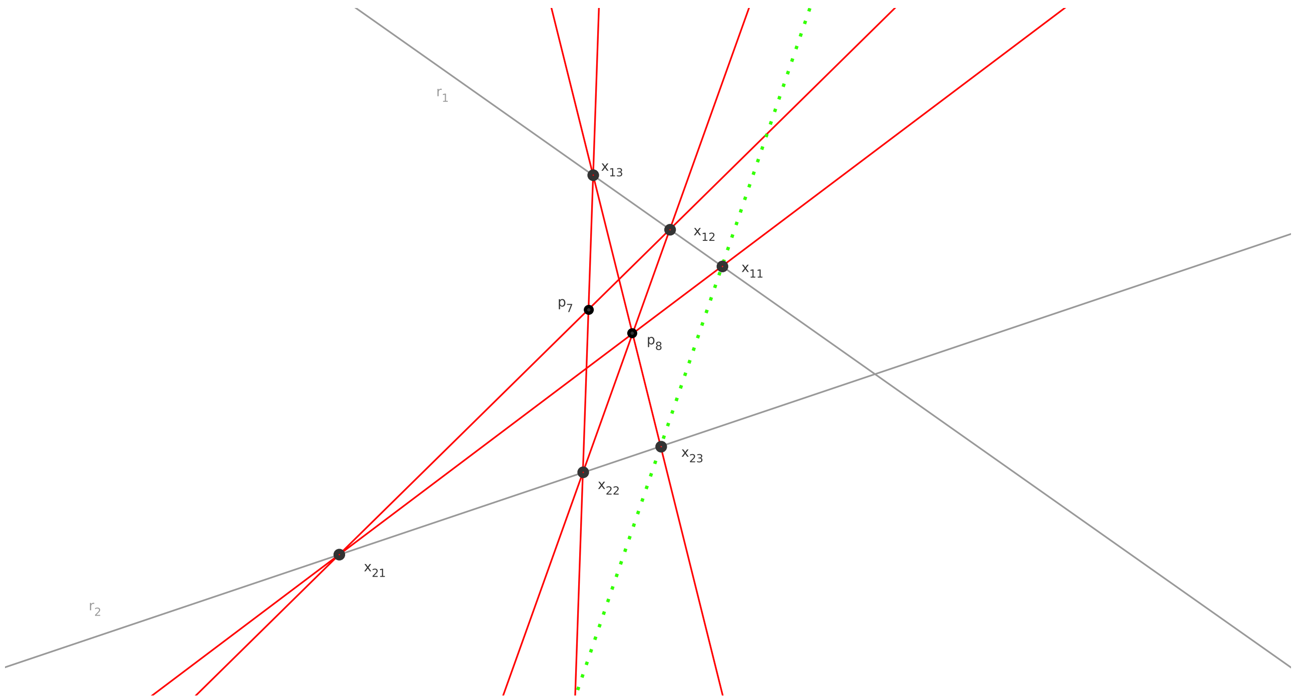


Figure 7: Configuration of points in the plane that satisfies all but one (in green) of the conditions

One way to easily see that the situation on the plane is impossible is to apply a projective transformation to the points to send the line $\langle p_7, p_8\rangle$ to the line at infinity.

After applying the transformation, the conditions of collinearity that we were using become conditions of parallelism. For example, the points $\{x_{11}, x_{21}\}$, $\{x_{12}, x_{22}\}$ and $\{x_{13}, x_{23}\}$ must be aligned with the point p_8. At sending that point to the line at infinity, this means that the lines generated by those points must be parallel and in the direction of the point p_8 at infinity.

With this simplification, we would have two triplets of three points in the two lines r_1 and r_2 such that the lines $\langle x_{11}, x_{21}\rangle$, $\langle x_{11}, x_{21}\rangle$ and $\langle x_{11}, x_{21}\rangle$ are parallel to each other, and the lines $\langle x_{11}, x_{21}\rangle$, $\langle x_{11}, x_{21}\rangle$ and $\langle x_{11}, x_{21}\rangle$ are also parallel.

This scenario is clearly impossible, so we conclude that the whole thing is impossible.

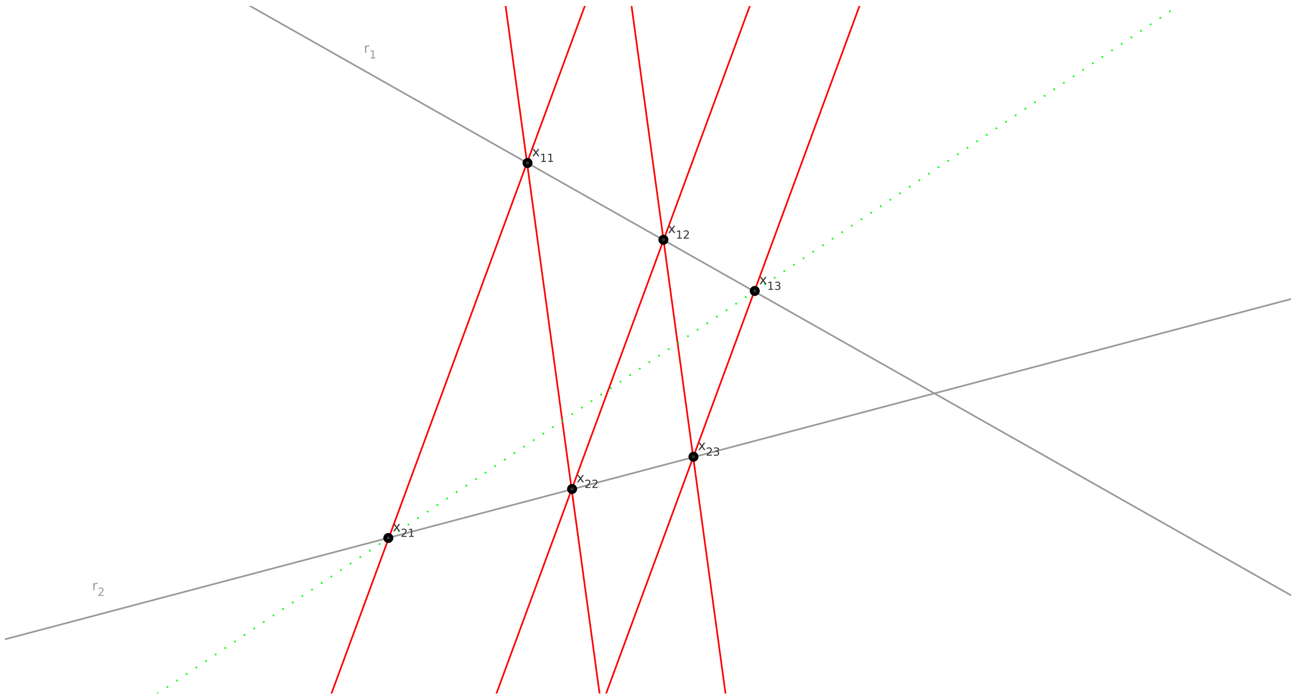


Figure 8: Configuration of points where all but one (in green) of the lines are parallel to their required direction

Going back to the beginning, as we have now proved that it is impossible for there to be a configuration of points with 8 5-hyperplanes, then $\tau_4 > 30$. As we saw that τ_4 had to be a multiple of 5 and we had the upper bound $e_4(8) \leq 35$ we conclude that:

$$e_4(8) = 35 \tag{33}$$

4.2.3 Case $d = 5$ and $n = 9$

We want now to study the case $d = 5, n = 9$. Our objective here is to improve the bounds for this case, so first let us compute the bounds we have right now.

The trivial example and the lemma 4.3 give us the following upper and lower bound respectively:

$$54 = \binom{9}{5} - \frac{6}{7}\binom{9}{6} \leq e_5(9) \leq \binom{8}{4} = 70$$

With the work done until now we can already do better than this. Taking a look at the lemma 4.2, after determining the value of $e_4(8)$ in the previous section, we can improve the lower bound:

$$e_5(9) \geq \frac{9}{5}e_4(8) = \frac{9}{5}35 = 63$$

In the following we will be constructing a configuration of points with lesser number of ordinary hyperplanes than the trivial example, thus improving the upper bound for this case. We will be explaining the process by which we are building this example instead of just showing the points.

The equation 4.1 in this case becomes:

$$\tau_5 + 6\tau_6 + 21\tau_7 + 56\tau_8 = 126$$

In order to build our example we will use the case $\tau_8 = \tau_7 = 0$, so this means that we will be working only with 6-hyperplanes. In this scenario (5 dimensions) the 6-hyperplanes are the equivalent to the 5-hyperplanes in the 4 dimensional space, and we can represent them by the triplets of points not contained in them.

According to lemma 4.4 we could arrange the set of 9 points in up to 12 triplets of points such that any two of them share at most one points. But translating these into 6-hyperplanes adds geometric restrictions that make such arrangement of hyperplanes impossible. Just by the lower bound we have computed, we know that $\tau_6 \leq 10$.

We will be using a similar structure as the one we were proved impossible in the previous section for $d = 4$. The objective will be maximize the number of 6-hyperplanes on the model.

Let us begin with three disjoint triplets of points, which will correspond to three 6-hyperplanes such that any pair of them share only 3 points of P. If we denote the points of P by $\{p_1, ..., p_9\}$, then let us denote the three hyperplanes as:

$$H_1 = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$
$$H_2 = \langle p_1, p_2, p_3, p_7, p_8, p_9 \rangle$$
$$H_3 = \langle p_4, p_5, p_6, p_7, p_8, p_9 \rangle$$

These three hyperplanes intersect into a plane that we will name π. Also, these points form the planes:

$$\alpha_1 = \langle p_1, p_2, p_3 \rangle$$
$$\alpha_2 = \langle p_4, p_5, p_6 \rangle$$
$$\alpha_3 = \langle p_7, p_8, p_9 \rangle$$

Each of the planes α_i intersects the plane π into a line r_i (because both π and α_i are contained in the intersection of two of the H_j, which is a 3 space).

We will consider that all other 6-hyperplanes of the model will consist on two points from each of the $alpha_i$. We notice, as in the previous section, that if an hyperplane contains exactly two of the points of the plane $alpha_i$, then its intersection with $alpha_i$ will be the line generated by the two points and in particular it will contain the point of intersection of this line with the line r_i.

Let us denote the points by:

$$
\begin{aligned}
x_{11} &= \langle p_1, p_2\rangle \cap r_1 & x_{21} &= \langle p_4, p_5\rangle \cap r_2 & x_{31} &= \langle p_7, p_8\rangle \cap r_3 \\
x_{12} &= \langle p_1, p_3\rangle \cap r_1 & x_{22} &= \langle p_4, p_6\rangle \cap r_2 & x_{32} &= \langle p_7, p_9\rangle \cap r_3 \\
x_{13} &= \langle p_2, p_3\rangle \cap r_1 & x_{23} &= \langle p_5, p_6\rangle \cap r_2 & x_{33} &= \langle p_8, p_9\rangle \cap r_3
\end{aligned}
$$

This is an important simplification, because we can restrict ourselves to the study of these points inside the plane π and work just in two dimensions. As we are building an example, if we construct a configuration of hyperplanes that contains the required points x_{ij} we can easily find points in the planes α_i such that the lines they generate hit the points x_{ij}.

We will consider the three lines r_i are concurrent.

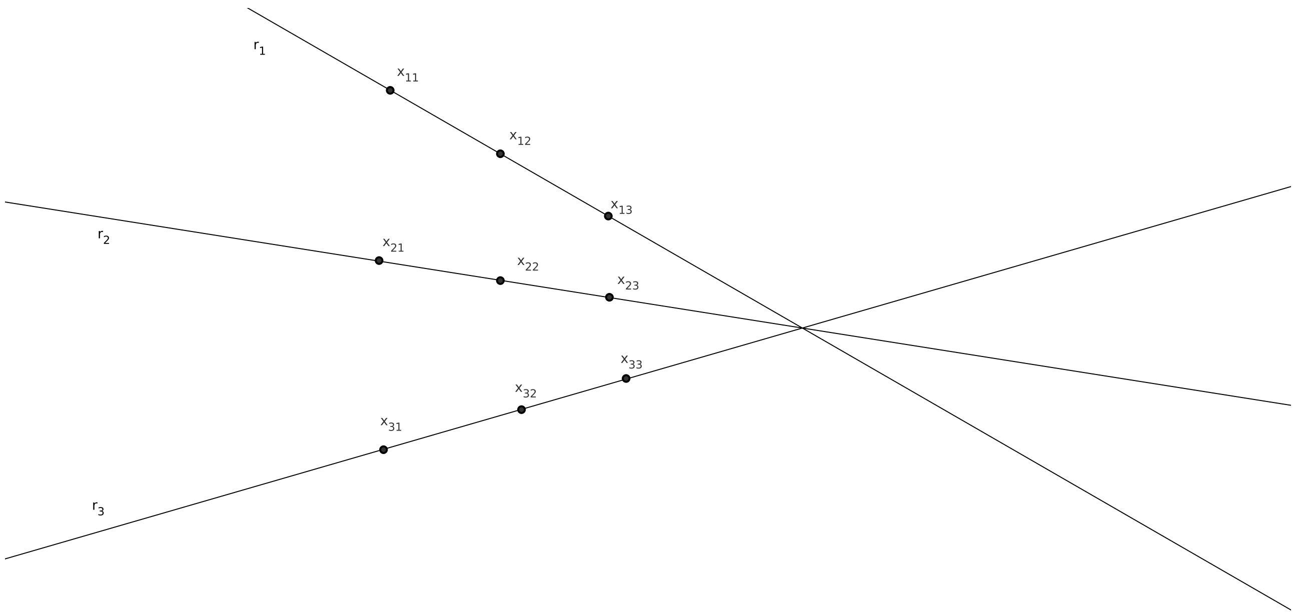


Figure 9: The nine points x_{ij} in the plane π

Now we want to find a configuration of the points x_{ij} that allow us to maximize the number of 6-hyperplanes of the model. We already have the three hyperplanes H_1, H_2 and H_3. Any other 6-hyperplane will have to contain one point of the x_{1i}, one of the x_{2j} and one of the x_{3k}, thus containing 6 points of P. Furthermore, any two of these hyperplanes can coincide in at most one of these three points x_{ij}, since otherwise they would share more than 4 points of P in common and that would mean they are the same.

Now to do this we will use the same trick as in the previous section and send one of the lines to infinity. If we send the line r_3 to infinity, then we are left with the lines r_1 and r_2, which are parallel now (because they were concurrent with r_3.

Now the problems transforms into finding the maximum number of lines that pass by one point of x_{1i}, one of x_{2j} and go in one of the three directions x_{3k}. We call a line meeting this criteria a g̈ood line: In the image 10 we show a configuration with 7 good lines.

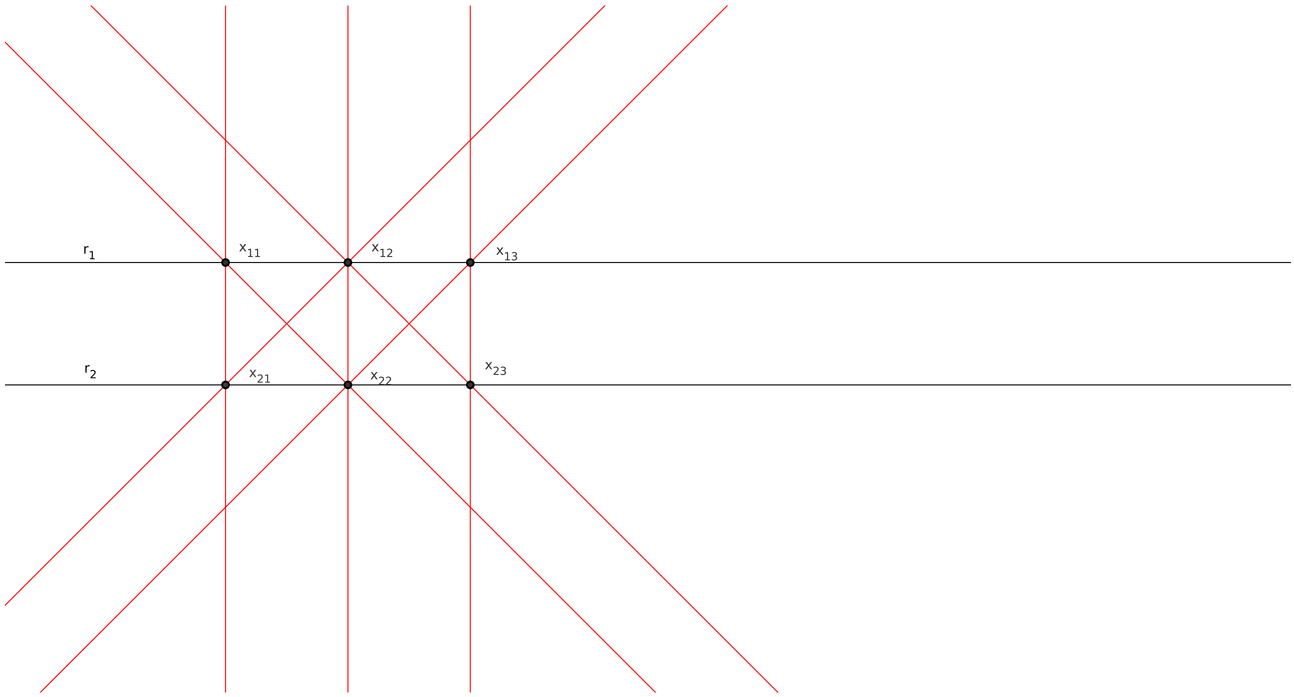


Figure 10: Configuration of the points x_{ij} in with the line r_3 at infinity spanning 7 good lines

We know that each good line will give us a 6-hyperplane when going back to the 5 dimensional model. As we have also the three hyperplanes H_1, H_2 and H_3, we will end up with $\tau_6 = 10$. Because of that, if we compute the number of ordinary hyperplanes, this would give us:

$$\tau_4 = \binom{9}{5} - 6\tau_6 = 126 - 60 = 66$$

Now the only thing left is to transform back these lines into the configuration of 9 points in 5 dimensions.

Let us denote a point in $\mathbb{P}^5_{\mathbb{R}}$ as $[X_0, X_1, X_2, X_3, X_4, X_5]$, where $[*, *, *, *, *, 0]$ is the hyperplane at infinity. We want to choose the coordinate of our objects in order to obtain the model on the picture 10 in the plane π.

We have some freedom to (for example) get rid of two dimensions by choosing carefully the planes α_1 and α_2. Also, in order to get the line r_3 at infinity, we would need the planes α_3 and π to be parallel to each other.

With these considerations, we can choose the objects defined so far as follows:

$$\begin{aligned}
\pi &= [0,0,1,*,*,*] \\
\alpha_1 &= [*,0,1,1,*,*], & \alpha_2 &= [0,*,1,0,*,*], & \alpha_3 &= [0,0,0,*,*,*] \\
r_1 &= [0,0,1,1,*,*], & r_2 &= [0,0,1,0,*,*], & r_3 &= [0,0,1,*,*,0] \\
x_{11} &= [0,0,1,1,0,1], & x_{12} &= [0,0,1,1,-1,1], & x_{13} &= [0,0,1,1,1,1] \\
x_{21} &= [0,0,1,0,0,1], & x_{22} &= [0,0,1,0,-1,1], & x_{23} &= [0,0,1,0,1,1]
\end{aligned}$$

Now we only need to find an adequate set of points P that gives matches this model. We can easily find the three points in each of the planes α_i that matches the corresponding x_{ij}. In the end we obtain the following set of points:

$$\begin{array}{lll} p_1 = [1,0,1,1,0,1] & p_2 = [3,0,1,1,0,1] & p_3 = [3/2,0,1,1,1/2,1] \\ p_4 = [0,1,1,0,0,1] & p_5 = [0,3,1,0,0,1] & p_6 = [0,3/2,1,0,1/2,1] \\ p_7 = [0,0,0,0,0,1] & p_8 = [0,0,0,2,0,1] & p_9 = [0,0,0,1,1,1] \end{array}$$

One can check that these points do match up with the objects defined above. Also one can check that this configuration of points indeed span exactly 66 ordinary hyperplanes.

After finding this example we can state that:

$$63 \leq e_5(9) \leq 66$$

In order to determine the exact value of $e_5(9)$ more work is needed. Though, we can say that if $e_5(9)$ was less than 66 then $\tau_7 = 1$ and $e_5(9) = 63$ (since τ_5 should be multiple of three).

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