Universitat Politècnica de Catalunya
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Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

The collinear four body problem.

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Contents

1. Introduction

The N body problem studies the movement of N bodies subject to gravitational laws. Since the publication of the first volume of the Isaac Newton's *Philosophiae Naturalis Principia Mathematica* in 1687 where there were established the principles of classical mechanics the N body problem has captivated a large number of astronomers and mathematicians for its obvious applications in celestial mechanics. The problem for $N = 2$ was solved by the Swiss mathematician Jean Bernoulli in the beginnings of the 18-th century. However, for $N \geq 3$ the problem becomes significantly more difficult. Since the ending of the 19-th century with the apparition of the theory of dynamical systems the N body problem has been an excellent framework where to apply the results of this discipline. In this master thesis we will study the collinear case of the 4 body problem from the numerical perspective. More precisely we will find different families of periodic, homoclinic and connection orbits and generalize the results. An homoclinic orbit, briefly speaking, is an orbit such that its limit forward in time and backward in time (ω and α limit) exist and are the same (in our case will be a periodic orbit). On the other hand a connection orbit is an orbit such that its limit forward in time is the total collision (collision of the four masses) and its limit backward in time is a periodic orbit. These orbits play an essential role in the dynamics of the system. For instance, the presence of homoclinic orbits is tightly related to chaotic phenomena. For finding these orbits we will proceed as follows:
Firstly we will deduce the equation of the system and we will get rid of the singularities of it using the ideas of Devaney described in [6] and applied by Martínez-Simó in the article [3]. Therefore, based on the article of Sekiguchi-Tanikawa [1] we will find four periodic orbits for a given value of $h = -1$ (conserved quantity) and $\alpha = 1$ (parameter of the system) using a suitable Poincaré map and extend the results for all $h \in \mathbb{R}^*$. Then, we will focus in one family of periodic orbits and, related to this one, we will find a family of homoclinic and connection orbits. We remark that the discovery of these families of connection and homoclinic orbits involve our own contribution to the study of the collinear four body problem.

2. The collinear four body problem

The aim of this master thesis is to discuss the behaviour of the collinear four body problem. We depart from the following situation: four bodies m_1, m_2, m_3 and m_4 aligned in this order move around the center of mass which we will fix on the origin. We suppose that the masses satisfy $m_2 = m_4 = \alpha$ and $m_3 = m_1 = 1$ and we define $\{x_i(t),\ t \in \mathbb{R}^+,\ i \in \{1,2,3,4\}\}$ as the position of the i-th body at time t. In the following image we illustrate this situation. Firstly, we define $x = x_1, \widetilde{y} = x_2$ (the bodies on the right side of the

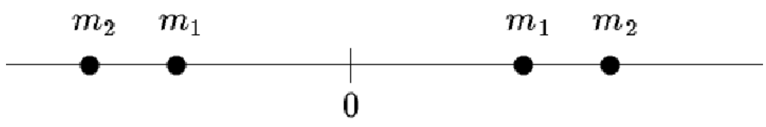

image in this respective order). Introducing the variable $y := \sqrt{\alpha}\widetilde{y}$ we apply Newton's laws to the objects m_1 and m_2:
Assuming $x, y \geq 0,\ \frac{y}{\sqrt{\alpha}} \geq x$:

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{\alpha(\frac{y}{\sqrt{\alpha}} - x)}{\|\frac{y-\sqrt{\alpha}x}{\sqrt{\alpha}}\|^3} - \frac{2x}{\|2x\|^3} + \frac{\alpha(\frac{-y}{\sqrt{\alpha}} - x)}{\|\frac{y}{\sqrt{\alpha}} + x\|^3} \\ \alpha\frac{d^2\widetilde{y}}{dt^2} = \frac{\alpha}{\sqrt{\alpha}}\frac{d^2y}{dt} = \frac{\alpha(x - \frac{y}{\sqrt{\alpha}})}{\|x - \frac{y}{\sqrt{\alpha}}\|^3} - \frac{\alpha(x + \frac{y}{\sqrt{\alpha}})}{\|\frac{y}{\sqrt{\alpha}} + x)\|^3} - \frac{2\frac{y}{\sqrt{\alpha}}\alpha^2}{\|2\frac{y}{\sqrt{\alpha}}\|^3}. \end{cases} \tag{1}$$

Simplifying the above equation we obtain:

$$\begin{cases} \frac{d^2x}{dt^2} = -\frac{1}{4x^2} + \frac{\alpha^2}{(y-\sqrt{\alpha}x)^2} - \frac{\alpha^2}{(y+\sqrt{\alpha}x)^2} \\ \frac{d^2y}{dt^2} = -\frac{\alpha^{3/2}}{(y-\sqrt{\alpha}x)} - \frac{\alpha^{3/2}}{(y+\sqrt{\alpha}x)^2} - \frac{\alpha^{5/3}}{4y^2} \end{cases} \tag{2}$$

Doing the change of variables $x' = \frac{p_x}{2},\ y' = \frac{p_y}{2}$ where x', y' denote the first derivatives of $x(t)$ and $y(t)$ we get the following system of four first order differential equations in the variables $\{x, y, p_x, p_y\}$:

$$\begin{cases} \frac{dx}{dt} = \frac{p_x}{2} \\ \frac{dy}{dt} = \frac{p_y}{2} \\ \frac{dp_x}{dt} = -\frac{1}{2x^2} + \frac{2\alpha^2}{(y-\sqrt{\alpha}x)^2} - \frac{2\alpha^2}{(y+\sqrt{\alpha}x)^2} \\ \frac{dp_y}{dt} = -\frac{-\alpha^{5/2}}{2y^2} - \frac{2\alpha^{3/2}}{(y-\sqrt{\alpha}x)^2} - \frac{2\alpha^{3/2}}{(y+\sqrt{\alpha})^2}. \end{cases} \tag{3}$$

We observe that the above system defines a Hamiltonian system with hamiltonian:

$$H(x, y, p_x, p_y) = \frac{1}{4}(p_x^2 + p_y^2) - U(x, y), \tag{4}$$

where

$$U(x, y) := \frac{1}{2x} + \frac{\alpha^{5/2}}{2y} + \frac{2\alpha^{3/2}}{y - \sqrt{\alpha}x} + \frac{2\alpha^3}{y + \sqrt{\alpha}x}.$$

Calling $h = H$ the value of the hamiltonian we check that

$$h + U(x, y) = \frac{1}{4}(p_x^2 + p_y^2) \geq 0$$

As the hamiltonian is a first integral of the system (i.e. its value is constant along an orbit) we have that for every orbit of the system (3) with hamiltonian h, the (x, y) coordinates of the orbit must satisfy:

$$\begin{gathered} U(x,y) + h \geq 0 \\ \frac{y}{\sqrt{\alpha}} > x. \end{gathered} \tag{5}$$

So, in other words. the (x, y) projection of all the orbits with hamiltonian h must belong to the region (5). In general these regions are called Hill's regions. In the following image we observe an example of a Hill's region of our problem.

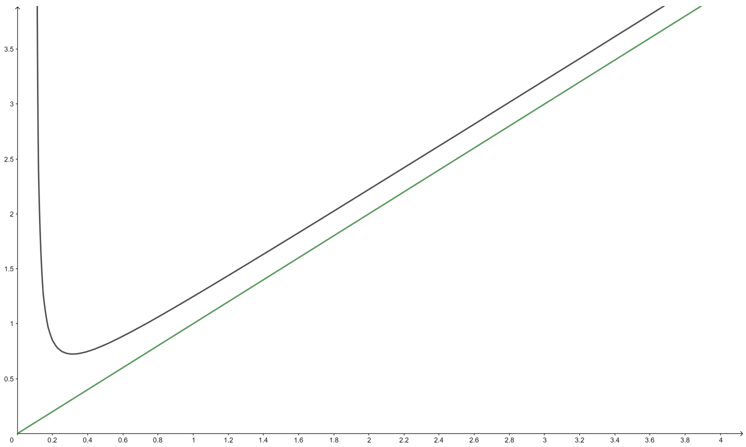

Figure 1: Hill's region of the system (3) (the region between the curves) for $h = -10$.

We realize that when $x = 0, \ \ y \geq 0$ there is a collision between the internal bodies. From now on, we will call these collisions $1-2-1$ collisions. On the other hand, when $y = \sqrt{\alpha}x$ we have two simultaneous collisions in the external bodies: one between the bodies m_1, m_2 and another between the bodies m_3 and m_4. From now on we will call this type of collisions $2-2$ collisions.
For studying properly the motion of the system we need to regularize the equations. In other words, we need to get rid of the singularities of (3) doing some changes of variables and time. We will apply the ideas of Devaney described in [6] . Firstly we define:

$$\begin{cases} x = \frac{r}{\sqrt{2}}\cos\theta \\ y = \frac{r}{\sqrt{2}}\sin\theta \\ p_x = \sqrt{2}p_x\cos\theta - \frac{\sqrt{2}p_\theta}{r}\sin\theta \\ p_y = \sqrt{2}p_r\sin\theta + \frac{\sqrt{2}p_\theta}{r}\cos\theta \end{cases} \tag{6}$$

We check that this change of variables is in fact a canonical change of variables. Let $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ be the vector field that defines (6). Defining $M := D_{(r,\theta,p_r,p_\theta)}\phi$ the Jacobian matrix of ϕ we observe that:

$$M^t J M = J,$$

when

$$M = \begin{pmatrix} \frac{\cos\theta}{\sqrt{2}} & -\frac{r\sin\theta}{\sqrt{2}} & 0 & 0 \\ \frac{\sin\theta}{\sqrt{2}} & \frac{r\cos\theta}{\sqrt{2}} & 0 & 0 \\ \frac{\sqrt{2}p_\theta\sin\theta}{r^2} & -\sqrt{2}p_r\sin(\theta) - \frac{\sqrt{2}p_\theta\sin\theta}{r} & \sqrt{2}\cos\theta & \frac{\sqrt{2}\sin\theta}{r} \\ -\sqrt{2}\frac{p_\theta}{r^2}\cos\theta & \sqrt{2}p_r\cos\theta - \frac{\sqrt{2}p_\theta}{r}\sin\theta & \sqrt{2}\sin\theta & \frac{\sqrt{2}}{r}\cos\theta \end{pmatrix} \tag{7}$$

and

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{8}$$

As J is a symplectic matrix we realize that the system (3) with respect to the new variables is hamiltonian with the following hamiltonian function.

$$H(r,\theta,p_r,p_\theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{1}{r}V(\theta), \tag{9}$$

where

$$V(\theta) := \frac{1}{\sqrt{2}\cos\theta} + \frac{\alpha^{5/2}}{\sqrt{2}\sin\theta} + \frac{2\sqrt{2}\alpha^{3/2}}{\sin\theta - \sqrt{\alpha}\cos\theta} + \frac{2\sqrt{2}\alpha^{3/2}}{\sin\theta + \sqrt{\alpha}\cos\theta}. \tag{10}$$

For more details about this property look at the Meyer's book [5]. So, we obtain the following equations of the motion:

$$\begin{cases} \frac{dr}{dt} = p_r \\ \frac{d\theta}{dt} = \frac{p_\theta}{r^2} \\ \frac{dp_r}{dt} = \frac{p_\theta^2}{r^3} - \frac{1}{r^2}V(\theta) \\ \frac{dp_\theta}{dt} = \frac{1}{r}V'(\theta) \end{cases} \tag{11}$$

We remark that the manifold $r = 0$ represents the total collision and the manifold $\theta = \frac{\pi}{2}$ and $\theta = \arctan\sqrt{\alpha}$ represent the collisions $1-2-1$ and $2-2$ respectively. We observe that the new vector field that defines the ode still have discontinuities when $r = 0$, $\theta = \arctan\sqrt{\alpha}$ and $\theta = \frac{\pi}{2}$. Firstly, for removing the singularity $r = 0$ (Total collision) we introduce the following variables, $p_r = \frac{v}{\sqrt{r}}$, $p_\theta = \sqrt{r}u$. Then the system (11) becomes

$$\begin{cases} \frac{dr}{dt} = r^{-1/2}v \\ \frac{dv}{dt} = r^{-3/2}(u^2 + \frac{v^2}{2} - V(\theta)) \\ \frac{d\theta}{dt} = r^{-3/2}u \\ \frac{du}{dt} = r^{-3/2}(-\frac{1}{2}uv + V'(\theta)). \end{cases} \tag{12}$$

We observe that

$$rH = \frac{1}{2}(u^2 + v^2) - V(\theta), \tag{13}$$

where H defined in (4) is a first integral of the system in these new variables. We define the following change of variables in time $\frac{dt}{d\tau} = r^{3/2}$. Calling $' = \frac{d}{d\tau}$ the system (12) becomes:

$$\begin{cases} r' = rv \\ v' = u^2 + \frac{v^2}{2} - V(\theta) \\ \theta' = u \\ u' = -\frac{1}{2}uv + V'(\theta) \end{cases} \tag{14}$$

Then, we will regularize the binary collisions $2-2$ and $1-2-1$. Firstly, we will introduce the variable:

$$w := \frac{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} u,$$

where,

$$W(\theta) = V(\theta)\cos(\theta)(\sin\theta - \sqrt{\alpha}\cos\theta). \tag{15}$$

We observe that $W(\theta)$ is regular in the Hill region. Then, the system becomes:

$$\begin{cases} r' = rv \\ v' = 2rh - \frac{v^2}{2} + \frac{W(\theta)}{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)} \\ \theta' = \frac{\sqrt{W(\theta)}}{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)} w \\ w' = -\frac{1}{2}vw + (\cos 2\theta + \sqrt{\alpha}\sin 2\theta)(\frac{2rh - v^2}{\sqrt{W(\theta)}} + \frac{\sqrt{W(\theta)}}{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)}) - \frac{W'(\theta)}{\sqrt{W(\theta)}}(\frac{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)}{2W(\theta)}(2rh - v^2). \end{cases} \tag{16}$$

Finally, doing the change of variables in time $\frac{d\tau}{ds} = \frac{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta}{\sqrt{W(\theta)}}$ we obtain:

$$\begin{cases} \frac{dr}{ds} = rv\frac{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} \\ \frac{dv}{ds} = \sqrt{W(\theta)}\left(1 + (2rh - \frac{v^2}{2})\frac{\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)}{W(\theta)}\right) \\ \frac{d\theta}{ds} = w \\ \frac{dw}{ds} = -\frac{1}{2}\frac{vw\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} + (\cos 2\theta + \sqrt{\alpha}\sin 2\theta)\left(\frac{2rh - v^2}{W(\theta)}\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta) + 1\right) + \\ \frac{W'(\theta)}{W(\theta)}(\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta) - \frac{w^2}{2}) \end{cases} \tag{17}$$

We remark that the system of ode (17) is free of singularities. In this new set of variables we can write the conserved quantity (13) as:

$$w^2 = (2rh - v^2)\frac{\cos^2\theta(\sin\theta - \sqrt{\alpha}\cos\theta)^2}{W(\theta)} + 2\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta) \tag{18}$$

where h is the Hamiltonian of the system . In the total collision $r = 0$ we obtain the following relation:

$$w^2 = -\frac{\cos^2\theta(\sin\theta - \sqrt{\alpha}\cos\theta)^2}{W(\theta)}v^2 + 2\cos\theta(\sin\theta - \sqrt{\alpha}\cos\theta). \tag{19}$$

3. Variational equations and the Poincaré map

One of the goals of this thesis is to find periodic orbits in some special cases and analyze its stability. For doing that it will be required to us to use two essential tools in dynamical systems: the Poincaré map and the variational equations. In this chapter we will introduce some basic properties that we will use during the development of this thesis. We remark that all the results that we expose for autonomous differential equations can be generalized for non autonomous systems.

Definition 3.1. Given an autonomous differential equation

$$x' = f(x) \tag{20}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function $(f \in \mathcal{C}^r,\ r \in \mathbb{N}, r \geq 1)$ we define the flow as the function $\phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(t, t_0, x_0)$ is the solution $x(t)$ of (20) at time t that satisfies $x(t_0) = x_0$.

We observe that as a consequence of the theorem of existence and uniqueness of differential equations the flow is well defined.
Once defined the flow we will introduce the Poincaré map.

Definition 3.2. Let Σ be a manifold in $\mathbb{R}^n$. We will say that it is Poincaré a transversal section of the system (20) if $\forall t_0, x_0$ such that $t_0 \in \mathbb{R}, x_0 \in \Sigma$ exists $t' \in \mathbb{R}$ $t' > t_0$ such that $\phi(t'; t_0, x_0) \in \Sigma$.

Definition 3.3. Given an autonomous system of ode defined as in (20) and Σ a Poincaré transversal section, we define the Poincaré map: $P : \Sigma \to \Sigma$ as

$$P(x) = \phi(\tau(x), 0, x_0)$$

where $\tau : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\tau(x) = \min\{t \in \mathbb{R},\ \ t > 0 \mid \phi(\tau(x), 0, x_0) \in \Sigma\}.$$

In other words, the Poincaré map follows the orbit of a point $x_0 \in \Sigma$ until this orbit intersects again Σ at time $\tau(x)$. We can visualize this behaviour in the following plot.

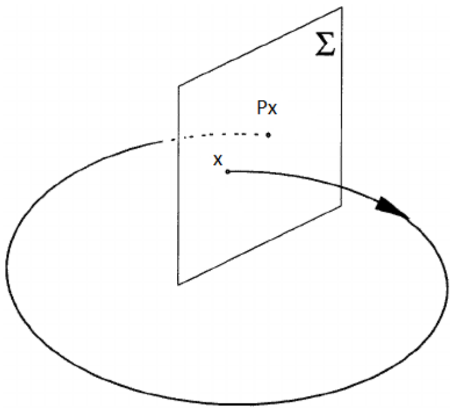

Figure 2: Scheme of the Poincaré map

We remark that if Σ is a transversal manifold then the Poincaré map is always well defined. Here below we state a basic property of the Poincaré map.

Proposition 3.4. *Given an autonomous system of ode defined as in* (20) *and a Poincaré map* $P : \Sigma \to \Sigma$ *then if* x_0 *is a fixed point of* P^n *for some* $n \in \mathbb{Z}^+$ *then* $\phi(t, t_0, x_0)$ *is a periodic orbit.*

Proof. It is a consequence of the theorem of existence and uniqueness of ordinary differential equations. □

In our case we will use these techniques for finding periodic orbits of the collinear four body problem using a suitable Poincaré map.
As the Poincaré map is an important tool in order to find periodic orbits, the variational equations are related to the stability of these periodic orbits. Here below we will introduce them and state some basic properties that we will use during this thesis:

Definition 3.5. Let $\phi(t, t_0, x_0)$ be the flow of an autonomous differential equation (20) we call variational equations of the system to the following system of ode:

$$\frac{d}{dt} D_{x_0}\phi(t, t_0, x_0) = D_x f(\phi(t, t_0, x_0)) D_{x_0}\phi(t, t_0, x_0), \tag{21}$$

As a first remark we observe that the system comes from the fact that the flow satisfies

$$\frac{d}{dt}\phi(t, t_0, x_0) = f(\phi(t, t_0, x_0)),$$

then we obtain:

$$\frac{d}{dt} D_{x_0}\phi(t, t_0, x_0) = D_x f(\phi(t, t_0, x_0)) D_{x_0}\phi(t, t_0, x_0).$$

The following theorem will allow us to describe properly the dynamic of a periodic orbit.

Theorem 3.6. *Let* $\phi(t, t_0, x_0)$ *be the flow of an autonomous differential equation* (20) *and* $z(t)$ *a periodic orbit of this system with period* T*. Let* $D_{x_0}\phi(t, t_0, x_0)$ *be the solution of the variational equations* (21)*. Then for all* $x_0 = z(r)$, $0 \leq r \leq T$ *there exists a stable invariant manifold* $W^s(x_0)$ *tangent to the eigenspace related to the eigenvectors* $\lambda \in \mathbb{C}$ *of* $D_{x_0}\phi(T, 0, x_0)$ *with* $|\lambda| < 1$ *and there exists an unstable manifold tangent to the eigenspace related to the eigenvectors* $\lambda \in \mathbb{C}$ *of* $D_{x_0}\phi(T, 0, x_0)$ *with* $|\lambda| > 1$.

Proof. The proof can be found in the book of Guckenheimer and Holmes [9]. □

For a periodic orbit $z(t)$ we will define the stable and unstable invariant manifolds of the whole orbit $z(t)$ as

$$W^s(z(t)) := \bigcup_{0 \le r \le T} W^s(z(r)),$$

$$W^u(z(t)) := \bigcup_{0 \le r \le T} W^u(z(r)).$$

In this thesis we will approximate these manifolds by its linear approximation.

4. Periodic orbits

Coming back to the system (17) we depart from the following system of ode. The following system is free of singularities in the Hill's region.

$$\begin{cases} \frac{dr}{ds} = rv\frac{\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} \\ \frac{dv}{ds} = \sqrt{W(\theta)}\left(1+(2rh-\frac{v^2}{2})\frac{\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)}{W(\theta)}\right) \\ \frac{d\theta}{ds} = w \\ \frac{dw}{ds} = -\frac{1}{2}vw\frac{vw\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} + (\cos 2\theta + \sqrt{\alpha}\sin 2\theta)\left(\frac{2rh-v^2}{W(\theta)}\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)+1\right) + \\ \frac{W'(\theta)}{W(\theta)}(\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)-\frac{w^2}{2}) \end{cases} \tag{22}$$

In this section we will apply the techniques defined in the article of Sekiguchi-Tanikawa [1] for finding periodic orbits using a suitable section in $\mathbb{R}^4$ where we will apply the Poincaré map. Firstly we will study only the cases $h = -1$ and $\alpha = 1$ and later we will generalize. In fact, in regards to h we can generalize easily the results that we will find for $h = -1$ using the following property of our system.

Lemma 4.1. *Let $x(t) = (r(t), v(t), \theta(t), w(t))$ be a solution of (17) with conserved energy $h_0 \in \mathbb{R}^*$ then $\forall h_1 \in \mathbb{R}^*$, $y_{h_1}(t) := (\frac{h_0}{h_1}r(t), v(t), \theta(t), w(t))$ is a solution of (17) with conserved energy energy h_1.*

Proof. Let $x(t) = (r(t), v(t), \theta(t), w(t))$ be a solution of (17) with first integral $h_0 \in \mathbb{R}^*$ we check easily that the function $y_{h_1}(t) := (\frac{h_0}{h_1}r(t), v(t), \theta(t), w(t))$ for $h_1 \in \mathbb{R}^*$ is a solution of

$$\begin{cases} \frac{dr}{ds} = rv\frac{\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} \\ \frac{dv}{ds} = \sqrt{W(\theta)}\left(1+(2rh_1-\frac{v^2}{2})\frac{\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)}{W(\theta)}\right) \\ \frac{d\theta}{ds} = w \\ \frac{dw}{ds} = -\frac{1}{2}vw\frac{vw\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)}{\sqrt{W(\theta)}} + (\cos 2\theta + \sqrt{\alpha}\sin 2\theta)\left(\frac{2rh_1-v^2}{W(\theta)}\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)+1\right) + \\ \frac{W'(\theta)}{W(\theta)}(\cos\theta(\sin\theta-\sqrt{\alpha}\cos\theta)-\frac{w^2}{2}). \end{cases} \tag{23}$$

Hence $y_{h_1}(t)$ is a solution of (23) with conserved quantity h_1. □

Corollary 4.2. *Let $x(t) = (r(t), v(t), \theta(t), w(t))$ be a periodic orbit of the system (17) with period T and conserved energy h_0, then we have a family of periodic orbits, varying with h,*

$$\mathcal{F} = \{x_h(t) |\ x_h(t) = (\frac{h_0}{h}r(t), v(t), \theta(t), w(t)), \forall h \in \mathbb{R}^*\},$$

where each orbit $x_h(t)$ has period T.

Proof. By the previous lemma we know that $\forall h \in \mathbb{R}^*$, $x_h(t)$ is a solution of (17) and it satisfies:

$$x_h(0) = x_h(T).$$

Hence $x_h(t)$ is a periodic orbit with period T. □

This corollary allows us to state that the problem of finding periodic orbits of (17) can be reduced to finding periodic orbits for a fixed conserved energy $h \in \mathbb{R}^*$. In our case we will study the case $h = -1$. This fact is also true in the case of homoclinic and connection orbits. We will talk about these topics in the next chapters.
Returning to our restricted problem ($h = -1, \alpha = 1$), we introduce the section $\theta = \theta_c$ where θ_c is the solution of

$$\frac{dV}{d\theta} = 0,$$

and $V(\theta)$ is a real function defined in (10). Looking at the conserved energy equation (18) restricted on the manifold $\{\theta = \theta_c\}$ we obtain the following relation:

$$w^2 = -(2r + v^2)\frac{\cos\theta_c^2(\sin\theta_c - \sqrt{\alpha}\cos\theta_c)^2}{W(\theta_c)} + 2\cos\theta_c(\sin\theta_c - \sqrt{\alpha}\cos\theta_c). \tag{24}$$

Calling

$$p(r) := \frac{-2r\cos^2\theta_c(\sin\theta_c - \sqrt{\alpha}\cos\theta_c)^2}{W(\theta_c)} + 2\cos\theta_c(\sin\theta_c - \sqrt{\alpha}\cos\theta_c),$$

we can write (24) as:

$$\frac{w^2}{p(r)} + \frac{v^2}{\frac{W(\theta_c)p(r)}{\cos^2\theta_c(\sin\theta_c - \sqrt{\alpha}\cos\theta_c)^2}} = 1. \tag{25}$$

We observe that as $W(\theta_c) > 0$, we have that in the cases $p(r) > 0$ the previous equation represents an ellipse with respect to w and v. For a similar reason we observe that

$$p(r) \leq 2\cos\theta_c(\sin\theta_c - \sqrt{\alpha}\cos\theta),$$

and the equality is satisfied if and only if $r = 0$. Hence, we observe that in the section $\{\theta = \theta_c\}$ all the orbits that satisfy (25) and $p(r) > 0$ satisfy also:

$$\frac{w^2}{2\cos\theta_c(\sin\theta_c - \sqrt{\alpha}\cos\theta_c)} + \frac{v^2}{\frac{2W(\theta_c)}{\cos\theta_c(\sin\theta_c - \sqrt{\alpha}\sin\theta_c)}} \leq 1 \tag{26}$$

where, in fact the boundary of this set represents the v, w components of the intersection of the total collision $r = 0$ with the section $\{\theta = \theta_c\}$. The strategy now will be the following: We will work in the (w, v) projection of the section $\theta = \theta_c$. We realize that applying the first integral relation we can write r as a function of the rest of the variables.
Taking $M = 100$ initial conditions on the x-axis of the interior of the ellipse (26) we will calculate the $N = 100$ iterations of the Poincaré map defined in the manifold $\{\theta = \theta_c\}$ and plot the results in the (w, v) plane. This will provide us an intuition of how our system behaves. Doing this simulation we obtain:

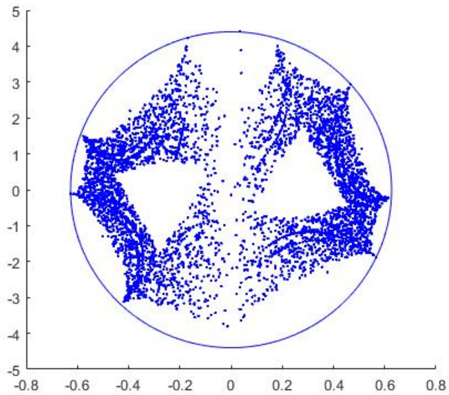

Figure 3: Iterations of the Poincaré map in the section $\theta = \theta_c$ in (w, v) coordinates.

Firstly we remark that the ellipse of the plot represents the intersection of the total collision with the section $\{\theta = \theta_c\}$ projected in the plane (w, v). Moreover, we can appreciate the two different regions described in the Sweetman's article [7]: one defined by the two gaps in the center of the plot that Sweetman defines as quasiperiodic region and another with a "chaotic" behaviour outside these two gaps.
Now we will proceed to find periodic orbits of the system. For simplicity, from now on we call Σ_c the section $\{\theta = \theta_c\}$. Defining P_{θ_c} the Poincaré map on the section Σ_c, we know from proposition 3.4 that every periodic orbit that intersects Σ_c will be a fixed point of $P^n_{\theta_c}$ for some $n \in \mathbb{Z}^+$. In other words, we want to find solutions $x \in \mathbb{R}^4$ of

$$P^n_{\theta_c}(x) = x \quad n \in \mathbb{N}.$$

Every solution $x \in \mathbb{R}^4$ will correspond to an initial condition of a periodic orbit. Taking into account the fact that $x \in \Sigma_c$ and the first integral relation we realize that (r, θ) can be uniquely determined by (w, v). So, in fact we can reduce our problem to find $x = (w, v) \in \mathbb{R}^2$ such that for some $n \in \mathbb{N}$

$$P^n_{\theta_c} x = x,$$

when P_{θ_c} is defined in the (w, v) plane. For finding numerically solutions of this problem we will use an iterative method taking initial conditions in some special regions that we can observe from the figure 5. Firstly we consider the regions with more density of points or where we can appreciate aligned points: these regions could represent an stable manifold of a periodic orbit and hence, it could be a good region for take the initial conditions of an iterative method. On the other side, the two gaps of the figure 5 could represent an stable region defined by an stable periodic orbit. Applying this procedure for taking the initial conditions using our iterative method we found the following periodic orbits:

Periodic orbits of the system for $\alpha = 1$ and $h = -1$			
Number of the periodic orbit	Initial conditions in (w, v)	Period	Number of intersections with Σ_c per period
1	$(0.295463117983, 0.159856987581)$	5.629520695466097	2
2	$(0.3338390474572, -1.0789608423286)$	15.154887900710179	4
3	$(-0.388798768702389, -0.658050218585429)$	47.371127699916727	16
4	$(-0.343932372061647, 1.327830372465703)$	17.909921477392498	4

We remark that all the results are obtained using double precision and an absolute and relative tolerances of 10^{-12}. Automatically, from corollary 4.2 we obtain a family of four periodic orbits depending on $h \in \mathbb{R}^*$. Now we are going to analyze better each periodic orbit. First of all, we represent each periodic orbit in the variables (x, y). We remark that the (x, y) variables represents the distance of the objects on the right of figure 1 with the center of mass.

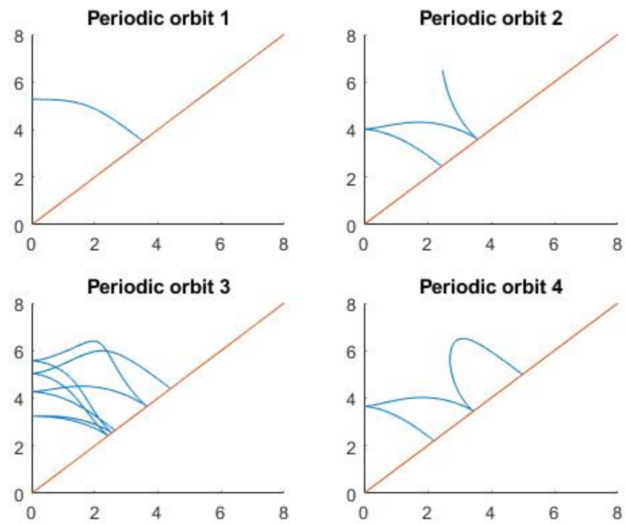

Figure 4: Periodic orbits 1 to 4 in (x, y) variables and the $2-2$ collision.

Solving the variational equations of the system applied in each periodic orbit we can conclude that the periodic orbit 1 on the table is stable, and the other three are unstable (there exists an unstable manifold). The periodic orbit 1 is an example of a Schubart like periodic orbit. A Schubart like periodic orbit is an orbit such that in one period it achieves a $1-2-1$ and a $2-2$ collision. We can appreciate this behaviour in the first image of the figure 4.

From now on we will focus on the periodic orbit 2 (unstable). For the later calculations it will be required to us to simulate the unstable invariant manifold of this orbit. For doing that we will proceed as follows: Firstly we take a random point x_0 of the periodic orbit 2 and the normalized eigenvector v_0 of eigenvalue $|\lambda| > 1$ of the solution of the variational equations taking initial condition the identity matrix. Then, we will consider the following intervals in $\mathbb{R}^4$:

$$I^+ = \{x \in \mathbb{R}^4 \mid x = x_0 + tsv_0,\ s = 10^{-2},\ t \in [0,1]\},$$

$$I^- = \{x \in \mathbb{R}^4 \mid x = x_0 - tsv_0,\ s = 10^{-2},\ t \in [0,1]\}.$$

Therefore, we discretize the intervals I^+ and I^- in the following way:

$$x_n = x_0 \pm \frac{n}{N} sv_0, \quad 1 \leq n \leq N.$$

In our case we will take $N = 100$. For each $x_n,\ 1 \leq n \leq N$ we will calculate $M = 100$ iterations of the Poincaré map P_{θ_c} and plot the results in the (w, v) plane. Doing that, we obtain the following simulation of the unstable invariant manifold of the periodic orbit 2.

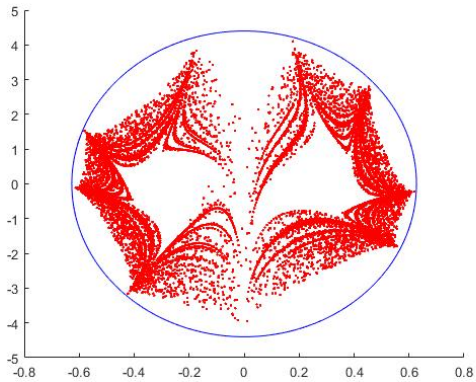

Figure 5: Unstable periodic orbit 2 defined on the section $\theta = \theta_c$.

5. Homoclinic orbits

In this chapter we will introduce a technique for finding homoclinic orbits connected to an unstable periodic orbit of the system (17) and we will apply it for finding an homoclinic connections to the periodic orbit 2. Firstly, we will restrict to the case $h = -1$ and $\alpha = 1$ and later we will generalize $\forall h \in \mathbb{R}^*$.

Definition 5.1. Let $x' = f(x)$ be an autonomous system of ode with a fixed point x_0, we will say that an orbit $x(t)$ is an homoclinic connection to x_0 if

$$\lim_{t\to\infty} x(t) = \lim_{t\to-\infty} x(t) = x_0.$$

Equivalently, $x(t) \in W^u(x_0) \cap W^s(x_0)$ where $W^u(x_0)$ and $W^s(x_0)$ denote the unstable and stable invariant manifolds of x_0 respectively.

As a first remark we can define homoclinic connections to periodic orbits in the following way:

Definition 5.2. Let $x' = f(x)$ be an autonomous system of ode with a periodic orbit $z(t)$. We will say that an orbit $w(t)$ is an homoclinic connection to $z(t)$ if and only if $w(t) \in W^u(z(t)) \cap W^s(z(t))$, where $W^u(z(t))$ and $W^s(z(t))$ denote the unstable and stable invariant manifolds of the orbit $z(t)$ respectively.

The following lemma states that in fact we can generalize in h easily as we did in the case of periodic orbits in 4.2.

Lemma 5.3. *Let $w(t) = (r(t), v(t), \theta(t), w(t))$ be an homoclinic orbit of (17) with conserved energy h_0 connected to a periodic orbit $z(t) = (\tilde{r}(t), \tilde{v}(t), \tilde{\theta}(t), \tilde{w}(t))$, then we have a family of homoclinic orbits*

$$\mathcal{F} = \{w_h(t) = (r(t)\frac{h_0}{h}, v(t), \theta(t), w(t)) \ \mid h \in \mathbb{R}^*\},$$

where each orbit $w_h(t)$ is connected to the periodic orbit $z_h(t) := (\tilde{r}(t)\frac{h_0}{h}, \tilde{v}(t), \tilde{\theta}(t), \tilde{w}(t))$.

Proof. As a consequence of the corollary (4.2) we observe that the orbits $z_h(t) = (\tilde{r}(t)\frac{h_0}{h}, \tilde{v}(t), \tilde{\theta}(t), \tilde{w}(t))$ are periodic. On the other hand, the function $w_h(t) = (\tilde{r}(t)\frac{h_0}{h}, \tilde{v}(t), \tilde{\theta}(t), \tilde{w}(t))$ is an orbit of the system (17) as a consequence of the lemma 4.1. We check easily that is $w_h(t)$ is homoclinically connected with $z_h(t)$. □

Returning to our problem, i.e taking the system (17) we will describe how to find numerically an homoclinic orbit given an unstable periodic orbit. First of all we must introduce the following proposition.

Proposition 5.4. *Let $x(t)$ be an orbit belonging to the unstable invariant manifold of a periodic orbit of the system (17). If $x(t)$ intersects the manifold $\mathcal{R} = \{(r, v, \theta, w) \in \mathbb{R}^4 \mid v = w = 0\}$ then $x(t)$ is an homoclinic symmetric orbit.*

Using this proposition we observe that finding an homoclinic connection of a given periodic orbit will be equivalent to find an orbit belonging to the unstable invariant manifold such that it intersects the manifold $\mathcal{R}$ described above. For doing this we will proceed as follows: Firstly we take a random point x_0 of the unstable periodic orbit and the normalized eigenvector v_0 of eigenvalue $|\lambda| > 1$ of the solution of the variational equations taking initial condition the identity matrix. Then, we will consider the following interval in $\mathbb{R}^4$:

$$I = \{x \in \mathbb{R}^4 \mid x = x_0 + tsv_0, \ s = 10^{-2}, \ t \in [0, 1]\}.$$

We realize that I is contained in the unstable invariant manifold of the periodic orbit (in fact it belongs to its linear approximation). In general for every s sufficiently small this interval is called fundamental interval. Firstly, we will discretize I in $N = 600$ equidistant points of the form:

$$x_n = x_0 + \frac{n}{N} s v_0, \quad 1 \le n \le N. \tag{27}$$

We will integrate each initial condition $x_n \in I$ up to the k-th crossing with the hiperplane $\{w = 0\}$ and we will consider its variable v of the intersection. For each k we will plot the variable v with respect to n.

As a consequence of proposition 5.4 we realize that if we find a value $t^* \in [0, 1]$ such that for a given $k \in \mathbb{N}$, the k-th intersection of the orbit with initial condition $x^* := x_0 + t^* s v_0 \in I$ with the hiperplane $\{w = 0\}$ has its component $v = 0$ then, x^* is the initial condition of an homoclinic orbit.
Now we will apply this methodology to the unstable periodic orbit 2. So, we restrict to the case $\alpha = 1$ and $h = -1$. Firstly we will calculate for each $1 \le k \le 20$ the plot of the component v with respect to n of the k-th crossing with the hiperplane $\{w = 0\}$ and look for the presence of zeros in the y-axis (component v). Concretely, for $k = 18$ we obtain the following plot: In this plot we appreciate taking into account the

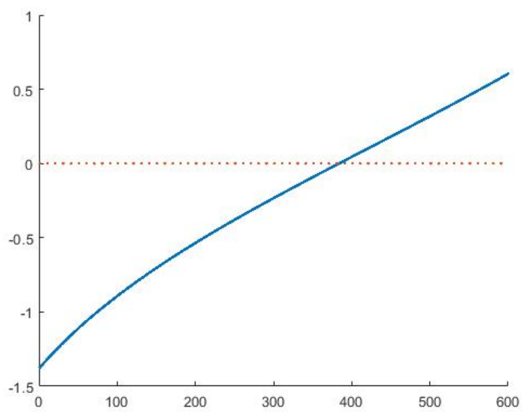

Figure 6: Component v in terms of n of the 18-th crossing of the elements in (27) with the hiperplane $\{w = 0\}$.

continuity of the flow with respect to the initial conditions that there exists a point x^* in the fundamental interval I such that its orbit intersects $\mathcal{R} = \{(r, v, \theta, w) \in \mathbb{R}^4 \mid v = w = 0\}$. Hence, the orbit of x^* is homoclinically connected to the periodic orbit 2. In fact, we can find explicitly this point x^* finding a zero of the following function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(r) = \phi_2(\tau_{r,18}, 0, x_r),$$

where, ϕ_2 denotes the second component of the flow, $x_r = x_0 + r s v_0 \in I$ and $\tau_{r,18}$ denotes the time the orbit with initial condition x_r takes for intersecting $\{w = 0\}$ for 18-th time. Applying this technique we obtain the following initial condition of an homoclinic orbit:

$$x^* = (6.346850623987035, -1.078420078535209, 1.263757439765094, 0.334412126197204).$$

Here below we observe the (x, y) projection of this orbit:

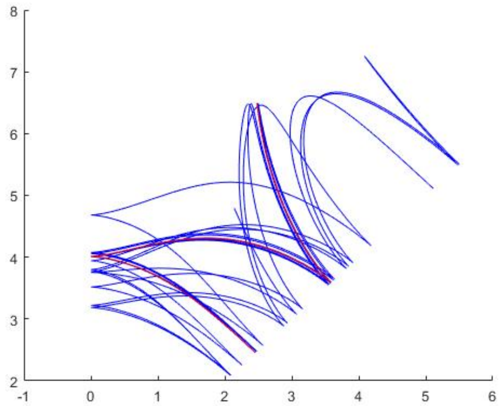

Figure 7: Homoclinic periodic orbit and the second unstable periodic orbit.

We observe that finding this homoclinic orbit in fact we obtain a family of homoclinic orbits depending on $h \in \mathbb{R}^*$ as a consequence of the lemma 5.3.

6. Connection orbits

In this section we will find a family of connected orbits from a given unstable periodic orbit to the total collision manifold $\mathcal{T} := \{(r, v, \theta, w) \in \mathbb{R}^4 \mid r = 0\}$. In other words, let $w(t)$ be a parametrization of the periodic orbit we will find an orbit $z(t)$ of our system such that $\lim_{t\to-\infty} z(t) = w(t)$ and $\lim_{t\to\infty} z(t) \in \mathcal{T}$. From now on, for simplicity we will call these orbits connection orbits. First of all we realize that, as in the cases of homoclinic and periodic orbits, if we find a connection orbit for a fixed $h \in \mathbb{R}^*$ we will obtain a family of connection orbits depending on h. The following proposition formalizes this fact:

Proposition 6.1. *Let $x(t) = (\tilde{r}(t), \tilde{v}(t), \tilde{\theta}(t), \tilde{w}(t))$ be a periodic orbit of the system (17). Therefore, if $z(t) = (r(t), v(t), \theta(t), w(t))$ is an orbit of (17) with conserved energy $h_0 \in \mathbb{R}^*$ that connects $x(t)$ and $\mathcal{T}$, then there is a family of connection orbits*

$$\mathcal{F} = \{z_h(t) = (\frac{h_0}{h} r(t), v(t), \theta(t), w(t)) \ \mid \ h \in \mathbb{R}^*\},$$

where each orbit $z_h(t)$ has conserved energy $h \in \mathbb{R}^*$ and it is a connection orbit between the periodic orbit $(\frac{h_0}{h} r(t), v(t), \theta(t), w(t))$ and $\mathcal{T}$.

Proof. The proof is essentially the same than the proof of the proposition 5.3. □

Now we observe that finding a connection orbit for $h = -1$ in fact we obtain a family of connection orbits depending in $h \in \mathbb{R}^*$. For finding it numerically we will proceed in a similar way than in the case of homoclinic orbits: Firstly we will use again the fundamental interval of the unstable manifold of the periodic orbit:

$$I = \{x \in \mathbb{R}^4 \mid x = x_0 + tsv_0, \ s = 10^{-2}, \ t \in [0, 1]\}.$$

where x_0 is the eigenvector of the eigenvalue $|\lambda| > 1$ of the solution of the variational equations taking initial condition the identity matrix. We observe, as in the previous chapter that I is a subset of the unstable invariant manifold (the linear approximation). Now we will define the following discretization of I:

$$x_n = x_0 + \frac{n}{N} sv_0, \quad 1 \le n \le N. \tag{28}$$

Now we define the following region:

$$\mathcal{T} = \{(r, v, \theta, w) \in \mathbb{R}^4 \mid \theta = \frac{\pi}{4}\} \cup \{(r, v, \theta, w) \in \mathbb{R}^4 \mid \theta = \frac{\pi}{2}\}.$$

We realize that the region $\mathcal{T}$ is the region of the $1-2-1$ and $2-2$ collisions defined before for $\alpha = 1$. Basically $\mathcal{T}$ defines all types of collisions except the total one. Taking $N = 600$ we will integrate each initial condition $x_n \in I$ up to the k-th crossing with a $\mathcal{T}$ and we will consider the value of $r(\theta - \theta_c)$. For doing that we implement the Poincaré map $P_{\mathcal{T}}$ defined on $\mathcal{T}$. Then, for a fixed k-th crossing with $\mathcal{T}$ we will represent each value $r(\theta - \theta_c)$ in terms of n (the index of the initial point x_n) and we will pay attention at the changes of sign of this plot. Concretely, for the particular example of the periodic orbit 2, $\alpha = 1$ and $h = -1$ applying the described methodology we obtain for $k = 16$ the following plot:

In this case we observe, taking into account the continuity with respect of the initial conditions of the solutions of the system (17) that between the elements x_{274} and x_{275} there must be an initial condition x^* such that

$$r(\theta - \theta_c) = 0 \tag{29}$$

in the 16-th intersection of the its orbit with $\mathcal{T}$. Here below we can a appreciate the change of signs of the expression $r(\theta - \theta_c)$ in the 16-th intersection with $\mathcal{T}$ of the orbits with initial condition x_{274} and x_{275}.

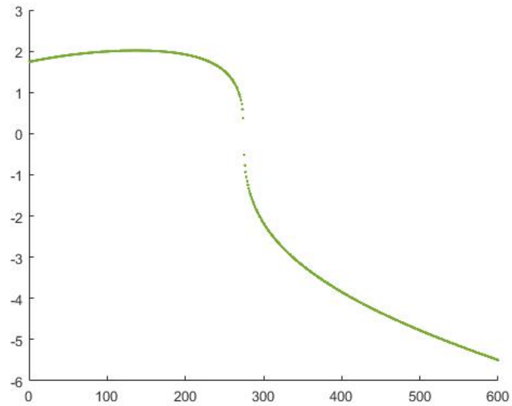

Figure 8: Value of $r(\theta - \theta_c)$ of the 16-th crossing with $\mathcal{T}$ of the orbits with initial condition in (28) in terms of the index n.

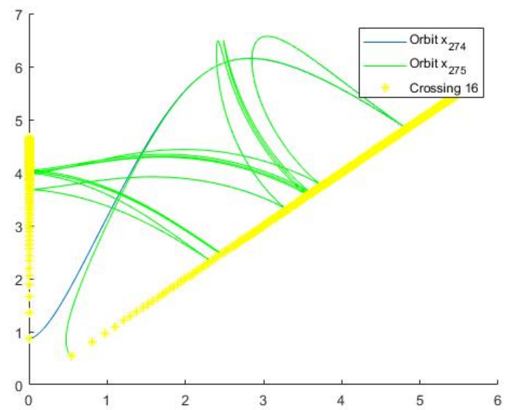

Figure 9: x_{274} and x_{275} orbits and the 16-th intersection of the elements of (28) with $\mathcal{T}$.

Looking at this plot we can appreciate intuitively that there must be an orbit with initial condition in the segment defined by x_{274} and x_{275} such that its limit achieve the total collision $(x, y) = (0, 0)$.

We will see that (29) implies $r = 0$. If not, it would imply that $\theta = \theta_c$ in the 16-th intersection of x^* with $\mathcal{T}$. Looking at the definition of $\mathcal{T}$ we see that θ must be either $\arctan \sqrt{\alpha} = \frac{\pi}{4}$ or $\frac{\pi}{2}$, both different that θ_c. Hence, $r = 0$ and of course, as $r = 0$ is an invariant manifold of the system, the time that takes x^* to achieve $\mathcal{T}$ is infinite. Therefore, we observe that the initial condition x^* defines a collision orbit. Here below we can appreciate the plot of the (x, y) projection of this orbit:

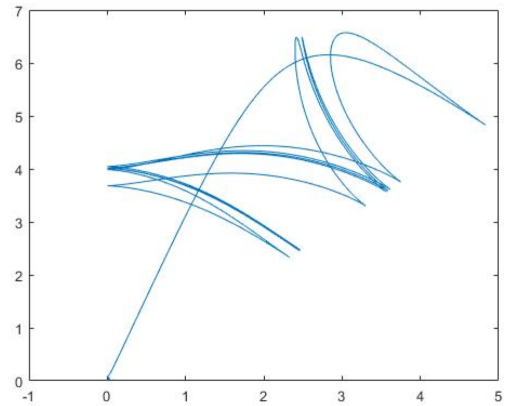

Figure 10: (x, y) projection of the connection orbit.

In fact doing the same process for the k-th crossing with $\mathcal{T}$ for $k = 20$ we obtain the following plot:

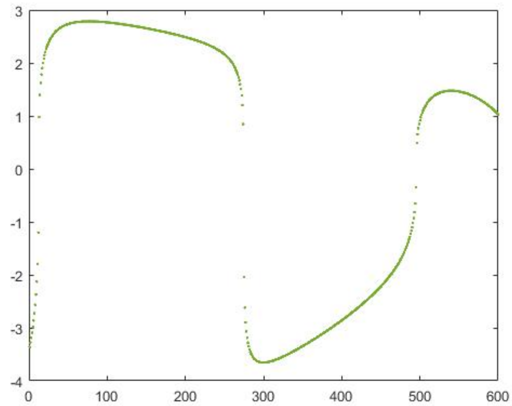

Figure 11: Value of $r(\theta - \theta_c)$ in the 20-th crossing with $\mathcal{T}$ in terms of the index n.

Looking at this plot we can ensure the presence of two more connection orbits with initial conditions in the fundamental interval I doing the same argument as before. Finally, applying the lemma 6.1 we obtain three different families of connection orbits depending on the first integral $h \in \mathbb{R}^*$.

7. Conclusions

In this master thesis we found four families of periodic orbits depending in $h \in \mathbb{R}^*$ and, related to one of them we found two families of homoclinic and connection orbits. For arriving to this conclusion we proceeded as follows: Firstly we deduced the equation of the system and we got rid of the singularities of it using the ideas of Devaney described in [6] and applied by Martínez-Simó in the article [3]. Therefore, based on the article of Sekiguchi-Tanikawa [1] we found four periodic orbits for a given value of $h = -1$ (conserved quantity) and $\alpha = 1$ (parameter of the system) using a suitable Poincaré map and extended the results for all $h \in \mathbb{R}^*$. Then, we described a methodology for finding homoclinic and connection orbits and we applied it to one periodic orbit. Finally, we obtained two different families of homoclinic and connection orbits depending in $h \in \mathbb{R}^*$. We remark that the discovery of these families of connection and homoclinic orbits involve our own contribution to the study of the collinear four body problem.

8. Numerical considerations

All the numerical simulations and calculations that we did in this thesis are implemented in Matlab. For integrating the differential equations we used the package ode45 (Explicit Runge Kutta method of orther 4). On the other hand, for implementing the Poincaré map we used the option "Events" of the package ode45 that allows us to fix an endpoint of and integration (in our case $\theta = \theta_c$). Finally, for finding zeros of functions we used the package "fsolve" for its simplicity and effectiveness. We remark that all the numerical results are obtained using double precision and an absolute and relative tolerances of 10^{-12}.

9. Codes

In this section we will introduce the most important codes that we used during this master thesis.

```
% System of ode r4bp
function [y] = f_2(t,x)
    global alpha;
    global h;
    r=x(1);
    v=x(2);
    theta=x(3);
    w=x(4);
    y = [r.*v.*cos(theta).*(sin(theta)-sqrt(alpha)*cos(theta))./sqrt(W(theta));
        sqrt(W(theta)).*(1+(2*r*h-(v.^2)/2).*cos(theta).*(sin(theta)-sqrt(alpha)*cos(
            theta))./W(theta));
        w;
      -(1/2)*v.*w.*cos(theta).*(sin(theta)-sqrt(alpha)*cos(theta))./sqrt(W(theta))+(cos
          (2*theta)+sqrt(alpha)*sin(2*theta)).*((2*r*h-v.^2).*cos(theta).*(sin(theta)-
          sqrt(alpha)*cos(theta))./W(theta)+1)+(W_prime(theta)./W(theta)).*(cos(theta)
          .*(sin(theta)-sqrt(alpha)*cos(theta))-(w.^2)/2);];
end
```

```
% Function W
function [y] = W(theta)
    global alpha;
    s=sin(theta);
    c=cos(theta);
  y = (sin(theta)-sqrt(alpha)*cos(theta))/sqrt(2)+(alpha^(5/2)*cos(theta)*(sin(theta)-
      sqrt(alpha)*cos(theta)))/(sqrt(2)*sin(theta))+2*sqrt(2)*alpha^(3/2)*cos(theta)+(2*
      sqrt(2)*alpha^(3/2)*cos(theta)*(sin(theta)-sqrt(alpha)*cos(theta)))/(sin(theta)+
      sqrt(alpha)*cos(theta));
end
```

```
% W' function
function [y] = W_prime(theta)
    global alpha;
    c=cos(theta);
    s=sin(theta);
    y= 1/sqrt(2)*(c+sqrt(alpha)*s)*(1+alpha^(5/2)*c/s)-alpha^(5/2)*(s-sqrt(alpha)*c)/((s
       ^2)*sqrt(2))-4*sqrt(2)*(s^2)/(s+sqrt(alpha)*c)+4*sqrt(2)*(alpha^2)*c/(s+sqrt(
       alpha)*c)^2;
end
```

```
% System variational equations
function [y]=var_f2(t,x)
  global alpha;
  global h;
  r=x(1);
  v=x(2);
  theta=x(3);
  w=x(4);
  s=sin(theta);
  c=cos(theta);
  z = [r.*v.*cos(theta).*(sin(theta)-sqrt(alpha)*cos(theta))./sqrt(W(theta));
```

```
        sqrt(W(theta)).*(1+(2*r*h-(v.^2)/2).*cos(theta).*(sin(theta)-sqrt(alpha)*cos(
            theta))./W(theta));
        w;
        -(1/2)*v.*w.*cos(theta).*(sin(theta)-sqrt(alpha)*cos(theta))./sqrt(W(theta))+(cos
            (2*theta)+sqrt(alpha)*sin(2*theta)).*((2*r*h-v.^2).*cos(theta).*(sin(theta)-
            sqrt(alpha)*cos(theta))./W(theta)+1)+(W_prime(theta)./W(theta)).*(cos(theta)
            .*(sin(theta)-sqrt(alpha)*cos(theta))-(w.^2)/2);
  ];
  A = [v*c*(s-sqrt(alpha)*c)/sqrt(W(theta)) r*c*(s-sqrt(alpha)*c)/sqrt(W(theta)) r*v*((
      cos(2*theta)+sqrt(alpha)*sin(2*theta))/sqrt(W(theta))-0.5*c*(s-sqrt(alpha)*c)*W(
      theta)^(-3/2)*W_prime(theta)) 0;
        2*h*c*(s-sqrt(alpha)*c)/sqrt(W(theta)) -v*c*(s-sqrt(alpha)*c)/sqrt(W(theta))
            W_prime(theta)/(2*sqrt(W(theta)))+(2*r*h-1/2*v^2)*c*(s-sqrt(alpha)*c)*W_prime(
            theta)/(2*W(theta)^(3/2))+(2*r*h-1/2*v^2)*((cos(2*theta)+sqrt(alpha)*sin(2*
            theta))/sqrt(W(theta))-c*(s-sqrt(alpha)*c)*W_prime(theta)/(W(theta)^(3/2))) 0;
        0 0 0 1;
       (cos(2*theta)+sqrt(alpha)*sin(2*theta))*2*h*c*(s-sqrt(alpha)*c)/W(theta) -0.5*w*c*(
           s-sqrt(alpha)*c)/sqrt(W(theta))-(cos(2*theta)+sqrt(alpha)*sin(2*theta))*2*v*c*(
           s-sqrt(alpha)*c)/W(theta) -0.5*v*w*(c*W_prime(theta)*(sqrt(alpha)*c-s)+2*W(
           theta)*(sqrt(alpha)*sin(2*theta)+cos(2*theta)))/(2*W(theta)^(3/2))+(-sin(2*
           theta)*2+sqrt(alpha)*cos(2*theta)*2)*((2*r*h-v^2)*c*(s-sqrt(alpha)*c)/W(theta)
           +1)+(cos(2*theta)+sqrt(alpha)*sin(2*theta))*(-W_prime(theta)/(W(theta)^2)*(2*r*
           h-v^2)*c*(s-sqrt(alpha)*c)+(2*r*h-v^2)*(cos(2*theta)+sqrt(alpha)*sin(2*theta))/
           W(theta))+(ddW(theta)*W(theta)-W_prime(theta)^2)/(W(theta)^2)*(c*(s-sqrt(alpha)
           *c)-1/2*w^2)+W_prime(theta)/W(theta)*(cos(2*theta)+sqrt(alpha)*sin(2*theta))
           -0.5*v*c*(s-sqrt(alpha)*c)/sqrt(W(theta))-w*W_prime(theta)/W(theta)];
  p=[x(5:8)';x(9:12)';x(13:16)';x(17:20)'];
  q=A*p;
  y=[z;q(1,:)';q(2,:)';q(3,:)';q(4,:)'];
end
```

```
% Second derivative of W
function [y] = ddW(theta)
    global alpha;
    s=sin(theta);
    c=cos(theta);
    y = -1/sqrt(2)*(s-sqrt(alpha)*c)*(1+alpha^(5/2)*c/s)+1/(sqrt(2))*(c+sqrt(alpha)*s)*(-
        alpha^(5/2)/s^2)-alpha^(5/2)/sqrt(2)*((c+sqrt(alpha)*s)/s^2-2*(s-sqrt(alpha)*c)*c
        /s^3)-8*sqrt(2)*alpha^(3/2)*s*c/(s+sqrt(alpha)*c)+4*sqrt(2)*alpha^(3/2)*s^2*(c-
        sqrt(alpha)*s)/(s+sqrt(alpha)*c)^2-4*sqrt(2)*alpha^2*s/(s+sqrt(alpha)*c)^2-8*sqrt
        (2)*alpha^2*c*(c-sqrt(alpha)*s)/(s+sqrt(alpha)*c)^3;
end
```

```
%  First integral h of the system of ode
function [y] = h2(x)
    global alpha;
    r = x(:,1);
    v = x(:,2);
    theta = x(:,3);
    w = x(:,4);
    y=(((w.^2-2*cos(theta).*(sin(theta)-sqrt(alpha)*cos(theta))).*W(theta))./((cos(theta)
        .^2).*(sin(theta)-sqrt(alpha)*cos(theta)).^2)+v.^2)./(2*r);
end
```

```
% Derivative of the function V
function [y] = dV(theta)
    global alpha;
    s=sin(theta);
    c=cos(theta);
    y= -s/(sqrt(2)*c^2)+alpha^(5/2)*c/(sqrt(2)*s^2)+2*sqrt(2)*alpha^(3/2)*(c+sqrt(alpha)*
        s)/((s-sqrt(alpha)*c)^2)+2*sqrt(2)*alpha^(3/2)*(c-sqrt(alpha)*s)/((s+sqrt(alpha)*
        c)^2);
end
```

```
% Plot Poincare Map section theta=theta_c
global alpha;
alpha=1;
global h;
h=-1;
global theta_c;
options1=optimoptions('fsolve','FunctionTolerance',1e-12);
theta_c=fsolve(@dV,pi/3,options1);
s=1;
p=cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c));
N=500;
iter=100;
B_c=2*p;
A_c=W(theta_c)/(p^2);
w=linspace(-sqrt(B_c),sqrt(B_c),1e6);
z1=w;
y1=zeros(1e6,1);
y2=zeros(1e6,1);
for i=1:1e6
   y1(i)=sqrt((1-(w(i)^2)/B_c)*A_c*B_c);
   y2(i)=-sqrt((1-(w(i)^2)/B_c)*A_c*B_c);
end
x=zeros(N,iter);
y=zeros(N,iter);
x(:,1)=linspace(-0.5,-0.1,N);
y(:,1)=zeros(N,1);
options2=odeset('AbsTol',1e-12,'RelTol',1e-12,'Events',@Evenfun);
options1=odeset('AbsTol',1e-12,'RelTol',1e-12);
hold on
plot(z1,y1,'b');
plot(z1,y2,'b');
for i=1:N
        t0=0;
        for k=1:iter-1
            r=W(theta_c)/(2*h*p^2)*(x(i,k)^2+y(i,k)^2*p^2/W(theta_c)-2*p);
            z=[r y(i,k) theta_c x(i,k)]';
            if pr>2*cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c))
                break
            end
            [tt,zz]=ode45(@f_2,[0 1e-1],z,options1);
            [t,zz]=ode45(@f_2,[1e-1 100],zz(end,:),options2);
            z=zz(end,:);
            r=z(1);
            pr=-(2*r*cos(theta_c)^2*(sin(theta_c)-sqrt(alpha)*cos(theta_c))^2)/W(theta_c)
                +2*cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c));
            if pr<=2*cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c))
```

```
                x(i,k+1)=z(4);
                y(i,k+1)=z(2);
                plot(x(i,k+1),y(i,k+1),'.b');
            end
        end
end
hold off
```

```
% Auxiliary function matlab (endpoint integration)
function [ value, isterminal,direction ] = Evenfun(t,y)
    global theta_c;
    value(1)=y(3)-theta_c;
    isterminal(1)=1;
    direction=[];
end
```

```
% Calculation unstable invariant manifold related to the periodic orbit 2 main
% alpha=1 h=-1
global alpha;
global h;
alpha=1;
h=-1;
global theta_c;
global v0;
options1=optimoptions('fsolve','FunctionTolerance',1e-13,'OptimalityTolerance',1e-13,'
    StepTolerance',1e-13);
theta_c=fsolve(@dV,pi/3,options1);
options2=odeset('Events',@Evenfun,'AbsTol',1e-12,'RelTol',1e-12);
options22=odeset('Events',@Evenfun2);
options3=odeset('AbsTol',1e-13,'RelTol',1e-13);
x3=theta_c;
intersec=50;
norb=1000;
s=1e-4;
% Initial condition periodic orbit 2
x0=[6.35315094974196;-1.07896084232869;1.26450466042599;0.333839047457243];
% Period
T=  15.154887900710179;
[tt,ZZ]=ode45(@var_f2,[0 T],[x0' 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1],options3);
B = [ZZ(end,5:8);ZZ(end,9:12);ZZ(end,13:16);ZZ(end,17:20)];
[V,D]=eig(B);
% Choose the eignevalue lambda, |lambda|>1 unstable
% Choose the eigenvalue lambda, |lambda|<1 stable
aux=abs(D(1,1));
k=1;
for i=1:4
    if abs(D(i,i))>aux
        aux=D(i,i);
        k=i;
    end
end
v0=V(:,k)/norm(V(:,k));
X=zeros(norb,25);
Y=zeros(norb,25);
titer=zeros(norb,25);
p=cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c));
```

```
B_c=2*p;
A_c=W(theta_c)/(p^2);
w=linspace(-sqrt(B_c),sqrt(B_c),1e6);
y1=zeros(1e6,1);
y2=zeros(1e6,1);
X=zeros(norb,25);
Y=zeros(norb,25);
 for i=1:1e6
   y1(i)=sqrt((1-(w(i)^2)/B_c)*A_c*B_c);
   y2(i)=-sqrt((1-(w(i)^2)/B_c)*A_c*B_c);
 end
 hold on
 plot(w,y1,'b'),
 plot(w,y2,'b');
 tol=1e-11;
 % Primera branca
for i=1:norb
    x_ini=x0+s*i/norb*v0;
    x=x_ini;
    h=h2(x_ini');
    for j=1:50
        [ttt,xxx]=ode45(@f_2,[0 100],x,options2);
        x=xxx(end,:);
        X(i,j)=x(4);
        Y(i,j)=x(2);
        if abs(x(3)-theta_c)<tol
            plot(x(4),x(2),'.r');
        else
            break;
        end
    end
end
% Segona branca
for i=1:norb
    x_ini=x0-s*i/norb*v0;
    x=x_ini;
    h=h2(x_ini');
    t0=0;
    for j=1:50
        [ttt,xxx]=ode45(@f_2,[0 10000],x,options2);
        x=xxx(end,:);
        if abs(x(3)-theta_c)<tol
            plot(x(4),x(2),'.r');
        else
            break;
        end
    end
end
hold off
```

```
% Homoclinic orbit main
global alpha;
global h;
alpha=1;
h=-1;
global theta_c;
global v0;
```

```
options1=optimoptions('fsolve','FunctionTolerance',
1e-13,'OptimalityTolerance',1e-13,'StepTolerance',1e-13);
theta_c=fsolve(@dV,pi/3,options1);
options2=odeset('Events',@Evenfun3,'AbsTol',1e-11,'RelTol',1e-11);
options22=odeset('Events',@Evenfun2);
options3=odeset('AbsTol',1e-13,'RelTol',1e-13);
x3=theta_c;
intersec=25;
norb=600;
s=1e-2;
x0=[6.35315094974196;-1.07896084232869;1.26450466042599;0.333839047457243];
T=  15.154887900710179;
[tt,ZZ]=ode45(@var_f2,[0 T],[x0' 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1],options3);
B = [ZZ(end,5:8);ZZ(end,9:12);ZZ(end,13:16);ZZ(end,17:20)];
[V,D]=eig(B);
v0=V(:,1)/norm(V(:,1));
hold on;
X=zeros(norb,25);
Y=zeros(norb,25);
titer=zeros(norb,25);
tol=1e-7;
options1=optimoptions('fsolve','FunctionTolerance',1e-13,'OptimalityTolerance',1e-13,'
    StepTolerance',1e-13);
x_ini=x0+s*380/norb*v0;
h=h2(x_ini');
x=fsolve(@findhomo,380,options1);
x_ini=x0+s*x/norb*v0;
t=0;
for j=1:13
[tt1,xx1]=ode45(@f_2,[0 100],x_ini,options2);
x_ini=xx1(end,:);
t=t+tt1(end);
end
for i=1:norb
    x_ini=x0+s*i/norb*v0;
    x=x_ini;
    h=h2(x_ini');
    t0=0;
    for j=1:intersec
        [tt1,xx1]=ode45(@f_2,[0 100],x,options2);
        x=xx1(end,:);
        t0=t0+tt1(end);
        X(i,j)=x(2);
        titer(i,j)=t0;
    end
end
hold on
for i=1:intersec
    plot([1:norb],X(:,i),'.');
end
hold off;
```

```
% Function for finding the zero for the Homoclinic orbit
function [y] = findhomo(t)
global alpha;
global h;
global v0;
s=1e-2;
intersec=18;
x0=[6.35315094974196;-1.07896084232869;1.26450466042599;0.333839047457243];
options2=odeset('Events',@Evenfun3,'AbsTol',1e-11,'RelTol',1e-11);
x=x0+s*v0*t/600;
h=h2(x');
for i=1:intersec
    [tt1,xx1]=ode45(@f_2,[0 100],x,options2);
    x=xx1(end,:);
end
y=x(2);
end
```

```
% Find periodic orbits main

% Example of initial points of the search
% Initial points search Schubart like periodic orbit
%x0=[0.2,0.1]';
% Initial points search unstable periodic orbit 4
%x0=[-0.35,1.3]';
global alpha;
global h;
alpha=1;
h=-1;
global theta_c;
options1=optimoptions('fsolve','FunctionTolerance',1e-13,'OptimalityTolerance',1e-13,'
    StepTolerance',1e-13);
theta_c=fsolve(@dV,pi/3,options1);
options2=odeset('AbsTol',1e-13,'RelTol',1e-13,'Events',@Evenfun);
options3=odeset('AbsTol',1e-13,'RelTol',1e-13);
x=fsolve(@Po,x0,options1);
x3=theta_c;
p=cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c));
r=W(theta_c)/(2*h*p^2)*(x(1)^2+x(2)^2*p^2/W(theta_c)-2*p);
% Initial condition periodic orbit
x=[r x(2) theta_c x(1)]';
% Find period T
%i=Crossing with the section theta=theta_c
T=0;
for i=1:4 %2 or 16 (depending on the p.o)
    [tt,xx]=ode45(@f_2,[0 1e-2],x,options3);
    T=T+tt(end);
    [tt,xx]=ode45(@f_2,[1e-2 100],xx(end,:),options2);
    T=T+tt(end);
    x=xx(end,:);
end
```

```
% Find periodic orbit with 16, 4 or 2 intersections with theta_c per period
function [y] = Po(x)
    global theta_c;
    global alpha;
```

```
    global h;
    t0=0;
    options2=odeset('AbsTol',1e-13,'RelTol',1e-13,'Events',@Evenfun);
    options1=odeset('AbsTol',1e-13,'RelTol',1e-13);
    x3=theta_c;
    p=cos(theta_c)*(sin(theta_c)-sqrt(alpha)*cos(theta_c));
    r=W(theta_c)/(2*h*p^2)*(x(1)^2+x(2)^2*p^2/W(theta_c)-2*p);
    z0=[r x(2) x3 x(1)]';
    for i=1:4
         [t,zz]=ode45(@f_2,[t0 t0+1e-2],z0,options1);
         t1=t(end);
         z1=zz(end,:);
        [t,zz]=ode45(@f_2,[t1 1e20+t1],z1',options2);
        t0=t(end);
        z0=zz(end,:);
    end
    y=[z0(4),z0(2)]-[x(1),x(2)];
end
```

```
% Plot connection orbit main
global alpha;
global h;
alpha=1;
h=-1;
global theta_c;
options1=optimoptions('fsolve','FunctionTolerance',1e-13,'OptimalityTolerance',1e-13,'
    StepTolerance',1e-13);
theta_c=fsolve(@dV,pi/3,options1);
options2=odeset('Events',@Evenfun3,'AbsTol',1e-11,'RelTol',1e-11);
options22=odeset('Events',@Evenfun2);
options3=odeset('AbsTol',1e-13,'RelTol',1e-13);
x3=theta_c;
% Crossings
intersec=20;
% Number of elements of the discretization
norb=600;
s=1e-2;
% Initial condition periodic orbit 2
x0=[6.35315094974196;-1.07896084232869;1.26450466042599;0.333839047457243];
% Period
T=  15.154887900710179;
[tt,ZZ]=ode45(@var_f2,[0 T],[x0' 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1],options3);
B = [ZZ(end,5:8);ZZ(end,9:12);ZZ(end,13:16);ZZ(end,17:20)];
[V,D]=eig(B);
v0=V(:,1)/norm(V(:,1));
hold on;
X=zeros(norb,22);
Y=zeros(norb,22);
titer=zeros(norb,22);
tol=1e-7;
for i=1:norb
    x_ini=x0+s*i/norb*v0;
    x=x_ini;
    h=h2(x_ini');
    t0=0;
    for j=1:intersec
        [tt1,xx1]=ode45(@f_2,[0 100],x,options2);
```

```
        x=xx1(end,:);
        t0=t0+tt1(end);
        while abs(x(3)-pi/2)>tol && abs(x(3)-pi/4)>tol
              [tt1,xx1]=ode45(@f_2,[0 100],x,options2);
              t0=t0+tt1(end);
              x=xx1(end,:);
        end
        X(i,j)=x(1);
        Y(i,j)=x(3);
        titer(i,j)=t0;
    end
end
hold on
for i=1:intersec
    plot([1:norb],X(:,i).*(Y(:,i)-theta_c*ones(1,norb)),'.');
end
hold off;
```

```
% Auxiliary matlab function. Stoping time integration
function [ value, isterminal,direction ] = Evenfun3(t,y)
    global alpha;
    value(1)=y(4);
    isterminal(1)=1;
    direction=0;
end
```

```
% Auxiliary matlab function. Stoping time integration 2
function [ value, isterminal,direction ] = Evenfun2(t,y)
    value(1)=y(3)-pi/2;
    isterminal(1)=1;
    direction=[];
end
```

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