A robust LMI approach on nonlinear feedback stabilization of continuous state-delay systems with Lipschitzian nonlinearities

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Abstract: This paper suggests a novel nonlinear state feedback stabilization control law using linear matrix inequalities for a class of time-delayed nonlinear dynamic systems with Lipschitz nonlinearity conditions. Based on the Lyapunov-Krasovskii stability theory, the asymptotic stabilization criterion is derived in the linear matrix inequality form and the coefficients of the nonlinear state-feedback controller are determined. Meanwhile, an appropriate criterion to find the proper feedback gain matrix F is also provided. The robustness purpose against nonlinear functions and time-delays is guaranteed in this scheme. Moreover, the problem of robust H_∞ performance analysis for a class of nonlinear time-delayed systems with external disturbance is studied in this paper. Simulations are presented to demonstrate the proficiency of the offered technique. For this purpose, an unstable nonlinear numerical system and a rotary inverted pendulum system have been studied in the simulation section. Moreover, an experimental study of the practical rotary inverted pendulum is provided. These results confirm the expected satisfactory performance of the suggested method.

Keywords: Nonlinear feedback stabilization, linear matrix inequalities, Lipschitz nonlinearity, Lyapunov-Krasoviskii functional, time-delay.

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1. Introduction

Time delays are often sources of instability and degradation of system efficiency in many control systems and are frequently encountered in a wide range of nonlinear dynamical systems, such as pneumatic systems [1], chemical engineering [2], hydraulic systems [3], biological systems [4], nuclear reactors [5] and population dynamics models [6]. The problem of stabilization of the time-delayed dynamical systems and synthesis of controllers for them has received a significant attention over the past years and different approaches have been proposed. Nevertheless, the offered methodologies remain restrictive to the specific classes of nonlinear systems, and there is any general technique to analyze and synthesize the general class of nonlinear systems [7-9]. This is the purpose of the current investigation on the analysis and control of the time-delayed nonlinear systems. For this purpose, selection of the predefined variables using a powerful computational design tool such as linear matrix inequality (LMI) technique is required.

LMIs have developed as an influential structure and design procedure for various control problems [10]. In the past years, this method has been applied to find solutions of minimization convex problems, for instance, H_2 control [11], H_∞ control [12] and guaranteed cost control [13]. Even though LMI is a convex optimization problem, such structure offers a numerically tractable mean for hard problems in the absence of analytical solutions. Moreover, some effective interior-point algorithms are now available to solve LMI problems. In [14], an LMI-based H_∞ state feedback stabilization problem for the uncertain switched impulsive linear systems with state-delays and nonlinear parametric uncertainties is proposed. In [15], a robust H_∞ fuzzy control method for TS-fuzzy time-delayed discrete-time bilinear systems with disturbances is proposed where the conditions of the system stability are formulated in the form of LMIs. In [16], the problem of stabilization analysis using LMIs and robust H_∞ controller design for time-delayed systems with stochastic disturbances and parametric uncertainties is investigated. In [17], an LMI-based Robust H_∞ state-feedback controller for the uncertain discrete-time systems with state-delays is proposed. In [18], using a quadratic Lyapunov functional and variation of parameters method, the problem of LMI-based delay-dependent BIBO stabilization control for the uncertain time-delayed systems is investigated. In [19], a nonlinear matrix inequality is employed as a stabilization condition for uncertain time-delayed linear systems. In [20], the robust exponential stabilization problem based on the Lyapunov parameter-dependent function and LMIs for a class of uncertain systems with time-varying delays is investigated. In [21], the stabilization problem of a two-dimensional Burgers equation around a stationary solution using nonlinear feedback boundary controller is investigated. In [22], the stabilization problem of uniform Euler-Bernoulli beam via nonlinear locally-distributed feedback controller is studied where the energy of the beam decays exponentially. In [23], chaotification technique based on the nonlinear time-delayed feedback control method for a two-dimensional vibration isolation floating raft structure is presented. In [24], the synchronization problem of the uncertain time-delay chaotic systems with the unknown inputs in a drive-response framework using robust adaptive observer-based controller is investigated. In [25], the nonlinear vibration control problem of the active vehicle suspension systems with the actuator delays using feedback linearization technique is studied. In [26], the impact of delays on the self-excited oscillations of single and two degrees of freedom systems via nonlinear feedback is considered and a bounded saturated feedback control technique with controllable time-delays is suggested to induce the self-excited oscillations. To the best of the author's information, very little attention has been paid for the nonlinear state-feedback stabilization problem of time-delayed nonlinear systems with Lipschitz nonlinearities using LMIs, which is still an open problem. This stimulates the present research.

Motivated by the above discussion, the problem of robust H_∞ performance analysis for a class of nonlinear systems with state-delay and external disturbance is investigated in this paper. This work presents a state feedback control law for the stability problem of Lipschitz nonlinear time-delayed systems. By constructing a Lyapunov-Krasoviskii functional, asymptotical stabilization conditions are prepared in LMI form and the coefficients of the nonlinear state-feedback control law are determined via LMIs. The proposed controller guarantees asymptotical stability of these systems even if the nonlinear part is non-zero. Unlike the former researches, the resultant LMI conditions have fewer pre-assumed design parameters, and consequently, the planned technique can yield less conservative conditions.

The presentation of this article is listed as follows: Sect. 2 develops the problem description and some required preliminaries. Sect. 3 proposes the analysis of the stability and design process of nonlinear state feedback controller based on LMIs for the nonlinear time-delayed systems. In Sect. 4, simulation results on two dynamical systems are illustrated. Moreover, experimental results on a rotary inverted pendulum (RIP) system are shown in Sect. 4. Finally, some concluding remarks are given in Sect. 5.

2. Problem description and required preliminaries

The nonlinear time-delayed system is considered as:

$$\begin{aligned} \dot{x}(t) &= f(x) + Ax(t) + A_1 x(t-\tau) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad (1)$$

where $u(t) \in R^n$, $x(t) \in R^n$, and $y(t) \in R^n$ represent the input to system, the state variables and the output of the system, respectively. The parameter τ is the time-delay and the nonlinear function $f(x) \in R^n$ is a time-varying vector. Moreover, matrices A, A_1, B, and C are some constant matrices with appropriate dimensions.

Assumption 1. The nonlinear function $f(x)$ is Lipschitz for all $x \in R^n$ and $\bar{x} \in R^n$ which satisfies [27, 28]:

$$\left\| f(x) - f(\bar{x}) \right\| \leq \left\| L(x - \bar{x}) \right\|, \quad (2)$$

where $L \in R^{n\times n}$ is a Lipschitz constant matrix. Equivalently, the Lipschitz inequality (2) is re-written as follows:

$$\left(f(x)-f(0)\right)^T I\left(f(x)-f(0)\right) \le x^T L^T Lx. \tag{3}$$

The nonlinear state feedback control input is specified by:

$$u(t) = Fx(t) - B^{-1} f(0), \tag{4}$$

where F is the state-feedback gain which will be calculated later using LMIs. The additional term $B^{-1} f(0)$ in (4) is necessary so that deal with systems possessing $f(0) \ne 0$.

Remark 1. The matrix B in (4) is assumed to be square and of full row rank. When the matrix B is non-square and has full rank, the nonlinear state-feedback control law can be expressed using the right inverse of B as:

$$u(t) = Fx(t) - B^T \left(BB^T\right)^{-1} f(0). \tag{5}$$

Therefore, this approach can be applied also on the situations in which the matrix B is non-square.

Lemma 1 (Schur complement) [29]. If there exist matrices S_1, S_2 and S_3 where $S_1 = S_1^T$ and $S_3 = S_3^T$, then the inequality

$$S_1 - S_2 S_3^{-1} S_2^T < 0 \tag{6}$$

is equivalent to:

$$\begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} < 0 \tag{7}$$

Lemma 2 (S-procedure) [30]. Let $T_0, \dots, T_p$ $T_0, \dots, T_p$ be symmetric matrices. Consider the following condition on $T_0, \dots, T_p$:

$$\zeta^T T_0 \zeta > 0 \text{ for all } \zeta \neq 0 \; \zeta \neq 0 \text{ such that } \zeta^T T_i \zeta \geq 0,\; i = 1, \ldots, p. \tag{8}$$

If there exist non-negative scalars τ_i for $i = 1, \ldots, p$, such that $T_0 - \sum_{i=1}^{p} \tau_i T_i > 0$ $T_0 - \sum_{i=1}^{p} \tau_i T_i > 0$is satisfied, then (8) holds.

3. Main results

Theorem 1. Consider the nonlinear system (1) and the control input (4) with $f(0) = 0$. Given a positive scalar τ_1, if there exist matrices M, $Q = Q^T > 0$ and $H = H^T > 0$ with appropriate dimensions such that the following LMI condition is satisfied:

$$\begin{bmatrix} QA^T + M^T B^T + AQ + BM + H & A_1 & \beta_1 I & QL^T \\ * & -P_1 & 0 & 0 \\ * & * & -\beta_1 I & 0 \\ * & * & * & -\beta_1 I \end{bmatrix} < 0, \tag{9}$$

then the asymptotical stability of the state trajectories is fulfilled and one can obtain the gain matrix F as $F = MQ^{-1}$.

Proof. If Eq. (4) (without $-B^{-1}f(0)$) is substituted into (1), we obtain:

$$\dot{x}(t) = f(x) + (A + BF)x(t) + A_1 x(t - \tau). \tag{10}$$

We construct the Lyapunov-Krasovsky function candidate as follows:

$$V(t) = x(t)^T P x(t) + \int_{t-\tau}^{t} x(s)^T P_1 x(s) ds, \tag{11}$$

with real symmetric matrices $P > 0$ and $P_1 > 0$ which are determined using the LMI. The derivative of (11) with respect to time is derived as:

$$\dot{V}(t) = \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) + x(t)^T P_1 x(t) - x(t-\tau)^T P_1 x(t-\tau), \quad (12)$$

Substituting (10) in (12) obtains:

$$\begin{aligned}\dot{V}(t) = f(x)^T Px(t) + x(t)^T (A^T + F^T B^T)Px(t) + x(t-\tau)^T A_1^T Px(t) + x(t)^T Pf(x) \\ + x(t)^T P(A+BF)x(t) + x(t)^T PA_1 x(t-\tau) + x(t)^T P_1 x(t) - x(t-\tau)^T P_1 x(t-\tau),\end{aligned} \quad (13)$$

which can be rewritten by:

$$\dot{V}(t) = \begin{pmatrix} x(t) \\ x(t-\tau) \\ f(x) \end{pmatrix}^T \begin{bmatrix} P(A+BF) + (A^T + F^T B^T)P + P_1 & PA_1 & P \\ * & -P_1 & 0 \\ * & * & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ x(t-\tau) \\ f(x) \end{pmatrix} < 0. \quad (14)$$

Note that the condition (3) can be restated as:

$$\begin{pmatrix} x(t) \\ x(t-\tau) \\ f(x) \end{pmatrix}^T \begin{bmatrix} L^T L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{pmatrix} x(t) \\ x(t-\tau) \\ f(x) \end{pmatrix} \geq 0. \quad (15)$$

By combining (14) and (15) with S-procedure (Lemma 1), the condition $\dot{V}(t) < 0$ is satisfied if there exist a scalar τ_1 such that:

$$\begin{bmatrix} P(A+BF) + (A^T + F^T B^T)P + P_1 + \tau_1 L^T L & PA_1 & P \\ * & -P_1 & 0 \\ * & * & -\tau_1 I \end{bmatrix} < 0. \quad (16)$$

Since the inequality (16) is not in the form of LMIs, assuming $Q = P^{-1}$, $M = FQ$, $P_1 = PHP$, and pre- and post- multiplying (16) by $diag(Q, I, \tau^{-1} I)$ yields:

$$\begin{bmatrix} QA^T + M^T B^T + AQ + BM + H + \tau_1 QL^T LQ & A_1 & \tau_1^{-1} I \\ * & -P_1 & 0 \\ * & * & -\tau_1^{-1} I \end{bmatrix} < 0. \quad (17)$$

Inequality (17) can be rewritten in the form of (6) as

$$\begin{bmatrix} QA^T + M^T B^T + AQ + BM + H & A_1 & \tau_1^{-1} I \\ * & -P_1 & 0 \\ * & * & -\tau_1^{-1} I \end{bmatrix} - \begin{bmatrix} QL^T \\ 0 \\ 0 \end{bmatrix} (-\tau_1 I) \begin{bmatrix} LQ & 0 & 0 \end{bmatrix} < 0. \tag{18}$$

By applying Schur complement on (18), the following inequality is obtained:

$$\begin{bmatrix} QA^T + M^T B^T + AQ + BM + H & A_1 & \tau_1^{-1} I & QL^T \\ * & -P_1 & 0 & 0 \\ * & * & -\tau_1^{-1} I & 0 \\ * & * & * & -\tau_1^{-1} I \end{bmatrix} < 0, \tag{19}$$

where defining $\beta_1 = \tau_1^{-1}$, LMI (9) is attained. This completes the proof. □

Theorem 2. Let consider the nonlinear time-delayed system (1) and the proposed control input (4). Assume that Assumption 1 is fulfilled. If there exist matrices M, $Q = Q^T > 0$, and $H = QP_1Q$ with appropriate dimensions such that:

$$\Pi = \begin{pmatrix} QA^T + M^T B^T + AQ + BM + H & A_1 Q & I & QL^T \\ * & -H & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0, \tag{20}$$

is fulfilled, then the control signal (4) confirms the asymptotical stability of states of the considered system and we can obtain F in (4) as $F = MQ^{-1}$.

Proof. If (4) is substituted into (1), one can achieve:

$$\dot{x}(t) = f(x) - f(0) + (A + BF)x(t) + A_1 x(t - \tau). \tag{21}$$

The Lyapunov-Krasovsky functional candidate is constructed as (11). The derivative of (11) with respect to time is derived as (12). Substituting (21) into (12) gives:

$$\dot{V}(t) = (f(x) - f(0))^T P x(t) + x(t)^T (A^T + F^T B^T) P x(t) + x(t-\tau)^T A_1^T P x(t) + x(t)^T P (f(x) - f(0)) + x(t)^T P_1 x(t) - x(t-\tau)^T P_1 x(t-\tau) + x(t)^T P (A + BF) x(t) + x(t)^T P A_1 x(t-\tau), \quad (22)$$

where considering (3) and (22), one can obtain:

$$\dot{V} \le (f(x) - f(0))^T P x(t) + x(t)^T (A^T + F^T B^T) P x(t) + x(t-\tau)^T A_1^T P x(t) + x(t)^T P (f(x) - f(0)) + x(t)^T P (A + BF) x(t) + x(t)^T P A_1 x(t-\tau) + x(t)^T P_1 x(t) - x(t-\tau)^T P_1 x(t-\tau) - (f(x) - \mathrm{f}(0))^T I (f(x) - \mathrm{f}(0)) + x(t)^T L^T L x(t), \quad (23)$$

which, further, can be written as:

$$\dot{V} \le \Psi^T \Gamma \Psi, \quad (24)$$

where

$$\Psi = \begin{bmatrix} x(t)^T & x(t-\tau)^T & (f(x) - f(0))^T \end{bmatrix}^T, \quad (25)$$

$$\Gamma = \begin{pmatrix} (A^T + F^T B^T) P + P(A + BF) + L^T L + P_1 & PA_1 & P \\ * & -P_1 & 0 \\ * & * & -I \end{pmatrix} < 0. \quad (26)$$

Inequality (26) can be written in the form of (6) as

$$\begin{pmatrix} (A^T + F^T B^T) P + P(A + BF) + P_1 & PA_1 & P \\ * & -P_1 & 0 \\ * & * & -I \end{pmatrix} - \begin{pmatrix} L^T \\ 0 \\ 0 \end{pmatrix} (-I) \begin{pmatrix} L & 0 & 0 \end{pmatrix} < 0. \quad (27)$$

Now, applying the Schur complement on (27) yields:

$$\begin{pmatrix} (A^T + F^T B^T) P + P(A + BF) + P_1 & PA_1 & P & L^T \\ * & -P_1 & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0. \quad (28)$$

Since the inequality (28) is non-LMI, assuming $Q = P^{-1}$, $M = FQ$, $H = QP_1Q$, and pre- and post-multiplying (28) by $\mathrm{diag}(Q, Q, I, I)$, LMI (20) is obtained. □

In what follows, the asymptotic stability and H_∞ performance analysis of system (1) with external disturbance are considered. Then, considering $u(t) = 0$, we have:

$$\begin{aligned} \dot{x}(t) &= f(x) + Ax(t) + A_1 x(t-\tau) + E_\omega \omega(t), \\ y(t) &= Cx(t), \end{aligned} \tag{29}$$

where $\omega(t)$ denotes the external disturbance and E_ω represent the constant matrix with suitable dimension.

Definition 1. The perturbed time-delayed system (29) is said to be robustly asymptotically stable with an H_∞ disturbance attenuation $\gamma > 0$ if the system (29) with $\omega(t) = 0$ is robustly stable and under zero initial condition, there exists:

$$\int_0^\infty \|y(t)\|^2 dt \le \gamma^2 \int_0^\infty \|\omega(t)\|^2 dt. \tag{30}$$

Theorem 3. Consider the nonlinear perturbed time-delayed system (29). Suppose that Assumption 1 is guaranteed. If there exist matrices $Q = Q^T > 0$, and $H = QP_1Q$ with suitable dimensions such that the LMI condition:

$$\begin{pmatrix} QA^T + AQ + H & A_1Q & I & E_\omega & QL^T & QC^T \\ * & -H & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{pmatrix} < 0 \tag{31}$$

is satisfied, then the perturbed time-delayed system (29) is asymptotically stable and fulfills the H_∞ performance condition (30).

Proof. The Lyapunov-Krasovsky candidate function is defined as (11). Substituting (29) into the time-derivative of the Lyapunov-Krasovsky function gives:

$$\begin{aligned}\dot{V}(t) = & f(x)^T P x(t) + x(t)^T P f(x) + x(t)^T \left[A^T P + PA + P_1\right] x(t) \\ & + x(t-\tau)^T A_1^T P x(t) + x(t)^T P A_1 x(t-\tau) - x(t-\tau)^T P_1 x(t-\tau) \\ & + x(t)^T P E_\omega \omega(t) + \omega(t)^T E_\omega^T P x(t).\end{aligned} \tag{32}$$

Now, considering (3) and (32), we have:

$$\begin{aligned}\dot{V} \le & f(x)^T P x(t) + x(t)^T P f(x) + x(t)^T \left[A^T P + PA + P_1\right] x(t) \\ & + x(t-\tau)^T A_1^T P x(t) + x(t)^T P A_1 x(t-\tau) - x(t-\tau)^T P_1 x(t-\tau) \\ & + x(t)^T P E_\omega \omega(t) + \omega(t)^T E_\omega^T P x(t) - f(x)^T I f(x) + x(t)^T L^T L x(t).\end{aligned} \tag{33}$$

The H_∞ disturbance attenuation in (30) can be written as:

$$x(t)^T C^T C x(t) \le \gamma^2 \omega(t)^T \omega(t). \tag{34}$$

From (33) and (34), one can obtain:

$$\dot{V} + x(t)^T C^T C x(t) - \gamma^2 \omega(t)^T \omega(t) \le \Omega^T \Theta \Omega \tag{35}$$

where using (33)-(35) gives:

$$\begin{aligned}& x(t)^T \left[A^T P + PA + P_1 + C^T C + L^T L\right] x(t) + f(x)^T P x(t) + x(t)^T P f(x) \\ & + x(t-\tau)^T A_1^T P x(t) + x(t)^T P A_1 x(t-\tau) - x(t-\tau)^T P_1 x(t-\tau) \\ & + x(t)^T P E_\omega \omega(t) + \omega(t)^T E_\omega^T P x(t) - f(x)^T I f(x) - \gamma^2 \omega(t)^T \omega(t) \le \Omega^T \Theta \Omega.\end{aligned} \tag{36}$$

where

$$\Omega = \left[x(t)^T \quad x(t-\tau)^T \quad f(x)^T \quad \omega(t)^T\right]^T, \tag{37}$$

$$\Theta = \begin{pmatrix} A^T P + PA + C^T C + P_1 + L^T L & PA_1 & P & PE_\omega \\ * & -P_1 & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\gamma^2 I \end{pmatrix} < 0 \tag{38}$$

Inequality (38) can be written in the form of (6) as

$$\begin{pmatrix} A^T P + PA + C^T C + P_1 + L^T L & PA_1 & P & PE_\omega \\ * & -P_1 & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\gamma^2 I \end{pmatrix} - \begin{pmatrix} L^T \\ 0 \\ 0 \\ 0 \end{pmatrix} (-I) \begin{pmatrix} L & 0 & 0 & 0 \end{pmatrix} < 0. \tag{39}$$

Applying the Schur complement on (39) gives:

$$\begin{pmatrix} A^T P + PA + C^T C + P_1 & PA_1 & P & PE_w & L^T \\ * & -P_1 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -I \end{pmatrix} < 0. \tag{40}$$

Similarly, if the Schur complement is applied on (40), one obtains:

$$\begin{pmatrix} A^T P + PA + P_1 & PA_1 & P & PE_\omega & L^T & C^T \\ * & -P_1 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{pmatrix} < 0. \tag{41}$$

Since Eq. (40) is non-LMI, assuming $Q = P^{-1}$, $H = QP_1Q$, and pre- and post- multiplying (40) by diag(Q, Q, I, I, I, I), LMI (31) is achieved. □

4. Simulation results

To illustrate the usefulness of the planned method, two simulation examples are considered. In *Example A*, an unstable nonlinear numerical system with state-delays is proposed. In *Example B*, the proposed control technique is applied on a practical RIP system with state-delays and nonlinearities.

Example A: Unstable Nonlinear Numerical System

The differential equations of this system are considered as:

$$\dot{x} = \begin{pmatrix} 0.2\cos x_1 \\ 0.3\sqrt{x_2^2+5} \\ 0.4\sin x_3 \end{pmatrix} + \begin{pmatrix} 0.1 & 0.2 & 0.3 \\ 0.1 & 0.1 & -0.5 \\ 0.3 & -0.4 & -0.3 \end{pmatrix} x + \begin{pmatrix} 0 & 0.01 & -0.01 \\ 0.05 & -0.04 & 0 \\ 0.01 & 0.02 & -0.03 \end{pmatrix} x(t-\tau)$$
$$+ \begin{pmatrix} 0.012 & 0.013 & 0.014 & 0.016 \\ 0.01 & 0.014 & 0.01 & 0 \\ 0.013 & 0.017 & 0.018 & 0.011 \end{pmatrix} u, \tag{42}$$
$$y = \begin{pmatrix} 1.5 & 2 & 1.25 \\ 0.84 & 0.5 & 0.2 \end{pmatrix} x.$$

For simulation, the initial states and time-delay value are initialized as: $x(0) = [10 \quad 4 \quad -6]^T$, $\tau = 2$. The Lipschitzian matrix is specified by: $L = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$. The solutions of LMI (20) are attained using LMI® toolbox in Matlab® software and YALMIP® solver as:

$$H = \begin{bmatrix} 0.0141 & 0.0316 & 0.0262 \\ 0.0316 & 0.1419 & 0.0649 \\ 0.0262 & 0.0649 & 0.1196 \end{bmatrix}, \; P = \begin{bmatrix} 17.238 & -4.7564 & -10.5622 \\ -4.7564 & 27.9164 & -20.9467 \\ -10.5622 & -20.9467 & 39.3667 \end{bmatrix}.$$

$$P_1 = \begin{bmatrix} 12.5486 & 1.6137 & -25.1278 \\ 1.6137 & 84.36 & -86.4733 \\ -25.1278 & -86.4733 & 134.515 \end{bmatrix}, \; F = \begin{bmatrix} -90.2711 & -93.498 & 189.909 \\ -74.8967 & -56.8381 & 75.6911 \\ 100.6461 & 107.5033 & -216.382 \\ -3.705 & 46.7269 & -7.824 \end{bmatrix}.$$

Figure 1 displays the states of the differential equations of system (42) by using the nonlinear state feedback controller (4). All of the state trajectories are appropriately convergent to the origin. The output responses of the system are demonstrated in Figure 2. Therefore, the simulations are robust in the presence of time-delays and indicate satisfactory and reasonable performance as well.

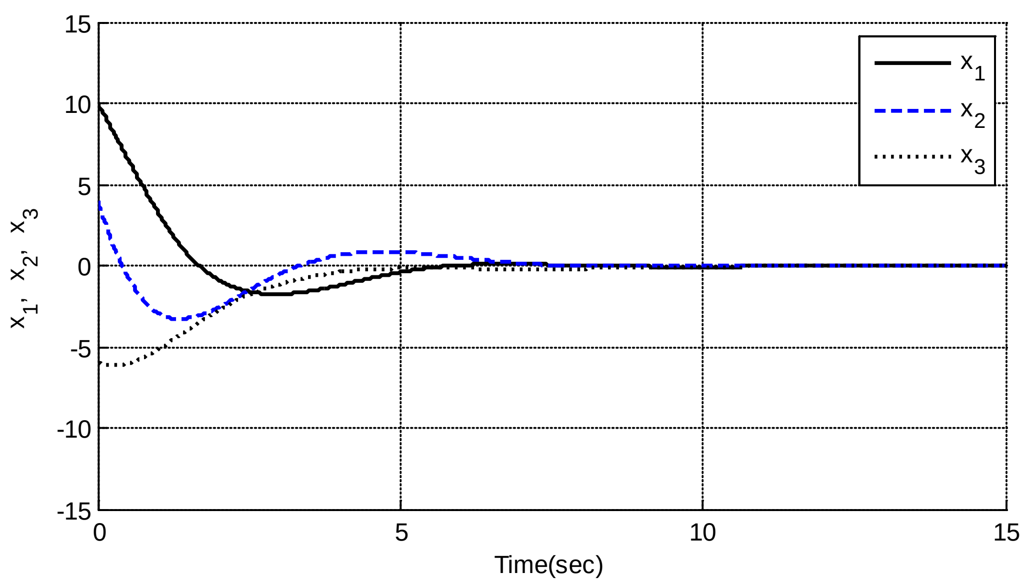

Figure 1. The trajectories of the system states.

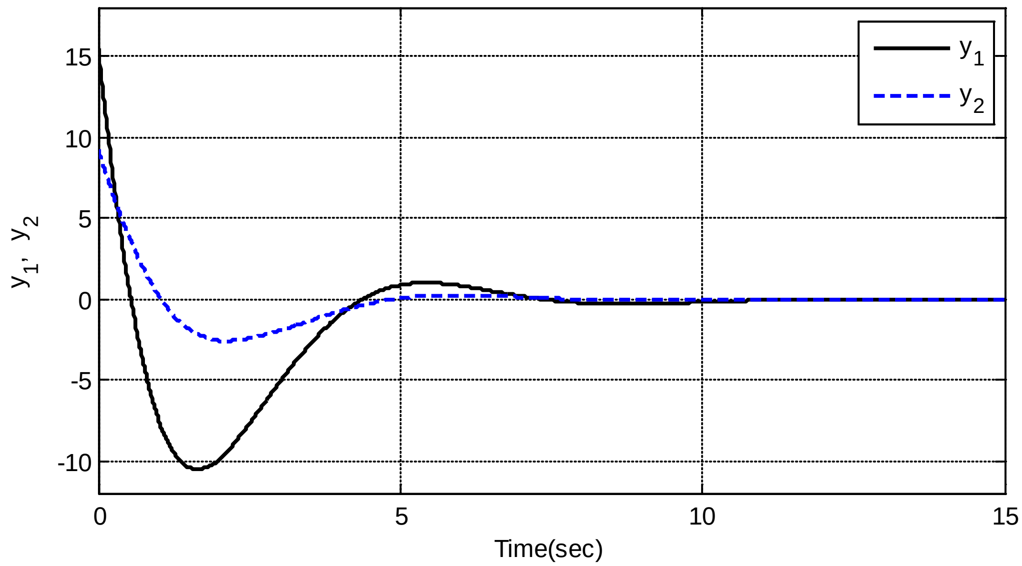

Figure 2. The output responses.

Example B: A Practical RIP System

RIP is a well-known test platform for evaluating control strategies. The control objective is to balance the pendulum in upright unstable equilibrium position. RIP system involves a rotational servo-motor which drives the output gear, rotational arm and an inverted pendulum. This system as an underactuated mechanical system has significant application in robotics, aerospace, marine vehicles and pointing control . In Figure 3, the schematic diagram of the RIP system is shown. Let α_p, θ_a, m_p, l_p, r_a, u, τ_a and J_b be the pendulum angle, drive disk angle (or arm angle), pendulum mass, pendulum length, arm length, control signal, motor torque and moment of inertia of the effective mass, correspondingly.

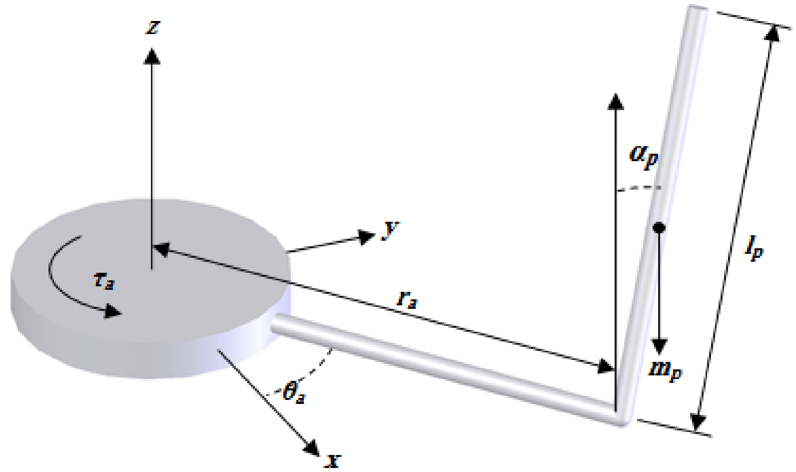

Figure 3. Schematic diagram of RIP system.

The dynamical equations of RIP with constant time-delays, friction and backlash effects are given by:

$$A_p\ddot{\theta}_a-(C_p\sin\alpha_p)\dot{\alpha}_p^2+(B_p\sin 2\alpha_p)\dot{\alpha}_p\dot{\theta}_a+F_p\dot{\theta}_a$$
$$+G_p\,\mathrm{sgn}(\dot{\theta}_a)+H_p\theta_a=I_pu+\frac{A_pB_p-C_p^2}{B_p}\begin{bmatrix}\alpha_p(t-\tau)\\ \dot{\alpha}_p(t-\tau)\\ \theta_a(t-\tau)\\ \dot{\theta}_a(t-\tau)\end{bmatrix}, \tag{43}$$

$$B_p\ddot{\alpha}_p-(B_p\sin\alpha_p\cos\alpha_p)\dot{\theta}_a^2-D_p\sin\alpha_p+E_p\dot{\alpha}_p=0 \tag{44}$$

where E_p, F_p, I_p, H_p and G_p are damping constant of the pendulum, damping constant of the arm, control input coefficient, elasticity coefficients and arm Coulomb friction, respectively. The parameters A_p, B_p, C_p and D_p are considered as [31]:

$$\begin{aligned}A_p&=m_pr_a^2+J_b,\\ B_p&=\frac{1}{3}m_pl_p^2,\\ C_p&=\frac{1}{2}m_pr_al_p,\\ D_p&=\frac{1}{2}m_pgl_p.\end{aligned} \tag{45}$$

The constant parameters of the nonlinear dynamical model (43)-(44) are set as:

$$A_p = 3.291, \quad D_p = 6.052, \quad G_p = 1.428,$$
$$B_p = 0.237, \quad E_p = 0.0132, \quad H_p = 1.72,$$
$$C_p = 0.237, \quad F_p = 14.283, \quad I_p = 6.38.$$

The nonlinear time-delayed model (43)-(44) with some reformations can be illustrated in the form of (1) as follows:

$$\dot{x} = \begin{pmatrix} 0 \\ (\sin x_1 \cos x_1)x_4^{\,2} + 25.54\sin x_1 \\ 0 \\ (0.072\sin x_1)x_2^{\,2} - (0.072\sin 2x_1)x_2 x_4 - 0.43\,\mathrm{sgn}(x_4) \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.056 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.52 & -4.34 \end{pmatrix} x + 0.93\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} x(t-\tau) + \begin{pmatrix} 0.2 \\ 0.1 \\ 0.1 \\ 1.94 \end{pmatrix} u, \qquad (46)$$
$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x,$$

where $x = \begin{bmatrix} \alpha_p & \dot{\alpha}_p & \theta_a & \dot{\theta}_a \end{bmatrix}^T$. For the simulation usage, the initial states are specified as: $x(0) = \begin{bmatrix} \pi & -1 & -4 & 2 \end{bmatrix}^T$, and the time-delay is chosen as $\tau = 2$. The Lipschitzian matrix is specified by: $L = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix}$. The solutions of matrices H, P, P_1, F are calculated using Matlab® LMI® toolbox and YALMIP® routine as a solver as:

$$H = \begin{bmatrix} 0.7213 & -0.1484 & 0.1765 & 1.7343 \\ -0.1484 & 0.7503 & 0.1916 & 1.9435 \\ 0.1765 & 0.1916 & 0.5853 & 2.2188 \\ 1.7343 & 1.9435 & 2.2188 & 12.6601 \end{bmatrix}, \quad P = \begin{bmatrix} 1.1107 & 0.4934 & 0.4132 & 0.4817 \\ 0.4934 & 1.7864 & 0.1968 & -0.1674 \\ 0.4132 & 0.1968 & 1.3576 & 0.6499 \\ 0.4817 & -0.1674 & 0.6499 & 2.2046 \end{bmatrix}.$$

$$P_1 = \begin{bmatrix} 7.8512 & 1.6463 & 9.5245 & 23.4526 \\ 1.6463 & 1.2598 & 2.0790 & 5.7018 \\ 9.5245 & 2.0790 & 12.1973 & 29.2457 \\ 23.4526 & 5.7018 & 29.2457 & 70.6682 \end{bmatrix}, \; F = \begin{bmatrix} -3.5140 & -3.2700 & -3.3280 & -2.0280 \end{bmatrix}.$$

The time trajectories of states of the RIP system by using the suggested control law are presented in Figure 4. The initial position is $\boldsymbol{x}(0) = \begin{bmatrix} -3 & 0 & 0 & 0 \end{bmatrix}^T$, related to the experimental part. It is observed from Figure 4 that the states of the RIP system can be regulated to the origin, irrespective of the time-delays and nonlinearities. The time response of the control signal is depicted in Figure 5 which displays the respectable efficiency of the suggested scheme. These simulations prove the robustness performance of the offered controller and show reasonable efficiency as well.

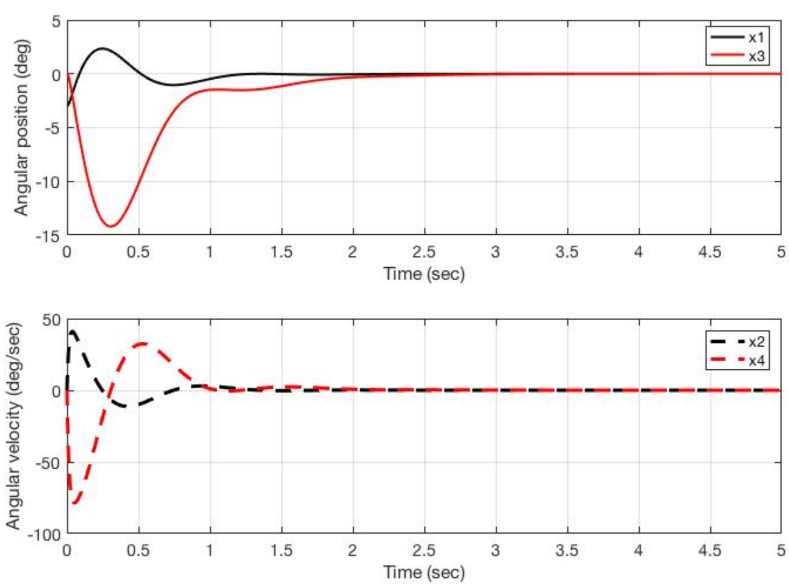

Figure 4. The trajectories of the RIP system states.

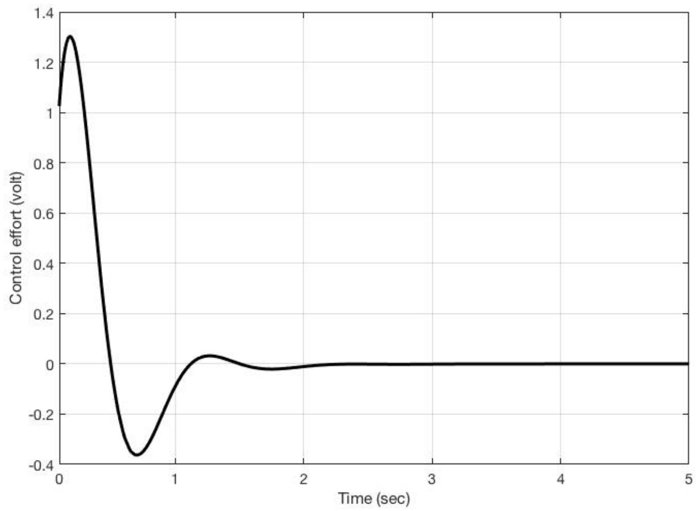

Figure 5. The control input.

In what follows, an experimental assessment of the proposed controller on the practical RIP system is presented. The experiment is performed on an ECP Model 220 industrial emulator with inverted pendulum, that includes a PC-based platform and DC brushless servo system [32]. The mechatronic system includes a motor used as servo actuator, a power amplifier and two encoders which provide accurate position measurements; i.e., 4000 lines per revolution with 4X hardware interpolation giving 16000 counts per revolution to each encoder; 1 count (equivalent to 0.000392 radians or 0.0225 degrees) is the lowest angular measurable [32]. The pendulum is fixed on the load disk (see figure 6).

Experimental results for the pendulum angle, load disk angle and time response of the applied control signal are demonstrated in Figure 7 and Figure 8, revealing that the suggested control method is indeed effective in practice.

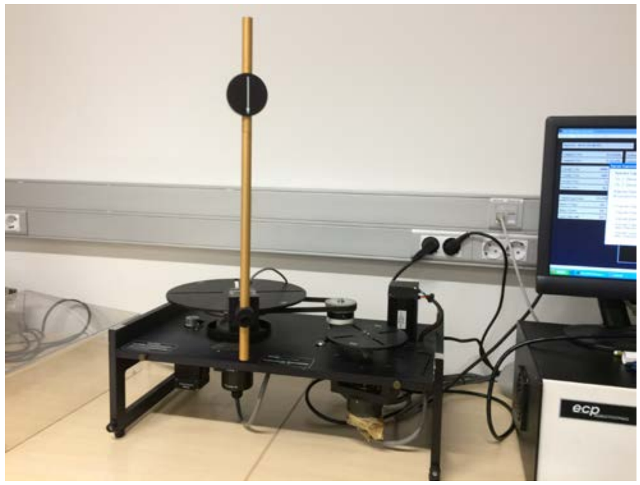

Figure 6. Practical RIP system, from CoDAlab laboratory (UPC).

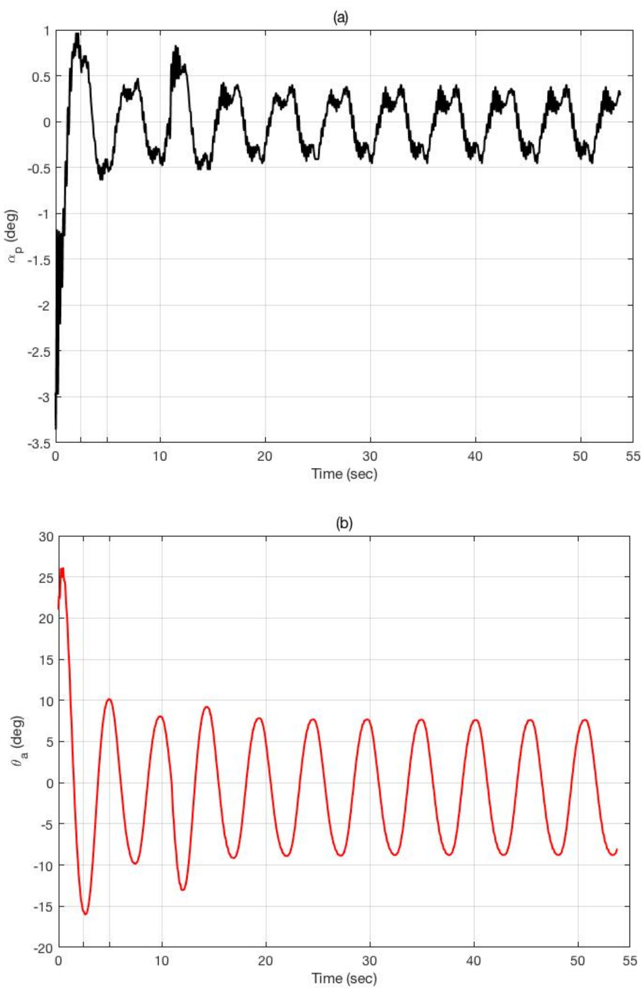

Figure 7. Experimental results of practical RIP system, (a) pendulum angle (α); (b) drive disk angle (θ).

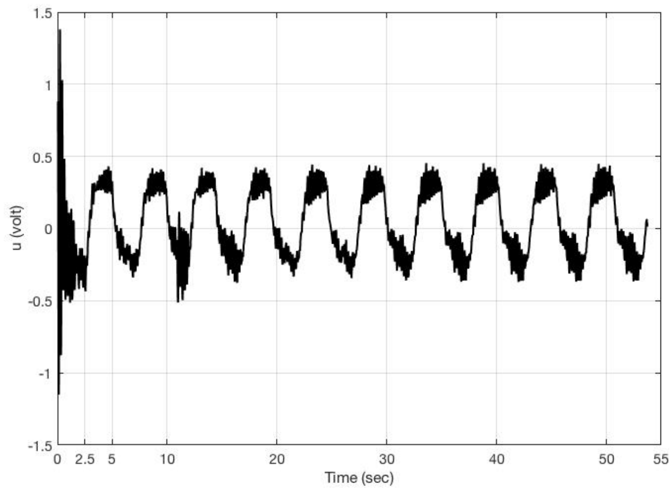

Figure 8. Control input of practical RIP system.

5. Conclusions

In this paper, the scheme of nonlinear feedback stabilization procedure is provided for the stabilization control of a class of nonlinear systems with time-delays and Lipschitz nonlinearities. Based on the Lyapunov-Krasovskii stability theory, the stability performance of the system is verified in the form of LMIs and the states are convergent uniformly asymptotically to the origin. The controller gains are specified by the sufficient conditions using LMIs. Furthermore, the problem of robust H_∞ performance analysis for a class of nonlinear perturbed time-delayed systems is investigated in this paper. The obvious simulation and experimental results are displayed to confirm the effectiveness of the presented technique and finally, some acceptable results are realized. The recommended control technique can attain favorable tracking performance for the higher-order nonlinear dynamical systems.

6. Asknowledgements

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