Eigenvalue and eigenmode synthesis in elastically coupled subsystems

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Abstract

A method to synthesize the modal characteristics of a system from the modal characteristics of its subsystems is proposed. The interest is focused on those systems with elastic links between the parts which is the main feature of the proposed method. An algebraic proof is provided for the case of arbitrary number of connections. The solution is a system of equations with a reduced number of degrees of freedom that correspond to the number of elastic links between the subsystems. In addition the method is also interpreted from a physical point of view (equilibrium of the interaction forces). An application to plates linked by means of springs shows how the global eigenfrequencies and eigenmodes are properly computed by means of the subsystems eigenfrequencies and eigenmodes.

Keywords: Subsystems, Synthesis, Eigenvalues, Eigenvectors, Sub-structuring, Eigenmodes

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Symbol list

$\mathbf{A}$	Dynamic matrix of the mechanical system
$\mathbf{B}$	Diagonal block matrix involving matrices $\mathbf{B}_1$ y $\mathbf{B}_2$
$\mathbf{C}$	Coupling matrix, non-zero valued only at antisymmetric positions connecting $\mathbf{B}_1$ and $\mathbf{B}_2$ coefficients
$c_{P_iQ_i}$	Matrix $\mathbf{C}$ coefficient linking degrees of freedom labelled as P_i and Q_i
$c_{P_iQ_i}$	Matrix $\mathbf{C}$ coefficient linking degrees of freedom labelled as P_i and Q_i
$\mathbf{D}_B$	Diagonal matrix of eigenvalues of $\mathbf{B}$
$\mathbf{e}_{P_k}$	Base vector, unity-valued at degree of freedom labelled P_k
$\mathbf{e}_{Q_k}$	Base vector, unity-valued at degree of freedom labelled Q_k
$\mathbf{H}$	$\mathbf{B}+\mathbf{C}$
$\mathbf{K}$	Stiffness matrix
$\mathbf{M}$	Mass matrix, diagonal an positive definite
m	Number of degrees of freedom of $\mathbf{B}$ connected by $\mathbf{C}$
m_j	jj coefficient of the mass matrix
n_ϵ	Dimension of a generic matrix ϵ
P_i	Label for the i'th connection on subsystem 1 may not correspond to degree of freedom i
Q_i	Label for the i'th connection on subsystem 2 may not correspond to degree of freedom i
$\mathbf{u}_{P_k}$	Column vector in $\mathbf{V}_B^{-1}$ on degree of freedom labelled P_k
$\mathbf{u}_{Q_k}$	Column vector in $\mathbf{V}_B^{-1}$ on degree of freedom labelled Q_k
$\mathbf{V}_B$	Matrix of eigenvectors of $\mathbf{B}$
$\mathbf{v}_{P_k}^T$	Row vector in $\mathbf{V}_B$ on degree of freedom labelled P_k
$\mathbf{v}_{Q_k}^T$	Row vector in $\mathbf{V}_B$ on degree of freedom labelled Q_k

Γ	Characteristic function whose zeroes are eigenvalues of $\mathbf{H}$
λ_A	Any eigenvalue of matrix $\mathbf{A}$
$\lambda_{B_1,k}$	k'th eigenvalue of matrix $\mathbf{B}_1$
$\lambda_{B_2,k}$	k'th eigenvalue of matrix $\mathbf{B}_2$
$\mathbf{\Phi}_i$	i'th eigenmode of subsystem 1.
$\phi_i(P_j)$	Coefficient in $\mathbf{\Phi}_i$ corresponding to degree of freedom P_j
μ_i	Modal mass of eigenmode i in subsystem 1, $\mathbf{\Phi}_i$
$\mathbf{\Psi}_i$	i'th eigenmode of subsystem 2.
$\psi_i(Q_j)$	Coefficient in $\mathbf{\Psi}_i$ corresponding to degree of freedom Q_j
σ_i	Modal mass of eigenmode i in subsystem 2, $\mathbf{\Psi}_i$
ω	Angular frequency

1 Introduction

Many products consist of an active device; e.g. engines, compressor units, frequency converters, etc. and a structure which is defined to meet a functionality target. Such structures may be for example trains, cars, refrigerators, etc. and the corresponding active devices which supply energy to the entire system are frequently categorized as auxiliary equipment.

Other type of products which may be partitioned in a similar way are all kind of machines installed on top of a supporting structure, as typically encountered in industrial facilities.

The assembly of the two aforementioned elements, typically through elastic joints (see Fig. 1), may generate noise and/or vibration complications due to the dynamic coupling between them. For instance, let an engine, whose dynamics are known, having no resonances at its fundamental excitation frequency. The engine is installed by means of elastic supports on a structure also not having resonances at excitation frequency. Then, the question that arises is whether the engine-structure assembly will resonate. A resonance of the assembly at the excitation frequency would be critical since it may cause noise issues as well as ill-functioning of the product.

Now, consider that the modal frequencies and mode shapes of each subsystem are known when the subsystems are subject to an infinite impedance condition through the elastic joints at their interface, c.f. Fig. 2. Then, the problem to be solved is determining the assembled system modal frequencies.

The problem of devising the behavior of a system from which one knows that of its two composing subsystems has been widely studied. The fields in which this problem has been addressed are widespread. As a matter of example one can cite in the field of electrical networks a method developed in ref. [1], the method is known as the Kron method. It was immediately applied to dynamics in [2], [3] and has been further developed recently in [4], [5], [6], [7]. One can also cite methods in the field of physics of composite systems such as those applied to sets of phonons [8] or to composite materials [9]. Eventually, one may refer to methods in the field of dynamics, for which a non-exhaustive review of methods may be found in [10].

The latter reference exposes an historic review of methods from which it summarizes the problem in just three equations and, subsequently, it examines the different approaches for its solution in the physical space, the modal space or the frequency space. The first equation is the dynamic equation for each of the subsystems subjected to external forces as well as coupling reaction forces. The second equation defines a linear dependency between coupling interface coordinates, usually trough a boolean matrix. Finally, the third equation defines the relation between the forces that each subsystem exerts on the other. Notice that the second equation assumes that the coupling interfaces are rigid or, generically, holonomic[1].

From this historical review one can note that the substructuring problem with elastic joints is not of frequent study. Yet, references [11], [12],[13] have studied the problem with elastic couplings through modal or FRF based methods. The first of these two references confirms the fact that the vast majority of analysis methods for substructures coupling consider the interface being of rigid coupling kind.

The present article addresses the synthesis of eigenmodes as an algebraic problem. It is based on solving the eigenvalues of a block matrix having one or more anti-diagonal elements in antisymmetric positions, from the knowledge of each matrix block's eigenvalues and eigenvectors and of the antidiagonal elements.

The solution to this algebraic problem is achieved through a *Divide and Conquer* approach. One of the basic articles of such solution method is that of Cuppen [14], which studies the synthesis of eigensolutions of a tridiagonal matrix through a first order perturbation approach. Also, an article by Arbenz, ref. [15], extends the method to modifications of higher order.

The references in the former paragraph solve a very similar problem to that posed in the present work, given that a tridiagonal matrix may be regarded as a block matrix having two antisymmetric elements in its anti-diagonal. However, opposite to the Divide and Conquer method, the method proposed here does not modify the matrix blocks in order to achieve the solution.

This article is divided into three parts. The first part demonstrates the equivalence between the dynamic and the algebraic problems and the solution to the eigenvalues' algebraic problem is described for one, two and multiple coupling terms (section 2 and section 3). In the second part (section 4) the algebraic solution is applied to the physical problem (section 4.1) and the physical interpretation of the solution is discussed (section 4.2). This physical interpretation substantiates the computation of the eigenvectors in subsection 4.3.

Thence, the first part gives a formal solution to the algebraic problem which is applicable to any linear system, regardless of its physical interpretation, and the second part provides a physical meaning to the eigenvalue solution and to the computation of the eigenmodes.

The third part (section 5) presents two examples, one on a simple system shows the basic mechanics of the method and a second example shows the performance of the method on a system with many degrees of freedom and multiple connections.

[1] A coupling interface is holonomic if one or more functions relating the degrees of freedom of the coupling interface exist $f(x_1, x_2.., x_n) = 0$. Consider a structure is split in two parts and the coupling interface coinciding degrees of freedom are labelled x_{1F} y x_{2F}, then the continuity condition is simply $x_{1F} - x_{2F} = 0$. Conversely, if the coupling interface degrees of freedom are non-coincident, e.g. they are linked by an elastic joint as in the example problem, it can only be said that $k(x_{1F} - x_{2F}) = f$ being f a force of unknown magnitude, so that the problem constraints are no longer holonomic.

2 Equivalence of the physical and algebraic problems

In the present article an algorithm which computes the eigenfrequencies and eigenvectors of a system consisting of two parts is developed from the eigenfrequencies and eigenvectors of each of the two parts and the characteristics of the elastic elements which join them.

The proposed problem is equivalent to obtaining the eigenvalues and eigenvectors of a block matrix in which the two composing blocks are connected at m degrees of freedom, hence having non-zero valued anti-diagonal elements.

Let a mechanical system consisting of two subsystems such as shown in Fig. 1

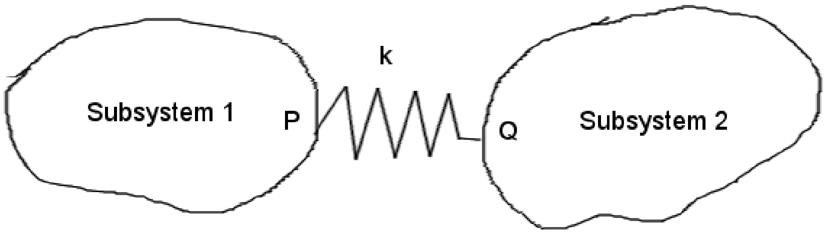

Figure 1: Subsystems 1 and 2 with one or more coupling spring

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be the dynamic matrices of the two subsystems in isolation as depicted in Fig. 2, i.e. including fixed boundary conditions at the coupling interface.

Then

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{K}_1 - \omega^2 \mathbf{M}_1 \\ \mathbf{A}_2 &= \mathbf{K}_2 - \omega^2 \mathbf{M}_2 \end{aligned} \tag{1}$$

where the mass matrices $\mathbf{M}$ are diagonal matrices and $\mathbf{K}$ are the stiffness matrices.

Let k be the stiffness of the coupling spring.

The assembled system dynamic matrix $\mathbf{A}$ may be described as:

$$\mathbf{A} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 \end{pmatrix} - \omega^2 \begin{pmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & k & \dots & 0 \\ 0 & \dots & k & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2}$$

Be it P and Q labels corresponding to the degrees of freedom linked by the elastic element, P in subsystem 1 and Q in subsystem 2. The system eigenmodes and eigenfrequencies correspond to the eigenvalues and eigenvectors of the matrix

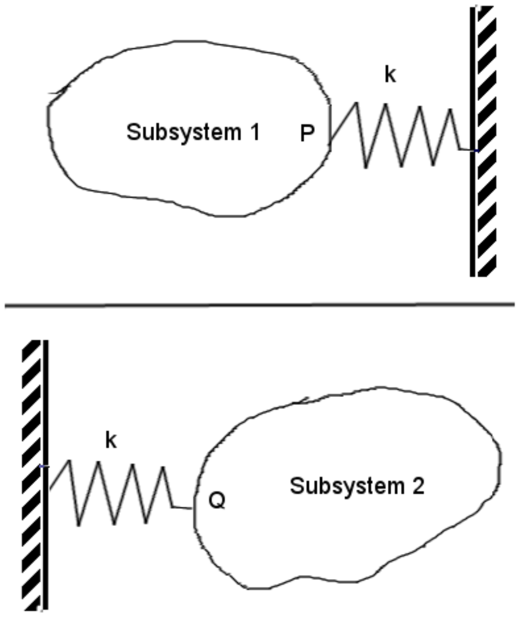

Figure 2: The total system is split into subsystems 1 and 2 as shown in the figure. This decomposition allows the blocks that define the subsystems not to vary by forming the total system.

$$\mathbf{H} = \begin{pmatrix} \mathbf{M}_1^{-1}\mathbf{K}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2^{-1}\mathbf{K}_2 \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{k}{m_P} & \dots & 0 \\ 0 & \dots & \frac{k}{m_Q} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3}$$

In this equation m_P is the diagonal coefficient of matrix $\mathbf{M}_1$ corresponding to the degree of freedom labelled as P and, likewise, m_Q is the diagonal coefficient of matrix $\mathbf{M}_2$ corresponding to the degree of freedom labelled as Q.

According to Eq. (3), matrix $\mathbf{H}$, from which the eigensolutions are seeked, is of the kind:

$$\mathbf{H} = \mathbf{B} + \mathbf{C} \tag{4}$$

where $\mathbf{B}$ is a block matrix, as it is explicit in Eq. (5), and $\mathbf{C}$ is a matrix with non-zero values in, at least, two antisymmetric positions in the two antisymmetric blocks, Eq. (6).

$$\mathbf{B} \equiv \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{pmatrix} \tag{5}$$

$$
\mathbf{C} \equiv \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & c_{PQ} & \dots & 0 \\ 0 & \dots & c_{QP} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \tag{6}
$$

The equivalent algebraic problem involves the calculation the eigenvalues and eigenvectors of matrix $\mathbf{H}$ from those of $\mathbf{B}_1$, $\mathbf{B}_2$ and from the coefficients of matrix $\mathbf{C}$.

3 Method

Under the framework of the problem raised in the introduction, the solution for the union of two diagonal blocks in $\mathbf{H}$ will be obtained here for the cases: A single connecting element (section 3.1), two connecting elements (section 3.2) and m connecting elements (section 3.3).

3.1 Single connection. Synthesis of eigenvalues

Let

$$
\mathbf{H} = \mathbf{B} + \mathbf{C} \tag{7}
$$

where the matrix $\mathbf{C}$ and the eigensolutions for $\mathbf{B}$ are known, the eigenvalues of $\mathbf{H}$ are sought. $\mathbf{C}$ may be expressed as:

$$
\mathbf{C} = \mathbf{e}_P c_{PQ} \mathbf{e}_Q^T + \mathbf{e}_Q c_{QP} \mathbf{e}_P^T \tag{8}
$$

where $\mathbf{e}_P$ is a basis vector unity valued at the degree of freedom labelled P and zero valued elsewhere. The same holds for $\mathbf{e}_Q$.

If $\mathbf{B}$ is diagonalizable (over the complex field C, any matrix either is diagonalizable or is arbitrarily close to a matrix with distinct eigenvalues that does), it may be formulated as:

$$
\mathbf{B} = \mathbf{V}_B \mathbf{D}_B \mathbf{V}_B^{-1} \tag{9}
$$

with $\mathbf{D}_B$ a diagonal matrix containing the eigenvalues of the isolated subsystems and $\mathbf{V}_B$ the corresponding matrix of eigenvectors. Matrix $\mathbf{V}_B$ has, naturally, the same block structure as $\mathbf{B}$.

Consequently, $\mathbf{H}$ is reformulated as:

$$
\mathbf{H} = \mathbf{V}_B(\mathbf{D}_B + \mathbf{V}_B^{-1}\mathbf{C}\mathbf{V}_B)\mathbf{V}_B^{-1} = \mathbf{V}_B\mathbf{H}'\mathbf{V}_B^{-1} \tag{10}
$$

where

$$
\mathbf{H}' = \mathbf{D}_B + \mathbf{V}_B^{-1}\mathbf{C}\mathbf{V}_B \tag{11}
$$

Equation (10) demonstrates that $\mathbf{H}$ and $\mathbf{H}'$ are similar matrices, ergo they certainly have identical eigenvalues and eigenvectors.

Let

$$\mathbf{u}_P = \mathbf{V}_B^{-1}\mathbf{e}_P \qquad\qquad \mathbf{v}_P^T = \mathbf{e}_P^T\mathbf{V}_B \tag{12}$$

where $\mathbf{u}_P$ is the column vector corresponding to degree of freedom P in matrix $\mathbf{V}_B^{-1}$ and $\mathbf{v}_P^T$ is the row vector corresponding to degree of freedom P in matrix $\mathbf{V}_B$. Analogous formulation holds for Q.

Substituting $\mathbf{C}$ by Eq. (8) in equation Eq. (11)

$$\mathbf{H}' = \mathbf{D}_B + c_{PQ}\mathbf{u}_P\mathbf{v}_Q^T + c_{QP}\mathbf{u}_Q\mathbf{v}_P^T \tag{13}$$

if $\mathbf{x}$ is an eigenvector of $\mathbf{H}'$ with associated eigenvalue λ_H it holds that:

$$\left(\mathbf{D}_B + c_{PQ}\mathbf{u}_P\mathbf{v}_Q^T + c_{QP}\mathbf{u}_Q\mathbf{v}_P^T\right)\mathbf{x} = \lambda_H\mathbf{x} \tag{14}$$

The inverse of $(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}$ exists provided that the eigenvalues of the coupled system do not coincide with those of the subsystems. For very weak coupling, the system eigenvalues will be very close to the eigenvalues of the isolated blocks, but they will not be strictly equal unless the blocks were fully decoupled. Therefore, in general, the inverse of $(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}$ must exist. Hence, equation (14) may be rearranged as follows

$$\mathbf{x} = (\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}(c_{PQ}\mathbf{u}_P\mathbf{v}_Q^T\mathbf{x} + c_{QP}\mathbf{u}_Q\mathbf{v}_P^T\mathbf{x}) \tag{15}$$

Then, taking the scalar product of Eq. (15) with $\mathbf{v}_Q^T$ and, separately, with $\mathbf{v}_P^T$, and given that $\mathbf{v}_P^T$ only has non zero valued components in subsystem 1 and $\mathbf{v}_Q^T$ only has non zero valued components in subsystem 2, one obtains:

$$\mathbf{v}_Q^T\mathbf{x} = c_{QP}\mathbf{v}_Q^T(\lambda_A\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_Q\mathbf{v}_P^T\mathbf{x} \tag{16}$$

$$\mathbf{v}_P^T\mathbf{x} = c_{PQ}\mathbf{v}_P^T(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_P\mathbf{v}_Q^T\mathbf{x} \tag{17}$$

Substituting Eq. (17) in Eq. (16) results in

$$\mathbf{v}_Q^T\mathbf{x} = c_{QP}c_{PQ}\mathbf{v}_Q^T(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_Q\mathbf{v}_P^T(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_P\mathbf{v}_Q^T\mathbf{x} \tag{18}$$

Assuming that $\mathbf{v}_Q^T\mathbf{x} \neq 0$, equation (18) proves that the eigenvalues of $\mathbf{H}$, λ_H, must satisfy:

$$1 = c_{PQ}c_{QP}\sum_{i=1}^{n_1}\frac{v_{P,i}^T u_{i,P}}{\lambda_H - \lambda_{B_1,i}}\sum_{j=1}^{n_2}\frac{v_{Q,j}^T u_{j,Q}}{\lambda_H - \lambda_{B_2,j}} \tag{19}$$

Thus, as a conclusion to this section, it can be asserted that there exists a characteristic function

$$\boxed{\Gamma(\lambda) = 1 - c_{PQ}c_{QP}\sum_{i=1}^{n_1}\frac{v_{P,i}^T u_{i,P}}{\lambda - \lambda_{B_1,i}}\sum_{j=1}^{n_2}\frac{v_{Q,j}^T u_{j,Q}}{\lambda - \lambda_{B_2,j}}} \tag{20}$$

whose zeroes correspond to the characteristic values (eigenvalues of matrix $\mathbf{H}$).

3.2 Two connections. Synthesis of eigenvalues

Let $\mathbf{H}$ and $\mathbf{B}$ be the same matrices as in section 3.1, but now assuming that $\mathbf{C}$ has two connection elements, one connecting P_1 with Q_1 and another connecting P_2 with Q_2. The degrees of freedom labelled P always correspond to block 1, and those labelled Q belong to block 2.

Under these assumptions, matrix $\mathbf{C}$ may be reformulated as:

$$\mathbf{C} = \mathrm{c}_{Q_1P_1}\mathbf{e}_{Q_1}\mathbf{e}^T_{P_1} + \mathrm{c}_{P_1Q_1}\mathbf{e}_{P_1}\mathbf{e}^T_{Q_1} + \mathrm{c}_{Q_2P_2}\mathbf{e}_{Q_2}\mathbf{e}^T_{P_2} + \mathrm{c}_{Q_2P_2}\mathbf{e}_{P_2}\mathbf{e}^T_{Q_2}$$

Similarly to section 3.1 derivations, one may define:

$$\mathbf{u}_{P_s} = \mathbf{V}_B^{-1}\mathbf{e}_{P_s} \qquad \mathbf{v}^T_{P_s} = \mathbf{e}^T_{P_s}\mathbf{V}_B \tag{21}$$

for $s = 1, 2$, and analogous definitions for Q.

If $\mathbf{x}$ is an eigenvector of $\mathbf{H}$ with associated eigenvalue λ_H it holds that

$$\left(\mathbf{D}_B + \mathrm{c}_{Q_1P_1}\mathbf{u}_{Q_1}\mathbf{v}^T_{P_1} + \mathrm{c}_{P_1Q_1}\mathbf{u}_{P_1}\mathbf{v}^T_{Q_1} + \mathrm{c}_{Q_2P_2}\mathbf{u}_{Q_2}\mathbf{v}^T_{P_2} + \mathrm{c}_{P_2Q_2}\mathbf{u}_{P_2}\mathbf{v}^T_{Q_2}\right)\mathbf{x} = \lambda_H\mathbf{x} \tag{22}$$

and taking scalar product with $\mathbf{v}^T_{P_1}$ and $\mathbf{v}^T_{P_2}$ yields

$$\begin{pmatrix}\mathbf{v}^T_{P_1}\mathbf{x}\\ \mathbf{v}^T_{P_2}\mathbf{x}\end{pmatrix} = \begin{pmatrix} c_{P_1Q_1}\mathbf{v}^T_{P_1}(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{P_1} & c_{P_2Q_2}\mathbf{v}^T_{P_1}(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{P_2}\\ c_{P_1Q_1}\mathbf{v}^T_{P_2}(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{P_1} & c_{P_2Q_2}\mathbf{v}^T_{P_2}(\lambda_H-\mathbf{D}_B\mathbf{I})^{-1}\mathbf{u}_{P_2}\end{pmatrix}\begin{pmatrix}\mathbf{v}^T_{Q_1}\mathbf{x}\\ \mathbf{v}^T_{Q_2}\mathbf{x}\end{pmatrix} \tag{23}$$

Likewise, it is straightforward to prove that

$$\begin{pmatrix}\mathbf{v}^T_{Q_1}\mathbf{x}\\ \mathbf{v}^T_{Q_2}\mathbf{x}\end{pmatrix} = \begin{pmatrix} c_{Q_1P_1}\mathbf{v}^T_{Q_1}(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{Q_1} & c_{Q_2P_2}\mathbf{v}^T_{Q_1}(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{Q_2}\\ c_{Q_1P_1}\mathbf{v}^T_{Q_2}(\lambda_A\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{Q_1} & c_{Q_2P_2}\mathbf{v}^T_{Q_2}(\lambda_H\mathbf{I}-\mathbf{D}_B)^{-1}\mathbf{u}_{Q_2}\end{pmatrix}\begin{pmatrix}\mathbf{v}^T_{P_1}\mathbf{x}\\ \mathbf{v}^T_{P_2}\mathbf{x}\end{pmatrix} \tag{24}$$

Let the matrices in (23) and (24) be named $\mathbf{E}$ and $\mathbf{F}$. Substituting the vector to the left of (24) in (23) one obtains

$$\begin{pmatrix}\mathbf{v}^T_{P_1}\mathbf{x}\\ \mathbf{v}^T_{P_2}\mathbf{x}\end{pmatrix} = \mathbf{EF}\begin{pmatrix}\mathbf{v}^T_{P_1}\mathbf{x}\\ \mathbf{v}^T_{P_2}\mathbf{x}\end{pmatrix} \Rightarrow (\mathbf{I}-\mathbf{EF})\begin{pmatrix}\mathbf{v}^T_{P_1}\mathbf{x}\\ \mathbf{v}^T_{P_2}\mathbf{x}\end{pmatrix} = \mathbf{0} \tag{25}$$

So that the non-trivial solution to (25), i.e. $\mathbf{x}$ not being a null vector, is:

$$\boxed{\det(\mathbf{I}-\mathbf{EF}) = 0} \tag{26}$$

This equation is equivalent to the one obtained for one degree of freedom in Eq. (20).

Equation (26) is a function of λ, whose zeroes are the eigenvalues λ_H of matrix $\mathbf{H}$.

3.3 Arbitrary number of connections. Synthesis of eigenvalues

Following section (3.2), the general solution may be readily formulated.

Let

$$\mathbf{E} = \begin{pmatrix} \mathbf{v}_{P_1}^T \mathbf{S}\mathbf{u}_{P_1} & \mathbf{v}_{P_1}^T \mathbf{S}\mathbf{u}_{P_2} & \mathbf{v}_{P_1}^T \mathbf{S}\mathbf{u}_{P_m} \\ \mathbf{v}_{P_2}^T \mathbf{S}\mathbf{u}_{P_1} & \mathbf{v}_{P_2}^T \mathbf{S}\mathbf{u}_{P_2} & \mathbf{v}_{P_1}^T \mathbf{S}\mathbf{u}_{P_m} \\ \vdots & \vdots & \vdots \\ \mathbf{v}_{P_m}^T \mathbf{S}\mathbf{u}_{P_1} & \mathbf{v}_{P_m}^T \mathbf{S}\mathbf{u}_{P_2} & \mathbf{v}_{P_1}^T \mathbf{S}\mathbf{u}_{P_m} \end{pmatrix} \begin{pmatrix} c_{P_1Q_1} & & & \\ & c_{P_2Q_2} & & \\ & & \ddots & \\ & & & c_{P_mQ_m} \end{pmatrix} \tag{27}$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{v}_{Q_1}^T \mathbf{S}\mathbf{u}_{Q_1} & \mathbf{v}_{Q_1}^T \mathbf{S}\mathbf{u}_{Q_2} & \mathbf{v}_{Q_1}^T \mathbf{S}\mathbf{u}_{Q_m} \\ \mathbf{v}_{Q_2}^T \mathbf{S}\mathbf{u}_{Q_1} & \mathbf{v}_{Q_2}^T \mathbf{S}\mathbf{u}_{Q_2} & \mathbf{v}_{Q_1}^T \mathbf{S}\mathbf{u}_{Q_m} \\ \vdots & \vdots & \vdots \\ \mathbf{v}_{Q_m}^T \mathbf{S}\mathbf{u}_{Q_1} & \mathbf{v}_{Q_m}^T \mathbf{S}\mathbf{u}_{Q_2} & \mathbf{v}_{Q_1}^T \mathbf{S}\mathbf{u}_{Q_m} \end{pmatrix} \begin{pmatrix} c_{Q_1P_1} & & & \\ & c_{Q_2P_2} & & \\ & & \ddots & \\ & & & c_{Q_mP_m} \end{pmatrix} \tag{28}$$

where $\mathbf{S} = (\lambda\mathbf{I} - \mathbf{D}_B)^{-1}$.

The eigenvalues of $\mathbf{H}$, for any given number of connections between the two blocks, are the zeroes of $\mathbf{\Gamma}(\lambda) = \det(\mathbf{I} - \mathbf{EF})$.

That is

$$\boxed{\mathbf{\Gamma}(\lambda_H) = 0} \tag{29}$$

Notice that the order of matrix $(\mathbf{I} - \mathbf{EF})$ is m, i.e. the number of connected degrees of freedom per subsystem, which is typically a much lower figure than the total number of degrees of freedom of the coupled system.

4 Application of the algebraic solution to the physical problem, physical meaning and eigenmodes

In the following, the formulation obtained in the algebraic problem in Section 3 is applied to a mechanical system consisting of two subsystems, the physical meaning of the equations is interpreted and the equations defining the system eigenmodes are derived.

4.1 Application to dynamics of mechanical systems

In order to apply the formulation in Eq. (29) to a mechanical problem one may interpret the elements in matrices $\mathbf{E}$ and $\mathbf{F}$ in accordance with the eigenvalues and eigenmodes of the mechanical subsystems in isolation.

To this effect, the inverse of the eigenmodes matrix must be known in order to get the $\mathbf{u}_{p_i}$ terms in Eq. (21).

Consider a mechanical system represented by the matrix $\mathbf{A}$ and consisting of two undamped subsystems 1 and 2 connected by m elastic elements. Suppose that subsystem 1 has n_1 degrees of freedom and subsystem 2 has n_2 degrees of freedom.

Recall that the connection points are labelled P_i in subsystem 1 and Q_i in subsystem 2, and that the subscripts indicate the elastic elements, i.e. element i joins points P_i and Q_i.

Let $c_{P_iQ_i}$ be the stiffness between P_i and Q_i divided by the mass in P_i.

Let $\mathbf{\Phi}_i$ $i = 1...n_1$ be an eigenmode of subsystem 1 and $\mathbf{\Psi}_j$ $j = 1...n_2$ an eigenmode of subsystem 2, let also $\lambda_{1,i}$ an eigenvalue of subsystem 1, $\lambda_{2,j}$ an eigenvalue of subsystem 2 and $\lambda_{A,s}$ an eigenvalue of the coupled system $\mathbf{A}$.

A discrete (or discretized) mechanical system may be described through analogous equations to those in Eq. (1).

If $\mathbf{M}$ is a positive-definite diagonal matrix with coefficients m_i and $\mathbf{K}$ is a symmetric matrix, the corresponding eigenmodes are orthogonal with respect to the scalar product defined by matrix $\mathbf{M}$. If the eigenmodes are normalised to have unity Euclidean norm it follows that

$$\mathbf{\Phi}_i^T \mathbf{M} \mathbf{\Phi}_j = \mu_i \delta_{ij} \tag{30}$$

where μ_i are the so called modal masses.

From Eq. (30) it can be inferred that the inverse of the eigenmodes matrix is just the transposed eigenmodes matrix right-scaled by the mass matrix and left-scaled by the inverse modal mass matrix. If $\mathbf{\Pi} = \mathrm{diag}(\mu_1, \mu_2 \ldots, \mu_{n1})$, then

$$\mathbf{\Phi}^{-1} = \mathbf{\Pi}^{-1} \mathbf{\Phi}^T \mathbf{M} \tag{31}$$

Same formulation holds for $\mathbf{\Psi}$, the eigenmodes matrix of subsystem 2.

The P_i, P_j terms of the elements in matrix $\mathbf{E}$ in subsection (3.3) read:

$$\mathbf{v}_{P_s}^T (\mathbf{D}_B - \lambda I)^{-1} \mathbf{u}_{P_l} \tag{32}$$

where

$$\mathbf{u}_{P_l} = \mathbf{V}_B^{-1} \mathbf{e}_{P_l} \text{ and } \mathbf{v}_{P_s}^T = \mathbf{e}_{P_s}^T \mathbf{V}_B \tag{33}$$

In consequence, for subsystem 1

$$\mathbf{v}_{P_s}^T = \begin{pmatrix} \boldsymbol{\phi}_1(P_s), \boldsymbol{\phi}_2(P_s) \ldots & , \boldsymbol{\phi}_{\boldsymbol{n1}}(P_s) \end{pmatrix} \text{ and } \mathbf{u}_{P_l} = \begin{pmatrix} \boldsymbol{\phi}_1(P_l) \frac{m_l}{\mu_1} \\ \boldsymbol{\phi}_2(P_l) \frac{m_l}{\mu_2} \\ \cdots \\ \boldsymbol{\phi}_{n1}(P_l) \frac{m_l}{\mu_{n1}} \end{pmatrix} \tag{34}$$

Finally, the P_i, P_j term of matrix $\mathbf{E}$ in Eq. (32) is reformulated as

$$m_j \sum_{s=1}^{n1} \frac{\boldsymbol{\phi}_s(P_i) \boldsymbol{\phi}_s(P_j)}{\lambda_s - \lambda} \frac{1}{\mu_s} \tag{35}$$

Other than the term m_j, Eq. (35) is a system receptance function or, in other words, the Green's function relating the forces applied at P_i and the displacement produced at P_j.

The connection terms in matrix $\mathbf{C}$ are

$$c_{P_j,Q_j} = \frac{k(P_j, Q_j)}{m_j} \tag{36}$$

where $k(P_j, Q_j)$ is the stiffness of the element connecting P_j and Q_j, and m_j is mass in subsystem 1 connected to the j^{th} elastic element.

Given that in the final solution the P_i, P_j coefficient in matrix E multiplies with $c(P_j, Q_j)$ (c.f. Eq. (27)), the elements of $\mathbf{E}$ will eventually be the P_i, Q_j receptance functions.

At this point, it is worth noting that from an experimental point of view it is easier to measure a receptance function than it is to compute it out of a (experimental) modal analysis which, in turn, would be of limited precision with respect to the measured receptance due to having a finite number of modes.

In order to obtain the coupled system eigenvalues λ, let

$$\mathbf{G}_1 = \begin{pmatrix} \sum_{i=1}^{n1} \frac{\phi_i(P_1)\phi_i(P_1)}{\mu_i(\lambda_i-\lambda)} & \dots & \sum_{i=1}^{n1} \frac{\phi_i(P_1)\phi_i(P_m)}{\mu_i(\lambda_i-\lambda)} \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{n1} \frac{\phi_i(P_m)\phi_i(P_1)}{\mu_i(\lambda_i-\lambda)} & \dots & \sum_{i=1}^{n1} \frac{\phi_i(P_m)\phi_i(P_m)}{\mu_i(\lambda_i-\lambda)} \end{pmatrix} \tag{37}$$

$$\mathbf{K}_c = \begin{pmatrix} k_{P_1Q_1} & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & k_{P_mQ_m} \end{pmatrix} \tag{38}$$

and $\mathbf{G}_2$ an equivalent expression to that of $\mathbf{G}_1$.

The system eigenvalues λ are the zeroes of the $\mathbf{I}-\mathbf{EF}$ determinant, with $\mathbf{E} = \mathbf{G}_1\mathbf{K}_c$, $\mathbf{F} = \mathbf{G}_2\mathbf{K}_c$ and being $\mathbf{I}$ an identity matrix of order m.

$$\boxed{\det(\mathbf{I} - \mathbf{EF}) = 0} \tag{39}$$

4.2 Physical meaning of the solution

As discussed above, the elements of $\mathbf{G}_1$ and $\mathbf{G}_2$ are frequency response functions relating force to displacement. Element ij in $\mathbf{G}_1$ is the displacement in P_j due to a unit force with frequency $f = \frac{\sqrt{\lambda}}{2\pi}$ applied at P_i.

Now, let $\mathbf{G} = (\mathbf{I} - \mathbf{EF})$, the elements γ of matrix $\mathbf{G}$ are

$$\gamma_{ij} = \sum_l g_1(P_iP_l)k(P_lQ_l)g_2(Q_lQ_j)k(Q_jP_j) \tag{40}$$

$$\gamma_{ii} = 1 - \sum_l g_1(P_iP_l)k(P_lQ_l)g_2(Q_lQ_i)k(Q_iP_i) \tag{41}$$

Figs. 3 and 4 show how a force applied at P_1 in subsystem 1 is transmitted into P_1 itself through the springs which couple subsystems 1 and 2.

Fig. 3 shows the displacements produced at points P_1 to P_4 by the force applied at P_1. These displacements are simply $g(P_1, P_1)$ for P_1, $g(P_1, P_2)$ for P_2, $g(P_1, P_3)$ for P_3 and $g(P_1, P_4)$ for P_4. Each of these displacements produce a force at points Q_1 to Q_4 in subsystem 2, these being $g(P_1, P_i)k_{P_i,Q_i}$ for $i = 1, 2, 3, 4$.

In Fig. 4 it is shown how the forces on points Q_1 to Q_4 produce a total displacement in Q_1 equal to $\sum_{i=1}^{4} g(P_1, P_i)k_{P_i,Q_i}g'(Q_i, Q_1)$. Finally, the coupled system reaction force on P_1 is $\sum_{i=1}^{4} g(P_1, P_i)k_{P_i,Q_i}g'(Q_i, Q_1)k_{Q_i,P_1}$.

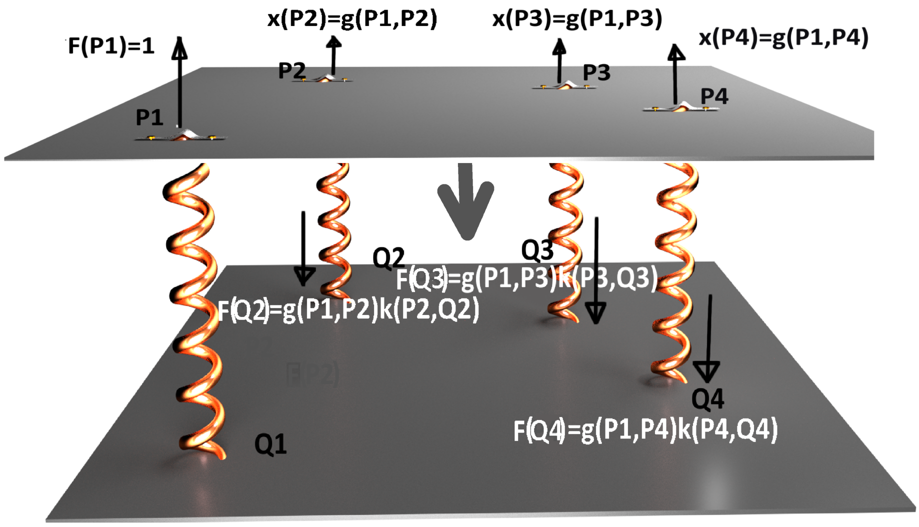

Figure 3: A force in P1 produces forces on the Q points

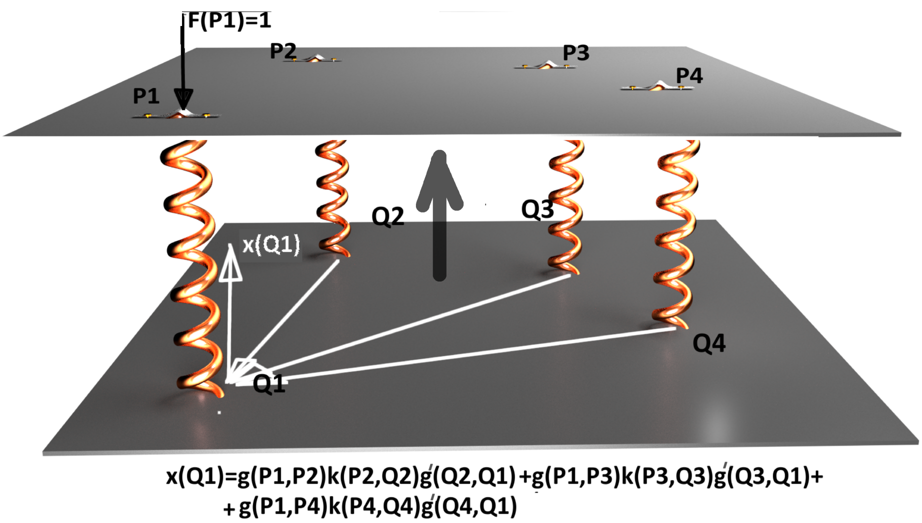

Figure 4: The forces in the Q points produces a reaction force on P1

Thus, the matrix element $\gamma_{i,i}$ is the reaction force over P_i as a result of a unit force applied at the same point P_i. Therefore, the resonance condition in Eq. (39) means that if a point unit force is applied only at P_i, the reaction force at P_i is unity, and zero elsewhere.

This means that the forces acting on $P1$ are self equilibrated at some of the frequencies without the need of external loading. It is, indeed, the resonance condition.

Formally, including forces in Eq. (39) it holds that

$$\det(\mathbf{G}) = 0 \Leftrightarrow \exists \mathbf{f} \mid \mathbf{G}\mathbf{f} = \mathbf{0} \Longleftrightarrow \mathbf{R}\mathbf{f} = \mathbf{f} \text{ being } \mathbf{R} \neq \mathbf{I} \text{ and } \mathbf{f} \neq \mathbf{0} \tag{42}$$

where $\mathbf{R} = \mathbf{I} - \mathbf{G}$.

For any given λ and $\mathbf{f}$ it holds that $\mathbf{Rf} = \mathbf{f}'$. Under the hypothesis that a force $\mathbf{f}$ is applied to the connection points of one of the subsystems, such force may be decomposed into two parts: the reaction force of the other subsystem, i.e. $\mathbf{Rf}$, and the external forces. Thereof, $\mathbf{f} = \mathbf{Rf} + \mathbf{f}_{\text{Exterior}}$. Solving $\mathbf{Rf} = \mathbf{f}$ implies finding a set of forces that may exist without the need of any external loading. These forces correspond to the reaction $\mathbf{Rf}$ at the couplings in absence of external forces, thus, at resonance. In consequence, Eq. (39) is a resonance condition.

The conclusion is therefore that the function derived here from algebraic considerations is a resonance condition from a physical point of view, which states that the coupling forces at resonance are the result of the action of same forces through the connections.

4.3 Eigenmodes calculation

Given that the resonance condition requires $\mathbf{Rf} = \mathbf{f}$, it can be ascertained that matrix $\mathbf{R}(\lambda_A)$, whose eigenvalues λ_A are zeroes of $\det \mathbf{G}$, has at least one eigenvector with eigenvalue equal to 1. Name this eigenvector $\mathbf{f}_A$.

The components of $\mathbf{f}_A$ over subsystem 1 are the coupling forces that apply on subsystem 1 at system resonance with eigenvalue λ_A.

Considering that $\mathbf{G}$ is of order m, i.e. the number of connections, which is a very small dimension as compared to the number of degrees of freedom of the subsystems specifically in the present target problem, the calculation of its eigenvectors will be very simple in terms of computation burden.

Once obtained the eigenvector $\mathbf{f}_A$ with associated eigenvalue 1, one must just apply the force vector $\mathbf{f}_A$ to the matrix of subsystem 1 in order to obtain the corresponding displacement that the applied forces produce. The computed displacements are those of the system eigenmode with eigenvalue λ_A over subsystem 1.

Proceeding in the same way on subsystem 2, displacements for the corresponding eigenmode are found.

5 Examples

In this section two examples are considered. The first example is a simple system so to explicitly illustrate the method steps towards the final solution. Also, the resemblance of this example to that in reference [3] is exploited so to briefly remark the differences between both methods with regards to the subsystems selection.

The second example is a system with many degrees of freedom, having 5 connections between two composing subsystems, so to illustrate the method performance in a much more complex problem.

5.1 Simple discrete system

The first example is a discrete system involving three masses and four springs, whose values are defined without magnitude units for brevity.

This is a very similar problem to the one exposed in [3] as an application example of Kron's method.

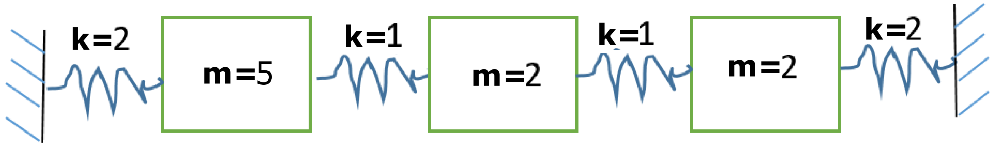

Figure 5: Simple discrete system example. k and m are stiffness and masses values.

In order to apply Kron's method, or any other method applicable to rigid connexions, the system must be split by dividing the masses elements. See for example Fig. 6.

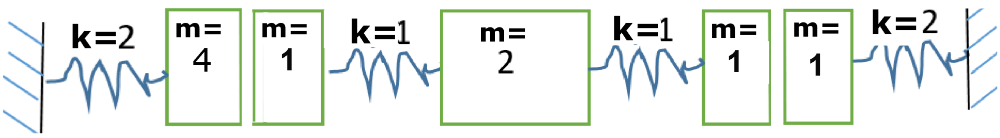

Figure 6: Subsystems division in order to apply Kron's method [3].

Notice that masses are split in Kron's method in order to make the connections holonomic. This is optimal to subdivide an structure, but it is virtually of no use for an auxiliary equipment.

The method proposed here decomposes the problem into two smaller problems with the coupling interface set at the springs (see Fig. 7).

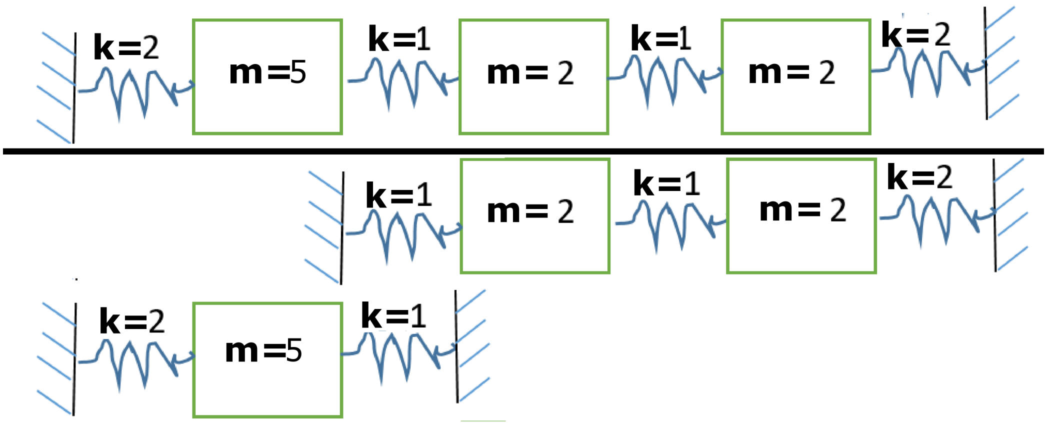

Figure 7: Top (above black line): Problem to solve. Middle: Subsystem 1. Bottom: Subsystem 2

The subsystems' mass matrices are:

$$\mathbf{M}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \; ; \; \mathbf{M}_2 = \begin{pmatrix} 5 \end{pmatrix} \tag{43}$$

and the subsystems' stiffness matrices:

$$\mathbf{K}_1 = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \; ; \; \mathbf{K}_2 = \begin{pmatrix} 3 \end{pmatrix} \tag{44}$$

The eigenmodes of subsystems 1 and 2 are

$$\mathbf{\Phi}_1 = \begin{pmatrix} -0.5257 & -0.8507 \\ -0.8507 & 0.5257 \end{pmatrix} \; ; \; \mathbf{\Phi}_2 = \begin{pmatrix} 1 \end{pmatrix} \tag{45}$$

and their corresponding eigenvalues

$$\mathbf{\Lambda}_1 = \begin{pmatrix} 0.6910 & 0 \\ 0 & 1.8090 \end{pmatrix} \; ; \; \mathbf{\Lambda}_2 = \begin{pmatrix} 0.6000 \end{pmatrix} \tag{46}$$

The coupled system stiffness and mass matrices are:

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \; ; \; \mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{47}$$

Since the mass matrix is positive definite and diagonal the generalized eigenvalues problem may be converted to an standard eigenvalues problem.Thence we have to find the eigenvalues and eigenvectors of:

$$\mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1 & 0.5 \\ 0 & 0.2 & 0.6 \end{pmatrix} \tag{48}$$

Eq. (48) may be decomposed as

$$\mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 0.6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.2 & 0 \end{pmatrix} \tag{49}$$

Now, identifying the decomposition terms as

$$\mathbf{H} = \begin{pmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1 & 0.5 \\ 0 & 0.2 & 0.6 \end{pmatrix} ; \mathbf{B} = \begin{pmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 0.6 \end{pmatrix} ; \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.2 & 0 \end{pmatrix} \tag{50}$$

it is clear that this decomposition corresponds with the proposed problem approach, where the **H** matrix eigenvectors and eigenvalues correspond to the eigenmodes and eigenfrequencies of the full system, the **B** matrix contains the decoupled subsystem matrices and **C** is the coupling matrix.

Notice that the coefficients in matrix $\mathbf{C}$, $c_{PQ} = 0.5$ and $c_{QP} = 0.2$, are normalised by the contiguous masses.

Then, following the notation used in the derivations in Section 3.1

$$\mathbf{V}_B = \begin{pmatrix} -0.5257 & -0.8507 & 0 \\ -0.8507 & 0.5257 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \mathbf{D}_B = \begin{pmatrix} 0.6910 & 0 & 0 \\ 0 & 1.8090 & 0 \\ 0 & 0 & 0.6 \end{pmatrix} \tag{51}$$

$$\mathbf{u} = \begin{pmatrix} -0.8507 \\ 0.5257 \\ 0 \end{pmatrix}; \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \mathbf{v}^T = \mathbf{w}^T; \mathbf{y}^T = \mathbf{u}^T \tag{52}$$

the solution are the zeroes of:

$$\begin{aligned} 1 =&(0.5)(0.2) \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda-0.6910} & 0 & 0 \\ 0 & \frac{1}{\lambda-1.8} & 0 \\ 0 & 0 & \frac{1}{\lambda-0.6} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &\begin{pmatrix} -0.8507 & 0.5257 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda-0.6910} & 0 & 0 \\ 0 & \frac{1}{\lambda-1.8} & 0 \\ 0 & 0 & \frac{1}{\lambda-0.6} \end{pmatrix} \begin{pmatrix} -0.8507 \\ 0.5257 \\ 0 \end{pmatrix} \end{aligned} \tag{53}$$

that is, the zeroes of:

$$1 = 0.1\frac{1}{\lambda - 0.6}\left(\frac{0.8507^2}{\lambda - 0.6910} + \frac{0.5257^2}{\lambda - 1.8}\right) \tag{54}$$

In Fig. 8 it can be observed that the zeroes of Eq. (54) coincide with the eigenvalues of the system, namely 0.3614, 0.9060 and 1.8327. In this figure black circles represent subsystem eigenvalues and black squares represent coupled system eigenvalues.

Given the significant frequency shift of the eigenvalues for coupled system vs. that of the decoupled subsystems, the interest in knowing the resonant frequencies of the coupled system is justified, since the excitation frequency may then coincided with a resonant frequency of the coupled system.

5.2 Two plates connected by five springs

The following example illustrates the performance of the method on a continuous system which is discretised, giving a mesh of 13x23 degrees of freedom.

The feature of having several degrees of freedom connected trough springs is also introduced in this example having 5 connections.

The model involves two plates with dimensions: 1.3 m length, 2.3 m width and 20 cm thickness and material properties as indicated in Table 1.

In Fig. 10, the continuous line is $\Gamma(\lambda)$ as defined in Eq. (39), computed over a λ sweep from 0 to 4 with $1E-3$ incremental step. In a real case the zone to explore will be defined by the nature of the problem, engine RPM, fan characteristics, etc. As it is known already, its zero value crossings correspond to the modal frequencies of the system, $*$ symbols indicate these zero crossings, which are the eigenvalues of the system.

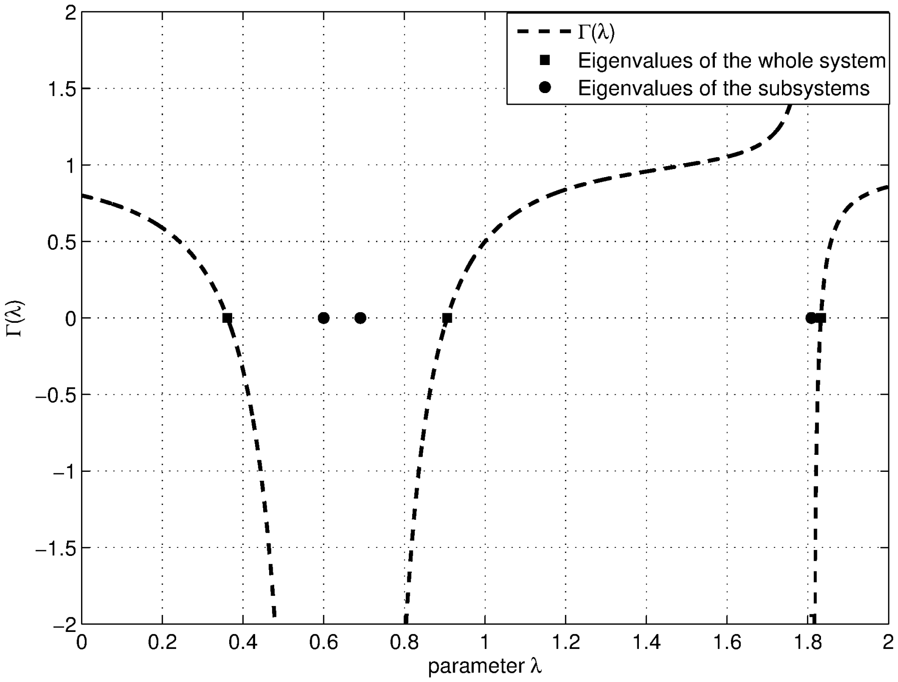

Figure 8: The total system eigenvalues calculated using standard methods (black squares) correspond to the zeroes of the $\Gamma(\lambda)$ function. The eigenvalues of the subsystems (black circles) are also represented.

Material	ρ_v (kg/m^3)	ν	E (Pa)	L_x (m)	L_y (m)	h (m)
Concrete	2000	0.2	$2 \cdot 10^{10}$	2.3	1.3	0.2
Concrete	2000	0.2	$2 \cdot 10^{10}$	2.3	1.3	0.2

Table 1: Geometric and mechanical properties of the two plates (ρ_v is the volumetric density, ν is the Poisson's ratio, E is the Young modulus, L_x, L_y the plate dimensions and h the plate thickness).

Spring	K ($\frac{\mathrm{N}}{\mathrm{m}}$)	x (m)	y (m)
1	$2 \cdot 10^5$	0.3	0.4
2	$2 \cdot 10^5$	0.6	0.1
3	$2 \cdot 10^5$	0.2	0.4
4	$2 \cdot 10^5$	0.5	0.5
5	$2 \cdot 10^5$	0.1	0.1

Table 2: Stiffness K and position of each spring that connects the plates.

The accuracy of the obtained result is analysed in Fig. 11, which shows on top the eigenvalues computed through $\Gamma(\lambda)$ (abscissa axis) versus the eigenvalues of the

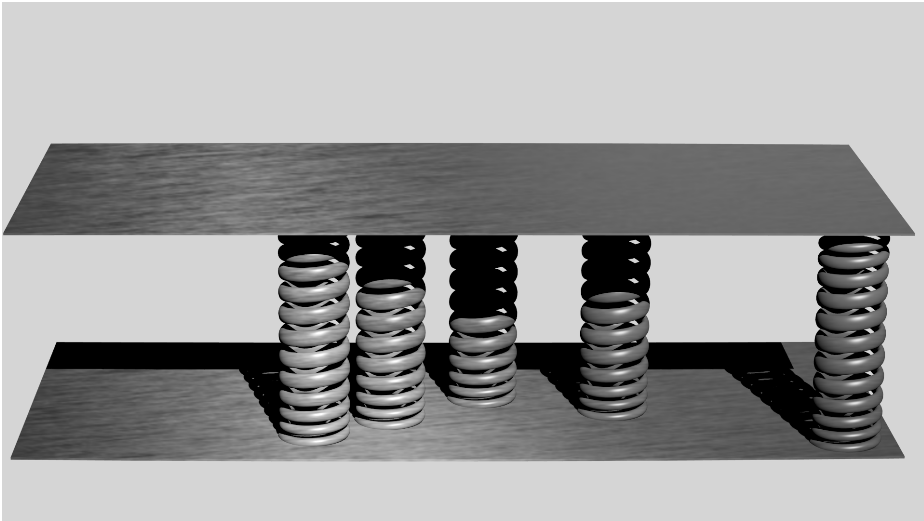

Figure 9: Example of continuous system with several non-holonomic coupling connections. Two plates connected through 5 springs.

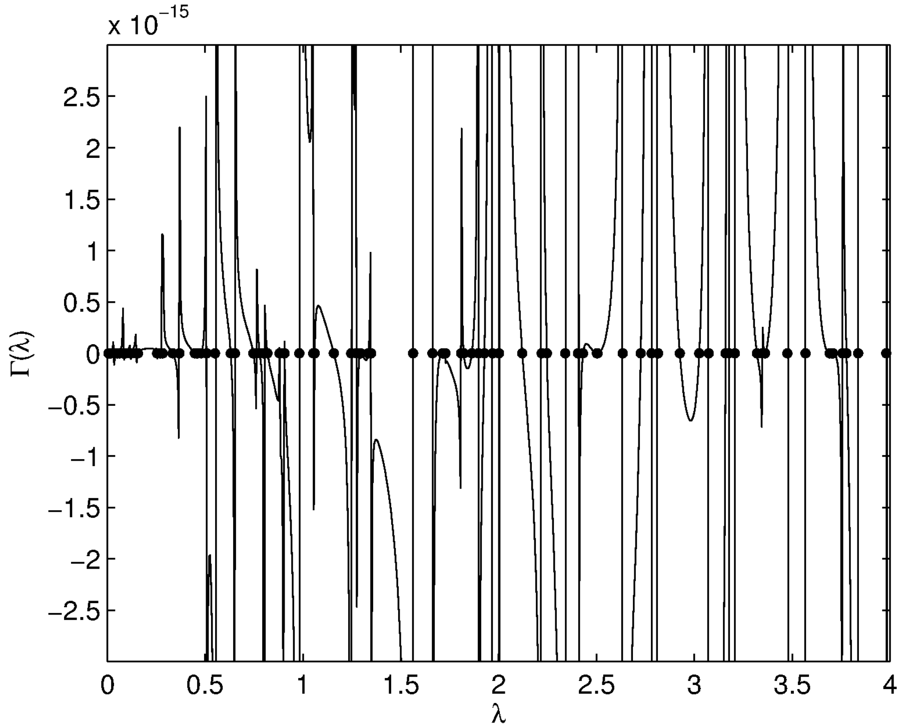

Figure 10: ——$\Gamma(\lambda)$ characteristic function whose zeros are the system eigenvalues, $*$ Eigenvalues of the system calculated using standard numerical methods

system directly compute with standard eigenvalue solver (ordinate axis). The scatter of resulting points is then least squares approximated by a straight line with equation y = 0.99999x + 0.00066, meaning that the correlation of the method results to those of a standard eigensolver is 0.99998, which is regarded as an optimal result.

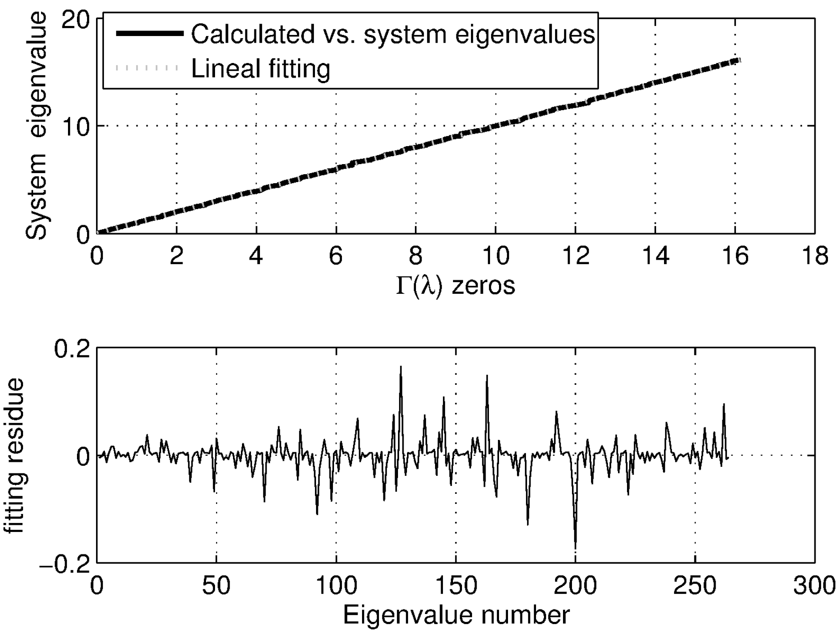

Figure 11: Linear correlation between the system eigenvalues and the $\Gamma(\lambda)$ zeros. Top curve: Linear approximation. Bottom curve: Differences between the linear approximation and the value calculated with this paper method (fitting residue).

Fig. 12 shows the relative error in the eigenmodes. Such relative error is computed in vector form as the difference between the eigenvectors computed through the gamma function and the eigenvectors obtained with standard solver, each of them normalised by the norm of the (standardly computed) eigenvector. In all cases the error is below $1E-9$.

6 Conclusions

For two coupled subsystems, a characteristic function $\Gamma(\lambda)$ whose zeroes correspond to the characteristic values, the eigenfrequencies of the coupled system has been derived, in which $\Gamma(\lambda)$ is the determinant of a matrix of order equal to the number of subsystem connections, typically a smaller number than the subsystems' number of degrees of freedom.

The characteristic function $\Gamma(\lambda)$ depends on the eigenvalues and eigenvectors of each of the two subsystems as well as on the elastic constants of the springs which connect them. Yet, in the solution procedure of the proposed method, the subsystem eigenvalues and eigenvectors get combined into receptance functions. Therefore, for an experimental application of the method, it is not needed to know the eigenmodes and eigenfrequencies of the subsystems. Knowing the receptances between the coupling degrees of freedom suffices.

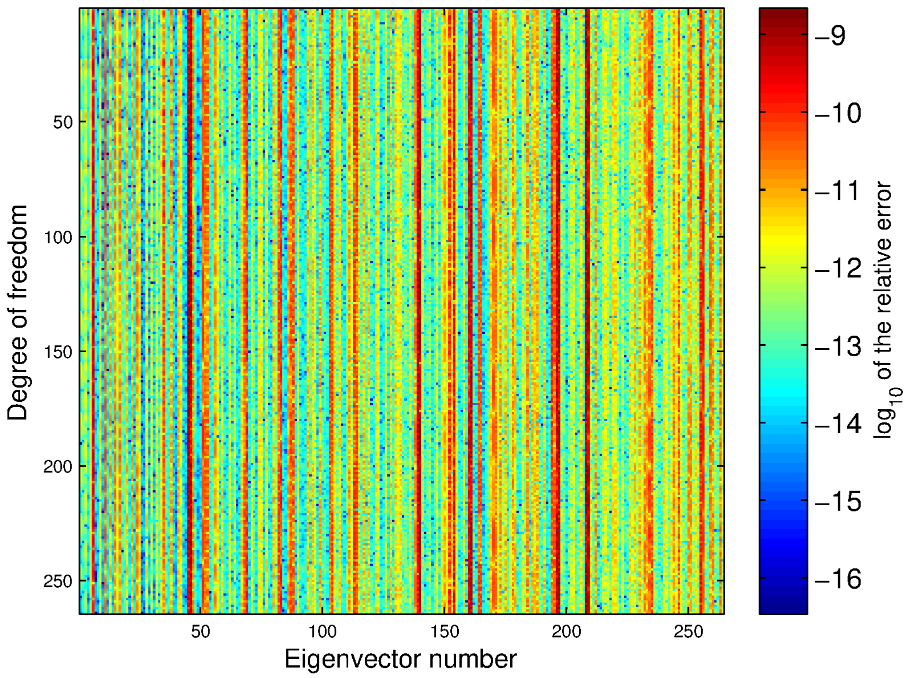

Figure 12: Relative error on the obtained eigenvectors vs. system ones

Furthermore, it has been demonstrated the procedure for obtaining the corresponding eigenvectors based on the matrix whose determinant generates $\Gamma(\lambda)$.

Finally, the method has been applied to two examples, a discrete system with one single connection and a continuous system with 5 connections. In both examples the proposed method shows fully satisfactory performance.

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