A NOTE ON FINITE LATTICES WITH MANY CONGRUENCES

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n-element lattices with exactly 2^{n-2} congruences. Finally, we point out that if the congruence lattice of an n-element algebra A is distributive, then A has at most 2^{n-1} congruences; furthermore, if this maximum number is reached, then the congruence lattice of A is boolean.

1. Introduction and motivation

It follows from Lagrange's Theorem that the size $|S|$ of an arbitrary subgroup S of a finite group G is either $|G|$, or it is at most the half of the maximum possible value, $|G|/2$. Furthermore, if the size of S is the half of its maximum possible value, then S has some special property since it is normal. Our goal is to prove something similar on the size of the congruence lattice $\mathrm{Con}(L)$ of an n-element lattice L.

For a *finite* lattice L, the relation between $|L|$ and $|\mathrm{Con}(L)|$ has been studied in some earlier papers, including Freese [5], Grätzer and Knapp [11], Grätzer, Lakser, and Schmidt [12], Grätzer, Rival, and Zaguia [13]. In particular, part (i) of Theorem 1 below is due to Freese [5]. Although Czédli and Mureşan [4] and Mureşan [14] deal only with infinite lattices, they are also among the papers motivating the present one. We will conclude the paper with some remarks on finite algebras distinct from lattices.

2. Our result on lattices and its proof

Mostly, we follow the terminology and notation of Grätzer [8]. In particular, the *glued sum* $L_0 \dot{+} L_1$ of finite lattices L_0 and L_1 is their Hall–Dilworth gluing along $L_0 \cap L_1 = \{1_{L_0}\} = \{0_{L_1}\}$; see, for example, Grätzer [8, Section IV.2]. Note that $\dot{+}$ is an associative operation. Our result is the following.

Theorem 1. *If L is a finite lattice of size $n = |L|$, then the following hold.*

(i) *L has at most 2^{n-1} many congruences. Furthermore, $|\mathrm{Con}(L)| = 2^{n-1}$ if and only if L is a chain.*

(ii) *If L has* less than *2^{n-1} congruences, then it has at most $2^{n-1}/2 = 2^{n-2}$ congruences.*

(iii) *$|\mathrm{Con}(L)| = 2^{n-2}$ if and only if L is of the form $C_1 \dot{+} B_2 \dot{+} C_2$ such that C_1 and C_2 are chains and B_2 is the four-element Boolean lattice.*

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For $n = 8$, part (iii) of this theorem is illustrated in Figure 1. Note that part (i) of the theorem is due to Freese [5, page 3458]; however, as a by-product of our approach leading to parts (ii) and (iii) of Theorem 1, this paper also includes a proof of part (i).

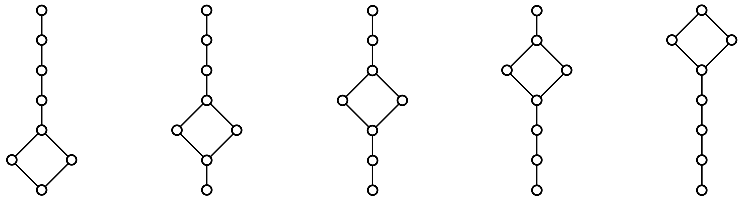

FIGURE 1. The full list of 8-element lattices with exactly $64 = 2^{8-2}$ many congruences

Proof of Theorem 1. We prove the theorem by induction on $n = |L|$. Since the case $n = 1$ is clear, assume as an induction hypothesis that $n > 1$ is a natural number and all the three parts of the theorem hold for every lattice with size less than n. Let L be a lattice with $|L| = n$. For $\langle a, b\rangle \in L^2$, the least congruence collapsing a and b will be denoted by $\mathrm{con}(a, b)$. A *prime interval* or an *edge* of L is an interval $[a, b]$ with $a \prec b$. For later reference, note that

$$\text{Con}(L) \text{ has an atom, and every of its atoms is of the form } \mathrm{con}(a, b) \text{ for some prime interval } [a, b]; \tag{2.1}$$

this follows from the finiteness of $\mathrm{Con}(L)$ and from the fact that every congruence on L is the join of congruences generated by *covering pairs* of elements; see also Grätzer [10, page 39] for this folkloric fact.

Based on (2.1), pick a prime interval $[a, b]$ of L such that $\Theta = \mathrm{con}(a, b)$ is an atom in $\mathrm{Con}(L)$. Consider the map $f\colon \mathrm{Con}(L) \to \mathrm{Con}(L)$ defined by $\Psi \mapsto \Theta \vee \Psi$. We claim that, with respect to f,

$$\text{every element of } f(\mathrm{Con}(L)) \text{ has at most two preimages.} \tag{2.2}$$

Suppose to the contrary that there are pairwise distinct $\Psi_1, \Psi_2, \Psi_3 \in \mathrm{Con}(L)$ with the same f-image. Since the $\Theta \wedge \Psi_i$ belong to the two-element principal ideal $\downarrow\Theta := \{\Gamma \in \mathrm{Con}(L) : \Gamma \le \Theta\}$ of $\mathrm{Con}(L)$, at least two of these meets coincide. So we can assume that $\Theta\wedge\Psi_1 = \Theta\wedge\Psi_2$ and, of course, we have that $\Theta\vee\Psi_1 = f(\Psi_1) = f(\Psi_2) = \Theta \vee \Psi_2$. This means that both Ψ_1 and Ψ_2 are relative complements of Θ in the interval $[\Theta \wedge \Psi_1, \Theta \vee \Psi_1]$. According to a classical result of Funayama and Nakayama [7], $\mathrm{Con}(L)$ is distributive. Since relative complements in distributive lattices are well-known to be unique, see, for example, Grätzer [8, Corollary 103], it follows that $\Psi_1 = \Psi_2$. This is a contradiction proving (2.2).

Clearly, f is a retraction map onto the filter $\uparrow\Theta$. It follows from (2.2) that $|{\uparrow\Theta}| \ge |\mathrm{Con}(L)|/2$. Also, by the well-known Correspondence Theorem, see Burris and Sankappanawar [2, Theorem 6.20], or see Theorem 5.4 (under the name Second Isomorphism Theorem) in Nation [15], $|{\uparrow\Theta}| = |\mathrm{Con}(L/\Theta)|$ holds. Hence, it follows that

$$|\mathrm{Con}(L/\Theta)| \ge \frac{1}{2} \cdot |\mathrm{Con}(L)|. \tag{2.3}$$

Since Θ collapses at least one pair of distinct elements, $\langle a, b\rangle$, we conclude that $|L/\Theta| \leq n-1$. Thus, it follows from part (i) of the induction hypothesis that $|\mathrm{Con}(L/\Theta)| \leq 2^{(n-1)-1} = 2^{n-2}$. Combining this inequality with (2.3), we obtain that $|\mathrm{Con}(L)| \leq 2 \cdot |\mathrm{Con}(L/\Theta)| \leq 2^{n-1}$. This shows the first half of part (i).

For later reference, note that we have not used that $[a, b]$ is a prime interval; this will be used only later. We only needed that $\mathrm{con}(a, b) \in \mathrm{Con}(L)$ was an atom and $\mathrm{Con}(L)$ was distributive. Hence, the same proof as above gives that

$$\text{if } A \text{ is an } n\text{-element algebra such that } \mathrm{Con}(A) \text{ is a distributive lattice, then } |\mathrm{Con}(A)| \leq 2^{n-1}. \tag{2.4}$$

If L is a chain, then $\mathrm{Con}(L)$ is known to be the 2^{n-1}-element boolean lattice; see, for example, Grätzer [10, Corollaries 3.11 and 3.12]. Hence, we have that $|\mathrm{Con}(L)| = 2^{n-1}$ if L is a chain. Conversely, assume the validity of $|\mathrm{Con}(L)| = 2^{n-1}$, and let $k = |L/\Theta|$. By the induction hypothesis, $|\mathrm{Con}(L/\Theta)| \leq 2^{k-1}$. On the other hand, $|\mathrm{Con}(L/\Theta)| \geq |\mathrm{Con}(L)|/2 = 2^{n-2}$ holds by (2.3). These two inequalities and $k < n$ yield that $k = n - 1$ and also that $|\mathrm{Con}(L/\Theta)| = 2^{n-2} = 2^{k-1}$. Hence, the induction hypothesis implies that L/Θ is a chain. For the sake of contradiction, suppose that L is not a chain, and pick a pair $\langle u, v\rangle$ of incomparable elements of L. The Θ-blocks u/Θ and v/Θ are comparable elements of the chain L/Θ, whence we can assume that $u/\Theta \leq v/\Theta$. It follows that $u/\Theta = u/\Theta \wedge v/\Theta = (u \wedge v)/\Theta$ and, by duality, $v/\Theta = (u \vee v)/\Theta$. Thus, since u, v, $u \wedge v$ and $u \vee v$ are pairwise distinct elements of L and Θ collapses both of the pairs $\langle u \wedge v, u\rangle$ and $\langle v, u \vee v\rangle$, we have that $k = |L/\Theta| \leq n - 2$, which is a contradiction. This proves part (i) of the theorem.

As usual, for a lattice K, let $J(K)$ and $M(K)$ denote the set of nonzero *join-irreducible elements* and the set of *meet-irreducible elements* distinct from 1, respectively. By a *narrows* we will mean a prime interval $[a, b]$ such that $a \in M(L)$ and $b \in J(L)$. Using Grätzer [9], it follows in a straightforward way that

$$\text{if } [a, b] \text{ is a narrows, then } \{a, b\} \text{ is the only non-singleton block of } \mathrm{con}(a, b). \tag{2.5}$$

Now, in order to prove part (ii) of the theorem, assume that $|\mathrm{Con}(L)| < 2^{n-1}$. By (1), we can pick a prime interval $[a, b]$ such that $\Theta := \mathrm{con}(a, b)$ is an atom in $\mathrm{Con}(L)$. There are two cases to consider depending on whether $[a, b]$ is a narrows or not; for later reference, some parts of the arguments for these two cases will be summarized in (2.6) and (2.7) redundantly. First, we deal with the case where $[a, b]$ is a narrows. We claim that

$$\text{if } |\mathrm{Con}(L)| < 2^{n-1},\ [a, b] \text{ is a narrows, and } \Theta = \mathrm{con}(a, b) \text{ is an atom in } \mathrm{Con}(L), \text{ then } L/\Theta \text{ is not a chain.} \tag{2.6}$$

By (2.5), $|L/\Theta| = n - 1$. By the already proved part (i), L is not a chain, whence there are $u, v \in L$ such that $u \parallel v$. We claim that u/Θ and v/Θ are incomparable elements of L/Θ. Suppose the contrary. Since u and v play a symmetric role, we can assume that $u/\Theta \vee v/\Theta = v/\Theta$, i.e., $(u \vee v)/\Theta = v/\Theta$. But $u \vee v \neq v$ since $u \parallel v$, whereby (2.5) gives that $\{v, u \vee v\} = \{a, b\}$. Since $a < b$, this means that $v = a$ and $u \vee v = b$. Thus, $u \vee v \in J(L)$ since $[a, b]$ is a narrows. The membership $u \vee v \in J(L)$ gives that $u \vee v \in \{u, v\}$, contradicting $u \parallel v$. This shows that $u/\Theta \parallel v/\Theta$, whence L/Θ is not a chain. We have shown the validity of (2.6). Using part (i) and $|L/\Theta| = n-1$, it follows that $|\mathrm{Con}(L/\Theta)| < 2^{(n-1)-1}$. By the induction hypothesis, we can apply (ii) to L/Θ to conclude that $|\mathrm{Con}(L/\Theta)| \le 2^{(n-1)-2}$. This inequality and (2.3) yield that $|\mathrm{Con}(L)| \le 2 \cdot |\mathrm{Con}(L/\Theta)| \le 2^{n-2}$, as required.

Second, assume that $[a, b]$ is not a narrows. Our immediate plan is to show that

$$\text{if a prime interval } [a,b] \text{ of } L \text{ is not a narrows and } \Theta = \mathrm{con}(a,b), \text{ then } |L/\Theta| \le n-2. \tag{2.7}$$

By duality, we can assume that a is meet-reducible. Hence, we can pick an element $c \in L$ such that $a \prec c$ and $c \ne b$. Clearly, $c \ne b \vee c$ and $\Theta = \mathrm{con}(a, b)$ collapses both $\langle a, b\rangle$ and $\langle c, b \vee c\rangle$, which are distinct pairs. Thus, we obtain that $|L/\Theta| \le n - 2$, proving (2.7). Hence, $\mathrm{Con}(L/\Theta) \le 2^{n-3}$ by part (i) of the induction hypothesis. Combining this inequality with (2.3), we obtain the validity of the required inequality $\mathrm{Con}(L) \le 2^{n-2}$. This completes the induction step for part (ii).

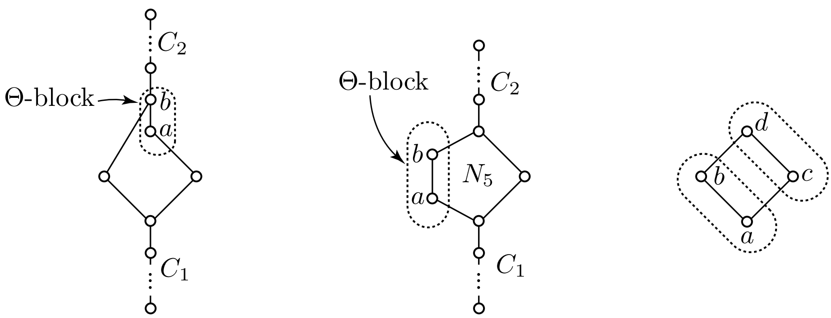

FIGURE 2. Illustrations for the proof

Next, in order to perform the induction step for part (iii), we assume that $|\mathrm{Con}(L)| = 2^{n-2}$. Again, there are two cases to consider. First, we assume that there exists a narrows $[a, b]$ in L such that $\Theta := \mathrm{con}(a, b)$ is an atom in $\mathrm{Con}(L)$. Then $|L/\Theta| = n - 1$ by (2.5) and L/Θ is not a chain by (2.6). By the induction hypothesis, parts (i) and (ii) hold for L/Θ, whereby we have that $|\mathrm{Con}(L/\Theta)| \le 2^{(n-1)-2} = 2^{n-3}$. On the other hand, it follows from (2.3) that $|\mathrm{Con}(L/\Theta)| \ge |\mathrm{Con}(L)|/2 = 2^{n-3}$. Hence, $|\mathrm{Con}(L/\Theta)| = 2^{n-3} = 2^{|L/\Theta|-2}$. By the induction hypothesis, L/Θ is of the form $C_1 \dot{+} B_2 \dot{+} C_2$. We know from (2.5) that $\{a, b\} = [a, b]$ is the unique non-singleton Θ-block. If this Θ-block is outside B_2, then L is obviously of the required form. If the Θ-block $\{a, b\}$ is in $C_2 \cap B_2$, then L is of the required form simply because the situation on the left of Figure 2 would contradict the fact that $[a, b]$ is a narrows. A dual treatment applies for the case $\{a, b\} \in C_1 \cap B_2$. If the Θ-block $\{a, b\}$ is in $B_2 \setminus (C_1 \cup C_2)$, then L is of the form $C_1 \dot{+} N_5 \dot{+} C_2$, where N_5 is the "pentagon"; see the middle part of Figure 2. For an arbitrary bounded lattice K and the two-element chain $\mathbf{2}$, it is straightforward to see that

$$\mathrm{Con}(K \dot{+} \mathbf{2}) \cong \mathrm{Con}(\mathbf{2} \dot{+} K) \cong \mathrm{Con}(K) \times \mathbf{2}. \tag{2.8}$$

A trivial induction based on (2.8) yields that $|\mathrm{Con}(C_1 \dot{+} N_5 \dot{+} C_2)|$ is divisible by $5 = |\mathrm{Con}(N_5)|$. But 5 does not divide $|\mathrm{Con}(L)| = 2^{n-2}$, ruling out the case that the Θ-block $\{a, b\}$ is in $B_2 \setminus (C_1 \cup C_2)$. Hence, L is of the required form.

Second, we assume that no narrows in L generates an atom of $\mathrm{Con}(L)$. By (2.1), we can pick a prime interval $[a, b]$ such that $\Theta := \mathrm{con}(a, b)$ is an atom of $\mathrm{Con}(L)$. Since $[a, b]$ is not a narrows, (2.7) gives that $|L/\Theta| \le n - 2$. We claim that we have equality here, that is, $|L/\Theta| = n-2$. Suppose to the contrary that $|L/\Theta| \le n-3$. Then part (i) and (2.3) yield that

$$2^{n-2} = |\mathrm{Con}(L)| \le 2 \cdot |\mathrm{Con}(L/\Theta)| \le 2 \cdot 2^{(n-3)-1} = 2^{n-3},$$

which is a contradiction. Hence, $|L/\Theta| = n-2$. Thus, we obtain from part (i) that $|\mathrm{Con}(L/\Theta)| \le 2^{n-3}$. On the other hand, (2.3) yields that $|\mathrm{Con}(L/\Theta)| \ge |\mathrm{Con}(L)|/2 = 2^{n-3}$, whence $|\mathrm{Con}(L/\Theta)| = 2^{n-3} = 2^{|L/\Theta|-1}$, and it follows by part (i) that L/Θ is a chain. Now, we have to look at the prime interval $[a, b]$ closely. It is not a narrows, whereby duality allows us to assume that b is not the only cover of a. So we can pick an element $c \in L \setminus \{b\}$ such that $a \prec c$, and let $d := b \vee c$; see on the right of Figure 2. Since $\langle c, d\rangle = \langle c\vee a, c\vee b\rangle \in \mathrm{con}(a, b) = \Theta$, any two elements of $[c, d]$ are collapsed by Θ. Using $\langle a, b\rangle \in \Theta$, $\langle c, d\rangle \in \Theta$, and $|L/\Theta| = n-2 = |L|-2$, it follows that there is no "internal element" in the interval $[c, d]$, that is, $c \prec d$. Furthermore, $[a, b] = \{a, b\}$ and $[c, d] = \{c, d\}$ are the only non-singleton blocks of Θ. In order to show that $b \prec d$, suppose to the contrary that $b < e < d$ holds for some $e \in L$. Since $d = b \vee c \le e \vee c \le d$, we have that $e \vee c = d$, implying $e \nleq c$. Hence, $c \wedge e < e$. Since $\langle c \wedge e, e\rangle = \langle c \wedge e, d \wedge e\rangle \in \Theta$, the Θ-block of e is not a singleton. This contradicts the fact that $\{a, b\}$ and $\{c, d\}$ are the only non-singleton Θ-blocks, whereby we conclude that $b \prec d$. The covering relations established so far show that $S := \{a = b\wedge c, b, c, d = b\vee c\}$ is a covering square in L. We know that both non-singleton Θ-blocks are subsets of S and L/Θ is a chain. Consequently, $L \setminus S$ is also a chain.

Hence, to complete the analysis of the second case when $[a, b]$ is not a narrows, it suffices to show that for every $x \in L \setminus S$, we have that either $x \le a$, or $x \ge d$. So, assume that $x \in L \setminus S$. Since L/Θ is a chain, $\{a, b\}$ and $\{x\}$ are comparable in L/Θ. If $\{x\} < \{a, b\}$, then $\{x\} \vee \{a, b\} = \{a, b\}$ gives that $x \vee a \in \{a, b\}$. If $x \vee a$ happens to equal b, then $x \nleq a$ leads to $x \wedge a < x$ and $\langle x \wedge a, x\rangle = \langle x \wedge a, x \wedge b\rangle \in \Theta$, contradicting the fact the $\{a, b\}$ and $\{c, d\}$ are the only non-singleton Θ-blocks. So if $\{x\} < \{a, b\}$, then $x \vee a = a$ and $x < a$, as required. Thus, we can assume that $\{x\} > \{a, b\}$. If $\{x\} > \{c, d\}$, then the dual of the easy argument just completed shows that $x \ge d$. So, we are left with the case $\{a, b\} < \{x\} < \{c, d\}$. Then the equalities $\{a, b\} \vee \{x\} = \{x\}$ and $\{x\} = \{x\} \wedge \{c, d\}$ give that $b \vee x = x = d \wedge x$, that is, $b \le x \le d$. But $x \notin S$, so $b < x < d$, contradicting $b \prec d$. This completes the second case of the induction step for part (iii) and the proof of Theorem 1. □

3. Remarks and problems on other finite algebras

We conclude the paper with some remarks and problems on finite algebras that are not necessarily lattices. Part (a) below has been pointed out in (2.4).

Remark 2. If A is an n-element algebra such that $\mathrm{Con}(A)$ is distributive, then the following two statements hold.

(a) $|\mathrm{Con}(A)| \le 2^{n-1}$.
(b) If $|\mathrm{Con}(A)| = 2^{n-1}$, then $\mathrm{Con}(A)$ is a boolean lattice.

First proof of Remark 2. As mentioned above, part (a) follows from (2.4). With straightforward changes, the same argument is appropriate to prove part (b); we outline this possibility as follows. Again, we use induction on n. If $|\mathrm{Con}(A)| = 2^{n-1}$, then the induction hypothesis together with (2.3) yield that $\uparrow\Theta$ is a boolean lattice and $|\uparrow\Theta| = 2^{n-2}$, whence the "at most" in (2.2) turns into "exactly". Hence, for each $\Psi \in {\uparrow}\Theta$, there is exactly one $g(\Psi) \in \mathrm{Con}(A)$ such that $g(\Psi) \neq \Psi = f(g(\Psi))$. Using that $|\mathrm{Con}(A)| = 2^{n-1} = 2 \cdot |{\uparrow}\Theta|$, it follows that $g({\uparrow}\Theta) = \{g(\Psi) : \Psi \in {\uparrow}\Theta\}$ is disjoint from ${\uparrow}\Theta$ and so $\mathrm{Con}(A)$ is the disjoint union of ${\uparrow}\Theta$ and $g({\uparrow}\Theta)$. Furthermore, g and the restriction $f\rceil_{g({\uparrow}\Theta)}$ of f to the subset $g({\uparrow}\Theta) = \mathrm{Con}(A) \setminus {\uparrow}\Theta$ are reciprocal bijections. For $\Psi \in g({\uparrow}\Theta)$, we have that $f\rceil_{g({\uparrow}\Theta)}(\Psi) = \Theta \vee \Psi$, whereby it follows from distributivity that $f\rceil_{g({\uparrow}\Theta)}$ is a lattice homomorphism from $g({\uparrow}\Theta)$ onto ${\uparrow}\Theta$, so it is an isomorphism. Since $\Psi < f\rceil_{g({\uparrow}\Theta)}(\Psi)$ for every $\Psi \in g(\Theta)$, we conclude that $\mathrm{Con}(A)$ is the direct product of the two-element chain and the boolean lattice ${\uparrow}\Theta$. Consequently, $\mathrm{Con}(A)$ is also boolean, proving part (b). □

As a preparation for another remark, we also give an alternative proof.

Second proof of Remark 2. The equivalence lattice $\mathrm{Equ}(A)$ is semimodular by Ore [16]; see also Grätzer [8, Theorem 404]. Let $\emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_n = A$ be a maximal chain of subsets of A and denote by Δ_A the equality relation on A. Then $\{\Delta_A \cup (X_i \times X_i) : 1 \leq i \leq n\}$ is a maximal chain of length $n - 1$ in $\mathrm{Equ}(A)$. By semimodularity, $\mathrm{Equ}(A)$ has no longer chain, and neither has $\mathrm{Con}(A)$ since it is a sublattice of $\mathrm{Equ}(A)$. Finally, we know, say, from Grätzer [8, Lemma 170 and Corollaries 169 and 171] that a distributive lattice of length $n-1$ has at most 2^{n-1} elements and we have equality only in the boolean case. □

It follows easily from Freese and Nation [6] that parts (a) and (b) above hold even if A is an n-element semilattice, where $\mathrm{Con}(A)$ is not distributive in general; see also Czédli [3]. This fact and the second proof above raise the problem how to relax the assumption that $\mathrm{Con}(A)$ is distributive if we want to ensure the validity of parts (a) and (b) of Remark 2.

Denote by $B(n) = |\mathrm{Equ}(\{1, 2, \ldots, n\})|$ the n-th *Bell number*; see Bell [1] and Rota [17]. For example, $B(5) = 52$ and $B(6) = 203$; see [1, page 540]; these equalities show that $B(n)$ is much larger than 2^{n-1}. Hence, any meaningful generalization of Remark 2 must exclude that $\mathrm{Con}(A) = \mathrm{Equ}(A)$. Since, for n large enough, every element of $\mathrm{Equ}(A)$ is meet reducible or join-reducible with high multiplicity, we cannot leave only few elements from $\mathrm{Equ}(A)$ to get a proper sublattice. This means that the difference $B(n) - |\mathrm{Con}(A)|$ cannot be too small. This difference can be even larger than what the lattice theoretical analysis of $\mathrm{Equ}(A)$ gives, because many sublattices of $\mathrm{Equ}(A)$ cannot be congruence lattices of A; this follows easily from Zádori [18]. As a second problem, we are far from finding the largest number $m(n)$ in the set $\{|\mathrm{Con}(A)| : A \text{ is an } n\text{-element algebra and } \mathrm{Con}(A) \neq \mathrm{Equ}(A)\}$. All we know is a lower bound given in the following remark; this remark and the inequality $1 + B(6 - 1) = 53 > 2^{6-1}$ will show that $m(n)$ is much larger than 2^{n-1} in general.

Remark 3. For every integer $n \geq 2$, there exists an n-element algebra $\langle A; F\rangle$ such that $\mathrm{Con}(\langle A; F\rangle) \neq \mathrm{Equ}(A)$ and $|\mathrm{Con}(\langle A; F\rangle)| = 1 + B(n - 1)$.

Proof. Fix an element $u \in A$ and let $H := A \setminus \{u\}$. A pair $\langle a, b\rangle$ is *nontrivial* if $a \neq b$. For each nontrivial pair $\langle a, b\rangle \in H^2$, define the following unary operation:

$$f_{a,b} \colon A^2 \to A, \quad , x \mapsto \begin{cases} a & \text{if } x \neq u \\ b & \text{if } x = u. \end{cases}$$

Let $F := \{f_{a,b} : \langle a, b\rangle \in H^2$ is a nontrivial pair$\}$. We claim that

$$\text{for each } \Psi \in \mathrm{Equ}(H),\ \ \{\langle u, u\rangle\} \cup \Psi \in \mathrm{Con}(\langle A; F\rangle), \text{ and} \tag{3.1}$$

$$\text{if } \Theta \in \mathrm{Con}(\langle A; F\rangle) \text{ and the } \Theta\text{-block of } u \text{ is not a singleton, then } \Theta = A^2. \tag{3.2}$$

Every operation $f_{a,b}$ is constant on H and every nontrivial pair from $\langle u, u\rangle \cup \Psi$ belongs to H^2, whence (3.1) follows trivially. Assuming the premise of (3.2), pick an element $x \neq u$ in the Θ-block of u. Then $\langle a, b\rangle = \langle f_{a,b}(x), f_{a,b}(u)\rangle \in \Theta$ for every nontrivial pair $\langle a, b\rangle \in H^2$, implying $\Theta = A^2$ and (3.2). Finally, Remark 3 follows from (3.1) and (3.2). □

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