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Reading a Single Qubit System Using Weak Measurement with Variable Strength

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Abstract

The information contents of an unknown qubit system is usually read using sharp measurement. Sharp measurement is an irreversible operation that will cause the superposition to collapse to one of the two possible states in a probabilistic way. This paper will propose a quantum algorithm to read the information contents of an unknown qubit without applying sharp measurement on that qubit. A quantum feedback control scheme will be introduced where sharp measurement will be applied iteratively on an auxiliary qubit weakly entangled with the unknown qubit. It will shown that the information contents can be read by counting the outcomes from the sharp measurement on the auxiliary qubit which will make the amplitudes of the superposition move in a random walk manner. The effect of this operation on the unknown qubit can be reversed to decrease the disturbance introduced to the system. The strength of the weak measurement can then be defined and can be controlled using an arbitrary number of dummy qubits (virtual qubits) μ to be added to the system. This can slowdown the measurement process to an arbitrary scale to reach the effect of the sharp measurement after $O(\mu^2)$ measurements on the auxiliary qubit.

Keywords: Quantum algorithm; Sharp measurement; Weak measurement; Random walk; Quantum feedback control.
2010 MSC: 68Q12, 81P15, 81P16, 81P50

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1. Introduction

Reading the information contents of an unknown qubit system is essential during any computation process, e.g. examining the contents and quantum error corrections. The reading process of a quantum system is usually done by measurements. Quantum Measurement, strong measurement, or sharp measurement is widely believed to be an irreversible [9] operation that produce a probabilistic outcome by projecting the superposition of the possible states into a single state. Using strong measurement will destroy the original information contents of a qubit and might act as an error in this context.

It was shown in [17, 18, 19] that a measurement process can be logically or physically reversible. A measurement process is said to be logically reversible [17, 18] when the information about the pre-measurement state is preserved during the measurement [16] and can be recovered from the post-measurement state only if the post-measurement density operator and the outcome of the measurement can be used to fully calculate the pre-measurement density operator of the measured system, and so we can construct a logically reversible measurement for any sharp measurement that continuously approaches that sharp measurement with a decrease in the measurement error. A quantum measurement is said be physically reversible [18, 19] if the pre-measurement state can be restored from the post-measurement state in a probabilistic way using another reversing measurement so that the information about the system is preserved during the measurement process and the original state can be recovered using a physical process.

A physically reversible quantum measurement can be seen as a weak measurement where it was shown in [8, 6] that a quantum state post a partial-collapse measurement (weak measurement) can be recovered (uncollapsed) by adding a rotation and a second partial measurement with the same strength so that the extracted information from the partial-collapse measurement is erased, canceling the effect of both measurements. Physically reversible quantum mea-

surement has been used in [12] on a spin-1/2 system using a spin-1/2 probe trying to completely specify an unknown quantum state of a single system (see also erratum of Ref. [12]).

Quantum feedback control was first studied in quantum optics [21, 4, 14]. Quantum feedback control was shown to have many applications, e.g. cooling an atom in an optical cavity [15], measuring optical phase using adaptive measurements [11], the stabilization of a single qubit, prepared in one of two nonorthogonal states against dephasing noise [3], quantum error correction [7], entanglement generation using measurement[13], and quantum communication [5].

It was shown in [14, 3, 20, 1, 2] that to obtain information about a quantum system, quantum feedback control using weak measurement can be used where the timescale of the measurement process can be extended where it takes the form of a random walk towards the final outcome such that the more the system is disturbed by the measurement, the more information is obtained about that system.

In this paper, a quantum algorithm will be proposed to acquire an unknown qubit system in order to obtain information about it without applying sharp measurement. The algorithm will read the content of that qubit using a quantum feedback control scheme where the sharp measurement on an auxiliary qubit will give the effect of weak measurement on the unknown qubit due to weak entanglement. The algorithm will make the amplitudes of the superposition move in a random walk manner to decrease the disturbance on the system where the opposite steps of the random walk will have a reversal effect on that system. The proposed algorithm will show that the strength of the weak measurement can be controlled by controlling the amount of disturbance introduced by adding an arbitrary number of dummy qubits to the system. This can slowdown the measurement process to an arbitrary scale according to the amount of information needed such that the more we disturb the superposition, the more information we gain about it.

The paper is organized as follows: Section 2 defines the problem to be solved

by the proposed algorithm. Section 3 defines the partial negation operator that will be used to create weak entanglement between the unknown qubit and an auxiliary qubit. Section 4 proposes the algorithm to read the information contents of an unknown qubit without applying sharp measurement on that qubit. Section 5 shows that weak measurement applied on the unknown qubit by applying iterative measurements on the auxiliary qubit has a reversal effect when the random walk moves in opposite directions. Section 6 shows that the algorithm will preserve the stability state so that the random walk converges to the correct destination even if the random walk moves up to some specific number of steps in the wrong direction. Section 7 defines the strength of the weak measurement and shows that this strength can be controlled based on the number of dummy qubits added to the system. Section 8 discusses the case of partial gain of information about the unknown qubit. The paper ends up with a conclusion in Section 9.

2. Problem statement

Given a qubit $|\psi\rangle$ with unknown ϕ as follows,

$$|\psi\rangle = \cos(\phi)\,|0\rangle + \sin(\phi)\,|1\rangle\,. \tag{1}$$

It is required to know how close the qubit to either $|0\rangle$ or $|1\rangle$ without too much disturbance to the superposition, i.e. no projective measurement is allowed on that qubit since projective measurement will make the qubit collapses to either $|0\rangle$ with probability $\cos^2(\phi)$ or to $|1\rangle$ with probability $\sin^2(\phi)$.

3. Partial negation operator

Let X be the Pauli-X gate which is the quantum equivalent to the NOT gate. It can be seen as a rotation of the Bloch Sphere around the X-axis by π radians as follows,

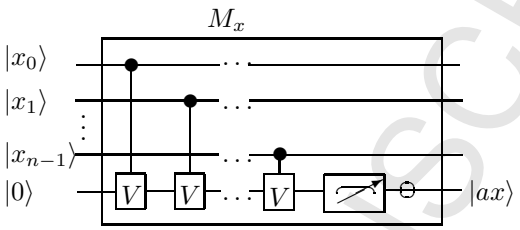

Figure 1: Quantum circuits for the M_x operator followed by a partial measurement then reset the auxiliary qubit $|ax\rangle$ to state $|0\rangle$.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

The c^{th} partial negation operator V is the c^{th} root of the X gate and can be calculated using diagonalization as follows,

$$V = \sqrt[c]{X} = \frac{1}{2} \begin{bmatrix} 1+t & 1-t \\ 1-t & 1+t \end{bmatrix}, \tag{3}$$

where $t = \sqrt[c]{-1}$, and applying V for d times on a qubit is equivalent to the operator,

$$V^d = \frac{1}{2} \begin{bmatrix} 1+t^d & 1-t^d \\ 1-t^d & 1+t^d \end{bmatrix}, \tag{4}$$

such that if $d = c$, then $V^d = X$.

The V gate will be used to define an operator M_x as follows [22], M_x is an operator on $n+1$ qubits register that applies V conditionally for n times on an auxiliary qubit initialized to state $|0\rangle$ and will be denoted as $|ax\rangle$. The number of times the V gate is applied on $|ax\rangle$ is based on the 1-density of a vector $|x_0 x_1 \dots x_{n-1}\rangle$, where the 1-density of a state vector is the number of qubits in state $|1\rangle$, as follows (as shown in Fig. 1),

$$M_x = Cont\_V(x_0; ax)Cont\_V(x_1; ax)\dots Cont\_V(x_{n-1}; ax), \tag{5}$$

where the $Cont\_V(x_j; ax)$ gate is a 2-qubit controlled gate with control qubit $|x_j\rangle$ and target qubit $|ax\rangle$. The $Cont\_V(x_j; ax)$ gate applies V conditionally on $|ax\rangle$ if $|x_j\rangle = |1\rangle$, so, if d is the 1-density of $|x_0x_1\dots x_{n-1}\rangle$ then,

$$M_x\left(|x_0x_1...x_{n-1}\rangle \otimes |0\rangle\right) = |x_0x_1...x_{n-1}\rangle \otimes \left(\frac{1+t^d}{2}|0\rangle + \frac{1-t^d}{2}|1\rangle\right), \tag{6}$$

and the probabilities of finding the auxiliary qubit $|ax\rangle$ in state $|0\rangle$ or $|1\rangle$ when measured is respectively as follows,

$$\begin{aligned} Pr(|ax\rangle = |0\rangle) &= \left|\tfrac{1+t^d}{2}\right|^2 = \cos^2\left(\tfrac{d\pi}{2c}\right), \\ Pr(|ax\rangle = |1\rangle) &= \left|\tfrac{1-t^d}{2}\right|^2 = \sin^2\left(\tfrac{d\pi}{2c}\right). \end{aligned} \tag{7}$$

4. The proposed algorithm

4.1. Register preparation

Given an unknown qubit $|\psi\rangle = \cos(\phi)|0\rangle + \sin(\phi)|1\rangle$, append a quantum register of $\mu+1$ qubits to $|\psi\rangle$, where the μ qubits are all initialized to state $|1\rangle$ and a single auxiliary qubit $|ax\rangle$ initialized to state $|0\rangle$ as follows,

$$\begin{aligned} |\psi_{ext}\rangle &= |\psi\rangle \otimes |1\rangle^{\otimes\mu} \otimes |0\rangle \\ &= \cos(\phi)\left(|0\rangle \otimes |1\rangle^{\otimes\mu} \otimes |0\rangle\right) + \sin(\phi)\left(|1\rangle \otimes |1\rangle^{\otimes\mu} \otimes |0\rangle\right) \\ &= \cos(\phi)\left(|\psi_0\rangle \otimes |0\rangle\right) + \sin(\phi)\left(|\psi_1\rangle \otimes |0\rangle\right). \end{aligned} \tag{8}$$

The number of the μ qubits is a free parameter that will be used to adjust the accuracy of the proposed algorithm according to our purposes as will be shown later.

4.2. The algorithm

When the operator M_x is applied on $|\psi_{ext}\rangle$, it gives,

$$M_x \left|\psi_{ext}\right\rangle = \cos\left(\phi\right)\left(\left|\psi_0\right\rangle \otimes \left(\frac{1+t^{d_0}}{2}\left|0\right\rangle + \frac{1-t^{d_0}}{2}\left|1\right\rangle\right)\right) + \sin\left(\phi\right)\left(\left|\psi_1\right\rangle \otimes \left(\frac{1+t^{d_1}}{2}\left|0\right\rangle + \frac{1-t^{d_1}}{2}\left|1\right\rangle\right)\right), \tag{9}$$

where d_0 is the 1-density of the state $|\psi_0\rangle$ and d_1 is the 1-density of the state $|\psi_1\rangle$, then $d_0 = \mu$ and $d_1 = \mu + 1$, the probabilities of finding the auxiliary qubit $|ax\rangle$ in state $|0\rangle$ or $|1\rangle$ when measured is respectively as follows,

$$\Pr_0\left(|\mathrm{ax}\rangle = |0\rangle\right) = \sin^2\left(\phi\right)\cos^2\left(\theta_1\right) + \cos^2\left(\phi\right)\cos^2\left(\theta_0\right), \tag{10}$$

$$\Pr_0\left(|\mathrm{ax}\rangle = |1\rangle\right) = \sin^2\left(\phi\right)\sin^2\left(\theta_1\right) + \cos^2\left(\phi\right)\sin^2\left(\theta_0\right), \tag{11}$$

where $\theta_0 = \frac{\pi d_0}{2c}$ and $\theta_1 = \frac{\pi d_1}{2c}$.

Applying the Algorithm on $|\psi_{ext}\rangle$ for $j \geq 1$ iterations with $j = j_0 + j_1$, such that j_0 counts how many times we found $|ax\rangle = |0\rangle$ and j_1 counts how many times we found $|ax\rangle = |1\rangle$, then the amplitudes of the system will be updated after each iteration according to the following recurrence relations, let the system at iteration $j \geq 1$ is as follows,

$$\left|\psi_{ext}^{j}\right\rangle = \alpha_j \left|\psi_0\right\rangle + \beta_j \left|\psi_1\right\rangle, \tag{12}$$

with $\alpha_0 = \cos(\phi)$ and $\beta_0 = \sin(\phi)$. The probability to find $|ax\rangle = |0\rangle$ or $|ax\rangle = |1\rangle$ is as follows,

$$\Pr_{\mathrm{j}}\left(|ax\rangle = |0\rangle\right) = \alpha_j^2\cos^2\left(\theta_0\right) + \beta_j^2\cos^2\left(\theta_1\right), \tag{13}$$

$$\Pr_{\mathrm{j}}\left(|ax\rangle = |1\rangle\right) = \alpha_j^2\sin^2\left(\theta_0\right) + \beta_j^2\sin^2\left(\theta_1\right). \tag{14}$$

When measurement is applied on $|ax\rangle$, if we find $|ax\rangle = |0\rangle$ then the amplitudes of the system will be updated as follows,

$$\alpha_{j+1} = \frac{\alpha_j \cos\left(\theta_0\right)}{\sqrt{\Pr_j\left(ax = 0\right)}}, \tag{15}$$

Algorithm 1 Measurement Based Quantum Random Walk

Prepare $|\psi_{ext}\rangle$
Let $j_0 = 0$
Let $j_1 = 0$
for $counter = 1 \rightarrow r$ **do**
    Apply the operator M_x on $|\psi_{ext}\rangle$.
    Measure $|ax\rangle$
    if $|ax\rangle = |1\rangle$ **then**
        $j_1 = j_1 + 1$
    else
        $j_0 = j_0 + 1$
    end if
    Reset $|ax\rangle$ to state $|0\rangle$
end for
if $j_1 > j_0$ **then**
    The qubit $|\psi\rangle$ is closer to state $|1\rangle$
else
    The qubit $|\psi\rangle$ is closer to state $|0\rangle$
end if

$$\beta_{j+1} = \frac{\beta_j \cos(\theta_1)}{\sqrt{\Pr_j(ax=0)}}, \tag{16}$$

and if we find $|ax\rangle = |1\rangle$ then the amplitudes of the system will be updated as follows,

$$\alpha_{j+1} = \frac{\alpha_j \sin(\theta_0)}{\sqrt{\Pr_j(ax=1)}}, \tag{17}$$

$$\beta_{j+1} = \frac{\beta_j \sin(\theta_1)}{\sqrt{\Pr_j(ax=1)}}. \tag{18}$$

The following equations are the closed forms of the above recurrence relations such that $\Pr_j(|\psi_0\rangle) = \alpha_j^2$ and $\Pr_j(|\psi_1\rangle) = \beta_j^2$. The probabilities of finding the auxiliary qubit $|ax\rangle$ in state $|0\rangle$ or $|1\rangle$ when measured is respectively as follows,

$$\Pr_j(|ax\rangle = |0\rangle) = \frac{\sin^2(\phi)\cos^{2(j_0+1)}(\theta_1)\sin^{2j_1}(\theta_1) + \cos^2(\phi)\cos^{2(j_0+1)}(\theta_0)\sin^{2j_1}(\theta_0)}{\sin^2(\phi)\cos^{2j_0}(\theta_1)\sin^{2j_1}(\theta_1) + \cos^2(\phi)\cos^{2j_0}(\theta_0)\sin^{2j_1}(\theta_0)} \tag{19}$$

$$\Pr_j(|ax\rangle = |1\rangle) = \frac{\sin^2(\phi)\cos^{2j_0}(\theta_1)\sin^{2(j_1+1)}(\theta_1) + \cos^2(\phi)\cos^{2j_0}(\theta_0)\sin^{2(j_1+1)}(\theta_0)}{\sin^2(\phi)\cos^{2j_0}(\theta_1)\sin^{2j_1}(\theta_1) + \cos^2(\phi)\cos^{2j_0}(\theta_0)\sin^{2j_1}(\theta_0)} \tag{20}$$

and the probabilities of states $|\psi_0\rangle$ and $|\psi_1\rangle$ will be changed according to the outcome of the measurement on $|ax\rangle$, i.e. j_1 will be incremented by 1 if $|ax\rangle = |1\rangle$, and j_0 will be incremented by 1 if $|ax\rangle = |0\rangle$, so the probabilities of states $|\psi_0\rangle$ and $|\psi_1\rangle$ after $j \geq 1$ iterations will be as follows,

$$\Pr_j(|\psi_0\rangle) = \frac{\cos^2(\phi)\cos^{2j_0}(\theta_0)\sin^{2j_1}(\theta_0)}{\sin^2(\phi)\cos^{2j_0}(\theta_1)\sin^{2j_1}(\theta_1) + \cos^2(\phi)\cos^{2j_0}(\theta_0)\sin^{2j_1}(\theta_0)} \tag{21}$$

$$\Pr_j(|\psi_1\rangle) = \frac{\sin^2(\phi)\cos^{2j_0}(\theta_1)\sin^{2j_1}(\theta_1)}{\sin^2(\phi)\cos^{2j_0}(\theta_1)\sin^{2j_1}(\theta_1) + \cos^2(\phi)\cos^{2j_0}(\theta_0)\sin^{2j_1}(\theta_0)} \tag{22}$$

The first aim of the algorithm is to make the measurement on $|ax\rangle$ has a weak effect on the probabilities of $|\psi\rangle$, i.e. weak measurement. This can be done by setting c in M_x such that $c > d_1$ so that finding $|ax\rangle = |0\rangle$ will not

make $|\psi_1\rangle$ disappear from the superposition. One more benefit from using weak measurement is that weak measurement can be reversed as will proved later.

The second aim is to get $j_0 > j_1$ with high probability if $\sin^2(\phi) < \cos^2(\phi)$, and vice versa, and since the value of ϕ is unknown, so we need to make $\Pr(|\mathrm{ax}\rangle = |1\rangle)$ and $\Pr(|\mathrm{ax}\rangle = |0\rangle)$ as close as possible to 0.5 so that the impact of ϕ appears on the probabilities of $|ax\rangle$. Setting the probabilities of $|ax\rangle$ as close as possible to 0.5 will also make the measurement on $|ax\rangle$ has a small impact on the probabilities of $|\psi\rangle$.

To satisfy the above two aims, we need to set $\theta_0 = \frac{\pi}{4} - \varepsilon$ and $\theta_1 = \frac{\pi}{4} + \varepsilon$ for small $\varepsilon > 0$. This can be done by setting the parameters as follows,

$$\begin{aligned} d_0 &= \mu, \\ d_1 &= \mu + 1, \\ c &= 2\mu + 1, \end{aligned} \tag{23}$$

so that,

$$\begin{aligned} \theta_0 &= \tfrac{\pi\mu}{2(2\mu+1)}, \\ \theta_1 &= \tfrac{\pi(\mu+1)}{2(2\mu+1)}. \end{aligned} \tag{24}$$

The μ dummy qubits appended to the system will be used later to define the strength of the weak measurement, so that the scale of the weak measurement can be extended by adding more dummy qubits. The effect of the number of the dummy qubits will appear only in the definition of the M_x operator, so, instead of adding the dummy qubits physically to the system, they can be added virtually as a parameter in the M_x operator to save the physical resources and the dummy qubits can be seen as virtual qubits, so, $|\psi_{ext}\rangle$ can be redefined as follows,

$$|\psi_{ext}\rangle = |\psi\rangle \otimes |0\rangle, \tag{25}$$

and the M_x operator can be redefined as follows,

$$M_x = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1, \tag{26}$$

where U_0 and U_1 are defined as follows,

$$U_0 = \frac{1}{2}\begin{bmatrix} 1+t^{d_0} & 1-t^{d_0} \\ 1-t^{d_0} & 1+t^{d_0} \end{bmatrix}, U_1 = \frac{1}{2}\begin{bmatrix} 1+t^{d_1} & 1-t^{d_1} \\ 1-t^{d_1} & 1+t^{d_1} \end{bmatrix}. \tag{27}$$

5. Reversibility of weak measurement

During the run of the proposed algorithm, repetitive measurement on $|ax\rangle$ will slightly change the probabilities of $|\psi_0\rangle$ and $|\psi_1\rangle$. If after an arbitrary measurement, we find $|ax\rangle = |0\rangle$, then the probability of $|\psi_0\rangle$ will increase, and if we find $|ax\rangle = |1\rangle$, then the probability of $|\psi_1\rangle$ will increase. This section will show that after arbitrary number of measurements on $|ax\rangle$, if the number of times we found $|ax\rangle = |0\rangle$ equals to the number of times we found $|ax\rangle = |1\rangle$, then the probabilities of $|\psi_0\rangle$ and $|\psi_1\rangle$ will be restored to the initial probabilities, i.e. finding $|ax\rangle = |0\rangle$ after any measurement on $|ax\rangle$ will reverse the effect of finding $|ax\rangle = |1\rangle$ after any other measurement and vice versa. To prove this, we need the following lemma.

Lemma 5.1. *Let* $\theta_0 = \frac{\pi\mu}{2(2\mu+1)}$ *and* $\theta_1 = \frac{\pi(\mu+1)}{2(2\mu+1)}$ *for any* $\mu \geq 1$*, then for any* $m \geq 0$*,*

$$\frac{\cos^m(\theta_1)\sin^m(\theta_1)}{\cos^m(\theta_0)\sin^m(\theta_0)} = 1. \tag{28}$$

Proof *Since* $\theta_0 = \frac{\pi\mu}{2(2\mu+1)}$ *and* $\theta_1 = \frac{\pi(\mu+1)}{2(2\mu+1)}$*, then* θ_0 *and* θ_1 *can be re-written as,*

$$\begin{aligned} \theta_0 &= \tfrac{\pi}{4} - \varepsilon, \\ \theta_1 &= \tfrac{\pi}{4} + \varepsilon, \end{aligned} \tag{29}$$

with $\varepsilon = \frac{\pi}{4(2\mu+1)}$*, then,*

$$\begin{aligned} \cos(\theta_1) &= \cos\left(\tfrac{\pi}{4} + \varepsilon\right) \\ &= \tfrac{1}{\sqrt{2}}(\cos(\varepsilon) - \sin(\varepsilon)) \\ &= \sin\left(\tfrac{\pi}{4} - \varepsilon\right) \\ &= \sin(\theta_0), \end{aligned} \tag{30}$$

and,

$$\begin{aligned}\sin(\theta_1) &= \sin\left(\tfrac{\pi}{4}+\varepsilon\right)\\ &= \tfrac{1}{\sqrt{2}}\left(\cos(\varepsilon)+\sin(\varepsilon)\right)\\ &= \cos\left(\tfrac{\pi}{4}-\varepsilon\right)\\ &= \cos(\theta_0),\end{aligned} \tag{31}$$

and so Eq.(28) holds.

Theorem 5.2. *Assume that the initial probabilities of* $|\psi_0\rangle$ *and* $|\psi_1\rangle$ *be* $\cos^2(\phi)$ *and* $\sin^2(\phi)$ *respectively. Let* j_0 *and* j_1 *be the number of times we find* $|ax\rangle = |0\rangle$ *and* $|ax\rangle = |1\rangle$ *respectively when measured. If* $j_0 = j_1$ *then the probabilities of* $|\psi_0\rangle$ *and* $|\psi_1\rangle$ *will be equal to the initial probabilities.*

Proof *Assume that* $|ax\rangle$ *is measured for* j *times, where* j *is an even number such that* $j = j_0 + j_1$ *and* $j \geq 0$*. If* $j_0 = j_1$ *then the proof holds directly using Lemma 5.1 in Eq.(21) and Eq.(22).*

6. Stability of the proposed algorithm

Due to the symmetry of the problem, we can consider only the case when $\sin^2(\phi) < \cos^2(\phi)$, and the case of $\sin^2(\phi) > \cos^2(\phi)$ can be deduced by similarity. It is clear from Eqs. (10) and (11) that before the first measurement on $|\mathrm{ax}\rangle$, we have $\Pr_0(|ax\rangle = |0\rangle) > \Pr_0(|ax\rangle = |1\rangle)$ if $\sin^2(\phi) < \cos^2(\phi)$, and from from Eqs. (19) and (20) we can see that the more we move in the correct direction, i.e. incrementing j_0 faster than j_1, the more we gain bias to $\Pr_{\mathrm{j}}(|ax\rangle = |0\rangle)$.

This section will show that even if the algorithm moves in the wrong direction, i.e. incrementing j_1 faster than j_0 when $\sin^2(\phi) < \cos^2(\phi)$, $\Pr_{\mathrm{j}}(|ax\rangle = |0\rangle)$ will stay greater than $\Pr_{\mathrm{j}}(|ax\rangle = |1\rangle)$ for a certain number of wrong measurements on $|ax\rangle$, i.e. $|ax\rangle = |1\rangle$, giving a high probability for the algorithm to recover from the effect of moving in the wrong direction.

Given that $\cos(\theta_0) = \sin(\theta_1)$, and $\sin(\theta_0) = \cos(\theta_1)$ as shown in Eqs. (30) and (31), then the four master equations of the system shown in Eqs. (19), (20), (21) and (22) can be re-written as follows,

$$\Pr_{\mathrm{j}}(|ax\rangle = |0\rangle) = \frac{\tan^2(\varphi)\sin^2(\theta_0) + \cos^2(\theta_0)\tan^{2\Delta j}(\theta_0)}{\tan^2(\varphi) + \tan^{2\Delta j}(\theta_0)}, \tag{32}$$

$$\Pr_{\mathrm{j}}(|ax\rangle = |1\rangle) = \frac{\tan^2(\varphi)\cos^2(\theta_0) + \sin^2(\theta_0)\tan^{2\Delta j}(\theta_0)}{\tan^2(\varphi) + \tan^{2\Delta j}(\theta_0)}, \tag{33}$$

$$\Pr_{\mathrm{j}}(|\psi_0\rangle) = \frac{\tan^{2\Delta j}(\theta_0)}{\tan^2(\varphi) + \tan^{2\Delta j}(\theta_0)}, \tag{34}$$

$$\Pr_{\mathrm{j}}(|\psi_1\rangle) = \frac{\tan^2(\varphi)}{\tan^2(\varphi) + \tan^{2\Delta j}(\theta_0)}, \tag{35}$$

where $\Delta j = j_1 - j_0$. For the algorithm to be stable, then $\Delta j < 0$ when $\sin^2(\phi) < \cos^2(\phi)$. We know that weak measurement is reversible, assume the random walk moves for $\Delta j > 0$ steps in the wrong direction. We need to know how far the random walk should go in the wrong direction while maintaining the stability condition $\Pr(|ax\rangle = |0\rangle) > \frac{1}{2}$, so we get,

$$\sin^2(\phi) < \cos^2(\phi)\tan^{2\Delta j}(\theta_0), \tag{36}$$

such that, if $\Delta j = 0$, so we get the initial probabilities of the system, i.e. $\sin^2(\phi) < \cos^2(\phi)$, and we have $\Pr(|ax\rangle = |0\rangle) > \frac{1}{2}$ as long as,

$$\Delta j \geq \frac{\log(\tan(\phi))}{\log(\tan(\theta_0))} \geq 0. \tag{37}$$

This means that the algorithm will maintain the stability condition even if the random walk goes in the wrong direction for at most $\frac{\log(\tan(\phi))}{\log(\tan(\theta_0))}$ steps. This gives the algorithm a chance to restore the random walk to move in the correct direction

7. The strength of weak measurement

The strength of the weak measurement can be understood as the distance that the random walk has to move from the initial state to the state that are ϵ-far from the projected state for small $\epsilon \geq 0$, so, the scale of a projective measurement is of length 1, i.e. it has the maximum strength, after which the state of the unknown qubit will be projected to one of the eigen vectors of the system in a probabilistic way. This section will show that the strength of the weak measurement can controlled by using an arbitrary number of dummy qubits μ in the system. It will be shown that the measurement process can be scaled to an arbitrary length based on the number of dummy qubits added to the system.

Assuming again the case where $\sin^2(\phi) < \cos^2(\phi)$, then the scale of the measurement process is based upon the number of steps that the random walk should move starting from $\Pr_0(|\psi_0\rangle) = \cos^2(\varphi)$ to reach after $j \geq 1$ steps to $\Pr_j(|\psi_0\rangle) = 1 - \epsilon$ for small $\epsilon > 0$, so

$$\Pr_{\mathrm{j}}(|\psi_0\rangle) = \frac{\tan^{2\Delta j}(\theta_0)}{\tan^2(\varphi) + \tan^{2\Delta j}(\theta_0)} \geq 1 - \epsilon, \tag{38}$$

then,

$$\begin{aligned} \Delta j &\geq \frac{\log\left(\tan^2(\varphi)\left(\frac{1-\epsilon}{\epsilon}\right)\right)}{\log(\tan^2(\theta_0))} \\ &\geq \frac{\log\left(\tan^2(\varphi)\left(\frac{1-\epsilon}{\epsilon}\right)\right)}{\log(\sin^2(\theta_0)) - \log(\cos^2(\theta_0))} \\ &\geq \frac{\log\left(\tan^2(\varphi)\left(\frac{1-\epsilon}{\epsilon}\right)\right)}{\log(\cos^2(\theta_1)) - \log(\cos^2(\theta_0))} \\ &\geq \frac{\log\left(\tan^2(\varphi)\left(\frac{1-\epsilon}{\epsilon}\right)\right)}{\log(\cos^2(\theta_1)) - \log(\cos^2(\theta_0))}, \end{aligned} \tag{39}$$

and since $\theta_0 = \frac{\pi\mu}{4\mu+2}$ and $\theta_1 = \frac{\pi(\mu+1)}{4\mu+2}$ then

$$\begin{aligned} \Delta j &\geq \frac{\log\left(\tan^2(\varphi)\left(\frac{1-\epsilon}{\epsilon}\right)\right)}{\left(\frac{\pi\mu}{4\mu+2}\right)^2 - \left(\frac{\pi(\mu+1)}{4\mu+2}\right)^2} \\ &\geq \left(\frac{2}{\pi}\right)^2 \log\left(\tan^2(\varphi)\left(\frac{1-\epsilon}{\epsilon}\right)\right)(2\mu+1). \end{aligned} \tag{40}$$

For suffiently large $\mu > 0$, $\Pr_{\mathrm{j}}(|ax\rangle = |0\rangle) = \frac{1}{2} + \delta$ and $\Pr_{\mathrm{j}}(|ax\rangle = |1\rangle) = \frac{1}{2} - \delta$ for small $\delta > 0$, then [10],

$$\Delta j = \sqrt{\frac{2}{\pi} j} \tag{41}$$

and since φ is unknown, then assume $\varphi = \frac{\pi}{2}$ as an upper bound for the total number of steps j and so the scale of the measurement process is,

$$\begin{aligned} j_{proj} &\geq \tfrac{\pi}{2} (\Delta j)^2 \\ &\geq \tfrac{2}{\pi} \left(\log \left(\tan^2 \left(\tfrac{\pi}{2} \right) \left(\tfrac{1-\epsilon}{\epsilon} \right) \right) (2\mu + 1) \right)^2 \\ &\geq O\left(\mu^2\right). \end{aligned} \tag{42}$$

This means that if the algorithm is iterated for j_{proj} iterations, then $\mathrm{Pr}_{\mathrm{proj}}(\mathrm{j}_0 > \mathrm{j}_1) = \sin^2(\varphi)$ similar to the case of the projective measurement.

8. Partial gain of information

Assume the case when we are given a certain number of dummy qubits μ and we do not want to iterate the algorithm for j_{proj} times, but we want to stop early at iteration $J < j_{proj}$ for not fully disturbing the superposition, then we need to find $\mathrm{Pr}_{\mathrm{J}}(\mathrm{j}_0 > \mathrm{j}_1)$ after J iterations.

Assume that the algorithm is iterated for $J \geq 2$ times such that $J = J_0 + J_1$ where J_0 is the number of times we read $|ax\rangle = |0\rangle$ and J_1 is the number of times we read $|ax\rangle = |1\rangle$. It is important to notice here that due to the reveribilty of weak measurement shown in Theorem 5.2, the final state of the system will be the same with any order of outcomes from $|ax\rangle$ as long as the values of J_0 and J_1 are fixed since ΔJ will be the same, so the probability to find $|ax\rangle = |0\rangle$ for J_0 after J iterations can be calculated as follows,

$$\mathrm{Pr}(\mathrm{j}_0 = \mathrm{J}_0) = \begin{pmatrix} J \\ J_0 \end{pmatrix} \left(\mathrm{Pr}_{\mathrm{J}}(|\mathrm{ax}\rangle = |0\rangle)\right)^{\mathrm{J}_0} \left(\mathrm{Pr}_{\mathrm{J}}(|\mathrm{ax}\rangle = |1\rangle)\right)^{\mathrm{J}-\mathrm{J}_0}. \tag{43}$$

When $\varphi < \frac{\pi}{4}$, the algorithm is assumed to be successful if $j_0 > j_1$ and vice verse. Without losing of generality, assume J is even, then the algorithm is assumed successful when we read $|ax\rangle = |0\rangle$ for at least $\frac{J}{2} + 1$ times, i.e. $j_0 > \frac{J}{2}$, then

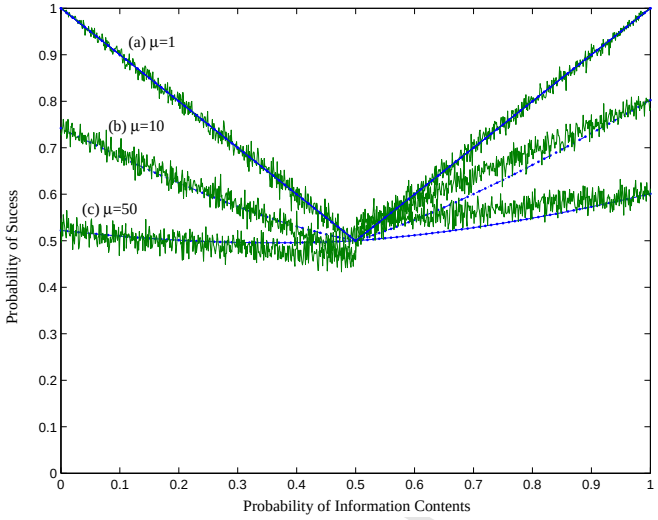

Figure 2: The probability of success for the measurement based quantum random walks with different values of μ, where the solid lines refers to the simulation results and the dotted lines is the probability of success shown in Eq.(45).

$$\Pr\left(j_0 > \frac{J}{2}\right) = \sum_{k=\frac{J}{2}+1}^{J} \begin{pmatrix} J \\ k \end{pmatrix} \left(\Pr_{\mathrm{J}}\left(|ax\rangle = |0\rangle\right)\right)^k \left(\Pr_{\mathrm{J}}\left(|\mathrm{ax}\rangle = |1\rangle\right)\right)^{J-k}, \quad (44)$$

where $\begin{pmatrix} J \\ k \end{pmatrix} = \frac{J!}{k!(J-k)!}$, and we know that $\Pr_{\mathrm{J}}\left(\mathrm{j}_0 > \mathrm{j}_1\right) = \frac{1}{2}$ as a trivial case when $\varphi = \frac{\pi}{4}$, i.e. when $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$, then the probability of success of the algorithm after J iterations with $\Delta J = -\sqrt{\frac{2}{\pi}J}$ is as follows,

$$\Pr_{\mathrm{J}}\left(j_0 > j_1\right) = \sin^2\left(\varphi\right) + \cos\left(2\varphi\right)\Pr\left(j_0 > \frac{J}{2}\right). \quad (45)$$

For $\varphi > \frac{\pi}{4}$, the same equation (Eq.(45)) can be used as the probability of success of the algorithm but with $\Delta J = \sqrt{\frac{2}{\pi}J}$. As an illustrative example, Fig. 2 shows simulation results of the algorithm compared with the probability of success shown in Eq.(45) by setting $J = 100$ for $\mu = 1$, $\mu = 10$ and $\mu = 50$. The simulation results shown in Fig. 2 is the average of the probability of success

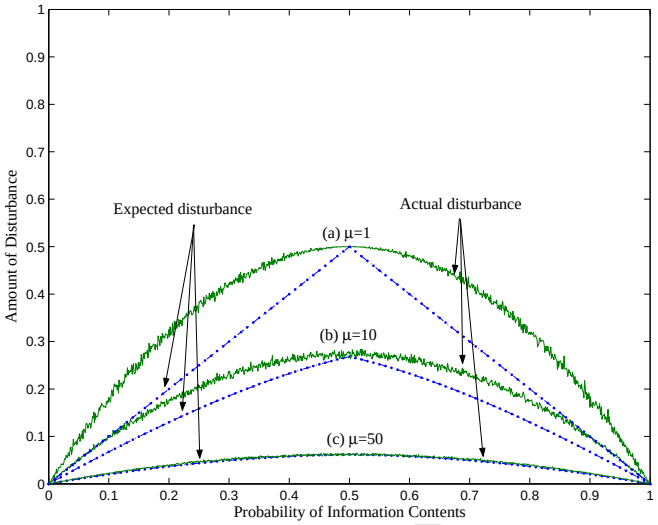

Figure 3: The amount of disturbance introduced to the system using the measurement based quantum random walks with different values of μ, where the solid lines refers to the simulation results and the dotted lines is expected amount of disturbance shown in Eq.(45).

to read the information of $|\psi\rangle$. The simulation results are collected by applying the algorithm iteratively for $0 \leq \sin^2(\phi) \leq 1$ with step 0.001 and each step is repeated 1000 times. Taking the probability of success of $\varphi = 0$ as a reference probability relevant to the probability of success of projective measurement, so iterating the algorithm for 100 items gives a probability of success of 1.0 using $\mu = 1$, 0.74224 using $\mu = 1$, and 0.52233 using $\mu = 50$ which is close to a random guess.

Based on the same example shown in Fig. 2, Fig. 3 shows the actual amount of disturbance introduced to the system using the proposed algorithm taken as the average disturbance from all the trials compared with the expected amount of disturbance d_e calculated as follows,

$$d_e = |\cos^2(\phi) - \Pr_j(|\psi_0\rangle)|. \tag{46}$$

It is possible to restrict the disturbance to be introduced to system to an arbitrary small $\delta > 0$ regardless to how many iterations of the algorithm is

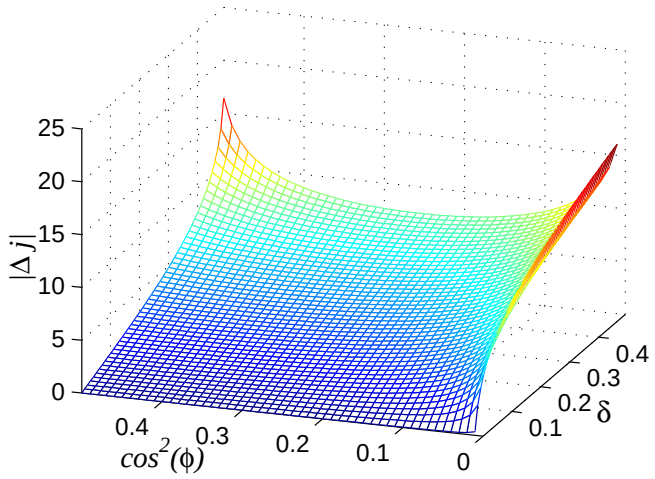

Figure 4: The upper bound of $|\Delta j|$ for not introducing disturbance $\delta > 0$ to the system such that $0 < \delta < \frac{1}{2}$ for $\phi \leq \frac{\pi}{4}$ using $\mu = 5$ dummy qubits.

applied as long as $d_e < \delta$. To achieve this, we have to make Δj satisfies the following condition,

$$\left| \cos^2(\phi) - \frac{\tan^{2\Delta j}(\theta_0)}{\tan^2(\phi) + \tan^{2\Delta j}(\theta_0)} \right| < \delta, \tag{47}$$

then,

$$|\Delta j| < \frac{\log\left(\frac{\tan^2(\phi)\left(\delta - \cos^2(\phi)\right)}{\cos^2(\phi) - 1 - \delta}\right)}{\log\left(\tan^2(\phi)\right)}. \tag{48}$$

This means that no disturbance more than $\delta > 0$ will be introduced to the system as long as Δj satisfies the condition shown in Eq.(48). Fig. 4 shows the required values of $|\Delta j|$ for $\phi \leq \frac{\pi}{4}$ and the allowed disturbance $0 < \delta < \frac{1}{2}$ using $\mu = 5$.

Fig. 5(a) shows a MBQRW with $\mu = 1$ where the $|\psi\rangle$ will collapse to either $|0\rangle$ or $|1\rangle$ very fast with probabilities close to $\cos^2(\phi)$ or $\sin^2(\phi)$ respectively. This gives high accuracy but will disturb the superposition in a way very close to the projective measurement.

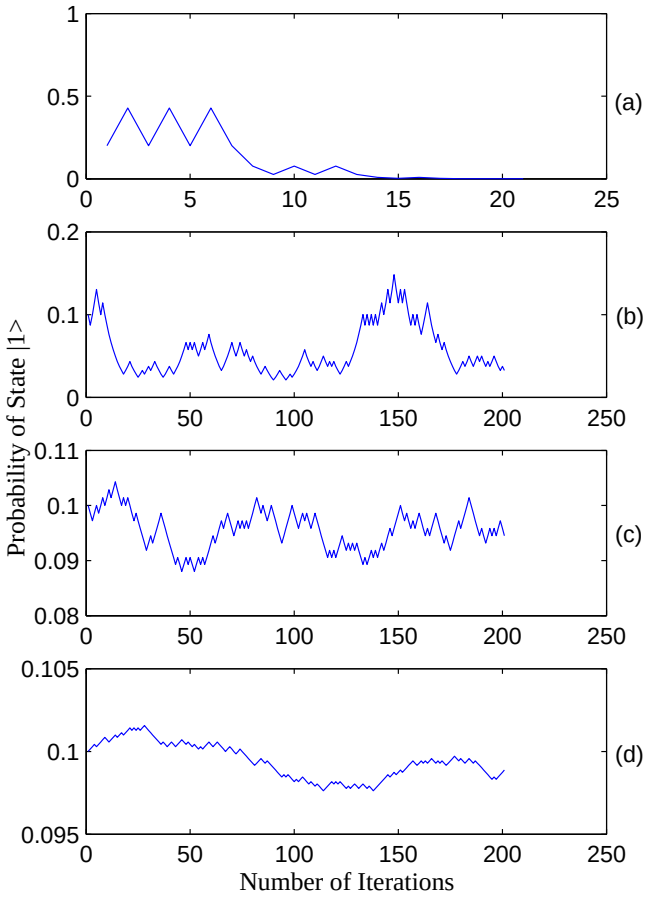

Figure 5: Measurement Based quantum random walks with (a) $\mu = 1$, (b)$\mu = 10$, (c)$\mu = 100$ and $\mu = 1000$.

Fig. 5(b) shows a MBQRW with $\mu = 10$ where $|\psi\rangle$ will not collapse to $|0\rangle$ or $|1\rangle$ but will make it move up or down with probabilities not far from $\cos^2(\phi)$ or $\sin^2(\phi)$ respectively. This gives acceptable accuracy and will not disturb the superposition very much.

Fig. 5(c) and Fig. 5(d) show MBQRWs with large number of dummy qubits μ where $|\psi\rangle$ will not collapse to either $|0\rangle$ or $|1\rangle$ and information gain about $|\psi\rangle$ will be no better than a random guess.

9. Conclusion

In this paper, a quantum algorithm has be proposed to read the information contents of an unknown qubit without using sharp measurement on that qubit. The proposed algorithm used a partial negation operator that creates a weak entanglement between the unknown qubit and the an auxiliary qubit. A quantum feedback control scheme is used where sharp measurement is applied iteratively on the auxiliary qubit. Counting the outcomes from the sharp measurement on the auxiliary qubit has been used to read the information contents on the unknown qubit. It has been shown that the iterative measurements on the auxiliary qubit makes the amplitudes of the superposition move in a random walk manner. The random walk has a reversal effect when moved in opposite directions, this helps to decrease the disturbance that will be introduced to the system during the run of the algorithm. The proposed algorithm defined the strength of the weak measurement as the distance the random walk has to move from the initial state to the state of the sharp measurement which can controlled by using an arbitrary number of dummy qubits (virtual qubits) μ in the system. Adding more dummy qubits to the system made the measurement process slower so that the effect of the sharp measurement will be reached after $O(\mu^2)$ measurements on the auxiliary qubit. It has been shown that the more we disturb the system, the more information we can get about that system.

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