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# ADAPTIVE FEM WITH COARSE INITIAL MESH GUARANTEES OPTIMAL CONVERGENCE RATES FOR COMPACTLY PERTURBED ELLIPTIC PROBLEMS

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ABSTRACT. We prove that for compactly perturbed elliptic problems, where the corresponding bilinear form satisfies a Gårding inequality, adaptive mesh-refinement is capable of overcoming the preasymptotic behavior and eventually leads to convergence with optimal algebraic rates. As an important consequence of our analysis, one does not have to deal with the *a priori* assumption that the underlying meshes are sufficiently fine. Hence, the overall conclusion of our results is that adaptivity has stabilizing effects and can overcome possibly pessimistic restrictions on the meshes. In particular, our analysis covers adaptive mesh-refinement for the finite element discretization of the Helmholtz equation from where our interest originated.

## 1. INTRODUCTION

1.1. **Adaptive mesh-refining algorithms.** *A posteriori* error estimation and related adaptive mesh-refinement is one fundamental column of finite element analysis. On the one hand, the *a posterori* error estimator allows to monitor whether the numerical solution is sufficiently accurate, even though the exact solution is unknown. On the other hand, its local contributions allow to adapt the underlying triangulation to resolve possible singularities most effectively. In recent years, the mathematical understanding of adaptive mesh-refinement has matured. It is now known that adaptive finite element methods (AFEM) of the type

$$(1) \qquad \boxed{\text{SOLVE}} \quad \Longrightarrow \quad \boxed{\text{ESTIMATE}} \quad \Longrightarrow \quad \boxed{\text{MARK}} \quad \Longrightarrow \quad \boxed{\text{REFINE}}$$

converge with optimal algebraic rate; see [Dör96, MNS00, BDD04, Ste07, CKNS08, FFP14] to mention some milestones for conforming AFEM for linear second-order elliptic PDEs and [CFPP14] for some axiomatic approach. Essentially, only problems satisfying the Lax-Milgram theorem have been treated [Dör96, MNS00, BDD04, Ste07, CKNS08]. In a more general case of compactly perturbed elliptic problems, existing results have the limitation that the initial mesh has to be sufficiently fine [MN05, CN12, FFP14]. On the other hand, numerical examples in the engineering literature suggest that adaptive mesh-refinement performs well even if the initial mesh is coarse (see, e.g., [SH96, BI98, BI99] in the case of the Helmholtz equation). The purpose of this work is to bridge this gap at least for conforming elements.

1.2. **Model problem.** Let $\Omega \subset \mathbb{R}^d$ be a polygonal resp. polyhedral Lipschitz domain, $d \geq 2$. Let $\langle f, g \rangle := \int_\Omega fg \, dx$ denote the $L^2(\Omega)$ scalar product. Suppose that $a(\cdot, \cdot)$ is a symmetric, continuous, and elliptic bilinear form on $\mathcal{H} := H_0^1(\Omega)$ and that $\mathcal{K} : H_0^1(\Omega) \to L^2(\Omega)$ is a continuous linear operator. Given $f \in L^2(\Omega)$, we suppose that the variational formulation

$$(2) \qquad b(u, v) := a(u, v) + \langle \mathcal{K}u, v \rangle = \langle f, v \rangle \quad \text{for all } v \in \mathcal{H}$$

admits a unique solution $u \in \mathcal{H}$. Possible examples include the weak formulation of the Helmholtz equation

$$(3) \qquad -\Delta u - \kappa^2 u = f \text{ in } \Omega \quad \text{subject to} \quad u = 0 \text{ on } \partial\Omega,$$

where $\kappa^2 \in \mathbb{R}$ is not an eigenvalue of $-\Delta$ and $\mathcal{K}u = -\kappa^2 u$, as well as more general diffusion problems with convection and reaction

$$(4) \qquad -\text{div}(A\nabla u) + b \cdot \nabla u + cu = f \text{ in } \Omega \quad \text{subject to} \quad u = 0 \text{ on } \partial\Omega,$$

for given coefficients $c \in L^\infty(\Omega)$, $b \in L^\infty(\Omega)^d$, and $A \in L^\infty(\Omega)^{d\times d}$, where $A(x) \in \mathbb{R}_{\text{sym}}^{d\times d}$ is symmetric and uniformly positive definite. We note that homogeneous Dirichlet conditions are only considered for the ease of presentation, while (inhomogeneous) mixed Dirichlet-Neumann-Robin boundary conditions can be included as in [FPP14, AFK+13, CFPP14].

We consider standard finite element spaces based on regular triangulations $\mathcal{T}_\bullet$ of $\Omega$. For some fixed polynomial degree $p \geq 1$, let

$$\mathcal{S}^p(\mathcal{T}_\bullet) := \left\{ V_\bullet \in C(\Omega) : \forall T \in \mathcal{T}_\bullet \quad V_\bullet|_T \text{ is a polynomial of degree } \leq p \right\}$$

be the usual finite element space of globally continuous piecewise polynomials and $\mathcal{X}_\bullet := \mathcal{S}^p(\mathcal{T}_\bullet) \cap H_0^1(\Omega)$ be the corresponding conforming subspace of $H_0^1(\Omega)$. Then, the discrete formulation reads as follows: Find $U_\bullet \in \mathcal{X}_\bullet$ such that

$$(5) \qquad b(U_\bullet, V_\bullet) = \langle f, V_\bullet \rangle \quad \text{for all } V_\bullet \in \mathcal{X}_\bullet.$$

Let $h_\bullet \in L^\infty(\Omega)$ denote the local mesh-size function defined by $h_\bullet|_T := |T|^{1/d}$ for all $T \in \mathcal{T}_\bullet$. Note that $h_\bullet|_T$ behaves like the diameter of the element $T \in \mathcal{T}_\bullet$ on shape-regular meshes. In general, (5) may fail to allow for a (unique) solution $U_\bullet \in \mathcal{X}_\bullet$. However, existence and uniqueness are guaranteed if $\mathcal{T}_\bullet$ is sufficiently fine (see Corollary 4), e.g., $\|h_\bullet\|_{L^\infty(\Omega)} \leq H \ll 1$. Therefore, we employ one step of uniform refinement if (5) does not allow for a unique solution $U_\bullet \in \mathcal{X}_\bullet$.

1.3. **Contributions of present work.** Given an initial triangulation $\mathcal{T}_0$, a typical adaptive algorithm (1) generates a sequence of refined meshes $\mathcal{T}_\ell$ with corresponding nested spaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset \mathcal{H}$ for all $\ell \geq 0$. We stress that unlike prior works [MN05, CN12, FFP14], our adaptive algorithm (Algorithm 7) will not be given any information on whether the current mesh is sufficiently fine to allow for a unique solution. In particular, we do not assume that the given initial mesh $\mathcal{T}_0$ (and, in fact, any adaptive mesh $\mathcal{T}_\ell$ generated by our algorithm) is sufficiently fine. Nevertheless, we derive similar results as for uniformly elliptic problems (see, e.g., [CKNS08, FFP14, CFPP14] and the references therein), i.e., we prove linear convergence (Theorem 19) with optimal algebraic convergence rates (Theorem 26). More precisely, the framework and the main contributions of the present work can be summarized as follows:

- We consider a fixed mesh-refinement strategy that satisfies certain abstract assumptions (Section 2.2 and Section 4.1) which are met, e.g., for newest vertex bisection [Ste08, KPP13].

- We consider a fixed *a posteriori* error estimation strategy which satisfies the *stability property on non-refined element domains* (A1), the *reduction property on refined element domains* (A2), and the *reliabilty property* (A3) as well as the *discrete reliability property* (A4).
- Under the above assumptions on the mesh-refinement and the error estimation strategy, we formulate our variant (Algorithm 7) of the adaptive loop (1), where marking is based on the Dörfler marking criterion introduced in [Dör96] with some adaptivity parameter $0 < \theta \leq 1$.
- If the "discrete" limit space $\mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell}$ satisfies an assumption (A5) which can be ensured by expanding the set of marked elements in the Dörfler marking strategy (Section 3.2), we prove linear convergence (Theorem 19) for any $0 < \theta \leq 1$.
- Starting from an index $L \in \mathbb{N}_0$, we prove that the Céa lemma is valid for the $a(\cdot, \cdot)$-induced energy norm and $\ell \geq L$, and the corresponding quasi-optimality constants converge to 1 as $\ell \to \infty$ (Theorem 20).
- If additionally $0 < \theta \ll 1$ is sufficiently small and $\mathcal{X}_\infty = \mathcal{H}$ (which can be ensured by the expanded Dörfler marking strategy mentioned above), we prove optimal algebraic convergence rates (Theorem 26). While our presentation employs the estimator-based approximation classes from [CFPP14], Section 4.2 also discusses the relation to the approximation classes based on the total error from [CKNS08].

We note that the entire analysis of this work applies to general situations, where $\mathcal{H}$ is a separable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{X}_\ell \subseteq \mathcal{H}$ are conforming subspaces, and $\mathcal{K} : \mathcal{H} \to \mathcal{H}^*$ is a compact operator; see Section 2.1.

1.4. **Outline.** Section 2 provides the abstract framework of our analysis (Section 2.1–2.2) and gives a precise statement of the adaptive algorithm (Section 2.4). Section 2.3 adapts [CFPP14] to the present setting and formulates certain properties of the error estimator. Section 3 proves convergence of the adaptive algorithm. Following [FFP14], we first prove plain convergence (Section 3.1) and then derive linear convergence (Section 3.3). Finally, we address the validity of the Céa lemma (Section 3.4). Optimal algebraic convergence rates are the topic of Section 4, where we also discuss the involved approximation classes (Section 4.2). In the final Section 5, we present numerical results for the 2D Helmholtz equation that underpin the developed theory.

**Notation.** We use $\lesssim$ to abbreviate $\leq$ up to some (generic) multiplicative constant which is clear from the context. Moreover, $\simeq$ abbreviates that both estimates $\lesssim$ and $\gtrsim$ hold. Throughout, the mesh-dependence of (discrete) quantities is explicitly stated by use of appropriate indices, e.g., $U_\bullet$ is the discrete solution for the triangulation $\mathcal{T}_\bullet$ and $\eta_\ell$ is the error estimator with respect to the triangulation $\mathcal{T}_\ell$.

## 2. ADAPTIVE ALGORITHM

2.1. **Abstract setting.** The model problem from Section 1.2 can be recast in the following abstract setting. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For each triangulation $\mathcal{T}_\bullet$ with local mesh-size $h_\bullet \in L^\infty(\Omega)$, let $\mathcal{X}_\bullet \subseteq \mathcal{H}$ be a conforming finite-dimensional subspace. Suppose that $a(\cdot, \cdot)$ is a hermitian, continuous, and elliptic sesquilinear form on $\mathcal{H}$, i.e., there exists some constant $\alpha > 0$ such that

$$(6) \qquad \alpha \|v\|_{\mathcal{H}}^2 \leq a(v, v) \quad \text{for all } v \in \mathcal{H}.$$

In particular, the $a(\cdot, \cdot)$-induced energy norm $\|v\|^2 := a(v, v)$ is an equivalent norm on $\mathcal{H}$, i.e., $\|v\| \simeq \|v\|_{\mathcal{H}}$ for all $v \in \mathcal{H}$. Let $\mathcal{H}^*$ be the dual space of $\mathcal{H}$, and let $\langle \cdot, \cdot \rangle$ denote the

corresponding duality pairing. Suppose that $\mathcal{K} : \mathcal{H} \to \mathcal{H}^*$ is a compact linear operator and $f \in \mathcal{H}^*$. In the remainder of this work, we consider the weak formulation (2) as well as its discretization (5) within the above abstract framework.

The next proposition is an improved version of [SS11, Theorem 4.2.9]. Even though the result appears to be well-known, we did not find the precise statement in the literature. We note that a similar result is proved in [BS08, Theorem 5.7.6] under additional regularity assumptions for the dual problem. Instead, our proof below proceeds without considering the dual problem, and hence no additional regularity assumptions are needed. For these reasons and for the convenience of the reader, we include the following statement together with its proof.

**Proposition 1.** *Suppose well-posedness of* (2), *i.e.,*

$$(7) \qquad \forall w \in \mathcal{H} \quad \left[ w = 0 \quad \Longleftrightarrow \quad \left( \forall v \in \mathcal{H} \quad b(w, v) = 0 \right) \right].$$

*Suppose that* $(\mathcal{X}_\ell)_{\ell \in \mathbb{N}_0}$ *is a dense sequence of discrete subspaces* $\mathcal{X}_\ell \subset \mathcal{H}$, *i.e.,*

$$(8) \qquad \lim_{\ell \to \infty} \min_{V_\ell \in \mathcal{X}_\ell} \|v - V_\ell\|_{\mathcal{H}} = 0 \quad \text{for all } v \in \mathcal{H}.$$

*Then, there exists some index* $\ell_\bullet \in \mathbb{N}_0$ *such that for all discrete subspaces* $\mathcal{X}_\bullet \subset \mathcal{H}$ *with* $\mathcal{X}_\bullet \supseteq \mathcal{X}_{\ell_\bullet}$, *the following holds: There exists* $\gamma > 0$ *which depends only on* $\mathcal{X}_{\ell_\bullet}$, *such that the* inf-sup *constant of* $\mathcal{X}_\bullet$ *is uniformly bounded from below, i.e.,*

$$(9) \qquad \gamma_\bullet := \inf_{W_\bullet \in \mathcal{X}_\bullet \setminus \{0\}} \sup_{V_\bullet \in \mathcal{X}_\bullet \setminus \{0\}} \frac{|b(W_\bullet, V_\bullet)|}{\|W_\bullet\|_{\mathcal{H}} \|V_\bullet\|_{\mathcal{H}}} \geq \gamma > 0.$$

*In particular, the discrete formulation* (5) *admits a unique solution* $U_\bullet \in \mathcal{X}_\bullet$. *Moreover, there holds uniform validity of the Céa lemma, i.e., there is a constant* $C > 0$ *which depends only on* $b(\cdot, \cdot)$ *and* $\gamma$ *but not on* $\mathcal{X}_\bullet$, *such that*

$$(10) \qquad \|u - U_\bullet\|_{\mathcal{H}} \leq C \min_{V_\bullet \in \mathcal{X}_\bullet} \|u - V_\bullet\|_{\mathcal{H}}.$$

*If the spaces* $\mathcal{X}_\ell$ *are nested, i.e.,* $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ *for all* $\ell \in \mathbb{N}_0$, *the latter guarantees convergence* $\|u - U_\ell\|_{\mathcal{H}} \to 0$ *as* $\ell \to \infty$.

*Proof.* The bilinear form $b(\cdot, \cdot)$ induces the linear and continuous operator

$$B_\bullet : \mathcal{X}_\bullet \to \mathcal{X}_\bullet^*, \qquad \langle B_\bullet W_\bullet, V_\bullet \rangle := b(W_\bullet, V_\bullet) \quad \text{for all } V_\bullet, W_\bullet \in \mathcal{X}_\bullet,$$

where $\mathcal{X}_\bullet$ is an arbitrary discrete subspace of $\mathcal{H}$ with dual space $\mathcal{X}_\bullet^*$.

*Step 1: Discrete inf-sup condition.* Since $\mathcal{X}_\bullet$ is finite dimensional and since we use the same discrete ansatz and test space, well-posedness of (5) is equivalent to the discrete inf-sup condition

$$(11) \qquad \gamma_\bullet = \inf_{W_\bullet \in \mathcal{X}_\bullet \setminus \{0\}} \sup_{V_\bullet \in \mathcal{X}_\bullet \setminus \{0\}} \frac{|b(W_\bullet, V_\bullet)|}{\|W_\bullet\|_{\mathcal{H}} \|V_\bullet\|_{\mathcal{H}}} = \inf_{W_\bullet \in \mathcal{X}_\bullet \setminus \{0\}} \frac{\|B_\bullet W_\bullet\|_{\mathcal{X}_\bullet^*}}{\|W_\bullet\|_{\mathcal{H}}} > 0.$$

(Note that (11) implies that $B_\bullet$ is injective, and surjectivity follows from finite dimensionality of $\mathcal{X}_\bullet$, i.e., $\dim \mathcal{X}_\bullet = \dim \mathcal{X}_\bullet^* < \infty$.) Moreover, in this case there holds inequality (10) with

$$C := 1 + \frac{M}{\gamma_\bullet}, \quad \text{where} \quad M := \sup_{\substack{v \in \mathcal{H} \setminus \{0\} \\ w \in \mathcal{H} \setminus \{0\}}} \frac{|b(w, v)|}{\|w\|_{\mathcal{H}} \|v\|_{\mathcal{H}}};$$

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see, e.g., [Bra01, Theorem 3.6, Lemma 3.7] or [Dem06, Section 3]. Therefore, it is sufficient to prove the following assertion:

$$(12) \qquad \exists \gamma > 0 \; \exists \ell_\bullet \in \mathbb{N}_0 \; \forall \mathcal{X}_\bullet \subset \mathcal{H} \text{ with } \mathcal{X}_\bullet \supseteq \mathcal{X}_{\ell_\bullet} \qquad \inf_{W_\bullet \in \mathcal{X}_\bullet \backslash \{0\}} \frac{\|B_\bullet W_\bullet\|_{\mathcal{X}_\bullet^*}}{\|W_\bullet\|_{\mathcal{H}}} \geq \gamma.$$

We will prove (12) by contradiction.

*Step 2:* Let us assume that (12) is wrong and hence

$$(13) \qquad \forall \gamma > 0 \; \forall \ell_\bullet \in \mathbb{N}_0 \; \exists \mathcal{X}_\bullet \subset \mathcal{H} \text{ with } \mathcal{X}_\bullet \supseteq \mathcal{X}_{\ell_\bullet} \qquad \inf_{W_\bullet \in \mathcal{X}_\bullet \backslash \{0\}} \frac{\|B_\bullet W_\bullet\|_{\mathcal{X}_\bullet^*}}{\|W_\bullet\|_{\mathcal{H}}} < \gamma.$$

For each $\ell_\bullet = \ell \geq 0$ and $\gamma = 1/\ell$, we can thus find a discrete subspace $\mathcal{X}_\bullet = \widehat{\mathcal{X}}_\ell \subset \mathcal{H}$ and an element $\widehat{W}_\ell \in \widehat{\mathcal{X}}_\ell$ such that

$$\widehat{\mathcal{X}}_\ell \supseteq \mathcal{X}_\ell, \quad \|\widehat{W}_\ell\|_{\mathcal{H}} = 1, \quad \text{and} \quad \|\widehat{B}_\ell \widehat{W}_\ell\|_{\widehat{\mathcal{X}}_\ell^*} < 1/\ell.$$

Since the sequence $\widehat{W}_\ell$ is bounded and without loss of generality, we may assume weak convergence $\widehat{W}_\ell \rightharpoonup w \in \mathcal{H}$ as $\ell \to \infty$.

*Step 3: There holds $w = 0$.* Let $\widehat{P}_\ell : \mathcal{H} \to \widehat{\mathcal{X}}_\ell$ be the orthogonal projection onto $\widehat{\mathcal{X}}_\ell$ and $v \in \mathcal{H}$. Then, weak convergence $\widehat{W}_\ell \rightharpoonup w$ and $b(\cdot, v) \in \mathcal{H}^*$ prove $b(\widehat{W}_\ell, v) \to b(w, v)$ as $\ell \to \infty$. Moreover, we employ $\|\widehat{W}_\ell\|_{\mathcal{H}} = 1$ and $\|\widehat{P}_\ell v\|_{\mathcal{H}} \leq \|v\|_{\mathcal{H}}$ to estimate

$$|b(\widehat{W}_\ell, v)| \leq |b(\widehat{W}_\ell, \widehat{P}_\ell v)| + |b(\widehat{W}_\ell, v - \widehat{P}_\ell v)| \leq \|\widehat{B}_\ell \widehat{W}_\ell\|_{\widehat{\mathcal{X}}_\ell^*} \|v\|_{\mathcal{H}} + M \|v - \widehat{P}_\ell v\|_{\mathcal{H}}.$$

Recall (8) and $\mathcal{X}_\ell \subseteq \widehat{\mathcal{X}}_\ell \subset \mathcal{H}$. This implies

$$\|v - \widehat{P}_\ell v\|_{\mathcal{H}} = \min_{\widehat{V}_\ell \in \widehat{\mathcal{X}}_\ell} \|v - \widehat{V}_\ell\|_{\mathcal{H}} \leq \min_{V_\ell \in \mathcal{X}_\ell} \|v - V_\ell\|_{\mathcal{H}} \xrightarrow{\ell \to \infty} 0.$$

Since $\|\widehat{B}_\ell \widehat{W}_\ell\|_{\widehat{\mathcal{X}}_\ell^*} \leq 1/\ell$, we thus conclude that $|b(\widehat{W}_\ell, v)| \to 0$ as $\ell \to \infty$. Altogether, $b(w, v) = 0$ for all $v \in \mathcal{H}$ and hence $w = 0$.

*Step 4: Assumption (13) yields a contradiction so that (12) follows.* Recall $\|\widehat{W}_\ell\|_{\mathcal{H}} = 1$. Ellipticity of $a(\cdot, \cdot)$ and the definition of $b(\cdot, \cdot)$ yield

$$\|\widehat{W}_\ell\|_{\mathcal{H}}^2 \overset{(6)}{\lesssim} a(\widehat{W}_\ell, \widehat{W}_\ell) \leq |b(\widehat{W}_\ell, \widehat{W}_\ell)| + |\langle \mathcal{K}\widehat{W}_\ell, \widehat{W}_\ell \rangle| \leq \|\widehat{B}_\ell \widehat{W}_\ell\|_{\widehat{\mathcal{X}}_\ell^*} + \|\mathcal{K}\widehat{W}_\ell\|_{\mathcal{H}^*}.$$

Recall that compact operators turn weak convergence into strong convergence. Hence $\widehat{W}_\ell \rightharpoonup w = 0$ in $\mathcal{H}$ implies $\|\mathcal{K}\widehat{W}_\ell\|_{\mathcal{H}^*} \to 0$ as $\ell \to \infty$. Together with $\|\widehat{B}_\ell \widehat{W}_\ell\|_{\widehat{\mathcal{X}}_\ell^*} \leq 1/\ell$, we thus obtain the contradiction $1 = \|\widehat{W}_\ell\|_{\mathcal{H}} \to 0$ as $\ell \to \infty$. $\qquad \square$

**Remark 2.** *To see that the model problem (2) fits into the abstract framework, recall that the Rellich theorem provides the compact inclusion $\mathcal{H} := H_0^1(\Omega) \Subset L^2(\Omega)$. Therefore, the Schauder theorem (see, e.g., [Rud91, Theorem 4.19]) implies the compact inclusion $L^2(\Omega) \Subset \mathcal{H}^*$, where duality is understood with respect to the $L^2(\Omega)$ scalar product. Therefore, the continuous linear operator $\mathcal{K} : H_0^1(\Omega) \to L^2(\Omega)$ turns out to be compact as an operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}^*$; see also the discussion in [FFP14].*

**Remark 3.** *The work [FFP14] considers problems, where the left-hand side of (2) is strongly elliptic on $\mathcal{H} = H_0^1(\Omega)$, i.e., there exists $\widetilde{\alpha} > 0$ such that*

$$(14) \qquad \widetilde{\alpha} \|v\|_{\mathcal{H}}^2 \leq \mathrm{Re}\left(a(v, v) + \langle \mathcal{K}v, v \rangle\right) \quad \text{for all } v \in \mathcal{H}.$$

*Suppose that $a(w, w) > 0$ for all $w \in \mathcal{H} \backslash \{0\}$. We note that (14) then already implies that $a(\cdot, \cdot)$ is elliptic in the sense of (6), so that the present work generalizes the analysis*

of [FFP14]. *To see that* (14) *implies* (6), *we argue by contradiction, i.e., we assume the following: For all* $\varepsilon > 0$, *there is some* $v \in \mathcal{H}$ *with* $|a(v,v)| < \varepsilon \|v\|_{\mathcal{H}}^2$. *Choosing* $\varepsilon = 1/n$, *we obtain a sequence* $(v_n)$ *in* $\mathcal{H}$ *with* $|a(v_n, v_n)| < \|v_n\|_{\mathcal{H}}^2/n$. *Define* $w_n := v_n/\|v_n\|_{\mathcal{H}}$. *Without loss of generality, we may thus suppose weak convergence* $w_n \rightharpoonup w$ *in* $\mathcal{H}$. *Weakly lower semicontinuity proves* $|a(w,w)| \leq \liminf_{n\to\infty} |a(w_n, w_n)| = 0$ *and hence* $w = 0$. *Therefore, compactness of* $\mathcal{K}$ *yields* $\|\mathcal{K}w_n\|_{\mathcal{H}^*} \to 0$ *as* $n \to \infty$. *Finally, ellipticity* (14) *gives* $\widetilde{\alpha} = \widetilde{\alpha} \|w_n\|_{\mathcal{H}}^2 \leq \mathrm{Re}\left(a(w_n, w_n) + \langle \mathcal{K}w_n, w_n \rangle\right) < 1/n + \|\mathcal{K}w_n\|_{\mathcal{H}^*} \xrightarrow{n\to\infty} 0$. *This contradicts* $\widetilde{\alpha} > 0$, *and we hence conclude that* (14) *implies* (6).

2.2. **Mesh-refinement.** From now on, suppose that $\mathcal{T}_0$ is a given initial mesh. Suppose that refine($\cdot$) is a fixed mesh-refinement strategy (e.g., newest vertex bisection [Ste08]) such that given a conforming triangulation $\mathcal{T}_\star$ and $\mathcal{M}_\star \subseteq \mathcal{T}_\star$, the call $\mathcal{T}_\bullet = \mathrm{refine}(\mathcal{T}_\star, \mathcal{M}_\star)$ returns the coarsest conforming refinement $\mathcal{T}_\bullet$ of $\mathcal{T}_\star$ such that all $T \in \mathcal{M}_\star$ have been refined, i.e.,

- $\mathcal{T}_\bullet$ is a conforming triangulation of $\Omega$;
- for all $T \in \mathcal{T}_\star$, it holds $T = \bigcup \{T' \in \mathcal{T}_\bullet : T' \subseteq T\}$;
- $\mathcal{M}_\star \subseteq \mathcal{T}_\star \backslash \mathcal{T}_\bullet$;
- the number of elements $\#\mathcal{T}_\bullet$ is minimal amongst all other triangulations $\mathcal{T}'$ which share the three foregoing properties.

Furthermore, we write $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$ if $\mathcal{T}_\bullet$ is obtained by a finite number of refinement steps, i.e., there exists $n \in \mathbb{N}_0$ as well as a finite sequence $\mathcal{T}^{(0)}, \ldots, \mathcal{T}^{(n)}$ of triangulations and corresponding sets $\mathcal{M}^{(j)} \subseteq \mathcal{T}^{(j)}$ such that

- $\mathcal{T}_\star = \mathcal{T}^{(0)}$,
- $\mathcal{T}^{(j+1)} = \mathrm{refine}(\mathcal{T}^{(j)}, \mathcal{M}^{(j)})$ for all $j = 0, \ldots, n-1$,
- $\mathcal{T}_\bullet = \mathcal{T}^{(n)}$.

In particular, $\mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}_\star)$. To abbreviate notation, we let $\mathbb{T} := \mathrm{refine}(\mathcal{T}_0)$ be the set of all possible triangulations which can be obtained from $\mathcal{T}_0$.

We suppose that the refinement strategy yields a contraction of the local mesh-size function on refined elements, i.e., there exists $0 < q_{\mathrm{mesh}} < 1$ such that $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$ implies $h_\bullet|_T \leq_{\mathrm{mesh}} h_\star|_T$ for all $T \in \mathcal{T}_\star \backslash \mathcal{T}_\bullet$. We note that $q_{\mathrm{mesh}} = 2^{-1/d}$ for newest vertex bisection [Ste08, CKNS08].

Finally, the following assumptions are clearly satisfied for the model problem from Section 1.2, but have to be supposed explicitly in the abstract framework of Section 2.1. First, each triangulation $\mathcal{T}_\star$ corresponds to a discrete subspace $\mathcal{X}_\star \subset \mathcal{H}$, and $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$ implies nestedness $\mathcal{X}_\star \subseteq \mathcal{X}_\bullet$. Second, iterated uniform mesh-refinement leads to a dense subspace of $\mathcal{H}$, i.e., for $\widehat{\mathcal{T}}_0 := \mathcal{T}_0$ and the inductively defined sequence $\widehat{\mathcal{T}}_{\ell+1} := \mathrm{refine}(\widehat{\mathcal{T}}_\ell, \widehat{\mathcal{M}}_\ell)$ with $\widehat{\mathcal{M}}_\ell \subseteq \widehat{\mathcal{T}}_\ell$ for all $\ell \in \mathbb{N}_0$, it holds the following: If $\#\{\ell \in \mathbb{N}_0 : \widehat{\mathcal{M}}_\ell = \widehat{\mathcal{T}}_\ell\} = \infty$ (i.e., there are infinitely many steps that perform uniform refinement), then $\mathcal{H} = \overline{\bigcup_{\ell=0}^\infty \widehat{\mathcal{X}}_\ell}$.

Under these assumptions, the following statement holds as an immediate consequence of Proposition 1.

**Corollary 4.** *Let* $\widehat{\mathcal{T}}_0 := \mathcal{T}_0$ *and* $\widehat{\mathcal{T}}_{\ell+1} := \mathrm{refine}(\widehat{\mathcal{T}}_\ell, \widehat{\mathcal{M}}_\ell)$ *with* $\widehat{\mathcal{M}}_\ell \subseteq \widehat{\mathcal{T}}_\ell$ *for all* $\ell \in \mathbb{N}_0$. *Suppose that* $\#\{\ell \in \mathbb{N}_0 : \widehat{\mathcal{M}}_\ell = \widehat{\mathcal{T}}_\ell\} = \infty$. *Then, there exists* $m \in \mathbb{N}_0$ *and* $\gamma > 0$ *such that for all discrete spaces* $\mathcal{X}_\bullet \subset \mathcal{H}$ *with* $\mathcal{X}_\bullet \supseteq \widehat{\mathcal{X}}_m$ *the related* inf *-sup constant* (9) *satisfies* $\gamma_\bullet \geq \gamma > 0$. *In particular,* $\mathcal{X}_\bullet$ *admits a unique solution* $U_\bullet \in \mathcal{X}_\bullet$ *of* (5) *which is quasi-optimal in the sense of inequality* (10). *Moreover, the Galerkin solutions* $\widehat{U}_\ell \in \widehat{\mathcal{X}}_\ell$, *for* $\ell \geq m$, *yield convergence* $\lim_{\ell\to\infty} \|u - \widehat{U}_\ell\|_{\mathcal{H}} = 0$. $\qquad\square$

2.3. **A posteriori error estimation.** Let $\mathcal{T}_\star \in \mathbb{T} = \mathrm{refine}(\mathcal{T}_0)$. We suppose that given the solution $U_\star \in \mathcal{X}_\star$ of (5) and $T \in \mathcal{T}_\star$, we can compute some local refinement indicators $\eta_\star(T) \geq 0$ as well as the related *a posteriori* error estimator

$$(15) \qquad \eta_\star := \eta_\star(\mathcal{T}_\star), \quad \text{where } \eta_\star(\mathcal{U}_\star) := \Big( \sum_{T \in \mathcal{U}_\star} \eta_\star(T)^2 \Big)^{1/2} \text{ for all } \mathcal{U}_\star \subseteq \mathcal{T}_\star.$$

To prove convergence with optimal algebraic rates for Algorithm 7, we rely on the following *axioms of adaptivity* which are slightly generalized when compared to those of [CFPP14], since we always have to suppose solvability of the related discrete problem (5).

(A1) **Stability on non-refined element domains:** There exists $C_{\mathrm{stb}} > 0$ such that for all $\mathcal{T}_\star \in \mathbb{T}$ and all $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$, the following implication holds: Provided that the discrete solutions $U_\star \in \mathcal{X}_\star$ and $U_\bullet \in \mathcal{X}_\bullet$ exist, it holds $\big| \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\star) - \eta_\star(\mathcal{T}_\bullet \cap \mathcal{T}_\star) \big| \leq C_{\mathrm{stb}} \|U_\bullet - U_\star\|_{\mathcal{H}}$.

(A2) **Reduction on refined element domains:** There exist $C_{\mathrm{red}} > 0$ and $0 < q_{\mathrm{red}} < 1$ such that for all $\mathcal{T}_\star \in \mathbb{T}$ and all $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$, the following implication holds: Provided that the discrete solutions $U_\star \in \mathcal{X}_\star$ and $U_\bullet \in \mathcal{X}_\bullet$ exist, it holds $\eta_\bullet(\mathcal{T}_\bullet \backslash \mathcal{T}_\star)^2 \leq q_{\mathrm{red}}\, \eta_\star(\mathcal{T}_\star \backslash \mathcal{T}_\bullet)^2 + C_{\mathrm{red}}^2 \|U_\bullet - U_\star\|_{\mathcal{H}}^2$.

(A3) **Reliability:** There exists $C_{\mathrm{rel}}' > 0$ such that for all $\mathcal{T}_\star \in \mathbb{T}$, the following implication holds: Provided that the discrete solution $U_\star \in \mathcal{X}_\star$ exists, it holds $\|u - U_\star\|_{\mathcal{H}} \leq C_{\mathrm{rel}}'\, \eta_\star$.

(A4) **Discrete reliability:** There exists $C_{\mathrm{rel}} > 0$ such that for all $\mathcal{T}_\star \in \mathbb{T}$ and all $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$, there exists a set $\mathcal{R}_{\star,\bullet} \subseteq \mathcal{T}_\star$ such that the following implication holds: Provided that the discrete solutions $U_\star \in \mathcal{X}_\star$ and $U_\bullet \in \mathcal{X}_\bullet$ exist, it holds $\|U_\bullet - U_\star\|_{\mathcal{H}} \leq C_{\mathrm{rel}}\, \gamma_\bullet^{-1}\, \eta_\star(\mathcal{R}_{\star,\bullet})$ as well as $\mathcal{T}_\star \backslash \mathcal{T}_\bullet \subseteq \mathcal{R}_{\star,\bullet}$ with $\#\mathcal{R}_{\star,\bullet} \leq C_{\mathrm{rel}}\, \#(\mathcal{T}_\star \backslash \mathcal{T}_\bullet)$, where $\gamma_\bullet > 0$ is the inf-sup constant (9) associated with $\mathcal{X}_\bullet$.

**Remark 5.** *For a general diffusion problem (4) with piecewise Lipschitz diffusion coefficient $A \in W^{1,\infty}(T_0)$ for all $T_0 \in \mathcal{T}_0$ and $\mathcal{X}_\star := \mathcal{S}^p(\mathcal{T}_\star) \cap H_0^1(\Omega)$, the local contributions of the usual residual error estimator read, for all $T \in \mathcal{T}_\star$,*

$$(16) \quad \eta_\star(T)^2 = h_T^2 \|f + \mathrm{div}(A \nabla U_\star) - b \cdot \nabla U_\star - c U_\star\|_{L^2(T)}^2 + h_T \|[(A \nabla U_\star) \cdot n]\|_{L^2(\partial T \cap \Omega)}^2,$$

*where $[(\cdot) \cdot n]$ denotes the normal jump over interior facets and $h_T := |T|^{1/d} \simeq \mathrm{diam}(T)$. For the Helmholtz problem (3), these local contributions simplify to*

$$(17) \qquad \eta_\star(T)^2 = h_T^2 \|f + \Delta U_\star + \kappa^2 U_\star\|_{L^2(T)}^2 + h_T \|[\partial_n U_\star]\|_{L^2(\partial T \cap \Omega)}^2.$$

*We note that in either case (A1)–(A4) are already known with $\mathcal{R}_{\star,\bullet} = \mathcal{T}_\star \backslash \mathcal{T}_\bullet$, and the corresponding constants depend only on uniform shape regularity of the triangulations $\mathcal{T}_\star \in \mathbb{T}$ and the well-posedness of the continuous problem (2); see [CKNS08, CN12, FFP14]. The error estimator can be extended to mixed Dirichlet-Neumann-Robin boundary conditions, where inhomogeneous Dirichlet conditions are discretized by nodal interpolation for $d = 2$ and $p = 1$, see [FPP14], or by Scott-Zhang interpolation for $d \geq 2$ and $p \geq 1$, see [CFPP14]. In any case (A1)–(A4) remain valid [FPP14, CFPP14], but $\mathcal{R}_{\star,\bullet}$ consists of a fixed patch of $\mathcal{T}_\star \backslash \mathcal{T}_\bullet$ [AFK+13, CFPP14].*

**Remark 6.** *In usual situations, reliability (A3) already follows from discrete reliability (A4); see Lemma 10 (d) below.*

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2.4. **Adaptive algorithm.** Based on the a *posteriori* error estimator from the previous section, we consider the following adaptive algorithm.

**Algorithm 7.** INPUT: *Parameters $0 < \theta \leq 1$ and $C_{\mathrm{mark}} \geq 1$ as well as initial triangulation $\mathcal{T}_0$ with $U_{-1} := 0 \in \mathcal{X}_0$ and $\eta_{-1} := 1$.*
ADAPTIVE LOOP: *For all $\ell = 0, 1, 2, \ldots$, iterate the following steps (i)–(v):*
   (i) *If (5) does not admit a unique solution in $\mathcal{X}_\ell$, define $U_\ell := U_{\ell-1} \in \mathcal{X}_\ell$ and $\eta_\ell := \eta_{\ell-1}$, let $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{T}_\ell)$ be the uniform refinement of $\mathcal{T}_\ell$, increase $\ell$ by 1, and continue with step (i).*
   (ii) *Compute the unique solution $U_\ell \in \mathcal{X}_\ell$ to (5).*
   (iii) *Compute the corresponding indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$.*
   (iv) *Determine a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of up to the multiplicative constant $C_{\mathrm{mark}}$ minimal cardinality such that $\theta \eta_\ell^2 \leq \eta_\ell(\mathcal{M}_\ell)^2$.*
   (v) *Compute $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, increase $\ell$ by 1, and continue with step (i).*
OUTPUT: *Sequences of successively refined triangulations $\mathcal{T}_\ell$, discrete solutions $U_\ell$, and corresponding estimators $\eta_\ell$.*

**Remark 8.**   • *Apart from step (i), Algorithm 7 is the usual adaptive loop based on the Dörfler marking strategy [Dör96] in step (iv) as used, e.g., in [CKNS08, FFP14, CFPP14].*
   • *While $C_{\mathrm{mark}} = 1$ requires to sort the indicators and hence leads to log-linear effort, Stevenson [Ste07] showed that $C_{\mathrm{mark}} = 2$ allows to determine $\mathcal{M}_\ell$ in linear complexity.*

To abbreviate notation, we define $\mathbb{T} := \mathrm{refine}(\mathcal{T}_0)$ as the set of all possible refinements of the given initial mesh $\mathcal{T}_0$ in Algorithm 7. The following lemma exploits the validity of Proposition 1 for uniform mesh-refinement (Corollary 4).

**Lemma 9.** *Let $(U_\ell)_{\ell \in \mathbb{N}_0}$ be the sequence of discrete solutions generated by Algorithm 7. Then, there exists a minimal index $\ell_0 \in \mathbb{N}_0$ such that (5) does not admit a unique solution in $\mathcal{X}_\ell$ for $0 \leq \ell < \ell_0$, but admits a unique solution $U_{\ell_0} \in \mathcal{X}_{\ell_0}$. In particular, the corresponding mesh $\mathcal{T}_{\ell_0}$ is the $\ell_0$-times uniform refinement of $\mathcal{T}_0$. Furthermore, there exists $\ell_1 \in \mathbb{N}_0$ such that (5) admits a unique solution $U_\ell \in \mathcal{X}_\ell$ for all steps $\ell \geq \ell_1$ of Algorithm 7.*

*Proof.* Thanks to Corollary 4, the uniform refinement in step (i) of Algorithm 7 will only be performed finitely many times. This concludes the proof. ☐

To prove convergence of Algorithm 7, we need an additional assumption (see (A5) below) which goes beyond the axioms in [CFPP14]. To that end, let us define the "discrete" limit space $\mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell}$. Because of nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ for all $\ell \geq 0$, $\mathcal{X}_\infty$ is a closed subspace of $\mathcal{H}$ and hence a Hilbert space.

   **(A5) Definiteness of $b(\cdot, \cdot)$ on $\mathcal{X}_\infty$:** For all $w \in \mathcal{X}_\infty$, the following implication holds: If $b(w, v) = 0$ for all $v \in \mathcal{X}_\infty$, then $w = 0$.

Clearly, (A5) is satisfied if $b(\cdot, \cdot)$ is elliptic (14). Moreover, note that well-posedness (7) of (2) implies that (A5) is satisfied, if $\mathcal{X}_\infty = \mathcal{H}$. In many generic situations, the identity $\mathcal{X}_\infty = \mathcal{H}$ is automatically satisfied, but it may also be enforced explicitly by expanding the set of marked elements in the Dörfler marking criterion in step (iv) of Algorithm 7; see Section 3.2 below.

   The following technical lemma exploits the validity of (A5).

**Lemma 10.** *Suppose (A1), (A2), (A4), and (A5). Employ the notation of Algorithm 7 for $0 < \theta \leq 1$. Then, there exists $\ell_2 \in \mathbb{N}_0$ and $\gamma > 0$ such that for all $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_{\ell_2})$ with $\mathcal{X}_\bullet \subseteq \mathcal{X}_\infty$, the following assertion (a) holds:*

(a) *The corresponding* inf-sup *constant* (9) *is bounded from below by* $\gamma_\bullet \geq \gamma > 0$. *In particular, there exists a unique Galerkin solution* $U_\bullet \in \mathcal{X}_\bullet$ *to* (5) *which is quasi-optimal in the sense of inequality* (10).

*Moreover, let* $\mathcal{T}_\star \in \mathbb{T}$ *and* $\mathcal{T}_\bullet \in \text{refine}(\mathcal{T}_\star) \cap \text{refine}(\mathcal{T}_{\ell_2})$ *and suppose that the Galerkin solution* $U_\star \in \mathcal{X}_\star$ *exists. Then, there hold the following assertions* (b)–(c) *with some additional constant* $C_{\text{mon}} > 0$ *which depends only on* $C_{\text{stb}}$, $C_{\text{red}}$, $C_{\text{rel}}$, *and* $\gamma$:

(b) uniform discrete reliability, *i.e.,* $\|U_\bullet - U_\star\|_\mathcal{H} \leq C_{\text{rel}} \gamma^{-1} \eta_\star(\mathcal{R}_{\star,\bullet})$.
(c) quasi-monotonicity of error estimator, *i.e.,* $\eta_\bullet \leq C_{\text{mon}} \eta_\star$.

*If in addition* $\mathcal{X}_\infty = \mathcal{H}$, *then the following assertion* (d) *holds:*

(d) *discrete reliability* (A4) *implies reliability* (A3), *i.e.,* $\|u - U_\bullet\|_\mathcal{H} \leq C_{\text{rel}} \gamma^{-1} \eta_\bullet$.

*Proof.* Employ Proposition 1 with $\mathcal{H}$ replaced by $\mathcal{X}_\infty$. This proves (a) and provides $\ell_2 \in \mathbb{N}_0$ and $\gamma > 0$ such that the inf-sup constant (9) for all discrete subspaces $\mathcal{X}_\bullet \subseteq \mathcal{X}_\infty$ with $\mathcal{X}_\bullet \supseteq \mathcal{X}_{\ell_2}$ is uniformly bounded from below by $\gamma_\bullet \geq \gamma > 0$. Together with (A4), this also proves (b). Moreover, (b) allows to apply [CFPP14, Lemma 3.5] to obtain the quasi-monotonicity (c). Finally, (d) follows from (b) and [CFPP14, Lemma 3.4], since uniform refinement yields convergence (see Corollary 4). $\qquad\square$

## 3. Convergence

### 3.1. Convergence of adaptive algorithm.
This section proves that Algorithm 7 guarantees convergence $\|u - U_\ell\|_\mathcal{H} \to 0$ as $\ell \to \infty$.

**Proposition 11.** *Suppose* (A1)–(A5) *and* $0 < \theta \leq 1$. *Employ the notation of Algorithm 7. Then, the "discrete" limit space* $\mathcal{X}_\infty = \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell}$ *contains the exact solution to problem* (2), *i.e.,* $u \in \mathcal{X}_\infty$. *Moreover,* $\lim_{\ell \to \infty} \|u - U_\ell\|_\mathcal{H} = 0 = \lim_{\ell \to \infty} \eta_\ell$.

The proof of Proposition 11 relies on the following estimator reduction which (in a weaker form) is first found in [CKNS08].

**Lemma 12** (generalized estimator reduction [FPZ16, Lemma 9]). *Stability* (A1) *and reduction* (A2) *together with the Dörfler marking strategy from step* (iv) *of Algorithm 7 imply the following perturbed contraction: For each* $\ell \in \mathbb{N}_0$ *and all* $\mathcal{T}_\star \in \text{refine}(\mathcal{T}_{\ell+1})$ *such that the discrete solutions* $U_\ell \in \mathcal{X}_\ell$ *and* $U_\star \in \mathcal{X}_\star$ *exist, it holds* $\eta_\star^2 \leq q_{\text{est}} \eta_\ell^2 + C_{\text{est}} \|U_\star - U_\ell\|_\mathcal{H}^2$. *The constants* $C_{\text{est}} > 0$ *and* $0 < q_{\text{est}} < 1$ *depend only on* (A1)–(A2) *and on* $0 < \theta \leq 1$. $\qquad\square$

*Proof of Proposition 11.* Let $\ell_2 \in \mathbb{N}_0$ be the index defined in Lemma 10. Without loss of generality, we may assume $\ell_2 = 0$ throughout the proof. In order to prove that $\eta_\ell \to 0$ as $\ell \to \infty$, we show that each subsequence $(\eta_{\ell_k})_{k \in \mathbb{N}_0}$ of the estimator sequence $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ contains a further subsequence $(\eta_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ with $\eta_{\ell_{k_j}} \to 0$ as $j \to \infty$. According to basic calculus, this is in fact equivalent to $\eta_\ell \to 0$ as $\ell \to \infty$.

*Step 1: Boundedness of estimator sequence.* We apply Lemma 10 with $\ell_2 = 0$. The quasi-monotonicity of the error estimator proves $\eta_\ell \leq C_{\text{mon}} \eta_0$ for all $\ell \in \mathbb{N}_0$.

*Step 2: Weak convergence of discrete solutions (subsequence).* Recall the $a(\cdot, \cdot)$-induced energy norm $\|\cdot\|$. From reliability (A3) and step 1, we infer that

$$\|U_\ell\| \leq \|u\| + \|u - U_\ell\| \lesssim \|u\| + \sup_{\ell \in \mathbb{N}_0} \eta_\ell < \infty,$$

i.e., the sequence of discrete solutions is uniformly bounded in $\mathcal{H}$. Let $(\eta_{\ell_k})_{k \in \mathbb{N}_0}$ be an arbitrary subsequence of $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ with corresponding discrete solutions $U_{\ell_k}$. Since $U_{\ell_k} \in \mathcal{X}_{\ell_k} \subseteq \mathcal{X}_\infty$, there exists a subsequence $(U_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ of $(U_{\ell_k})_{k \in \mathbb{N}_0}$ and some limit

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$w \in \mathcal{H}$ such that $U_{\ell_{k_j}} \rightharpoonup w$ weakly in $\mathcal{H}$ as $j \to \infty$. According to Mazur's lemma (see, e.g., [Rud91, Theorem 3.12]), convexity and closedness imply that $\mathcal{X}_\infty$ is also closed with respect to the weak topology and hence $w \in \mathcal{X}_\infty$. Let $v \in \mathcal{X}_\infty$. Let $P_\ell : \mathcal{H} \to \mathcal{X}_\ell$ be the orthogonal projection with respect to $\|\cdot\|$, i.e.,

$$\|v - P_\ell v\| = \min_{V_\ell \in \mathcal{X}_\ell} \|v - V_\ell\| \quad \text{for all } v \in \mathcal{H}.$$

By definition of $\mathcal{X}_\infty$, this also implies strong convergence $\|v - P_\ell v\| \to 0$ as $\ell \to \infty$. Recall that the product of a weakly convergent sequence and a strongly convergent sequence leads to convergence of the scalar product. Moreover, compact operators turn weak convergence into strong convergence, i.e., $\mathcal{K}U_{\ell_{k_j}} \to \mathcal{K}w$ strongly in $\mathcal{H}^*$ as $j \to \infty$. With these two observations, we derive

$$0 \overset{(5)}{=} \langle f , P_{\ell_{k_j}} v \rangle - a(U_{\ell_{k_j}}, P_{\ell_{k_j}} v) - \langle \mathcal{K}U_{\ell_{k_j}} , P_{\ell_{k_j}} v \rangle \xrightarrow{j \to \infty} \langle f , v \rangle - a(w, v) - \langle \mathcal{K}w , v \rangle.$$

This proves that the weak limit $w \in \mathcal{X}_\infty$ solves the Galerkin formulation

(18) $$a(w, v) + \langle \mathcal{K}w , v \rangle = \langle f , v \rangle \quad \text{for all } v \in \mathcal{X}_\infty.$$

*Step 3: Strong convergence of discrete solutions (subsequence).* Note that $\|w - U_{\ell_{k_j}}\|^2 = \|w\|^2 - 2\operatorname{Re} a(w, U_{\ell_{k_j}}) + \|U_{\ell_{k_j}}\|^2$. Therefore, strong convergence $\|w - U_{\ell_{k_j}}\| \to 0$ is equivalent to weak convergence $U_{\ell_{k_j}} \rightharpoonup w$ plus convergence of the norm $\|U_{\ell_{k_j}}\| \to \|w\|$. It thus only remains to prove the latter. With the previous observations, it holds

$$\|U_{\ell_{k_j}}\|^2 = a(U_{\ell_{k_j}}, U_{\ell_{k_j}}) \overset{(5)}{=} \langle f , U_{\ell_{k_j}} \rangle - \langle \mathcal{K}U_{\ell_{k_j}} , U_{\ell_{k_j}} \rangle$$
$$\xrightarrow{j \to \infty} \langle f , w \rangle - \langle \mathcal{K}w , w \rangle \overset{(18)}{=} a(w, w) = \|w\|^2.$$

*Step 4: Estimator reduction principle (subsequence).* Let $(\eta_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ denote the estimator subsequence corresponding to $(U_{\ell_{k_j}})_{j \in \mathbb{N}_0}$. With $\mathcal{T}_{\ell_{k_{j+1}}} \in \operatorname{refine}(\mathcal{T}_{\ell_{k_j}+1})$ and Lemma 12, it holds $\eta_{\ell_{k_{j+1}}}^2 \le q_{\mathrm{est}} \, \eta_{\ell_{k_j}}^2 + C_{\mathrm{est}} \|U_{\ell_{k_{j+1}}} - U_{\ell_{k_j}}\|_{\mathcal{H}}^2$. Moreover, step 3 implies convergence $\|U_{\ell_{k_{j+1}}} - U_{\ell_{k_j}}\|_{\mathcal{H}} \simeq \|U_{\ell_{k_{j+1}}} - U_{\ell_{k_j}}\| \to 0$ as $j \to \infty$. Hence, the subsequence $(\eta_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ is contractive up to a sequence that converges to zero. Therefore, basic calculus (see, e.g., [AFLP12, Lemma 2.3]) proves convergence $\eta_{\ell_{k_j}} \to 0$ as $j \to \infty$.

*Step 5: Estimator convergence (full sequence).* We have shown that each subsequence $(\eta_{\ell_k})_{k \in \mathbb{N}_0}$ of $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ has a further subsequence $(\eta_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ with $\eta_{\ell_{k_j}} \to 0$ as $j \to \infty$. As noted above, this yields $\eta_\ell \to 0$ as $\ell \to \infty$.

*Step 6: Strong convergence of discrete solutions (full sequence).* Finally, reliability (A3) yields $\|u - U_\ell\|_{\mathcal{H}} \lesssim \eta_\ell \to 0$ as $\ell \to \infty$ and hence concludes the proof. $\qquad\square$

**Remark 13.** *Note that the proof of Proposition 11 relies only on (A4)–(A5) to prove boundedness of the estimator sequence $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ (see step 1 of the proof). Instead, we can also modify the marking step (iv) of Algorithm 7 so that the assertion of Proposition 11 remains true, if (A1)–(A3) still hold, while (A4)–(A5) fail. To this end, consider the following new marking criterion:*

(iv) *If $\eta_\ell > \max_{j=0,\dots,\ell-1} \eta_j$, define $\mathcal{M}_\ell := \mathcal{T}_\ell$. Otherwise, determine a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of up to the multiplicative constant $C_{\mathrm{mark}}$ minimal cardinality such that $\theta \eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2$.*

*To see that this new marking criterion ensures that $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ is bounded, we argue as follows:*

Case 1: *Suppose that there exists an $M \in \mathbb{N}$ such that $\eta_\ell \leq \max_{j=0,\ldots,\ell-1} \eta_j$ for all $\ell \geq M$. Then, it even follows that $\eta_\ell \leq \max_{j=0,\ldots,M-1} \eta_j$ for all $\ell \in \mathbb{N}_0$.*

Case 2: *If the assumption of case 1 fails, the new step* (iv) *of Algorithm 7 enforces infinitely many steps of uniform refinement. Therefore, Corollary 4 applies and provides $m \in \mathbb{N}_0$ and $C > 0$ such that all discrete subspaces $\mathcal{X}_\star \subseteq \mathcal{H}$ with $\mathcal{X}_\star \supseteq \mathcal{X}_m$ admit a unique solution $U_\star \in \mathcal{X}_\star$ of* (5) *which is quasi-optimal in the sense of inequality* (10). *Since* (A1)–(A3) *hold,* [CFPP14, Lemma 3.5] *applies and proves quasi-monotonicity of the estimator, i.e.,*

$$\eta_\bullet \leq C_{\mathrm{mon}}\, \eta_\star \quad \text{for all } \mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}_m) \text{ and all } \mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star).$$

*In particular, this implies $\eta_\ell \leq C_{\mathrm{mon}}\, \eta_m$ for all $\ell \geq m$, and therefore $\eta_\ell \leq \max\{C_{\mathrm{mon}}, 1\} \max_{j=0,\ldots,m} \eta_j$ for all $\ell \in \mathbb{N}_0$.*

*Note that besides step 1 all steps of the proof of Proposition 11 rely only on* (A1)–(A3). *Therefore, we obtain $\eta_\ell \to 0$ as $\ell \to \infty$. In particular, this implies that Case 1 above is the generic case and that optimal convergence rates will not be affected by the new marking strategy.*

## 3.2. **Definiteness on the "discrete" limit space (A5).**

While (A1)–(A4) only rely on the a posteriori error estimation strategy, the property (A5) involves the "discrete" limit space $\mathcal{X}_\infty = \overline{\bigcup_{\ell=0}^{\infty} \mathcal{X}_\ell}$ generated by Algorithm 7 and is hence less accessible for the numerical analysis. However, recall that $\mathcal{H} = \mathcal{X}_\infty$ is sufficient to ensure (A5). For $\mathcal{H} = H_0^1(\Omega)$, the following lemma provides a simple criterion for the latter identity.

**Lemma 14.** *Let $\mathcal{H} = H_0^1(\Omega)$ and $\mathcal{X}_\ell = \mathcal{S}_0^p(\mathcal{T}_\ell)$ for some $p \geq 1$. Suppose that the triangulations $\mathcal{T}_\ell$ generated by Algorithm 7 are uniformly shape regular with $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ as $\ell \to \infty$. Then, $\mathcal{X}_\infty = \mathcal{H}$ and hence assumption* (A5) *is satisfied.*

*Proof.* For $w \in \mathcal{D} := H^2(\Omega) \cap H_0^1(\Omega)$, recall the approximation property $\inf_{V_\ell \in \mathcal{X}_\ell} \|w - V_\ell\|_{\mathcal{H}} \lesssim \|h_\ell\|_{L^\infty(\Omega)} \|D^2 w\|_{L^2(\Omega)}$ from, e.g., [BS08]. This proves

$$(19) \qquad \lim_{\ell \to \infty} \inf_{V_\ell \in \mathcal{X}_\ell} \|w - V_\ell\|_{\mathcal{H}} = 0 \quad \text{for all } w \in \mathcal{D}.$$

Let $v \in \mathcal{H}$ and $\varepsilon > 0$. Since $\mathcal{D}$ is dense within $H_0^1(\Omega)$, choose $w \in \mathcal{D}$ with $\|v - w\|_{\mathcal{H}} \leq \varepsilon/2$. According to (19), there exists an index $\ell_\star \in \mathbb{N}_0$ such that $\inf_{V_\ell \in \mathcal{X}_\ell} \|w - V_\ell\|_{\mathcal{H}} \leq \varepsilon/2$ for all $\ell \geq \ell_\star$. In particular, the triangle inequality concludes

$$\inf_{V_\ell \in \mathcal{X}_\ell} \|v - V_\ell\|_{\mathcal{H}} \leq \|v - w\|_{\mathcal{H}} + \inf_{V_\ell \in \mathcal{X}_\ell} \|w - V_\ell\|_{\mathcal{H}} \leq \varepsilon \quad \text{for all } \ell \geq \ell_\star.$$

This proves $v \in \mathcal{X}_\infty = \overline{\bigcup_{\ell=0}^{\infty} \mathcal{X}_\ell}$ and hence concludes $\mathcal{X}_\infty = \mathcal{H}$. $\qquad\square$

**Remark 15.** *In many generic situations, $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ and hence* (A5) *with $\mathcal{X}_\infty = \mathcal{H}$ is automatically verified: Let $p \geq 1$ and $q \geq 0$ be polynomial degrees. Suppose that $\mathcal{H} = H_0^1(\Omega)$ and $\mathcal{X}_\ell = \mathcal{S}_0^p(\mathcal{T}_\ell)$. For $f \in L^2(\Omega)$, let $f_\ell \in \mathcal{P}^q(\mathcal{T}_\ell) := \{V_\ell \in L^2(\Omega) : \forall T \in \mathcal{T}_\ell \; V_\ell|_T \text{ is a polynomial of degree } \leq q\}$ be the $L^2$-best approximation of $f$ in $\mathcal{P}^q(\mathcal{T}_\ell)$. Suppose that the error estimator is even reliable in the sense of*

$$(20) \qquad \|u - U_\ell\|_{H^1(\Omega)} + \|h_\ell(f - f_\ell)\|_{L^2(\Omega)} \leq C_{\mathrm{rel}}\, \eta_\ell \quad \text{for all } \ell \geq 0,$$

*where $C_{\mathrm{rel}} > 0$ is independent of $\ell$. Note that* (20) *is well-known for residual error estimators and elliptic PDEs with polynomial coefficients. Suppose that for all $\ell \in \mathbb{N}$ and all $T \in \mathcal{T}_\ell$ it holds $u|_T \notin \mathcal{P}^p(T)$ or $f|_T \notin \mathcal{P}^q(T)$, i.e., the continuous solution or the given data are not locally polynomial. Then, one can argue by contradiction to see that convergence $\eta_\ell \to 0$ as $\ell \to \infty$ (see, e.g., Remark 13) implies $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ as*

$\ell \to \infty$. In particular, it follows from Lemma 14 that assumption (A5) is satisfied with $\mathcal{X}_\infty = \mathcal{H}$. $\qquad\square$

The next proposition shows that $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ and hence (A5) with $\mathcal{X}_\infty = \mathcal{H}$ can be guaranteed by employing an expanded Dörfler marking strategy in step (iv) of Algorithm 7. We stress that this does not affect optimal convergence behaviour in the sense of Theorem 26 below.

**Proposition 16.** *Suppose $0 < \theta \le 1$. Employ the notation of Algorithm 7. Let $C'_{\text{mark}} > 0$. For all $\ell \in \mathbb{N}_0$, we suppose that the set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ in step (iv) of Algorithm 7 is selected as follows:*

- *Let $\mathcal{M}'_\ell \subseteq \mathcal{T}_\ell$ be a set of up to the multiplicative constant $C'_{\text{mark}}$ minimal cardinality such that $\theta \eta_\ell^2 \le \eta_\ell(\mathcal{M}'_\ell)^2$.*
- *Suppose that $\mathcal{T}_\ell = \{T_1, \ldots, T_N\}$ is sorted such that $|T_1| \ge |T_2| \ge \cdots \ge |T_N|$.*
- *With arbitrary $1 \le n \le \#\mathcal{M}'_\ell$, define $\mathcal{M}_\ell := \mathcal{M}'_\ell \cup \{T_1, \ldots, T_n\}$.*

*Then, $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ is a set of up to the multiplicative constant $C_{\text{mark}} := 2C'_{\text{mark}}$ minimal cardinality such that $\theta \eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2$. Moreover, Algorithm 7 guarantees $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ as $\ell \to \infty$. In particular, assumption (A5) with $\mathcal{X}_\infty = \mathcal{H}$ is satisfied for $\mathcal{H} = H_0^1(\Omega)$ and $\mathcal{X}_\ell = \mathcal{S}_0^p(\mathcal{T}_\ell)$.*

*Proof.* The claims on $\mathcal{M}_\ell$ are obvious. Recall that refinement leads to a uniform contraction of the mesh-size, i.e., $h_{\ell+1}|_T \le q_{\text{mesh}} h_\ell|_T$ for all $T \in \mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$. Since each mesh $\mathcal{T}_\ell$ is a finite set and each step of the adaptive algorithm guarantees that (at least) the element $T \in \mathcal{T}_\ell$ with the largest size $|T| \simeq (h_\ell|_T)^d$ is refined, this implies necessarily $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ as $\ell \to \infty$. Lemma 14 concludes the proof. $\qquad\square$

3.3. **Linear convergence of adaptive algorithm.** The analysis in this section adapts and extends some ideas from [FFP14]. We note that the latter work uses strong ellipticity (14) of $b(\cdot,\cdot)$, while we only rely on ellipticity (6) of $a(\cdot,\cdot)$.

**Lemma 17** ([FFP14, Lemma 3.5]). *Suppose (A1)–(A5) and $0 < \theta \le 1$. Employ the notation of Algorithm 7. Then, the sequences $(e_\ell)_{\ell \in \mathbb{N}}$ and $(E_\ell)_{\ell \in \mathbb{N}}$ defined by*

$$e_\ell := \begin{cases} \frac{u - U_\ell}{\|u - U_\ell\|_{\mathcal{H}}} & \text{for } u \ne U_\ell, \\ 0 & \text{else}, \end{cases}$$

$$E_\ell := \begin{cases} \frac{U_{\ell+1} - U_\ell}{\|U_{\ell+1} - U_\ell\|_{\mathcal{H}}} & \text{for } U_{\ell+1} \ne U_\ell, \\ 0 & \text{else}, \end{cases}$$

*converge weakly to zero, i.e., $\lim_{\ell \to \infty} \langle \phi, e_\ell \rangle = 0 = \lim_{\ell \to \infty} \langle \phi, E_\ell \rangle$ for all $\phi \in \mathcal{H}^*$.*

*Proof.* We consider the sequence $(e_\ell)_{\ell \in \mathbb{N}_0}$ and note that the claim for $(E_\ell)_{\ell \in \mathbb{N}_0}$ follows along the same lines. To prove $e_\ell \rightharpoonup 0$ as $\ell \to \infty$, we show that each subsequence $(e_{\ell_k})_{k \in \mathbb{N}_0}$ admits a further subsequence $(e_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ such that $e_{\ell_{k_j}} \rightharpoonup 0$ as $j \to \infty$. Let $(e_{\ell_k})_{k \in \mathbb{N}_0}$ be a subsequence of $(e_\ell)_{\ell \in \mathbb{N}_0}$. Due to boundedness $\|e_{\ell_k}\|_{\mathcal{H}} \le 1$, there exists a further subsequence $(e_{\ell_{k_j}})_{j \in \mathbb{N}_0}$ such that $e_{\ell_{k_j}} \rightharpoonup w \in \mathcal{H}$ as $j \to \infty$. It remains to show that $w = 0$. Note that $U_\ell, u \in \mathcal{X}_\infty$ (see Proposition 11) implies $e_\ell \in \mathcal{X}_\infty$ and hence $w \in \mathcal{X}_\infty$. Note the Galerkin orthogonality

(21) $\qquad 0 = b(u - U_\star, V_\star) = a(u - U_\star, V_\star) + \langle \mathcal{K}(u - U_\star), V_\star \rangle \quad$ for all $V_\star \in \mathcal{X}_\star$.

Let $n \in \mathbb{N}$ and $V_n \in \mathcal{X}_n$. If $\ell_{k_j} \ge n$ and $e_{\ell_{k_j}} \ne 0$, the Galerkin orthogonality proves

$$b(e_{\ell_{k_j}}, V_n) = b(u - U_{\ell_{k_j}}, V_n) / \|u - U_{\ell_{k_j}}\|_{\mathcal{H}} = 0$$

and hence $b(e_{\ell_{k_j}}, V_n) = 0$ for all $\ell_{k_j} \geq n$. With weak convergence, this yields

$$b(w, V_n) = \lim_{j \to \infty} b(e_{\ell_{k_j}}, V_n) = 0 \quad \text{for all } V_n \in \mathcal{X}_n \text{ and all } n \in \mathbb{N}_0.$$

Let $v \in \mathcal{X}_\infty$. By definition of $\mathcal{X}_\infty$, there exists a sequence $(V_n)_{n \in \mathbb{N}_0}$ with $V_n \in \mathcal{X}_n$ and $\|v - V_n\|_\mathcal{H} \to 0$ as $n \to \infty$. Therefore the preceding identity implies $b(w, v) = 0$ for all $v \in \mathcal{X}_\infty$. Finally, assumption (A5) concludes $w = 0$. $\qquad\square$

The following quasi-orthogonality (22) is a consequence of Lemma 17 and the Galerkin orthogonality (21). For elliptic $b(\cdot, \cdot)$, it is proved in [FFP14, Proposition 3.6]. Our proof essentially follows those ideas, but we use the norm $\|\cdot\|$ induced by $a(\cdot, \cdot)$ instead of the quasi-norm induced by $b(\cdot, \cdot)$, if $b(\cdot, \cdot)$ was elliptic. For the convenience of the reader, we include the most important steps of the proof.

**Lemma 18.** *Suppose* (A1)–(A5) *and* $0 < \theta \leq 1$. *Employ the notation of Algorithm 7. Then, for any* $0 < \varepsilon < 1$, *there exists* $\ell_3 \in \mathbb{N}_0$ *such that*

$$(22) \qquad \|u - U_{\ell+1}\|^2 + \|U_{\ell+1} - U_\ell\|^2 \leq \frac{1}{1 - \varepsilon} \|u - U_\ell\|^2 \quad \text{for all } \ell \geq \ell_3.$$

*Proof.* Let $\varepsilon > 0$. Let $\delta > 0$ be a free parameter which is fixed later. Consider the sequences $(e_\ell)_{\ell \in \mathbb{N}_0}$ and $(E_\ell)_{\ell \in \mathbb{N}_0}$ of Lemma 17. Recall that the compact operator $\mathcal{K}$ turns weak convergence $e_\ell, E_\ell \rightharpoonup 0$ in $\mathcal{H}$ into strong convergence $\mathcal{K}e_\ell, \mathcal{K}E_\ell \to 0$ in $\mathcal{H}^*$ as $\ell \to \infty$. For any $\delta > 0$, this provides some $\ell_3 \in \mathbb{N}$ such that

$$\|\mathcal{K}e_\ell\|_{\mathcal{H}^*} + \|\mathcal{K}E_\ell\|_{\mathcal{H}^*} \leq \delta \quad \text{for all } \ell \geq \ell_3.$$

For any $w \in \mathcal{H}$, this gives

$$|\langle \mathcal{K}(u - U_\ell), w \rangle| = |\langle \mathcal{K}e_\ell, w \rangle| \, \|u - U_\ell\|_\mathcal{H} \leq \delta \, \|u - U_\ell\|_\mathcal{H} \|w\|_\mathcal{H}$$

as well as

$$|\langle \mathcal{K}(U_{\ell+1} - U_\ell), w \rangle| = |\langle \mathcal{K}E_\ell, w \rangle| \, \|U_{\ell+1} - U_\ell\|_\mathcal{H} \leq \delta \, \|U_{\ell+1} - U_\ell\|_\mathcal{H} \|w\|_\mathcal{H}.$$

Algebraic computations with the Galerkin orthogonality (21) show

$$b(u - U_{\ell+1}, u - U_{\ell+1}) + b(U_{\ell+1} - U_\ell, U_{\ell+1} - U_\ell) + b(U_{\ell+1} - U_\ell, u - U_{\ell+1}) = b(u - U_\ell, u - U_\ell).$$

Since $\|v\|^2 = a(v, v) = b(v, v) - \langle \mathcal{K}v, v \rangle$ for all $v \in \mathcal{H}$, this translates to

$$\|u - U_{\ell+1}\|^2 + \|U_{\ell+1} - U_\ell\|^2 + \langle \mathcal{K}(u - U_{\ell+1}), u - U_{\ell+1} \rangle + \langle \mathcal{K}(U_{\ell+1} - U_\ell), U_{\ell+1} - U_\ell \rangle$$
$$+ b(U_{\ell+1} - U_\ell, u - U_{\ell+1}) = \|u - U_\ell\|^2 + \langle \mathcal{K}(u - U_\ell), u - U_\ell \rangle.$$

The remaining bilinear form $b(U_{\ell+1} - U_\ell, u - U_{\ell+1})$ is estimated as follows

$$
\begin{aligned}
|b(U_{\ell+1} - U_\ell, u - U_{\ell+1})| \;&=\; |a(u - U_{\ell+1}, U_{\ell+1} - U_\ell) + \langle \mathcal{K}(U_{\ell+1} - U_\ell), u - U_{\ell+1} \rangle| \\
&\overset{(21)}{=}\; |-\langle \mathcal{K}(u - U_{\ell+1}), U_{\ell+1} - U_\ell \rangle + \langle \mathcal{K}(U_{\ell+1} - U_\ell), u - U_{\ell+1} \rangle| \\
&\leq\; 2\delta \, \|u - U_{\ell+1}\|_\mathcal{H} \|U_{\ell+1} - U_\ell\|_\mathcal{H}.
\end{aligned}
$$

With norm equivalence $\|v\|_\mathcal{H}^2 \leq C \, \|v\|^2$ for all $v \in \mathcal{H}$, we thus see

$$(1 - \delta C) \|u - U_{\ell+1}\|^2 + (1 - \delta C) \|U_{\ell+1} - U_\ell\|^2$$
$$\leq (1 + \delta C) \|u - U_\ell\|^2 + 2\delta C \|u - U_{\ell+1}\| \|U_{\ell+1} - U_\ell\|.$$

Finally, the Young inequality $2cab \leq ca^2 + cb^2$ for all $a, b, c \geq 0$, yields

$$(1 - 2\delta C) \|u - U_{\ell+1}\|^2 + (1 - 2\delta C) \|U_{\ell+1} - U_\ell\|^2 \leq (1 + \delta C) \|u - U_\ell\|^2.$$

For sufficiently small $\delta > 0$ and $\dfrac{1 + \delta C}{1 - 2\delta C} \leq \dfrac{1}{1 - \varepsilon}$, this proves (22). $\qquad\square$

The following result was proved in [FFP14] for strongly elliptic problems (14). Here, we generalize the result by extending it to a more general class of problems. Our proof follows the ideas of [CKNS08].

**Theorem 19.** *Suppose (A1)–(A5) and $0 < \theta \le 1$. Then, there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that the output of Algorithm 7 satisfies $\eta_{\ell+n} \le C_{\mathrm{lin}} q_{\mathrm{lin}}^n \eta_\ell$ for all $\ell, n \in \mathbb{N}_0$ with $\ell \ge \ell_3$, where $\ell_3 \in \mathbb{N}_0$ is the index from Lemma 18.*

*Proof.* Due to norm equivalence $\|\cdot\|_{\mathcal{H}} \simeq \|\cdot\|$, reliability (A3) and estimator reduction (Lemma 12) also hold with respect to the $a(\cdot, \cdot)$-induced energy norm $\|\cdot\|$. To simplify the notation and without loss of generality, we therefore suppose $\|\cdot\|_{\mathcal{H}} = \|\cdot\|$ throughout the proof.

*Step 1:* In this step, we prove that there exist $0 < q_{\mathrm{lin}}, \lambda < 1$ and $\ell_3 \in \mathbb{N}_0$ such that

(23) $\qquad \Delta_{\ell+1} \le q_{\mathrm{lin}} \Delta_\ell \quad$ for all $\ell \ge \ell_3, \quad$ where $\quad \Delta_\star^2 := \|u - U_\star\|^2 + \lambda \eta_\star^2.$

Let $\varepsilon, \lambda > 0$ be free parameters which are fixed later. With Lemma 12 and Lemma 18, we see for $\ell \ge \ell_3 = \ell_3(\varepsilon)$

$$\Delta_{\ell+1}^2 = \|u - U_{\ell+1}\|^2 + \lambda \eta_{\ell+1}^2 \le \frac{1}{1-\varepsilon} \|u - U_\ell\|^2 + \lambda q_{\mathrm{est}} \eta_\ell^2 + (\lambda C_{\mathrm{est}} - 1) \|U_{\ell+1} - U_\ell\|^2.$$

For sufficiently small $\lambda$ (i.e., $\lambda C_{\mathrm{est}} \le 1$) and an additional free parameter $\delta > 0$, reliability (A3) yields that

$$\Delta_{\ell+1}^2 \le \frac{1}{1-\varepsilon} \|u - U_\ell\|^2 + \lambda q_{\mathrm{est}} \eta_\ell^2 \le \left( \frac{1}{1-\varepsilon} - \delta\lambda \right) \|u - U_\ell\|^2 + \lambda(q_{\mathrm{est}} + C_{\mathrm{rel}}^2 \delta) \eta_\ell^2$$

$$\le \max\left\{ \frac{1}{1-\varepsilon} - \delta\lambda, \, q_{\mathrm{est}} + C_{\mathrm{rel}}^2 \delta \right\} \Delta_\ell^2.$$

Since $0 < q_{\mathrm{est}} < 1$, we may choose $\delta > 0$ sufficiently small such that $0 < q_{\mathrm{est}} + C_{\mathrm{rel}}^2 \delta < 1$. Finally choose $\varepsilon > 0$ sufficiently small such that $0 < 1/(1-\varepsilon) - \delta\lambda < 1$. This concludes (23).

*Step 2:* We employ the notation of step 1. Induction on $n$ proves $\Delta_{\ell+n} \le_{\mathrm{lin}}^n \Delta_\ell$ for all $\ell \ge \ell_3$ and all $n \in \mathbb{N}_0$. Note that reliability (A3) yields $\eta_\star^2 \simeq \Delta_\star^2$. Combining these two observations, we conclude the proof. $\qquad \square$

3.4. **Validity of the Céa lemma.** In this section, we show that the discrete solutions computed in Algorithm 7 are quasi-optimal in the sense of the Céa lemma.

**Theorem 20.** *Suppose (A1)–(A5) and $0 < \theta \le 1$. Then, there exist $C_\ell \ge 1$ with $\lim_{\ell \to \infty} C_\ell = 1$ and $\ell_4 > 0$ such that the output of Algorithm 7 satisfies*

(24) $\qquad \|u - U_\ell\| \le C_\ell \min_{V_\ell \in \mathcal{X}_\ell} \|u - V_\ell\| \quad$ for all $\ell \ge \ell_4.$

*Proof.* Consider the sequences $(e_\ell)$ and $(E_\ell)$ of Lemma 17. We follow the arguments of the proof of Lemma 18. Let $V_\ell \in \mathcal{X}_\ell$. With the Galerkin orthogonality (21), it holds

$$
\begin{aligned}
\|u - U_\ell\|^2 &= b(u - U_\ell, u - U_\ell) - \langle \mathcal{K}(u - U_\ell), \, u - U_\ell \rangle \\
&\stackrel{(21)}{=} b(u - U_\ell, u - V_\ell) - \langle \mathcal{K}(u - U_\ell), \, u - U_\ell \rangle \\
&= a(u - U_\ell, u - V_\ell) + \langle \mathcal{K}(u - U_\ell), \, u - V_\ell \rangle - \langle \mathcal{K}(u - U_\ell), \, u - U_\ell \rangle \\
&\le \|u - U_\ell\| \|u - V_\ell\| + \|\mathcal{K}e_\ell\|_{\mathcal{H}^*} \|u - U_\ell\|_{\mathcal{H}} \|u - V_\ell\|_{\mathcal{H}} + \|\mathcal{K}e_\ell\|_{\mathcal{H}^*} \|u - U_\ell\|_{\mathcal{H}}^2.
\end{aligned}
$$

With norm equivalence $\|v\|_{\mathcal{H}}^2 \le C \|v\|^2$ for all $v \in \mathcal{H}$, we thus see

$$\|u - U_\ell\| \le (1 + C \|\mathcal{K}e_\ell\|_{\mathcal{H}^*}) \|u - V_\ell\| + C \|\mathcal{K}e_\ell\|_{\mathcal{H}^*} \|u - U_\ell\|.$$

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Rearranging this estimate, we prove

$$\|u - U_\ell\| \leq \frac{1 + C \left\|\mathcal{K}e_\ell\right\|_{\mathcal{H}^*}}{1 - C \left\|\mathcal{K}e_\ell\right\|_{\mathcal{H}^*}} \|u - V_\ell\|$$

and conclude (24), since $\|\mathcal{K}e_\ell\|_{\mathcal{H}^*} \to 0$ as $\ell \to \infty$. $\qquad\square$

## 4. Optimal Convergence Rates

4.1. **Fine properties of mesh-refinement.** The proof of optimal convergence rates requires further properties of the mesh-refinement. First, we suppose that each refined element is split in at most $C_{\mathrm{son}}$ and at least 2 sons. In particular, it holds

$$\#(\mathcal{T}_\star \setminus \mathcal{T}_\bullet) + \#\mathcal{T}_\star \leq \#\mathcal{T}_\bullet \quad \text{for all } \mathcal{T}_\star \in \mathbb{T} \text{ and all } \mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star). \tag{25}$$

Second, we require the mesh-closure estimate

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leq C_{\mathrm{mesh}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \quad \text{for all } \ell \in \mathbb{N}, \tag{26}$$

where the constant $C_{\mathrm{mesh}} \geq 1$ depends only on the initial mesh $\mathcal{T}_0$. Finally, we need the overlay estimate, i.e., for all triangulations $\mathcal{T} \in \mathbb{T}$ and all $\mathcal{T}_\bullet, \mathcal{T}_\star \in \mathrm{refine}(\mathcal{T})$ there exists a common refinement $\mathcal{T}_\bullet \oplus \mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}_\bullet) \cap \mathrm{refine}(\mathcal{T}_\star) \subseteq \mathrm{refine}(\mathcal{T})$ which satisfies

$$\#(\mathcal{T}_\bullet \oplus \mathcal{T}_\star) \leq \#\mathcal{T}_\bullet + \#\mathcal{T}_\star - \#\mathcal{T}. \tag{27}$$

For newest vertex bisection (NVB), the mesh-closure estimate has first been proved for $d = 2$ in [BDD04] and later for $d \geq 2$ in [Ste08]. While both works require an additional admissibility assumption on $\mathcal{T}_0$, [KPP13] proved that this condition is unnecessary for $d = 2$. The proof of the overlay estimate is found in [CKNS08, Ste07]. We note that NVB ensures $2 \leq C_{\mathrm{son}} < \infty$, where $C_{\mathrm{son}}$ depends only on $\mathcal{T}_0$ and $d$; see [GSS14]. For $d = 2$, it holds $C_{\mathrm{son}} = 4$ (see, e.g., [KPP13]). For other mesh-refinement strategies than NVB which satisfy (25)–(27), we refer to [BN10, MP15] as well as to [CFPP14, Section 2.5].

**Lemma 21.** *NVB guarantees the following properties* (a)–(c) *which are exploited in our analysis of optimal convergence rates:*

(a) *There exists $m \in \mathbb{N}$ such that the $m$-times uniform refinement $\widehat{\mathcal{T}}_0$ of $\mathcal{T}_0$ satisfies the assertions of Lemma 10 (with $\mathcal{T}_{\ell_2}$ replaced by $\widehat{\mathcal{T}}_0$). In particular, there holds the quasi-monotonicity of the estimator, i.e., there exists an independent constant $C_{\mathrm{mon}} > 0$ such that*

$$\eta_\bullet \leq C_{\mathrm{mon}} \eta_\star \quad \textit{for all } \mathcal{T}_\star \in \mathbb{T} \textit{ and all } \mathcal{T}_\bullet \in \mathrm{refine}(\widehat{\mathcal{T}}_0) \cap \mathrm{refine}(\mathcal{T}_\star),$$

*provided that the Galerkin solution $U_\star \in \mathcal{X}_\star$ exists.*

(b) *Moreover, for all $\mathcal{T}_\star \in \mathbb{T}$, the $m$-times uniform refinement $\widehat{\mathcal{T}}_\star$ of $\mathcal{T}_\star$ guarantees $\widehat{\mathcal{T}}_\star \in \mathrm{refine}(\widehat{\mathcal{T}}_0)$ and $\#\widehat{\mathcal{T}}_\star \leq C_{\mathrm{son}}^m \#\mathcal{T}_\star$.*

(c) *Suppose that $\mathcal{X}_\infty = \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell} = \mathcal{H}$ (e.g., the expanded Dörfler marking strategy from Proposition 16 is used). Then, there exists an index $\ell_5 \in \mathbb{N}_0$ such that $\mathcal{T}_\ell \in \mathrm{refine}(\widehat{\mathcal{T}}_0)$ for all $\ell \geq \ell_5$.*

*Proof.* Assertion (a) is a direct consequence of Corollary 4, if we argue as in the proof of Lemma 10. Assertions (b)–(c) follow from the fact that NVB is based on a binary refinement rule, where the order of the refinements does not matter [Ste08]. $\qquad\square$

4.2. **Approximation classes.** For $N \in \mathbb{N}_0$ and $\mathcal{T} \in \mathbb{T}$, we define

(28) $\quad \mathbb{T}_N(\mathcal{T}) := \{ \mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}) \ : \ \#\mathcal{T}_\star - \#\mathcal{T} \leq N \text{ and solution } U_\star \in \mathcal{X}_\star \text{ to (5) exists} \}$.

We note that $\mathbb{T}_N(\mathcal{T})$ is finite, but may be empty. However, according to Lemma 21, it holds $\mathbb{T}_N(\mathcal{T}) \neq \emptyset$ for all sufficiently large $N$, e.g., $N \geq C_{\mathrm{son}}^m \#\mathcal{T}$. We use the convention $\min_{\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T})} \eta_\star = 0$, if $\mathbb{T}_N(\mathcal{T}) = \emptyset$. For $s > 0$, we then define

(29) $$\|u\|_{\mathbb{A}_s(\mathcal{T})} := \sup_{N \in \mathbb{N}_0} \Big( (N+1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T})} \eta_\star \Big),$$

where $\eta_\star$ is the error estimator corresponding to the optimal triangulation $\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T})$. Note that $\|u\|_{\mathbb{A}_s(\mathcal{T})} < \infty$ means that starting from $\mathcal{T}$, a convergence behaviour of $\eta_\star = \mathcal{O}\big( (\#\mathcal{T}_\star)^{-s} \big)$ is possible, if the optimal meshes are chosen. To abbreviate notation, we let

(30) $$\mathbb{T}_N := \mathbb{T}_N(\mathcal{T}_0) \quad \text{and} \quad \|u\|_{\mathbb{A}_s} := \|u\|_{\mathbb{A}_s(\mathcal{T}_0)}.$$

**Lemma 22.** *For all $\mathcal{T} \in \mathbb{T}$ and $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T})$, it holds*

(31) $$\#\mathcal{T}_\bullet - \#\mathcal{T} + 1 \leq \#\mathcal{T}_\bullet \leq \#\mathcal{T} \, (\#\mathcal{T}_\bullet - \#\mathcal{T} + 1).$$

*Proof.* Note that $\big( \#\mathcal{T}_\bullet - \#\mathcal{T} + 1 \big) - \#\mathcal{T}_\bullet / \#\mathcal{T} = \big( \#\mathcal{T}_\bullet - \#\mathcal{T} \big) \big( 1 - 1/\#\mathcal{T} \big) \geq 0$. Rearranging the terms, we conclude the upper bound in (31), while the lower bound is obvious. $\square$

**Lemma 23.** *There exists $C_3 > 0$ which depends only on $C_{\mathrm{son}}$, $m$ from Lemma 21, and $\mathcal{T}_0$, such that for all $s > 0$ and all $\mathcal{T} \in \mathbb{T}$, it holds*

(32) $$\sup_{N \geq C_3 \#\mathcal{T}} \Big( (N+1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_N} \eta_\star \Big) \leq 2^s \, \|u\|_{\mathbb{A}_s(\mathcal{T})}$$

*as well as*

(33) $$\sup_{N \geq C_3 \#\mathcal{T}} \Big( (N+1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T})} \eta_\star \Big) \leq C_{\mathrm{mon}} \, 2^s \, \|u\|_{\mathbb{A}_s}.$$

*In particular, there holds equivalence*

(34) $$\|u\|_{\mathbb{A}_s(\mathcal{T})} < \infty \quad \Longleftrightarrow \quad \|u\|_{\mathbb{A}_s} < \infty.$$

*Proof. Step 1: The estimates (32)–(33) imply (34).* For any $M > 0$, the sets $\bigcup_{N=0}^M \mathbb{T}_N$ and $\bigcup_{N=0}^M \mathbb{T}_N(\mathcal{T})$ are finite. Hence, (32) provides an upper bound to $\|u\|_{\mathbb{A}_s}$ in terms of $\|u\|_{\mathbb{A}_s(\mathcal{T})}$, up to some finite summand which depends on $M = C_3 \#\mathcal{T} - 1$. Therefore, $\|u\|_{\mathbb{A}_s(\mathcal{T})} < \infty$ implies $\|u\|_{\mathbb{A}_s} < \infty$. The converse implication follows analogously.

*Step 2: Verification of (32).* Let $N \geq 0$. Apply Lemma 21 to see that the $m$-times uniform refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}$ satisfies $\#\mathcal{T} \leq \#\widehat{\mathcal{T}} \leq C_{\mathrm{son}}^m \#\mathcal{T} =: C$ and $\widehat{\mathcal{T}} \in \mathbb{T}_C \subseteq \mathbb{T}_{C+N}(\mathcal{T})$, i.e., $\mathbb{T}_{C+N}(\mathcal{T}) \neq \emptyset$. Choose $\mathcal{T}_\bullet \in \mathbb{T}_{C+N}(\mathcal{T})$ with $\eta_\bullet = \min_{\mathcal{T}_\star \in \mathbb{T}_{C+N}(\mathcal{T})} \eta_\star$. Then, we estimate

$$\#\mathcal{T}_\bullet - \#\mathcal{T}_0 = (\#\mathcal{T}_\bullet - \#\mathcal{T}) + (\#\mathcal{T} - \#\mathcal{T}_0) \leq (C+N) + \#\mathcal{T} \leq 2C + N,$$

i.e., $\mathcal{T}_\bullet \in \mathbb{T}_{2C+N}$. By choice of $\mathcal{T}_\bullet \in \mathbb{T}_{C+N}(\mathcal{T})$ and the definition of $\|u\|_{\mathbb{A}_s(\mathcal{T})}$, it follows

$$(2C + N + 1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_{2C+N}} \eta_\star \leq \Big( \frac{2C + N + 1}{C + N + 1} \Big)^s (C + N + 1)^s \, \eta_\bullet \leq 2^s \, \|u\|_{\mathbb{A}_s(\mathcal{T})}.$$

Since this estimate holds for all $N \geq 0$, we obtain (32) with $C_3 = 2 C_{\mathrm{son}}^m$.

*Step 3: Verification of (33).* Let $N \geq 0$. Adopt the notation from step 2 and recall that $\widehat{\mathcal{T}} \in \mathbb{T}_C \subseteq \mathbb{T}_{C+N}$. Choose $\mathcal{T}_+ \in \mathbb{T}_{C+N}$ with $\eta_+ = \min_{\mathcal{T}_\star \in \mathbb{T}_{C+N}} \eta_\star$. Define $\mathcal{T}_\bullet := \widehat{\mathcal{T}} \oplus \mathcal{T}_+$ to ensure that the discrete solution $U_\bullet \in \mathcal{X}_\bullet$ exists. Then,

$$\#\mathcal{T}_\bullet - \#\mathcal{T} \overset{(27)}{\leq} (\#\widehat{\mathcal{T}} + \#\mathcal{T}_+ - \#\mathcal{T}_0) - \#\mathcal{T} \leq \#\widehat{\mathcal{T}} + C + N \leq 2C + N,$$

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i.e., $\mathcal{T}_\bullet \in \mathbb{T}_{2C+N}(\mathcal{T})$. Moreover, quasi-monotonicity of the estimator (Lemma 21) yields

$$(2C+N+1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_{2C+N}(\mathcal{T})} \eta_\star \le (2C+N+1)^s \eta_\bullet$$

$$\le C_{\mathrm{mon}} \left(\frac{2C+N+1}{C+N+1}\right)^s (C+N+1)^s \eta_+ \le C_{\mathrm{mon}} \, 2^s \, \|u\|_{\mathbb{A}_s}.$$

Since this estimate holds for all $N \ge 0$, we obtain (33) again with $C_3 = 2C_{\mathrm{son}}^m$. $\qquad\square$

In the spirit of [CKNS08], one can also consider approximation classes based on the so-called *total error*. Suppose that the Galerkin solution $U_\star \in \mathcal{X}_\star$ of (5) exists. Suppose that $\mathrm{osc}_\star : \mathcal{X}_\star \to \mathbb{R}$ are so-called oscillation terms such that the error estimator is reliable and efficient in the sense of

$$(35) \qquad C_{\mathrm{rel}}^{-1} \|u - U_\star\|_\mathcal{H} \le \eta_\star \le C_{\mathrm{eff}} \left(\|u - U_\star\|_\mathcal{H} + \mathrm{osc}_\star(U_\star)\right).$$

Then, [CKNS08] considers

$$(36) \quad \|u\|_{\mathbb{E}_s(\mathcal{T})} := \sup_{N \in \mathbb{N}_0} \left((N+1)^s \min_{\substack{\mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}) \\ \#\mathcal{T}_\star - \#\mathcal{T} \le N}} \inf_{V_\star \in \mathcal{X}_\star} \left(\|u - V_\star\|_\mathcal{H} + \mathrm{osc}_\star(V_\star)\right)\right) \quad \text{for } \mathcal{T} \in \mathbb{T}.$$

Note that the definition of $\|u\|_{\mathbb{E}_s(\mathcal{T})}$ also involves meshes for which the existence of the discrete solution may fail. Adapting [CFPP14, Theorem 4.4], we derive the following result which states that the total error (starting from some arbitrary initial mesh $\mathcal{T}$) converges with the same algebraic rate as the error estimator.

**Lemma 24.** *Let* $\mathrm{osc}_\star : \mathcal{X}_\star \to \mathbb{R}$ *satisfy* (35). *Suppose that there exists* $C_{\mathrm{osc}} > 0$ *such that for all* $\mathcal{T}_\star \in \mathbb{T}$ *for which the discrete solution* $U_\star \in \mathcal{X}_\star$ *of* (5) *exists, it holds the following:*

- $\mathrm{osc}_\star := \mathrm{osc}_\star(U_\star) \le C_{\mathrm{osc}} \, \eta_\star$,
- $C_{\mathrm{osc}}^{-1} \mathrm{osc}_\star(V_\star) \le \mathrm{osc}_\star(W_\star) + \|V_\star - W_\star\|_\mathcal{H}$ *for all* $V_\star, W_\star \in \mathcal{X}_\star$.

*Then, for all* $s > 0$ *and all* $\mathcal{T} \in \mathbb{T}$, *it holds*

$$\|u\|_{\mathbb{E}_s(\mathcal{T})} < \infty \quad \Longleftrightarrow \quad \|u\|_{\mathbb{A}_s} < \infty.$$

*Proof.* We show that $\|u\|_{\mathbb{E}_s(\mathcal{T})} < \infty$ if and only if $\|u\|_{\mathbb{A}_s(\mathcal{T})} < \infty$. Then, Lemma 23 will conclude the proof.

*Step 1.* Let $\mathcal{T} \in \mathbb{T}$ and $\widehat{\mathcal{T}}_0 \in \mathbb{T}$ from Lemma 21. With $C := (C_{\mathrm{son}}^m - 1)\#\widehat{\mathcal{T}}_0$, the triangulation $\mathcal{T}_\star := \mathcal{T} \oplus \widehat{\mathcal{T}}_0$ satisfies $\#\mathcal{T}_\star \le \#\mathcal{T} + \#\widehat{\mathcal{T}}_0 - \#\mathcal{T}_0 \le \#\mathcal{T} + C$ and hence $\mathcal{T}_\star \in \mathbb{T}_C(\mathcal{T})$. This proves $\mathbb{T}_N(\mathcal{T}) \ne \emptyset$ for $N \ge C$.

*Step 2.* We prove that $\|u\|_{\mathbb{A}_s(\mathcal{T})} < \infty$ implies $\|u\|_{\mathbb{E}_s(\mathcal{T})} < \infty$ by showing

$$(37) \qquad \sup_{N \ge C} \left((N+1)^s \min_{\substack{\mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}) \\ \#\mathcal{T}_\star - \#\mathcal{T} \le N}} \inf_{V_\star \in \mathcal{X}_\star} \left(\|u - V_\star\|_\mathcal{H} + \mathrm{osc}_\star(V_\star)\right)\right) \lesssim \|u\|_{\mathbb{A}_s(\mathcal{T})} :$$

For $N \ge C$, choose $\mathcal{T}_\bullet \in \mathbb{T}_N(\mathcal{T})$ with $\eta_\bullet = \min_{\mathcal{T}_\star \in \mathcal{T}_N(\mathcal{T})} \eta_\star$. Then,

$$\min_{\substack{\mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}) \\ \#\mathcal{T}_\star - \#\mathcal{T} \le N}} \inf_{V_\star \in \mathcal{X}_\star} \left(\|u - V_\star\|_\mathcal{H} + \mathrm{osc}_\star(V_\star)\right) \le \|u - U_\bullet\|_\mathcal{H} + \mathrm{osc}_\bullet(U_\bullet) \simeq \eta_\bullet = \min_{\mathcal{T}_\star \in \mathcal{T}_N(\mathcal{T})} \eta_\star.$$

This proves (37).

*Step 3.* We prove that $\|u\|_{\mathbb{E}_s(\mathcal{T})} < \infty$ implies $\|u\|_{\mathbb{A}_s(\mathcal{T})} < \infty$ by showing

$$(38) \qquad \sup_{N \ge C} \left((N+1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T})} \eta_\star\right) \le (C+1)^s \|u\|_{\mathbb{E}_s} :$$

Let $N \ge 0$. Choose $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T})$ with $\#\mathcal{T}_\bullet - \#\mathcal{T} \le N$ and $\left(\|u - V_\bullet\|_\mathcal{H} + \mathrm{osc}_\bullet(V_\bullet)\right) = \inf_{V_\star \in \mathcal{X}_\star} \left(\|u - V_\star\|_\mathcal{H} + \mathrm{osc}_\star(V_\star)\right)$. Define $\mathcal{T}_\circ := \mathcal{T}_\bullet \oplus \widehat{\mathcal{T}}_0$ and note that $\mathcal{T}_\circ \in \mathcal{T}_{N+C}(\mathcal{T})$.

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Together with the Céa lemma (10) and our assumptions on the data oscillations, we obtain for all $V_\circ \in \mathcal{X}_\circ$,

$$\eta_\circ \simeq \|u - U_\circ\|_\mathcal{H} + \mathrm{osc}_\circ(U_\circ) \lesssim \|u - U_\circ\|_\mathcal{H} + \mathrm{osc}_\circ(V_\circ) + \|U_\circ - V_\circ\|_\mathcal{H}$$

$$\lesssim \|u - U_\circ\|_\mathcal{H} + \mathrm{osc}_\circ(V_\circ) + \|u - V_\circ\|_\mathcal{H} \overset{(10)}{\lesssim} \|u - V_\circ\|_\mathcal{H} + \mathrm{osc}_\circ(V_\circ).$$

This reveals $\eta_\circ \simeq \inf_{V_\circ \in \mathcal{X}_\circ} \left( \|u - V_\circ\|_\mathcal{H} + \mathrm{osc}_\circ(V_\circ) \right)$. Together with $\mathcal{X}_\circ \supseteq \mathcal{X}_\bullet$, we derive

$$(N + C + 1)^s \min_{\mathcal{T}_\star \in \mathbb{T}_{N+C}(\mathcal{T})} \eta_\star \leq (N + C + 1)^s \inf_{V_\circ \in \mathcal{X}_\circ} \left( \|u - V_\circ\|_\mathcal{H} + \mathrm{osc}_\circ(V_\circ) \right)$$

$$\leq \left( \frac{N + C + 1}{N + 1} \right)^s (N + 1)^s \inf_{V_\bullet \in \mathcal{X}_\bullet} \left( \|u - V_\bullet\|_\mathcal{H} + \mathrm{osc}_\bullet(V_\bullet) \right)$$

$$\leq (C + 1)^s \|u\|_{\mathbb{E}_s(\mathcal{T})}.$$

This proves (38). $\qquad\square$

**Remark 25.** *The assumptions of Lemma 24 are satisfied for residual-based error estimators in the frame of FEM with $\mathcal{X}_\star := \mathcal{S}^p(\mathcal{T}_\star) \cap H^1_0(\Omega)$; see [CKNS08, CN12, FFP14]. For each element $T \in \mathcal{T}_\star \in \mathbb{T}$, let $\mathcal{F}_T$ denote the set of its facets (i.e., edges for $d = 2$). For arbitrarily chosen $q \geq p - 1$, the data oscillations*

$$(39a) \qquad\qquad \mathrm{osc}_\star(V_\star)^2 := \sum_{T \in \mathcal{T}_\star} \mathrm{osc}_\star(T, V_\star)^2$$

*corresponding to the indicators from (16) read, for all $T \in \mathcal{T}_\star$,*

$$\mathrm{osc}_\star(T, V_\star)^2 = h_T^2 \min_{Q \in \mathcal{P}^q(T)} \|f + \mathrm{div}(A \nabla V_\star) - b \cdot \nabla V_\star - c V_\star - Q\|^2_{L^2(T)}$$

$$(39b)$$

$$+ h_T \sum_{F \in \mathcal{F}_T} \min_{Q \in \mathcal{P}^q(T)} \|[(A \nabla V_\star) \cdot n] - Q\|^2_{L^2(F \cap \Omega)}.$$

*The constant $C_{\mathrm{osc}}$ in Lemma 24 then depends on $q$ and $p$. If $A, b, c$ are piecewise polynomial and if $q$ is chosen sufficiently large, the local contributions simplify to the well-known data oscillations $\mathrm{osc}_\star(T, V_\star)^2 = h_T^2 \min_{f_T \in \mathcal{P}^q(T)} \|f - f_T\|^2_{L^2(T)}$ as for the Laplace problem.*

4.3. **Main result.** The following theorem is the main result of this work. It states that Algorithm 7 does not only guarantee (linear) convergence, but also the best possible algebraic convergence rate for the error estimator. In explicit terms, suppose that $\|u\|_{\mathbb{A}_s} < \infty$ for some $s > 0$. By definition (29) of the approximation class, there exists a sequence of meshes $\widehat{\mathcal{T}}_\ell \in \mathbb{T} = \mathrm{refine}(\mathcal{T}_0)$ and corresponding error estimators $\widehat{\eta}_\ell$ such that $\widehat{\eta}_\ell \lesssim \left( \#\widehat{\mathcal{T}}_\ell - \#\mathcal{T}_0 + 1 \right)^{-s}$ for all $\ell \in \mathbb{N}_0$. Note that these "optimal" triangulations are not necessarily successive refinements but in general even totally unrelated. Therefore, the important implication of the following theorem is that indeed the adaptively generated triangulations $\mathcal{T}_\ell$ yield the same algebraic decay $s > 0$ if the marking parameter $0 < \theta \ll 1$ is sufficiently small. Overall, Algorithm 7 thus guarantees that the error estimator decays asymptotically with any possible algebraic rate $s > 0$.

**Theorem 26.** *Suppose (A1)–(A5) with $\mathcal{X}_\infty = \mathcal{H}$ (which can, for instance, be enforced by the expanded Dörfler marking strategy from Proposition 16). Employ the notation of Algorithm 7. Let $\widehat{\gamma}_0 > 0$ be the lower-bound of the inf-sup constant (9) for the uniform refinement $\widehat{\mathcal{T}}_0$ from Lemma 21. Let $\ell_3, \ell_5 \in \mathbb{N}_0$ be the indices from Lemma 18 and Lemma 21, respectively. Define $\ell_6 := \max\{\ell_3, \ell_5\}$. Let $0 < \theta < \theta_{\mathrm{opt}} := (1 + C^2_{\mathrm{stb}} C^2_{\mathrm{rel}} / \widehat{\gamma}_0^2)^{-1}$.*

*Then, for all $s > 0$, it holds that*

(40) $\qquad \|u\|_{\mathbb{A}_s} < \infty \quad \Longleftrightarrow \quad \exists C_{\mathrm{opt}} > 0 \; \forall \ell \geq \ell_6 \quad \eta_\ell \leq C_{\mathrm{opt}} \left( \#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \right)^{-s}.$

*The constant $C_{\mathrm{opt}}$ depends only on $\#\mathcal{T}_{\ell_6}$, $\mathcal{T}_0$, $\theta$, $s$, and validity of (A1)–(A5).*

**Lemma 27** (optimality of Dörfler marking). *Under the assumptions of Theorem 26 and for all $0 < \theta < \theta_{\mathrm{opt}}$, there exists some $0 < \kappa_{\mathrm{opt}} < 1$ such that for all $\mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}_{\ell_5})$ and all $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_\star)$, it holds*

(41) $\qquad\qquad\qquad \eta_\bullet \leq \kappa_{\mathrm{opt}} \eta_\star \quad \Longrightarrow \quad \theta \eta_\star^2 \leq \eta_\star(\mathcal{R}_{\star,\bullet})^2,$

*where $\mathcal{R}_{\star,\bullet}$ is the (enlarged) set of refined elements from (A4).*

*Proof.* According to Lemma 21, the discrete solutions $U_\star \in \mathcal{X}_\star$ and $U_\bullet \in \mathcal{X}_\bullet$ exist, and the discrete reliability property (A4) holds with the uniform constant $C_{\mathrm{rel}}/\widehat{\gamma}_0$. Since stability (A1) holds, we can apply [CFPP14, Proposition 4.12], and the statement of the lemma follows. $\qquad\square$

**Lemma 28.** *Under the assumptions of Theorem 26, there exist constants $C_1, C_2 > 0$ such that for all $\ell \geq \ell_5$ and all $s > 0$, there exists a set $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ such that the following holds: If $\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})} < \infty$, then it holds*

(42) $\qquad\qquad\qquad\qquad \#\mathcal{R}_\ell \leq C_1 (C_2 \|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})})^{1/s} \eta_\ell^{-1/s}$

*as well as the Dörfler marking criterion*

(43) $\qquad\qquad\qquad\qquad\qquad \theta \eta_\ell^2 \leq \eta_\ell(\mathcal{R}_\ell)^2.$

*The contant $C_2$ depends only on $\theta$, $\widehat{\gamma}_0$, and (A1)–(A4), while $C_1$ additionally depends on $\#\mathcal{T}_{\ell_5}$ and $\mathcal{T}_0$.*

*Proof.* If $\eta_\ell = 0$, the claim (42)–(43) is satisfied with $\mathcal{R}_\ell := \mathcal{T}_\ell$. Thus, we suppose $\eta_\ell > 0$.

*Step 1: Construction of mesh $\mathcal{T}_\star$ and $\mathcal{R}_\ell := \mathcal{R}_{\ell,\star}$.* Let $\varepsilon := C_{\mathrm{mon}}^{-1} \kappa_{\mathrm{opt}} \eta_\ell > 0$. Due to $\ell \geq \ell_5$, quasi-monotonicity of the estimator (Lemma 21) yields $\varepsilon \leq \kappa_{\mathrm{opt}} \eta_{\ell_5} < \|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})} < \infty$. Choose the minimal $N \in \mathbb{N}$ such that $\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})} \leq \varepsilon(N + 1)^s$. This implies $\varepsilon < \|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})} \leq \varepsilon(N+1)^s$ and hence $N \geq 1$. Note that $\mathcal{T}_{\ell_5} \in \mathbb{T}_N(\mathcal{T}_{\ell_5})$ and hence $\mathbb{T}_N(\mathcal{T}_{\ell_5}) \neq \emptyset$. Choose $\mathcal{T}_\varepsilon \in \mathbb{T}_N(\mathcal{T}_{\ell_5})$ with $\eta_\varepsilon = \min_{\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T}_{\ell_5})} \eta_\star$. Define $\mathcal{T}_\star := \mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell$. Recall that all $\mathcal{T}_\bullet \in \mathrm{refine}(\mathcal{T}_{\ell_5})$ and corresponding spaces $\mathcal{X}_\bullet \supseteq \mathcal{X}_{\ell_5}$ provide unique solutions of the discrete formulation (5). Therefore, we obtain $\mathcal{T}_\star \in \mathbb{T}_N(\mathcal{T}_{\ell_5})$ with Galerkin solution $U_\star \in \mathcal{X}_\star$. Let $\mathcal{R}_\ell := \mathcal{R}_{\ell,\star}$ be the set provided by discrete reliability (A4).

*Step 2: Optimality of Dörfler marking yields (43).* With the quasi-monotonicity of the estimator (Lemma 21) and the definition of the approximation class (29), the choice of $N$ yields

$$\eta_\star \leq C_{\mathrm{mon}} \eta_\varepsilon \overset{(29)}{\leq} C_{\mathrm{mon}}(N + 1)^{-s} \|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})} \leq C_{\mathrm{mon}} \varepsilon = \kappa_{\mathrm{opt}} \eta_\ell.$$

This implies $\eta_\star \leq \kappa_{\mathrm{opt}} \eta_\ell$ and hence Lemma 27 proves (43).

*Step 3: Verification of (42).* The choice $\mathcal{R}_\ell = \mathcal{R}_{\ell,\star}$ together with $\mathcal{T}_\ell, \mathcal{T}_\varepsilon \in \mathrm{refine}(\mathcal{T}_{\ell_5})$ yields

(44) $\qquad \#\mathcal{R}_\ell \overset{(\mathrm{A4})}{\leq} C_{\mathrm{rel}} \#(\mathcal{T}_\ell \setminus \mathcal{T}_\star) \overset{(25)}{\leq} C_{\mathrm{rel}}(\#\mathcal{T}_\star - \#\mathcal{T}_\ell) \overset{(27)}{\leq} C_{\mathrm{rel}}(\#\mathcal{T}_\varepsilon - \#\mathcal{T}_{\ell_5}) \leq C_{\mathrm{rel}} N.$

Finally, minimality of $N$ implies

$$N < \|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})}^{1/s} \varepsilon^{-1/s} = C_3 \eta_\ell^{-1/s},$$

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with $C_3 := \|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})}^{1/s}(C_{\mathrm{mon}}^{-1}\kappa_{\mathrm{opt}})^{-1/s} = (C_{\mathrm{mon}}\kappa_{\mathrm{opt}}^{-1}\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})})^{1/s}$. Altogether, we thus see

$$\#\mathcal{R}_\ell \overset{(44)}{\lesssim} C_{\mathrm{rel}}N < C_{\mathrm{rel}}C_3\eta_\ell^{-1/s}.$$

This proves (42) with $C_1 = C_{\mathrm{rel}}$ and $C_2 = C_{\mathrm{mon}}\kappa_{\mathrm{opt}}^{-1}$. $\qquad\square$

*Proof of Theorem 26.* The implication "$\Longleftarrow$" in (40) follows by definition of the approximation class (cf. [CFPP14, Proposition 4.15]). We thus focus on the implication "$\Longrightarrow$" in (40). To this end, suppose that $\|u\|_{\mathbb{A}_s} < \infty$. Lemma 23 then implies $\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})} < \infty$. For $\ell \geq \ell_6 = \max\{\ell_3, \ell_5\}$, let $\mathcal{M}_\ell$ be the set of marked elements in the $\ell$-th step of Algorithm 7. According to Lemma 28, there exists $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ with (42)–(43). According to the minimality of $\mathcal{M}_\ell$ (see step (iv) in Algorithm 7), it follows

$$\#\mathcal{M}_\ell \overset{(43)}{\leq} C_{\mathrm{mark}}\#\mathcal{R}_\ell \overset{(42)}{\leq} C_{\mathrm{mark}}C_1(C_2\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})})^{1/s}\eta_\ell^{-1/s}.$$

With the mesh-closure estimate (26) and $C_{\mathrm{mesh}} \geq 1$, we further obtain

$$(45) \qquad \#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_6} + 1 \leq C_{\mathrm{mesh}}\sum_{j=\ell_6}^{\ell}\#\mathcal{M}_j \leq C_{\mathrm{mesh}}C_{\mathrm{mark}}C_1(C_2\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})})^{1/s}\sum_{j=\ell_6}^{\ell}\eta_j^{-1/s}.$$

Linear convergence (Theorem 19) implies

$$(46) \qquad\qquad\qquad \eta_\ell \leq C_{\mathrm{lin}}q_{\mathrm{lin}}^{\ell-j}\eta_j \quad \text{for all } \ell_3 \leq j \leq \ell$$

and hence

$$\eta_j^{-1/s} \leq C_{\mathrm{lin}}^{1/s}q_{\mathrm{lin}}^{(\ell-j)/s}\eta_\ell^{-1/s}.$$

Since there holds $0 < q := q_{\mathrm{lin}}^{1/s} < 1$, the geometric series applies and yields

$$\sum_{j=\ell_6}^{\ell}\eta_j^{-1/s} \leq C_{\mathrm{lin}}^{1/s}\eta_\ell^{-1/s}\sum_{j=0}^{\ell}q^{(\ell-j)} \leq \frac{C_{\mathrm{lin}}^{1/s}}{1-q_{\mathrm{lin}}^{1/s}}\eta_\ell^{-1/s}.$$

Combining this estimate with (45), we derive

$$\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_6} + 1 \leq \frac{C_{\mathrm{mesh}}C_{\mathrm{mark}}C_1}{1-q_{\mathrm{lin}}^{1/s}}(C_{\mathrm{lin}}C_2\|u\|_{\mathbb{A}_s(\mathcal{T}_{\ell_5})})^{1/s}\eta_\ell^{-1/s}.$$

Rearranging these terms, we see $\eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_6} + 1)^{-s}$. Lemma 22 yields

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \overset{(31)}{\leq} \#\mathcal{T}_\ell \overset{(31)}{\leq} \#\mathcal{T}_{\ell_6}\left(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_6} + 1\right).$$

This concludes the important implication of (40). $\qquad\square$

## 5. Numerical experiments

In this section, we present two numerical experiments for the 2D Helmholtz equation (3) that underpin our theoretical findings. We use the lowest-order FEM with $\mathcal{X}_\star := \mathcal{S}^1(\mathcal{T}_\star) \cap H_0^1(\Omega)$ and a residual a posteriori error estimator (see [BISG97] for a first systematic a posteriori error analysis for the Helmholtz equation and [OPD05] for a survey of available error estimation techniques for this problem). In the experiments, we compare the performance of Algorithm 7 with respect to

- different values of $\kappa \in \{1, 2, 4, 8, 16\}$,
- different values of $\theta \in \{0.1, 0.2, \ldots, 0.9\}$,
- standard Dörfler marking strategy (with $C_{\mathrm{mark}} = 1$) as well as the expanded Dörfler marking strategy of Proposition 16 (with $C_{\mathrm{mark}} = 2$).
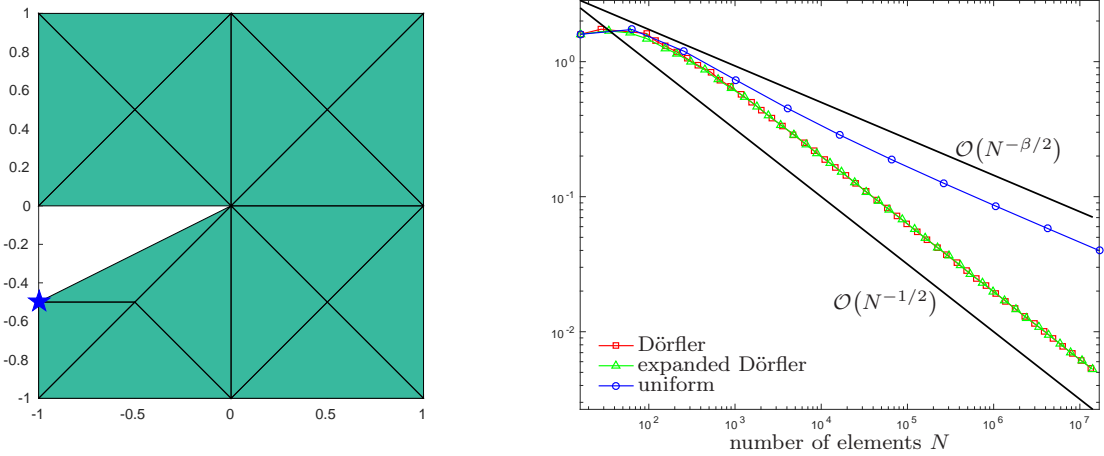
FIGURE 1. Geometry and initial partition $\mathcal{T}_0$ in the experiment from Section 5.1 (left), where the blue star indicates the node $(-1, -t) = (-1, -0.5)$. For $\kappa = 2$, we compare the error estimator for uniform vs. adaptive mesh-refinement with $\theta = 0.2$ (right). Uniform mesh-refinement leads to a sub-optimal convergence rate, while Algorithm 7 with Dörfler marking and expanded Dörfler marking recovers the optimal convergence rate.
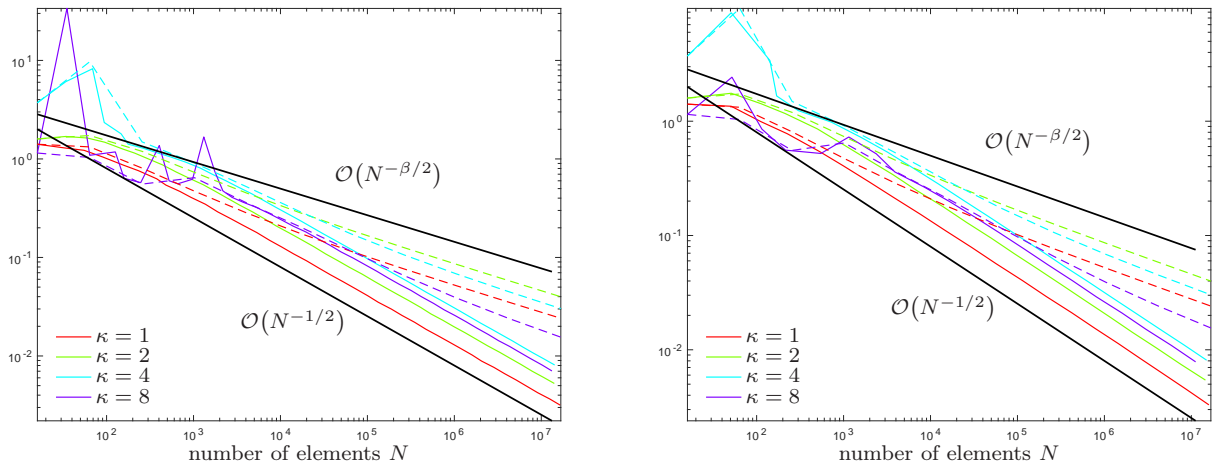


FIGURE 2. Convergence rates for the error estimator in the experiment from Section 5.1 for different values of $\kappa$ and for marking parameter $\theta = 0.2$ (left) and $\theta = 0.5$ (right). Dashed lines mark uniform refinement, while solid lines mark the output of Algorithm 7 with expanded Dörfler marking. The latter recovers optimal convergence rates, while uniform mesh-refinement does not.

We consider domains $\Omega \subset \mathbb{R}^2$ with a single re-entrant corner and corresponding interior angle $\alpha > \pi$. Note that elliptic regularity thus predicts a generic convergence order $\mathcal{O}(N^{-\beta/2})$ for the error on uniform meshes with $N$ elements, where $\beta = \pi/\alpha < 1$. On the other hand, the optimal convergence behavior for lowest-order elements is $\mathcal{O}(N^{-1/2})$ if the mesh is appropriately refined.

5.1. **Experiment with unknown solution.** We consider the Z-shaped domain $\Omega \subset \mathbb{R}^2$ from Figure 1. The marked node has the coordinates $(-1, -t) = (-1, -0.5)$ and determines the angle $\alpha$ at the re-entrant corner $(0, 0)$ which reads $\alpha = 2\pi - \arcsin\left(t/\sqrt{1+t^2}\right)$,
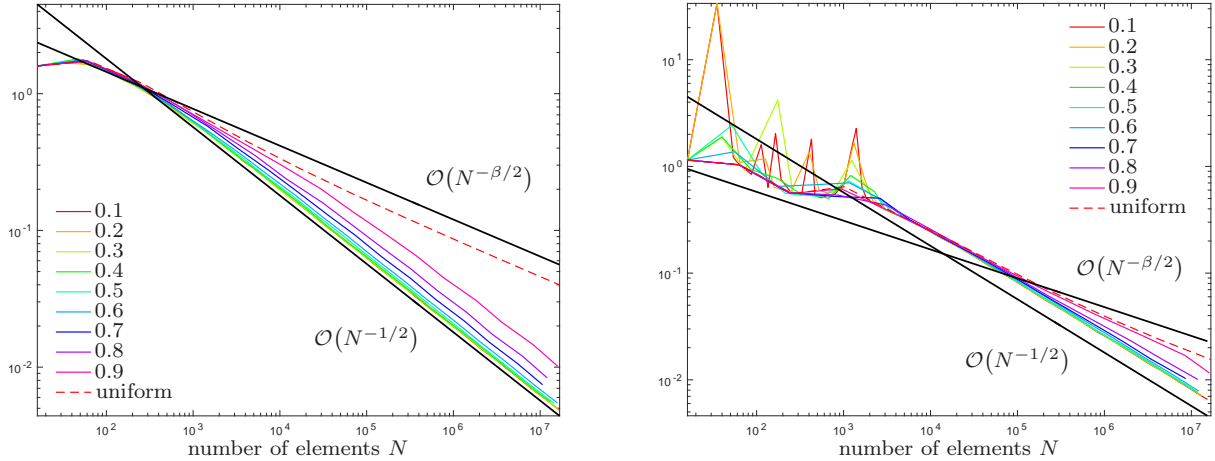
FIGURE 3. Convergence rates for the error estimator in the experiment from Section 5.1 for uniform and adaptive mesh-refinement with different values of $\theta \in \{0.1, \ldots, 0.9\}$ for $\kappa = 2$ (left) and $\kappa = 8$ (right). For all $\theta$, adaptive mesh-refinement leads to optimal convergence behavior, while the preasymptotic behavior increases with $\kappa$.

i.e., $\beta \approx 0.5398$. Consider the constant right-hand side $f = 1$ in (3) so that the residual error estimator is equivalent to the actual error, i.e., $\eta_\star \simeq \|u - U_\star\|_{H^1(\Omega)}$. For $\kappa = 2$, Figure 1 shows a generically reduced convergence rate for the error estimator on uniform meshes, while Algorithm 7 with $\theta = 0.2$ regains the optimal convergence rate. Empirically, the results generated by employing the standard Dörfler marking are of no difference to the results generated by employing the expanded Dörfler marking from Propositon 16. The same observation is made for other choices of $\theta$ (not displayed), so that we only consider the expanded Dörfler marking. Figure 2 compares uniform vs. adaptive mesh-refinement for fixed $\theta \in \{0.2, 0.5\}$ but various $\kappa \in \{1, 2, 4, 8\}$. As expected, the preasymptotic phase increases with $\kappa$. However, adaptive mesh-refinement results in asymptotically optimal convergence behavior. Figure 3 compares uniform vs. adaptive mesh-refinement for fixed $\kappa \in \{2, 8\}$ but various $\theta \in \{0.1, \ldots, 0.9\}$. Although Theorem 26 predicts optimal convergence rates only for small marking parameters $0 < \theta < \theta_{\mathrm{opt}} := (1 + C_{\mathrm{stb}}^2 C_{\mathrm{rel}}^2)^{-1}$, we observe that Algorithm 7 is stable in $\theta$, and any choice of $\theta \leq 0.9$ leads to the optimal convergence behavior. Finally, we observed that Algorithm 7 did never enforce uniform mesh-refinement in step (i), i.e., throughout the resulting discrete linear systems were indefinite but regular.

5.2. **Experiment with mixed boundary conditions.** We consider a Z-shaped domain with a symmetric opening at the re-entrant corner, see Figure 7. The marked nodes read $(-1, \pm t) = (-1, \pm 0.25)$. Analogously to the previous example, we expect a reduced convergence order $\mathcal{O}(N^{-\beta/2})$ for uniform mesh-refinement with $\beta \approx 0.5423$. We prescribe the exact solution of the Helmholtz equation in polar coordinates $(r, \phi)$ by

$$(47) \qquad u(x, y) = r^\beta \cos(\beta\phi)$$

and define $f := -\kappa^2 u$ in $\Omega$ and $g := \partial_n u$ on $\Gamma$. Note that $u$ has a generic singularity at the re-entrant corner $(0, 0)$ of $\Omega$ and that $u|_{\Gamma_D} = 0$ with the Dirichlet boundary $\Gamma_D := \mathrm{conv}\{(-1, \pm t), (0, 0)\}$. Define the Neumann boundary $\Gamma_N := \partial\Omega \backslash \Gamma_D$ and note that $u$ is the unique weak solution of the mixed boundary value problem

$$(48) \qquad -\Delta u - \kappa^2 u = f \text{ in } \Omega \quad \text{subject to} \quad u = 0 \text{ on } \Gamma_D \text{ and } \partial_n u = g \text{ on } \Gamma_N.$$
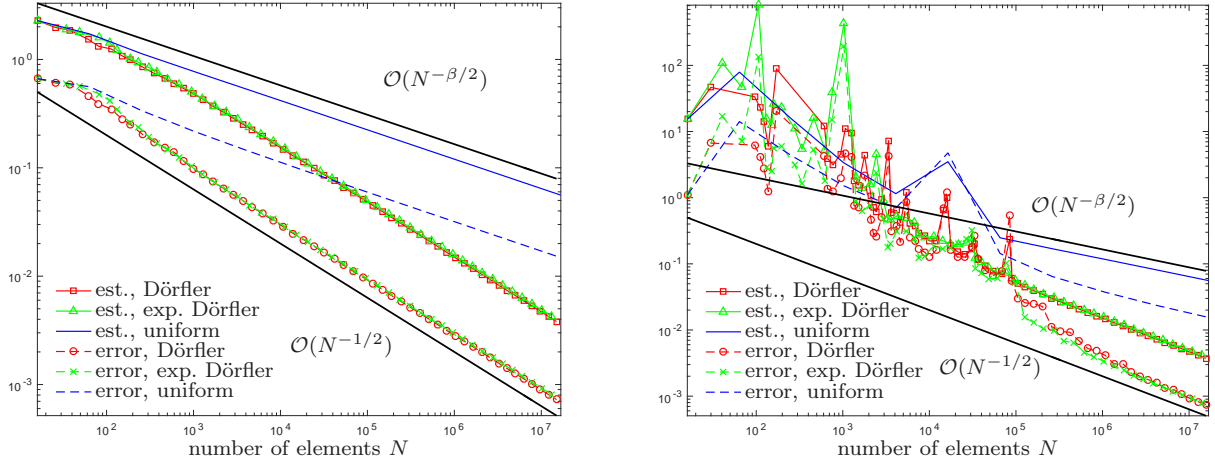
FIGURE 4. Error and error estimator in the experiment from Section 5.2 for $\kappa = 2$ (left) and $\kappa = 16$ (right) for uniform vs. adaptive mesh-refinement with $\theta = 0.2$. In particular, we compare Dörfler marking and expanded Dörfler marking.
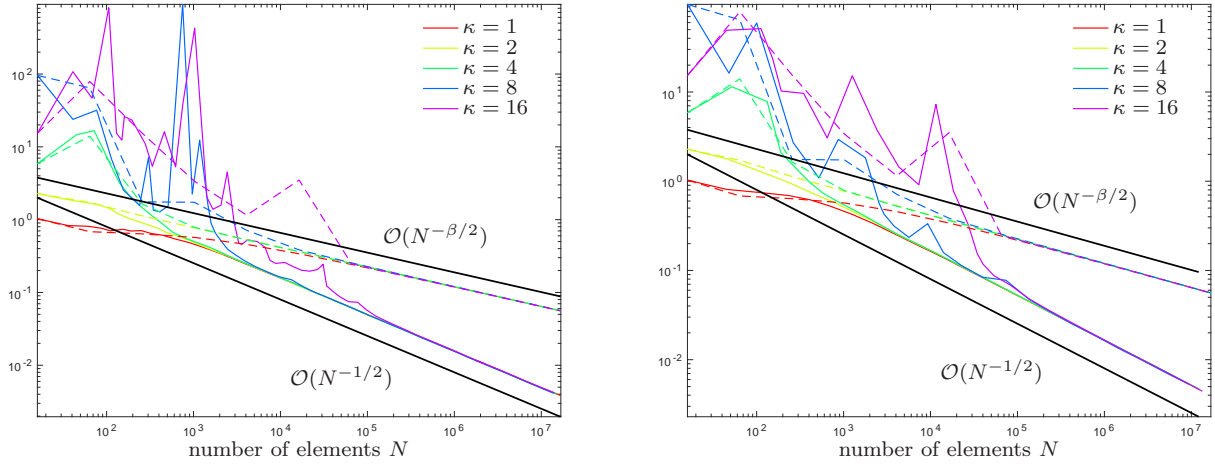


FIGURE 5. Convergence rates for the error estimator in the experiment from Section 5.2 for different values of $\kappa$ and for marking parameter $\theta = 0.2$ (left) and $\theta = 0.5$ (right). Dashed lines mark uniform refinement, while solid lines mark the output of Algorithm 7 with expanded Dörfler marking. The latter recovers optimal convergence rates, while uniform mesh-refinement does not.

The weak formulation of this problem can be written in the variational formulation (2) with $\mathcal{H} := H_D^1(\Omega) = \left\{ v \in H^1(\Omega) \ : \ v|_{\Gamma_D} = 0 \text{ in the sense of traces} \right\}$. Moreover, since the exact solution $u$ is given, we can compute the error $\|u - U_\star\|_{H^1(\Omega)}$ besides the corresponding error estimator $\eta_\star$.

The empirical observations are similar to those of Section 5.1; see Figure 4–6. Uniform mesh-refinement leads to suboptimal convergence behavior for both the error and the error estimator. Adaptive mesh-refinement resolves the geometric singularity at the re-entrant corner (see, e.g., Figure 7) and recovers the optimal convergence rate. Algorithm 7 appears to be stable for all $\theta \in \{0.1, \ldots, 0.9\}$. Different choices of $\kappa \in \{1, 2, 4, 8, 16\}$ affect only the preasymptotic phase. Finally, there is no empirical difference between the standard Dörfler marking and the expanded Dörfler marking.
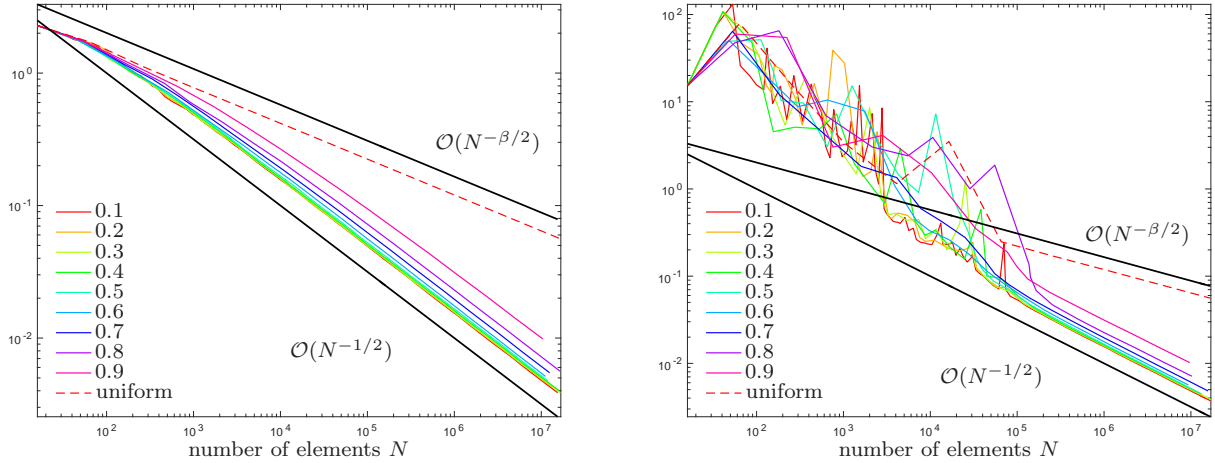
FIGURE 6. Convergence rates for the error estimator in the experiment from Section 5.2 for uniform and adaptive mesh-refinement with different values of $\theta \in \{0.1, \dots, 0.9\}$ for $\kappa = 2$ (left) and $\kappa = 16$ (right). For all $\theta$, adaptive mesh-refinement leads to optimal convergence behavior, while the preasymptotic behavior increases with $\kappa$.
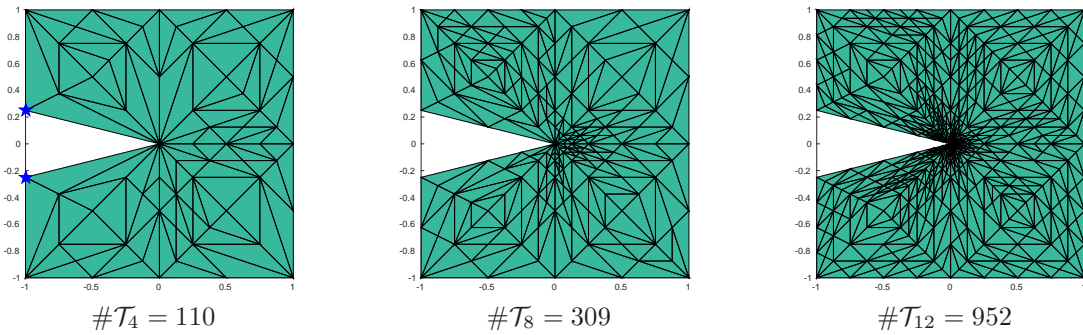


$\#\mathcal{T}_4 = 110$ $\qquad$ $\#\mathcal{T}_8 = 309$ $\qquad$ $\#\mathcal{T}_{12} = 952$

FIGURE 7. Adaptively generated meshes $\mathcal{T}_\ell$ in the experiment from Section 5.2 for $\kappa = 2$ and $\theta = 0.2$.

## REFERENCES

[AFK+13] Markus Aurada, Michael Feischl, Josef Kemetmüller, Marcus Page, and Dirk Praetorius. Each $H^{1/2}$-stable projection yields convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data in $\mathbb{R}^d$. *ESAIM Math. Model. Numer. Anal.*, 47:1207–1235, 2013.

[AFLP12] Markus Aurada, Samuel Ferraz-Leite, and Dirk Praetorius. Estimator reduction and convergence of adaptive BEM. *Appl. Numer. Math.*, 62(6):787–801, 2012.

[BDD04] Peter Binev, Wolfgang Dahmen, and Ronald DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97(2):219–268, 2004.

[BI98] Philippe Bouillard and Frank Ihlenburg. Error estimation and adaptivity for the finite element method in acoustics. In *Advances in adaptive computational methods in mechanics (Cachan, 1997)*, volume 47 of *Stud. Appl. Mech.*, pages 477–492. Elsevier, Amsterdam, 1998.

[BI99] Philippe Bouillard and Frank Ihlenburg. Error estimation and adaptivity for the finite element method in acoustics: 2d and 3d applications. *Comput. Methods Appl. Mech. Engrg.*, 176:147–163, 1999.

[BISG97] Ivo Babuška, Frank Ihlenburg, Theofanis Strouboulis, and Srihari K. Gangaraj. A posteriori error estimation for finite element solutions of Helmholtz' equation. I. The quality of local indicators and estimators. *Internat. J. Numer. Methods Engrg.*, 40(18):3443–3462, 1997.

[BN10] Andrea Bonito and Ricardo H. Nochetto. Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method. *SIAM J. Numer. Anal.*, 48(2):734–771, 2010.

[Bra01]    Dietrich Braess. *Finite elements.* Cambridge University Press, Cambridge, second edition, 2001.

[BS08]     Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics.* Springer, New York, third edition, 2008.

[CFPP14]   Carsten Carstensen, Michael Feischl, Marcus Page, and Dirk Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.

[CKNS08]   J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, and Kunibert G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[CN12]     J. Manuel Cascon and Ricardo H. Nochetto. Quasioptimal cardinality of AFEM driven by nonresidual estimators. *IMA J. Numer. Anal.*, 32(1):1–29, 2012.

[Dem06]    Leszek Demkowicz. Babuška ⇔ Brezzi?? *ICES Report*, 06-08, 2006.

[Dör96]    Willy Dörfler. A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal.*, 33(3):1106–1124, 1996.

[FFP14]    Michael Feischl, Thomas Führer, and Dirk Praetorius. Adaptive FEM with optimal convergence rates for a certain class of nonsymmetric and possibly nonlinear problems. *SIAM J. Numer. Anal.*, 52(2):601–625, 2014.

[FPP14]    Michael Feischl, Marcus Page, and Dirk Praetorius. Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data. *J. Comput. Appl. Math.*, 255:481–501, 2014.

[FPZ16]    Michael Feischl, Dirk Praetorius, and Kristoffer G. van der Zee. An abstract analysis of optimal goal-oriented adaptivity. *SIAM J. Numer. Anal.*, 54:1423–1448, 2016.

[GSS14]    Dietmar Gallistl, Mira Schedensack, and Rob P. Stevenson. A remark on newest vertex bisection in any space dimension. *Comput. Methods Appl. Math.*, 14(3):317–320, 2014.

[KPP13]    Michael Karkulik, David Pavlicek, and Dirk Praetorius. On 2D newest vertex bisection: optimality of mesh-closure and $H^1$-stability of $L_2$-projection. *Constr. Approx.*, 38(2):213–234, 2013.

[MN05]     Khamron Mekchay and Ricardo H. Nochetto. Convergence of adaptive finite element methods for general second order linear elliptic PDEs. *SIAM J. Numer. Anal.*, 43(5):1803–1827, 2005.

[MNS00]    Pedro Morin, Ricardo H. Nochetto, and Kunibert G. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal.*, 38(2):466–488, 2000.

[MP15]     Philipp Morgenstern and Daniel Peterseim. Analysis-suitable adaptive T-mesh refinement with linear complexity. *Comput. Aided Geom. Design*, 34:50–66, 2015.

[OPD05]    J. Tinsley Oden, Serge Prudhomme, and Leszek Demkowicz. A posteriori error estimation for acoustic wave propagation problems. *Arch. Comput. Methods Engrg.*, 12(4):343–389, 2005.

[Rud91]    Walter Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[SH96]     James R. Stewart and Thomas J.R. Hughes. A posteriori error estimation and adaptive finite element computation of the Helmholtz equation in exterior domains. *Finite Elem. Anal. Des.*, 22(1):15–24, 1996.

[SS11]     Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, 2011.

[Ste07]    Rob Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7(2):245–269, 2007.

[Ste08]    Rob Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77(261):227–241, 2008.

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