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# THE WIDOM-ROWLINSON MODEL, THE HARD-CORE MODEL AND THE EXTREMALITY OF THE COMPLETE GRAPH

EMMA COHEN, PÉTER CSIKVÁRI, WILL PERKINS, AND PRASAD TETALI

ABSTRACT. Let  $H_{\text{WR}}$  be the path on 3 vertices with a loop at each vertex. D. Galvin [4, 5] conjectured, and E. Cohen, W. Perkins and P. Tetali [2] proved that for any  $d$ -regular simple graph  $G$  on  $n$  vertices we have

$$\text{hom}(G, H_{\text{WR}}) \leq \text{hom}(K_{d+1}, H_{\text{WR}})^{n/(d+1)}.$$

In this paper we give a short proof of this theorem together with the proof of a conjecture of Cohen, Perkins and Tetali [2]. Our main tool is a simple bijection between the Widom-Rowlinson model and the hard-core model on another graph. We also give a large class of graphs  $H$  for which we have

$$\text{hom}(G, H) \leq \text{hom}(K_{d+1}, H)^{n/(d+1)}.$$

In particular, we show that the above inequality holds if  $H$  is a path or a cycle of even length at least 6 with loops at every vertex.

## 1. INTRODUCTION

For graphs  $G$  and  $H$ , with vertex and edge sets  $V_G, E_G, V_H$ , and  $E_H$  respectively, a map  $\varphi : V_G \rightarrow V_H$  is a homomorphism if  $(\varphi(u), \varphi(v)) \in E_H$  whenever  $(u, v) \in E_G$ . The number of homomorphisms from  $G$  to  $H$  is denoted by  $\text{hom}(G, H)$ . When  $H = H_{\text{ind}}$ , an edge with a loop at one end, homomorphisms from  $G$  to  $H_{\text{ind}}$  correspond to independent sets in the graph  $G$ , and so  $\text{hom}(G, H_{\text{ind}})$  counts the number of independent sets in  $G$ .

For a given  $H$ , the set of homomorphisms from  $G$  to  $H$  correspond to valid configurations in a corresponding statistical physics model with *hard constraints* (forbidden local configurations). The independent sets of  $G$  are the valid configurations of the *hard-core model* on  $G$ , a model of a random independent set from a graph. Another notable case is when  $H = H_{\text{WR}}$ , a path on 3 vertices with a loop at each vertex. In this case, we can imagine a homomorphism from  $G$  to  $H_{\text{WR}}$  as a 3-coloring of the vertex set of  $G$  subject to the requirement that a blue and a red vertex cannot be adjacent (with white vertices considered unoccupied); such a coloring is called a *Widom-Rowlinson configuration* of  $G$ , from the Widom-Rowlinson model of two particle types which repulse each other [12, 1]. See Figure 1.

For a fixed graph  $H$ , it is natural to study the normalized graph parameter

$$p_H(G) := \text{hom}(G, H)^{1/|V_G|},$$

where  $V_G$  denotes the number of vertices of the graph  $G$ .

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FIGURE 1. The target graphs for the Widom-Rowlinson model and the hard-core model.

For  $H = H_{\text{ind}}$ , J. Kahn [7] proved that for any  $d$ -regular bipartite graph  $G$ ,

$$p_{H_{\text{ind}}}(G) \leq p_{H_{\text{ind}}}(K_{d,d}),$$

where  $K_{d,d}$  is the complete bipartite graph with classes of size  $d$ . Y. Zhao [10] showed that one could drop the condition of bipartiteness in Kahn's theorem. That is, he showed that  $p_{H_{\text{ind}}}(G) \leq p_{H_{\text{ind}}}(K_{d,d})$ , for *any*  $d$ -regular graph  $G$ . Y. Zhao proved his result by reducing the general case to the bipartite case with a clever trick. He proved that

$$p_{H_{\text{ind}}}(G) \leq p_{H_{\text{ind}}}(G \times K_2),$$

where  $G \times K_2$  is the bipartite graph obtained by replacing every vertex  $u$  of  $V_G$  by a pair of vertices  $(u, 0)$  and  $(u, 1)$  and replacing every edge  $(u, v) \in E_G$  by the pair of edges  $((u, 0), (v, 1))$  and  $((u, 1), (v, 0))$ . This is clearly a bipartite graph, and if  $G$  is  $d$ -regular then  $G \times K_2$  is still  $d$ -regular.

D. Galvin [4, 5] conjectured a different behavior for  $H = H_{\text{WR}}$ : that instead of  $K_{d,d}$ , the complete graph  $K_{d+1}$  maximizes  $p_{H_{\text{ind}}}(G)$  among  $d$ -regular graphs  $G$ . E. Cohen, W. Perkins and P. Tetali [2] proved that this was indeed the case:

**Theorem 1.1.** [2] *For any  $d$ -regular simple graph  $G$  on  $n$  vertices we have*

$$p_{H_{\text{WR}}}(G) \leq p_{H_{\text{WR}}}(K_{d+1});$$

*in other words,*

$$\text{hom}(G, H_{\text{WR}}) \leq \text{hom}(K_{d+1}, H_{\text{WR}})^{n/(d+1)}.$$

One of the goals of this paper is to give a very simple proof of this fact<sup>1</sup>, along with a slight generalization. We use a trick similar to that used by Y. Zhao [10, 11]. We will need the following definition:

**Definition 1.2.** The *extended line graph*  $\tilde{H}$  of a (bipartite) graph  $H$  has  $V_{\tilde{H}} = E_H$ ; two edges  $e$  and  $f$  of  $H$  are adjacent in  $\tilde{H}$  if

- (a)  $e = f$ ,
- (b)  $e$  and  $f$  share a common vertex, or
- (c)  $e$  and  $f$  are opposite edges of a 4-cycle in  $G$ .

Throughout,  $V_H$  and  $E_H$  refer to the vertex-set and edge-set, respectively, of the graph  $H$ . If  $H$  is bipartite, we use  $A_H$  and  $B_H$  to refer to the parts of a fixed bipartition. Now we can give a generalization of Theorem 1.1:

**Theorem 1.3.** *If  $\tilde{H}$  is the extended line graph of a bipartite graph  $H$ , then for any  $d$ -regular simple graph  $G$  on  $n$  vertices we have*

$$p_{\tilde{H}}(G) \leq p_{\tilde{H}}(K_{d+1}),$$

<sup>1</sup>In fact, Theorem 1.1 follows from a stronger result in [2] that the Widom-Rowlinson *occupancy fraction* is maximized by  $K_{d+1}$ . We note that this stronger result also follows from the transformation below and Theorem 1 of [3].

or in other words,

$$\text{hom}(G, \tilde{H}) \leq \text{hom}(K_{d+1}, \tilde{H})^{n/(d+1)}.$$

To see that Theorem 1.3 is a generalization of Theorem 1.1 it suffices to check that  $H_{\text{WR}}$  is precisely the extended line graph of the path on 4 vertices. In Section 3 we will prove a slight generalization of Theorem 1.3 which allows for weights on the vertices of  $H$ .

## 2. SHORT PROOF OF THEOREM 1.1

We are not the first to notice the following connection between the Widom-Rowlinson model and the hardcore model (see, e.g., Section 5 of [1]): Given a graph  $G$ , let  $G'$  be the bipartite graph with vertex set  $V_{G'} = V_G \times \{0, 1\}$ , where  $(u, 0)$  and  $(v, 1)$  are adjacent in  $G'$  whenever either  $(u, v) \in E_G$  or  $u = v$ . That is,  $G'$  is  $G \times K_2$  with the extra edges  $((u, 0), (u, 1))$  for all  $u \in V_G$ . We will show that

$$\text{hom}(G, H_{\text{WR}}) = \text{hom}(G', H_{\text{ind}}).$$

Indeed, consider an independent set  $I$  in  $G'$ . Color  $u \in V_G$  blue if  $(u, 1) \in I$ , red if  $(u, 0) \in I$ , and white if it is neither red or blue. Note that since  $I$  was an independent set and  $((u, 0), (u, 1)) \in E_{G'}$ , the color of vertex  $u$  is well-defined and this coloring is in fact a Widom-Rowlinson coloring of  $G$ . This same construction also works in the other direction, so

$$\text{hom}(G, H_{\text{WR}}) = \text{hom}(G', H_{\text{ind}}).$$

If  $G$  is  $d$ -regular then  $G'$  is  $(d+1)$ -regular, and  $K'_{d+1} = K_{d+1, d+1}$ . Applying J. Kahn's result [7] for  $(d+1)$ -regular bipartite graphs, we see that if  $G$  has  $n$  vertices then

$$\begin{aligned} \text{hom}(G, H_{\text{WR}}) &= \text{hom}(G', H_{\text{ind}}) \\ &\leq \text{hom}(K_{d+1, d+1}, H_{\text{ind}})^{2n/(2(d+1))} = \text{hom}(K_{d+1}, H_{\text{WR}})^{n/(d+1)}. \end{aligned}$$

We remark that the transformation  $G \rightarrow G'$  is also mentioned in [8].

## 3. EXTENSION

In this section we would like to point out that for every graph  $H$  there is an  $\tilde{H}$  such that

$$\text{hom}(G, \tilde{H}) = \text{hom}(G', H),$$

where  $G'$  is the bipartite graph defined in the previous section. Exactly the same argument we used for  $H_{\text{WR}}$  will work for any graph  $\tilde{H}$  constructed in this manner. Actually, the situation is even better. To give the most general version we need a definition.

**Definition 3.1.** Let  $G$  be a bipartite graph. Let  $H$  be another bipartite graph equipped with a weight function  $\nu : V_H \rightarrow \mathbb{R}_+$ . Let  $\mathbb{I}_{E_H} : A_H \times B_H \rightarrow \{0, 1\}$  denote the characteristic function of  $E_H$ . Define

$$Z_b(G, H) = \sum_{\substack{\varphi: V_G \rightarrow V_H \\ \varphi(A_G) \subseteq A_H \\ \varphi(B_G) \subseteq B_H}} \prod_{(a, b) \in E_G} \mathbb{I}_{E_H}(\varphi(a), \varphi(b)) \prod_{w \in V_G} \nu(\varphi(w)),$$

(The subscript  $b$  stands for bipartite.) If  $G$  and  $H$  are not necessarily bipartite graphs, but  $H$  is a weighted graph we can still define

$$Z(G, H) = \sum_{\varphi: V_G \rightarrow V_H} \prod_{(u,v) \in E_G} \mathbb{I}_{E_H}(\varphi(u), \varphi(v)) \prod_{w \in V_G} \nu(\varphi(w)).$$

In the language of statistical physics,  $Z_b(G, H)$  and  $Z(G, H)$  are *partition functions*.

Somewhat surprisingly, J. Kahn's result holds even in this general case, as shown by D. Galvin and P. Tetali [6].

**Theorem 3.2.** [6] *For any bipartite graph  $H$  equipped with the weight function  $\nu : V_H \rightarrow \mathbb{R}_+$  and  $\mathbb{I}_{E_H} : A_H \times B_H \rightarrow \{0, 1\}$ , and for any  $d$ -regular simple graph  $G$  on  $n$  vertices,*

$$Z_b(G, H) \leq Z_b(K_{d,d}, H)^{n/(2d)}.$$

The key observation is that for a bipartite graph  $H$  equipped with the weight function  $\nu : V_H \rightarrow \mathbb{R}_+$  and characteristic function  $\mathbb{I}_{E_H} : A_H \times B_H \rightarrow \{0, 1\}$ , we can define a weighted graph  $\tilde{H}$  with weight function  $\tilde{\nu}$  and characteristic function  $\mathbb{I}_{E_{\tilde{H}}}$  such that

$$(3.1) \quad Z(G, \tilde{H}) = Z_b(G', H),$$

for any graph  $G$  (where  $G'$  is the modification of  $G$  defined in the previous section). Indeed, construct  $\tilde{H}$  with vertex set  $A_H \times B_H$ , edges

$$\mathbb{I}_{E_{\tilde{H}}}((a_1, b_1), (a_2, b_2)) = \mathbb{I}_{E_H}(a_1, b_2) \mathbb{I}_{E_H}(a_2, b_1),$$

and weight function

$$\tilde{\nu}(a, b) = \nu(a) \nu(b) \mathbb{I}_{E_H}(a, b).$$

In effect, the vertex set of  $\tilde{H}$  is only the edges of  $H$  (since non-edge pairs are given weight 0). Now, for a map  $\varphi : G' \rightarrow H$ , we can consider the map  $\tilde{\varphi} : G \rightarrow \tilde{H}$  given by

$$\tilde{\varphi}(u) = (\varphi((u, 0)), \varphi((u, 1))).$$

By the construction of the graphs  $G'$  and  $\tilde{H}$ , the contribution of  $\varphi$  to  $Z_b(G, H)$  is the same as the contribution of  $\tilde{\varphi}$  to  $Z(G, \tilde{H})$ , and the result (3.1) follows.

Finally, applying Theorem 3.2 to the  $(d+1)$ -regular graph  $G'$  yields

$$Z(G, \tilde{H}) = Z_b(G', H) \leq Z_b(K_{d,d}, H)^{2n/(2(d+1))} = Z(K_{d+1}, \tilde{H})^{n/(d+1)}.$$

Hence we have proved the following theorem.

**Theorem 3.3.** *For a bipartite graph  $H = (A, B, E)$  with vertex weight function  $\nu : V_H \rightarrow \mathbb{R}_+$  let  $\tilde{H}$  be the following weighted graph: its vertex set is  $E(H)$ , its edge set is defined by  $((a_1, b_1), (a_2, b_2)) \in E(\tilde{H})$  if and only if  $(a_1, b_2) \in E(H)$  and  $(a_2, b_1) \in E(H)$ , and the weight function on the vertex set is  $\tilde{\nu}(a, b) = \nu(a) \nu(b)$  for  $(a, b) \in E(H)$ . Then for any  $d$ -regular simple graph  $G$  on  $n$  vertices we have*

$$Z(G, \tilde{H}) \leq Z(K_{d+1}, \tilde{H})^{n/(d+1)}.$$

We can obtain Conjecture 3 of [2] as a corollary by applying this theorem in the case where  $H$  is the path on 4 vertices,  $a_1 b_1 a_2 b_2$ , with appropriate vertex weights. Indeed, if  $\nu(a_1) = 1$ ,  $\nu(b_1) = \lambda_b$ ,  $\nu(a_2) = \frac{\lambda_w}{\lambda_b}$ ,  $\nu(b_2) = \frac{\lambda_r \lambda_b}{\lambda_w}$  then  $\tilde{H}$  is precisely the

Widom-Rowlinson graph with vertex weights  $\lambda_b, \lambda_r, \lambda_w$ . This proves that even for the vertex-weighted Widom-Rowlinson graph we have

$$Z(G, H_{\text{WR}}) \leq Z(K_{d+1}, H_{\text{WR}})^{n/(d+1)}.$$

Hence we have proved the following theorem.

**Theorem 3.4.** *Let  $H_{\text{WR}}$  be the Widom-Rowlinson graph with vertex weights  $\lambda_b, \lambda_w, \lambda_r$ . Then for any  $d$ -regular simple graph  $G$  on  $n$  vertices we have*

$$Z(G, H_{\text{WR}}) \leq Z(K_{d+1}, H_{\text{WR}})^{n/(d+1)}.$$

Now let us consider the special case when  $H$  is unweighted ( $\nu \equiv 1$ ). In this case  $\tilde{\nu}$  is just  $\mathbb{I}_{E_H}$ , so we can think of  $\tilde{H}$  as an unweighted graph with vertex set  $V_{\tilde{H}} = E_H$ . There is an edge in  $\tilde{H}$  between edges  $e = (a_1, b_1)$  and  $f = (a_2, b_2)$  of  $H$  whenever  $(a_1, b_2)$  and  $(a_2, b_1)$  are both also edges of  $H$ . This is always the case when either  $a_1 = a_2$  or  $b_1 = b_2$ , so in particular every edge  $e \in E_H = V_{\tilde{H}}$  has a self-loop in  $\tilde{H}$ , and every pair of incident edges in  $H$  are adjacent in  $\tilde{H}$ . We also get an edge  $(e, f) \in E_{\tilde{H}}$  if four vertices  $a_1 b_1 a_2 b_2$  are all distinct and form a 4-cycle with  $e$  and  $f$  as opposite edges. In other words,  $\tilde{H}$  is precisely the extended line graph of  $H$ . Hence as a corollary of Theorem 3.3 we have proved Theorem 1.3.

If  $H$  does not contain any 4-cycle, then  $\tilde{H}$  is simply the line graph of  $H$  with loops at every vertex. In particular, if  $H$  is a path (or even cycle of length at least 6) then  $\tilde{H}$  is again a path (or even cycle of length at least 6), but now with a loop at every vertex. Letting  $H^\circ$  denote the graph obtained by adding a loop at every vertex of the graph  $H$ , we can write the corollary

**Corollary 3.5.** *If  $H = C_k^\circ$  (for  $k \geq 6$  even) or if  $H = P_k^\circ$  (for any  $k$ ), then for any  $d$ -regular graph  $G$*

$$p_H(G) \leq p_H(K_{d+1}).$$

It is a good question how to characterize all of the graphs  $\tilde{H}$  which can be obtained this way. Note that since  $\tilde{H}$  is always fully-looped, this class has no intersection with the class of graphs found by Galvin [4]: the set of graphs  $H_q^\ell$  obtained from a complete looped graph on  $q$  vertices with  $\ell \geq 1$  loops deleted.

**Remark 3.6.** Let  $S_k$  be the star on  $k$  vertices. One can show (for details see [4]) that, for large enough  $d$ ,

$$p_{S_k^\circ}(K_{d+1}) < p_{S_k^\circ}(K_{d,d})$$

for  $k \geq 6$ . From this example we can see that in order to have  $p_H(G) \leq p_H(K_{d+1})$  it is not sufficient merely for  $H$  to have a loop at every vertex.

L. Sernau [9] introduced many ideas for extending certain inequalities to a larger class of graphs. For instance, recall that the  $H_1 \times H_2$  has  $V_{H_1 \times H_2} = V_{H_1} \times V_{H_2}$  and  $((a_1, b_1), (a_2, b_2)) \in E_{H_1 \times H_2}$  if and only if  $(a_1, a_2) \in E_{H_1}$  and  $(b_1, b_2) \in E_{H_2}$ . Sernau noted that if  $H_1$  and  $H_2$  are graphs such that

$$p_{H_i}(G) \leq p_{H_i}(K_{d+1}),$$

for  $i = 1, 2$ , then it is also true that

$$p_{H_1 \times H_2}(G) \leq p_{H_1 \times H_2}(K_{d+1}).$$

This inequality simply follows from the identity

$$\text{hom}(G, H_1 \times H_2) = \text{hom}(G, H_1) \text{hom}(G, H_2),$$

which is explained in [9]. Surprisingly, this observation does not allow us to extend our result to any new graphs, because the product of two extended line graphs is again an extended line graph:

$$\tilde{H}_1 \times \tilde{H}_2 = \tilde{H}_{12},$$

where  $H_{12} = (A_{H_1} \times A_{H_2}, B_{H_1} \times B_{H_2}, E_{H_1} \times E_{H_2})$ .

#### 4. ON A THEOREM OF L. SERNAU

Theorem 3 of [9] also provides a class of graphs for which  $K_{d+1}$  is the maximizing graph. Below we explain the relationships between our results and his theorem.

**Definition 4.1.** Let  $H$  and  $A$  be graphs. Then the graph  $H^A$  is defined as follows: its vertices are the maps  $f : V(A) \rightarrow V(H)$  and the  $(f_1, f_2) \in E(H^A)$  if  $(f_1(u), f_2(v)) \in E(H)$  whenever  $(u, v) \in E(A)$ .

Then Sernau proved the following theorem.

**Theorem 4.2.** [9] *Let  $G$  be a  $d$ -regular graph, and let  $F = l(H^B)$ , where  $H$  is an arbitrary graph,  $B$  is a bipartite graph, and  $l(H^B)$  is the graph induced by the vertices of  $H^B$  which have a loop. Then*

$$p_F(G) \leq p_F(K_{d+1}).$$

When  $H = H_{ind}$ ,  $B = K_2$  then  $l(H^B) = H_{WR}$  so this also proves the conjecture of D. Galvin. Note that when  $B = K_2$  then  $l(H^B)$  is the extended line graph of  $H \times K_2$ . It is not a great surprise that these results are similar, even the proofs behind these results are strongly related to each other.

#### 5. CONJECTURES

Let  $H$  be a simple graph, i.e., with no multiple edges or loops. Let  $H^o$  denote the graph obtained by adding a loop at each vertex of  $H$  (so for instance  $C_n^o$  denotes the  $n$ -cycle with a loop at each vertex).

**Conjecture 5.1.** Let  $G$  be a  $d$ -regular simple graph. Then for any  $n \geq 4$

$$p_{C_n^o}(G) \leq p_{C_n^o}(K_{d+1}).$$

**Conjecture 5.2.** Let  $G$  be a  $d$ -regular simple graph. Then for any  $d \geq 4$

$$p_{S_4^o}(G) \leq p_{S_4^o}(K_{d+1}).$$

Furthermore, for  $k \geq 6$

$$p_{S_k^o}(G) \leq p_{S_k^o}(K_{d,d}).$$

Finally, for an arbitrary graph  $H$  it is not clear how to characterize the maximizers over all  $d$ -regular graphs  $G$  of  $p_H(G)$ . If we restrict to bipartite  $G$ , however, D. Galvin and P. Tetali proved that  $p_H(G) \leq p_H(K_{d,d})$  [6]. We conjecture that this can be extended to the class of triangle-free graphs.

**Conjecture 5.3.** Let  $G$  be a  $d$ -regular triangle-free graph. Then for any graph  $H$  we have

$$p_H(G) \leq p_H(K_{d,d}).$$

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