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SOLUTION TO A PROBLEM OF BOLLOBÁS AND HÄGGKVIST ON HAMILTON CYCLES IN REGULAR GRAPHS

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Abstract. We prove that, for large n, every 3-connected D-regular graph on n vertices with $D \geq n/4$ is Hamiltonian. This is best possible and verifies the only remaining case of a conjecture posed independently by Bollobás and Häggkvist in the 1970s. The proof builds on a structural decomposition result proved recently by the same authors.

1. Introduction

In this paper we give an exact solution to a longstanding conjecture on Hamilton cycles in regular graphs, posed independently by Bollobás and Häggkvist: every sufficiently large 3-connected regular graph on n vertices with degree at least $n/4$ contains a Hamilton cycle. The history of this problem goes back to Dirac's classical result that $n/2$ is the minimum degree threshold for Hamiltonicity. This is certainly best possible – consider e.g. the almost balanced complete bipartite graph or the disjoint union of two equally-sized cliques. The following natural question arises: can we reduce the minimum degree condition by making additional assumptions on G? The extremal examples above suggest that the family of regular graphs with some connectivity condition might have a lower minimum degree threshold for Hamiltonicity. Indeed, Bollobás [1] as well as Häggkvist (see [7]) independently made the following conjecture: *Every t-connected D-regular graph G on n vertices with $D \geq n/(t+1)$ is Hamiltonian.*[1] The case $t = 2$ was first considered by Szekeres (see [7]), and after partial results by several authors including Nash-Williams [14], Erdős and Hobbs [4] and Bollobás and Hobbs [2], it was finally settled in the affirmative by Jackson [7]. His result was extended by Hilbig [6] who showed that one can reduce D to $n/3-1$ unless G is the Petersen graph P or the 3-regular graph P' obtained by replacing one vertex of P with a triangle.

However, Jung [9] and independently Jackson, Li and Zhu [8] found a counterexample to the conjecture for $t \geq 4$. Until recently, the only remaining case $t = 3$ was wide open. Fan [5] and Jung [9] independently showed that every 3-connected D-regular graph contains a cycle of length at least $3D$, or a Hamilton cycle. Li and Zhu [13] proved the conjecture for $t = 3$ in the case when $D \geq 7n/22$[2] and Broersma, van den Heuvel, Jackson and Veldman [3] proved it for $D \geq 2(n+7)/7$. In [8], Jackson, Li and Zhu prove that if G satisfies the conditions of the conjecture, any longest cycle C in G is dominating provided that n is not too small. (In other words, the vertices not in C form an independent set.) Recently, in [10], we proved an approximate version of the conjecture, namely that for all $\varepsilon > 0$, whenever n is sufficiently large, any 3-connected D-regular graph on n vertices with $D \geq (1/4+\varepsilon)n$ is Hamiltonian. Here, we prove the exact version (for large n).

exact **Theorem 1.1.** *There exists $n_0 \in \mathbb{N}$ such that every 3-connected D-regular graph on $n \geq n_0$ vertices with $D \geq n/4$ is Hamiltonian.*

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[1]Bollobás's conjecture was stronger, with $D \geq n/(t+1) - 1$.

[2]there's a mistake in Li's survey where this result is quoted – it has $3n/22$ which is of course smaller than $n/4$!

Our proof builds on the results in [10]. In particular, it relies on a structural decomposition result which holds for any dense regular graph: it gives a partition into (bipartite) robust expanders with few edges between these[3] (see Section 3 and Theorem 4.4). [10] also contains further applications of this partition result.

There are several natural analogues of these questions for directed and bipartite graphs. For example, the following conjecture of Kühn and Osthus [11] is a directed analogue of Jackson's theorem [7]. Further open problems are discussed in [10]. We say that a digraph G is *D-regular* if every vertex has both in- and out-degree D.

Conjecture 1.2. *Every strongly 2-connected D-regular digraph on n vertices with $D \geq n/3$ contains a Hamilton cycle.*

This paper is organised as follows. In Section 2, we discuss the extremal examples which show that Theorem 1.1 is best possible. Section 3 contains a sketch of the proof of Theorem 1.1. Section 4 lists some notation, definitions and tools from [10] which will be used throughout the paper. The proof of Theorem 1.1 is split into three cases, and these are considered in Sections 5–7 respectively. Finally, we derive Theorem 1.1 in Section 8.

example

2. The extremal examples

In this section we show that Theorem 1.1 is best possible in the sense that neither the minimum degree condition nor the connectivity condition can be reduced. The example of Jung [9] and Jackson, Li and Zhu [8] shows that the minimum degree condition cannot be reduced for graphs with $n \equiv 1 \mod 8$ vertices; for completeness we extend this to all possible n in the following proposition. An illustration of their example may be found in Figure 1(i).

extremalex **Proposition 2.1.** *Let $n \geq 5$[4] and let D be the largest integer such that $D \leq \lceil n/4 \rceil - 1$ and nD is even. Then there is an $(\lfloor n/8 \rfloor - 1)$-connected D-regular graph G_n on n vertices which does not contain a Hamilton cycle.*

Proof. Recall that a D-regular graph on n vertices exists if and only if $n \geq D+1$ and nD is even. For each $n \geq 5$, we define a graph G_n on n vertices as follows. Let V_1, V_2, A, B be disjoint independent sets where $|A| = D$, $|B| = D-1$, and the other classes have sizes according to the table below. Let A_1, A_2 be a partition of A so that $\Big| D/2 - |A_1| \Big|$ is minimal subject to the parity conditions below being satisfied:

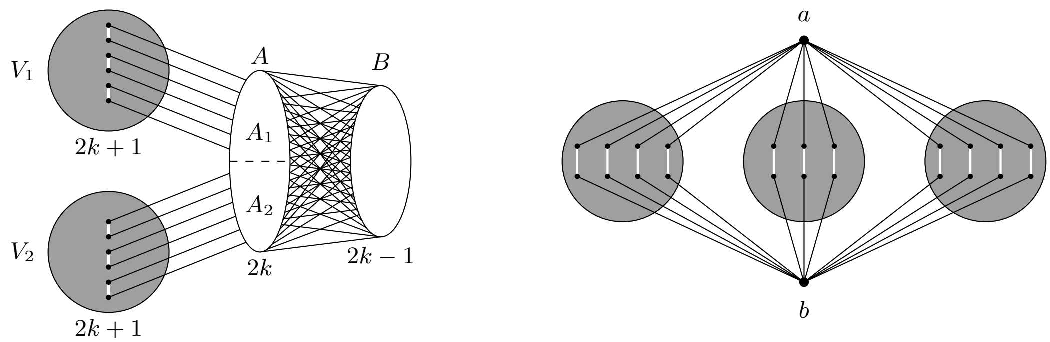
n	D	$\lvert V_1\rvert$	$\lvert V_2\rvert$	$\lvert A_1\rvert$	$\lvert A_2\rvert$
$8k+1$	$2k$	$2k+1$	$2k+1$	even	even
$8k+2$	$2k$	$2k+2$	$2k+1$	even	even
$8k+3$	$2k$	$2k+2$	$2k+2$	even	even
$8k+4$	$2k$	$2k+3$	$2k+2$	even	even
$8k+5$	$2k$	$2k+3$	$2k+3$	even	even
$8k+6$	$2k+1$	$2k+3$	$2k+2$	odd	even
$8k+7$	$2k$	$2k+4$	$2k+4$	even	even
$8k+8$	$2k+1$	$2k+4$	$2k+3$	even	odd

Note that $|V_i| \geq D+1$ for $i = 1, 2$. Add every edge between A and B. First consider the cases when $D = 2k$. Then $|A_i|$ is even for $i = 1, 2$. For each $i = 1, 2$, add edges so that $G_n[V_i]$ is D-regular. Let M_i be a matching of size $|A_i|/2$ in $G_n[V_i]$ and remove it. Let $V_i' := V(M_i)$. So $|V_i'| = |A_i|$. Add a perfect matching between V_i' and A_i.

Now consider the case when $D = 2k+1$. Then, by our choice of A_i and V_i we have that $|A_i| \equiv |V_i| \mod 2$. Fix $V_i' \subseteq V_i$ with $|V_i'| := |A_i|$. Define the edge set of $G_n[V_i]$ so that for all

[3]Deryk changed this sentence

[4]$|B| = D-1$ so $D - 1 = \lceil n/4 \rceil - 2 \geq 0$ so $n \geq 5$.



(i) (ii)

Figure 1. Extremal examples for Theorem 1.1.
(i) is an illustration for the case $n = 8k + 1$. Here, each V_i is a clique of order $2k + 1$ with a matching of size k removed.

$x \in V_i'$ we have $d_{V_i}(x) = D-1$ and for all $y \in V_i \setminus V_i'$ we have $d_{V_i}(y) = D$.[5] Add a perfect matching between V_i' and A_i.

Then G_n has n vertices, is D-regular and has connectivity $\min\{|A_1|, |A_2|\} \geq \lfloor n/8 \rfloor - 1$.[6] Moreover, G_n does not contain a Hamilton cycle because it is not 1-tough ($G_n \setminus A$ contains more than $|A|$ components). □

fig:exactex

There also exist non-Hamiltonian 2-connected regular graphs on n vertices with degree close to $n/3$ (see Figure 1(ii)). Indeed, we can construct such a graph G as follows. Start with three disjoint cliques on $3k$ vertices each. In the ith clique choose disjoint sets A_i and B_i with $|A_i| = |B_i|$ and $|A_1| = |A_3| = k$ and $|A_2| = k - 1$. Remove a perfect matching between A_i and B_i for each i. Add two new vertices a and b, where a is connected to all vertices in the sets A_i and b is connected to all vertices in all the sets B_i. Then G is a $(3k-1)$-regular 2-connected graph on $n = 9k + 2$ vertices. However, G is not Hamiltonian because $G \setminus \{a, b\}$ has three components. One can construct similar examples for all $n \in \mathbb{N}$.

Altogether this shows that none of the conditions — degree or connectivity — of Theorem 1.1 can be relaxed.

sketch

3. Sketch of the proof

3.1. **Robust partitions of dense regular graphs.** The main tool in our proof is a structural result on dense regular graphs that we proved in [10]. Roughly speaking, this allows us to partition the vertex set of such a graph G into a small number of 'robust components', each of which has strong expansion properties and sends few edges to the rest of the graph.

[5] Case 1: $|V_i|$ odd. Since $|V_i| \geq D + 1$ and we have that $|V_i| \geq D + 2$. Add every edge with both endpoints in V_i. Find a Hamilton decomposition (i.e. $(|V_i| - 1)/2 \geq (D + 1)/2$ edge-disjoint HCs). Choose $(D + 1)/2$ edge-disjoint Hamilton cycles $H_1, \dots, H_{(D+1)/2}$. Each H_i contains a matching of size $\lfloor |V_i|/2 \rfloor$. Let $M \subseteq H_{(D+1)/2}$ be a matching of size $(|V_i| - |A_i|)/2$. Let $G_n[V_i] := H_1 \cup \ldots \cup H_{(D-1)/2} \cup M$. Case 2: $|V_i|$ even. Add every edge with both endpoints in V_i. Find a 1-factorisation. Choose D edge-disjoint perfect matchings $M_1, \dots, M_D$ in this factorisation. Let $M \subseteq M_D$ have size $(|V_i| - |A_i|)/2$. Let $G_n[V_i] := M_1 \cup \ldots \cup M_{D-1} \cup M$.

[6] AL: calculation changed slightly. If $n \neq 8k + 8$, $\min\{|A_1|, |A_2|\} \geq k - 1 \geq \lfloor n/8 \rfloor - 1$. If $n = 8k + 8$, $\min\{|A_1|, |A_2|\} = k = n/8 - 1$.

There are two types of robust components: *robust expander components* and *bipartite robust expander components*. A robust expander component $G[U]$ is characterised by the following properties:

- for each $S \subseteq U$ which is neither too small nor too large, the 'robust neighbourhood' $RN(S)$ of S is significantly larger than S itself;
- G contains few edges between U and $V(G) \setminus U$.

Here the robust neighbourhood of S is the set of all vertices in U with linearly many neighbours in S. A bipartite robust expander component $G[W]$ has slightly more structure: $G[W]$ can be made into a balanced bipartite graph by removing a small number of vertices and edges, and sets in the first class expand robustly into the second class. More precisely, if W has bipartition A, B and $S \subseteq A$ is neither too large nor too small, then $RN(S) \cap B$ is significantly larger than S. (Note that we do not require that sets in both vertex classes expand.)

We say that $\mathcal{V} = \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$ is a *robust partition of G with parameters k, ℓ* if it is a partition of $V(G)$ such that $G[V_i]$ is a robust expander component for all $1 \leq i \leq k$ and $G[W_j]$ is a bipartite robust expander component for all $1 \leq j \leq \ell$. In [10] we proved the following:

($\star$) For all $r \in \mathbb{N}$ and $\varepsilon > 0$ and n sufficiently large, every D-regular graph G on n vertices with $D \geq (\frac{1}{r+1} + \varepsilon)n$ has a robust partition with parameters k, ℓ, where $k + 2\ell \leq r$.

In particular, the number of edges between robust components is $o(n^2)$ (see Theorem 4.4 for the precise statement).

3.2. **Finding a Hamilton cycle using a robust partition.** Now suppose that G is a D-regular graph on n vertices with $D \geq n/4$, where n is sufficiently large. Then ($\star$) applied with $r = 4$ implies that G has a robust partition $\mathcal{V}$ with parameters k, ℓ, where $k + 2\ell \leq 4$. This gives eight possible structures, parametrised by $(k, \ell) \in S_{\leq 3} \cup S_4$, where

$$S_{\leq 3} := \{(1,0), (2,0), (3,0), (0,1), (1,1)\} \quad \text{and } S_4 := \{(4,0), (0,2), (2,1)\}.$$

Note that the extremal example in Figure 1(i) corresponds to the case $(2, 1)$ and the one in (ii) corresponds to the case $(3, 0)$. Also note that when $D \geq (1/4 + \varepsilon)n$, we have $k + 2\ell \leq 3$ and so $(k, \ell) \in S_{\leq 3}$. In [10], we proved that if G is 3-connected and has a robust partition $\mathcal{V}$ with parameters k, ℓ where $(k, \ell) \in S_{\leq 3}$, then G is Hamiltonian. In particular, this implies an approximate version of Theorem 1.1. The proof proceeded by considering each possible structure separately. Therefore, to prove Theorem 1.1, it remains to show that if G is 3-connected and has a robust partition $\mathcal{V}$ with parameters k, ℓ where $(k, \ell) \in S_4$, then G is Hamiltonian (see Theorem 4.6). So the current paper does not supersede our previous result but rather uses it as an essential ingredient. Again, we consider each structure separately in Sections 5, 6 and 7 respectively.

In each case we adopt the following strategy. Let $\mathcal{V}$ be a robust partition of G with parameters k, ℓ. Kühn, Osthus and Treglown [12] proved that every large robust expander H with linear minimum degree contains a Hamilton cycle. This can be strengthened (see [10]) to show that one can cover all the vertices of a robust expander with a set of paths with prescribed endvertices. More precisely, one can show that each robust expander component $G[V_i]$ is Hamilton p-linked for each small p and all $1 \leq i \leq k$. (Here a graph H is *Hamilton p-linked* if, whenever $X := \{x_1, y_1, \dots, x_p, y_p\}$ is a collection of distinct vertices, there exist vertex-disjoint paths $P_1, \dots, P_p$ such that P_j connects x_j to y_j, and such that together the paths $P_1, \dots, P_p$ cover all vertices of H.) Balanced bipartite robust expanders have the same property, provided X is distributed equally between the bipartition classes. This means that we can hope to reduce the problem of finding a Hamilton cycle in G to finding a suitable set of *external edges* E_{ext}, where an edge is external if it has endpoints in different members of $\mathcal{V}$. We then apply the Hamilton p-linked property to each robust component to join up the external edges into a Hamilton cycle. The assumption of 3-connectivity is crucial for finding E_{ext}.

However, several problems arise. When $(k,\ell)=(4,0)$, we have four robust components and only the assumption of 3-connectivity, which makes it difficult to find a suitable set $E_{\rm ext}$ joining all four components directly. However, we can appeal to the dominating cycle result in [8] mentioned in the introduction, giving us a fairly short argument for this case. Note that the condition that $D \geq n/4$ is essential in this case — 3-connectivity on its own is not sufficient.[7]

Now suppose that $\ell \geq 1$, i.e. $\mathcal{V}$ contains a bipartite robust expander component. These cases are challenging since a bipartite graph does not contain a Hamilton cycle if it is not balanced. So as well as a suitable set $E_{\rm ext}$, we need to find a set $E_{\rm bal}$ of *balancing edges* incident to the bipartite robust expander component. Suppose for example that $(k,\ell)=(0,2)$ and G consists of two bipartite robust expander components W_1, W_2 such that W_i has vertex classes A_i, B_i where $|A_1|=|B_1|$ and $|A_2|=|B_2|+1$. Then we could choose $E_{\rm bal}$ to be a single edge with both endpoints in A_2. A second example would be $E_{\rm bal}=\{a_1a_2, b_1a_2'\}$ where $a_1 \in A_1$, $b_1 \in B_1$ and $a_2, a_2' \in A_2$ are distinct. (Note that these are also external edges and in this case we can actually take $E_{\rm ext} \cup E_{\rm bal} = \{a_1a_2, b_1a_2'\}$.)[8] Observe that we need at least $\Big||A_1|-|B_1|\Big| + \Big||A_2|-|B_2|\Big|$ balancing edges.[9]

Our robust partition guarantees that the vertex classes of any bipartite robust expander component differ by at most $o(n)$, so we must potentially find a similar number of balancing edges. This must be done in such a way that $\mathcal{P} := E_{\rm ext} \cup E_{\rm bal}$ can be extended into a Hamilton cycle. So in particular $\mathcal{P}$ must be a collection of vertex-disjoint paths. We use the Hamilton p-linkedness of the (bipartite) robust expander components to find these edges which extend $\mathcal{P}$ into a Hamilton cycle. Consider the second example above, with $\mathcal{P} = \{a_1a_2, b_1a_2'\}$. Choose a neighbour b_2 of a_2 in B_2 and let $\mathcal{P}' := \{a_1a_2b_2, b_1a_2'\}$. Then the Hamilton 1-linkedness of W_1, W_2 implies that we can find a path P_1 with endpoints a_1, b_1 which spans W_1, and a path P_2 with endpoints a_2', b_2 which spans $W_2 \setminus \{a_2\}$. Then the edges of $P_1, P_2, \mathcal{P}'$ together form a Hamilton cycle.[10]

It turns out that the condition that $D \geq n/4$ is crucial in the case when $(k,\ell)=(2,1)$ (see Section 2) but its full strength is not required in the case when $(k,\ell)=(0,2)$.[11] A sketch of the proof in each of the three cases can be found at the beginning of Sections 5, 6 and 7 respectively.

4. Notation, definitions and general tools

4.1. **General notation.** Given a graph G and $X \subseteq V(G)$, complements are always taken within G, so that $\overline{X} := V(G) \setminus X$. We write $G \setminus X$ to mean $G[V(G) \setminus X]$. Given $H \subseteq V(G)$, we write $G \setminus E(H)$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$. We write $N(X) := \bigcup_{x \in X} N(x)$. Given $x \in V(G)$ and $Y \subseteq V(G)$ we write $d_Y(x)$ for the number of edges xy with $y \in Y$.

If S, T are sets of vertices which are not necessarily disjoint and may not be subsets of $V(G)$, we write $e_G(S)$ for the number of edges of G with both endpoints in S, and $e_G(S,T)$ for the number of ST-edges of G, i.e. for the number of all edges with one endpoint in S and the other endpoint in T. Moreover, we set $G[S] := G[S \cap V(G)]$ and write $G[S,T]$ for the bipartite graph with vertex classes $S \cap V(G)$, $T \cap V(G)$ whose edge set consists of all the ST-edges of G. We omit the subscript G whenever the graph G is clear from the context.

Given[12] subsets X, Y of $V(G)$, we say that P is an *XY-path* if P has one endpoint in X and one endpoint in Y. We call a vertex-disjoint collection of non-trivial paths a *path system*. We will

[7] Probably $D \geq n/4 - 1$ or something a little smaller than $n/4$ is essential.

[8] I added this example to explain what balancing edges are.

[9] NEW big modulus

[10] continuing the previous example (optional?). A picky reader might not like the fact we use the Hamilton linkedness of W_2 not $W_2 \setminus \{a_2\}$.

[11] We use 3-connectivity in all cases. It may not be necessary when $(k,\ell)=(0,2)$ but we do use it. Should I mention this?

[12] NEW: previously disjoint subsets

often think of a path system $\mathcal{P}$ as a graph with edge set $\bigcup_{P\in\mathcal{P}} E(P)$, so that e.g. $V(\mathcal{P})$ is the union of the vertex sets of each path in $\mathcal{P}$, and $e_{\mathcal{P}}(X)$ denotes the number of edges on the paths in $\mathcal{P}$ having both endpoints in X, and $e_{\mathcal{P}}(X,Y)$ denotes the number of XY-edges in paths of $\mathcal{P}$.[13] By slightly abusing notation, given two vertex sets S and T and a path system $\mathcal{P}$, we write $\mathcal{P}[S]$ for the graph obtained from $\mathcal{P}[S]$ by deleting isolated vertices and define $\mathcal{P}[S,T]$ similarly.[14] We say that a vertex x is an *endpoint* of $\mathcal{P}$ if x is an endpoint of some path in $\mathcal{P}$. An *Euler tour* in a (multi)graph is a closed walk that uses each edge exactly once.[15]

We write $\mathbb{N}$ for the set of positive integers and write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative reals. Throughout we will omit floors and ceilings where the argument is unaffected. The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph), then there is a non-decreasing function $f : (0,1] \to (0,1]$ such that the result holds for all $0 < a,b,c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq f(b)$ and $1/n \leq f(a)$. Hierarchies with more constants are defined in a similar way. Given $0 < \varepsilon < 1$ and $x \in \mathbb{R}$, we write $\lceil x \rceil_\varepsilon := \lceil x - \varepsilon \rceil$.

sec:struct

4.2. **Robust partitions of regular graphs.** In this section we list the definitions which are required to state the structural result on dense regular graphs (Theorem 4.4) which is the main tool in our proof. As already indicated in Section 3, this involves the concept of 'robust expansion'.

Given a graph G on n vertices, $0 < \nu < 1$ and $S \subseteq V(G)$, we define the *ν-robust neighbourhood* $RN_{\nu,G}(S)$ of S to be the set of all those vertices with at least νn neighbours in S. Given $0 < \nu \leq \tau < 1$, we say that G is a *robust (ν,τ)-expander* if, for all sets S of vertices satisfying $\tau n \leq |S| \leq (1-\tau)n$, we have that $|RN_{\nu,G}(S)| \geq |S| + \nu n$. For $S \subseteq X \subseteq V(G)$ we write $RN_{\nu,X}(S) := RN_{\nu,G[X]}(S)$.

The next lemma (Lemma 4.8 in [10]) states that robust expanders are indeed robust, in the sense that the expansion property cannot be destroyed by adding or removing a small number of vertices.

expanderswallow

Lemma 4.1. *Let $0 < \nu \ll \tau \ll 1$. Suppose that G is a graph and $U, U' \subseteq V(G)$ are such that $G[U]$ is a robust (ν,τ)-expander and $|U \triangle U'| \leq \nu|U|/2$. Then $G[U']$ is a robust $(\nu/2, 2\tau)$-expander.*

We now introduce the concept of 'bipartite robust expansion'. Let $0 < \nu \leq \tau < 1$. Suppose that H is a (not necessarily bipartite) graph on n vertices and that A, B is a partition of $V(H)$. We say that H is a *bipartite robust (ν,τ)-expander with bipartition A, B* if every $S \subseteq A$ with $\tau|A| \leq |S| \leq (1-\tau)|A|$ satisfies $|RN_{\nu,H}(S) \cap B| \geq |S| + \nu n$. Note that the order of A and B matters here. We do not mention the bipartition if it is clear from the context.

Note that for $0 < \nu' \leq \nu \leq \tau \leq \tau' < 1$, any robust (ν,τ)-expander is also a robust (ν',τ')-expander (and the analogue holds in the bipartite case).

Given $0 < \rho < 1$, we say that $U \subseteq V(G)$ is a *ρ-component* of a graph G on n vertices if $|U| \geq \sqrt{\rho} n$ and $e(U, \overline{U}) \leq \rho n^2$. We will need the following simple observation (Lemma 4.1 in [10]) about ρ-components.

comp

Lemma 4.2. *Let $n, D \in \mathbb{N}$ and $\rho > 0$. Let G be a D-regular graph on n vertices and let U be a ρ-component of G. Then $|U| \geq D - \sqrt{\rho} n$.*

Suppose that G is a graph on n vertices and that $U \subseteq V(G)$. We say that $G[U]$ is *ρ-close to bipartite (with bipartition U_1, U_2)* if

(C1) U is the union of two disjoint sets U_1 and U_2 with $|U_1|, |U_2| \geq \sqrt{\rho} n$;

(C2) $\big||U_1| - |U_2|\big| \leq \rho n$;

(C3) $e(U_1, \overline{U_2}) + e(U_2, \overline{U_1}) \leq \rho n^2$.

[13] I wrote 'endpoints' here rather than 'endvertices' since that is what we use previously. NEW def

[14] DK changed this sentence

[15] I have removed the notions of $\mathcal{U}$-anchored and $\mathcal{U}$-extension.

(Recall that $\overline{U_1} = V(G) \setminus U_1$ and similarly for U_2.) Note that (C1) and (C3) together imply that U is a ρ-component. Suppose that G is a graph on n vertices and that $U \subseteq V(G)$. Let $0 < \rho \leq \nu \leq \tau < 1$. We say that $G[U]$ is a *(ρ, ν, τ)-robust expander component of G* if

(E1) U is a ρ-component;
(E2) $G[U]$ is a robust (ν, τ)-expander.

We say that $G[U]$ is a *bipartite (ρ, ν, τ)-robust expander component (with bipartition A, B) of G* if

(B1) $G[U]$ is ρ-close to bipartite with bipartition A, B;
(B2) $G[U]$ is a bipartite robust (ν, τ)-expander with bipartition A, B.

We say that U is a *(ρ, ν, τ)-robust component* if it is either a (ρ, ν, τ)-robust expander component or a bipartite (ρ, ν, τ)-robust expander component.

One can show that, after adding and removing a small number of vertices, a bipartite robust expander component is still a bipartite robust expander component, with slightly weaker parameters. This appears as Lemma 4.10 in [10] and the proof may be found in [15].[16]

BREadjust

Lemma 4.3. *Let $0 < 1/n \ll \rho \leq \gamma \ll \nu \ll \tau \ll \alpha < 1$ and suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$. Suppose that $G[A \cup B]$ is a bipartite (ρ, ν, τ)-robust expander component of G with bipartition A, B. Let $A', B' \subseteq V(G)$ be such that $|A \triangle A'| + |B \triangle B'| \leq \gamma n$. Then $G[A' \cup B']$ is a bipartite $(3\gamma, \nu/2, 2\tau)$-robust expander component of G with bipartition A', B'.*

Let $k, \ell, D \in \mathbb{N}_0$ and $0 < \rho \leq \nu \leq \tau < 1$. Given a D-regular graph G on n vertices, we say that $\mathcal{V}$ is a *robust partition of G with parameters ρ, ν, τ, k, ℓ* if the following conditions hold.

(D1) $\mathcal{V} = \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$ is a partition of $V(G)$;
(D2) for all $1 \leq i \leq k$, $G[V_i]$ is a (ρ, ν, τ)-robust expander component of G;
(D3) for all $1 \leq j \leq \ell$, there exists a partition A_j, B_j of W_j such that $G[W_j]$ is a bipartite (ρ, ν, τ)-robust expander component with bipartition A_j, B_j;
(D4) for all $X, X' \in \mathcal{V}$ and all $x \in X$, we have $d_X(x) \geq d_{X'}(x)$. In particular, $d_X(x) \geq D/m$, where $m := k + \ell$;
(D5) for all $1 \leq j \leq \ell$ we have $d_{B_j}(u) \geq d_{A_j}(u)$ for all $u \in A_j$ and $d_{A_j}(v) \geq d_{B_j}(v)$ for all $v \in B_j$; in particular, $\delta(G[A_j, B_j]) \geq D/2m$;
(D6) $k + 2\ell \leq \lfloor (1 + \rho^{1/3})n/D \rfloor$;
(D7) for all $X \in \mathcal{V}$, all but at most ρn vertices $x \in X$ satisfy $d_X(x) \geq D - \rho n$.

Note that (D7) implies that $|X| \geq D - \rho n$ for all $X \in \mathcal{V}$.

The following structural result (Theorem 3.1 in [10]) is our main tool. It states that any dense regular graph has a remarkably simple structure: a partition into a small number of (bipartite) robust expander components.

structure

Theorem 4.4. *For all $\alpha, \tau > 0$ and every non-decreasing function $f : (0, 1) \to (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that the following holds. For all D-regular graphs G on $n \geq n_0$ vertices where $D \geq \alpha n$, there exist ρ, ν with $1/n_0 \leq \rho \leq \nu \leq \tau$; $\rho \leq f(\nu)$ and $1/n_0 \leq f(\rho)$, and $k, \ell \in \mathbb{N}$ such that G has a robust partition $\mathcal{V}$ with parameters ρ, ν, τ, k, ℓ.*

Let $k, \ell \in \mathbb{N}_0$ and $0 < \rho \leq \nu \leq \tau \leq \eta < 1$. Given a graph G on n vertices, we say that $\mathcal{U}$ is a *weak robust partition of G with parameters $\rho, \nu, \tau, \eta, k, \ell$* if the following conditions hold.[17]

(D1′) $\mathcal{U} = \{U_1, \dots, U_k, Z_1, \dots, Z_\ell\}$ is a partition of $V(G)$;
(D2′) for all $1 \leq i \leq k$, $G[U_i]$ is a (ρ, ν, τ)-robust expander component of G;
(D3′) for all $1 \leq j \leq \ell$, there exists a partition A_j, B_j of Z_j such that $G[Z_j]$ is a bipartite (ρ, ν, τ)-robust expander component with bipartition A_j, B_j;
(D4′) $\delta(G[X]) \geq \eta n$ for all $X \in \mathcal{U}$;

[16] DK replaced 4.12 with 4.10

[17] no need to assume regularity. I have changed the definition to a *weak robust partition of G* instead of a *weak robust subpartition in G* as this is what we always have here.

(D5′) for all $1 \leq j \leq \ell$, we have $\delta(G[A_j, B_j]) \geq \eta n/2$.

Using Lemma 4.2 it is easy to check that whenever $\rho \leq \rho' \leq \nu$ and G is a D-regular graph on n vertices with $D \geq 5\sqrt{\rho'}n$, then any weak robust partition of G with parameters $\rho, \nu, \tau, \eta, k, \ell$ is also a weak robust partition with parameters $\rho', \nu, \tau, \eta, k, \ell$. A similar statement holds for robust partitions.[18]

A weak robust partition $\mathcal{U}$ is weaker than a robust partition in the sense that the graph is not necessarily regular, and we can make small adjustments to the partition while still maintaining (D1′)–(D5′) with slightly worse parameters. It is not hard to show the following (Proposition 5.1 in [10]).

WRSD-RD **Proposition 4.5.** *Let $k, \ell, D \in \mathbb{N}_0$ and suppose that $0 < 1/n \ll \rho \leq \nu \leq \tau \leq \eta \leq \alpha^2/2 < 1$. Suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$. Let $\mathcal{V}$ be a robust partition of G with parameters ρ, ν, τ, k, ℓ. Then $\mathcal{V}$ is a weak robust partition of G with parameters $\rho, \nu, \tau, \eta, k, \ell$.*

We also proved the following stability result (Theorem 6.11 in [10]). This implies that any sufficiently large 3-connected regular graph G on n vertices with degree at least a little larger than $n/5$ is either Hamiltonian, or has one of three very specific structures.

stability **Theorem 4.6.** *For every $\varepsilon, \tau > 0$ with $2\tau^{1/3} \leq \varepsilon$ and every non-decreasing function $g : (0, 1) \to (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that the following holds. For all 3-connected D-regular graphs G on $n \geq n_0$ vertices where $D \geq (1/5 + \varepsilon)n$, at least one of the following holds:*

- (i) *G has a Hamilton cycle;*
- (ii) *$D < (1/4 + \varepsilon)n$ and there exist ρ, ν with $1/n_0 \leq \rho \leq \nu \leq \tau$; $1/n_0 \leq g(\rho)$; $\rho \leq g(\nu)$, and $(k, \ell) \in \{(4, 0), (2, 1), (0, 2)\}$ such that G has a robust partition $\mathcal{V}$ with parameters ρ, ν, τ, k, ℓ.*

pathsystems

4.3. **Path systems and $\mathcal{V}$-tours.** Here we state some useful tools concerning path systems that we will need in our proof. All of these were proved in [10].

A simple double-counting argument gives the following proposition (Proposition 6.4 in [10]). We use it to guarantee the existence of edges in certain parts within a regular graph.

fact2 **Proposition 4.7.** *Let G be a D-regular graph with vertex partition A, B, U. Then*

- (i) $2(e(A) - e(B)) + e(A, U) - e(B, U) = (|A| - |B|)D$.

In particular,

- (ii) $2e(A) + e(A, U) \geq (|A| - |B|)D$.

Suppose that G is a graph containing a path system $\mathcal{P}$, and that $\mathcal{V}$ is a partition of $V(G)$. We define the *reduced multigraph $R_{\mathcal{V}}(\mathcal{P})$ of $\mathcal{P}$ with respect to $\mathcal{V}$* to be the multigraph with vertex set $\mathcal{V}$ in which we add a distinct edge between $X, X' \in \mathcal{V}$ for every path in $\mathcal{P}$ with one endpoint in X and one endpoint in X'. So $R_{\mathcal{V}}(\mathcal{P})$ might contain loops and multiple edges.

Given a graph G containing a path system $\mathcal{P}$, and $A \subseteq V(G)$, we write

F (4.1) $$F_{\mathcal{P}}(A) := (a_1, a_2)$$

when a_i is the number of vertices in A of degree i in $\mathcal{P}$ for $i = 1, 2$.[19] Note that, if $e_{\mathcal{P}}(A) = 0$, then[20]

edgecount (4.2) $$e_{\mathcal{P}}(A, \overline{A}) = a_1 + 2a_2.$$

The following lemma (Lemma 6.3 in [10]) is used in the case $(k, \ell) = (4, 0)$. An extension (Proposition 7.15) is used in the case $(k, \ell) = (2, 1)$.

[18] DK: new sentences, need $D \geq 5\sqrt{\rho'}n$ instead of $D \geq 2\sqrt{\rho'}n$ to check (C1)

[19] 'F' for forbidden. This is a measure of how hard it is to extend $\mathcal{P}$ using edges in A (so it's still an Euler tour, etc.

[20] So if this quantity is big, it is hard to add edges in A. But we need fewer such edges because the contribution from $\mathcal{P}$ itself is greater.

cliquetour **Lemma 4.8.** *Let G be a 3-connected graph and let $\mathcal{V}$ be a partition of $V(G)$ into at most three parts, where $|V| \geq 3$ for each $V \in \mathcal{V}$. Then G contains a path system $\mathcal{P}$ such that*

(i) *$e(\mathcal{P}) \leq 4$ and $\mathcal{P} \subseteq \bigcup_{V \in \mathcal{V}} G[V, \overline{V}]$;*
(ii) *$R_{\mathcal{V}}(\mathcal{P})$ has an Euler tour;*
(iii) *for each $V \in \mathcal{V}$, if $F_{\mathcal{P}}(V) = (c_1, c_2)$, then $c_1 + 2c_2 \in \{2, 4\}$ and $c_2 \leq 1$.*

Let $k, \ell \in \mathbb{N}_0$, let $0 < \rho \leq \nu \leq \tau \leq \eta < 1$ and let $0 < \gamma < 1$. Suppose that G is a graph on n vertices with a weak robust partition $\mathcal{V} = \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$ with parameters $\rho, \nu, \tau, \eta, k, \ell$, so that the bipartition of W_j specified by (D3′) is A_j, B_j. We say that a path system $\mathcal{P}$ is a *$\mathcal{V}$-tour with parameter γ* if

- $R_{\mathcal{V}}(\mathcal{P})$ has an Euler tour;
- for all $X \in \mathcal{V}$ we have $|V(\mathcal{P}) \cap X| \leq \gamma n$;
- for all $1 \leq j \leq \ell$ we have $|A_j \setminus V(\mathcal{P})| = |B_j \setminus V(\mathcal{P})|$.[21] Moreover, A_j, B_j contain the same number of endpoints of $\mathcal{P}$ and this number is positive.

We will often think of $R_{\mathcal{V}}(\mathcal{P})$ as a walk rather than a multigraph.[22] So in particular, we will often say that '$R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour'.

We will use the following lemma (a special case of Lemma 6.8 in [10])[23] to extend a path system into one that satisfies the third property above for all A, B forming a bipartite robust expander component.

balextend **Lemma 4.9.** *Let $n, k, \ell \in \mathbb{N}_0$ and $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \eta < 1$.*[24] *Let G be a graph on n vertices and suppose that $\mathcal{V} := \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$ is a weak robust partition of G with parameters $\rho, \nu, \tau, \eta, k, \ell$. For each $1 \leq j \leq \ell$, let A_j, B_j be the bipartition of W_j specified by* (D3′). *Let $\mathcal{P}$ be a path system such that for each $1 \leq j \leq \ell$,*

bal2 (4.3) $$2e_{\mathcal{P}}(A_j) - 2e_{\mathcal{P}}(B_j) + e_{\mathcal{P}}(A_j, \overline{W_j}) - e_{\mathcal{P}}(B_j, \overline{W_j}) = 2(|A_j| - |B_j|).$$

Suppose further that $|V(\mathcal{P}) \cap X| \leq \rho n$ for all $X \in \mathcal{V}$, and that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour. Then G contains a path system $\mathcal{P}'$ that is a $\mathcal{V}$-tour with parameter 9ρ.

The last result of this section (a special case of Lemma 5.2 in [10])[25] says that, in order to find a Hamilton cycle, it is sufficient to find a $\mathcal{V}$-tour.

HES **Lemma 4.10.** *Let $k, \ell, n \in \mathbb{N}_0$ and suppose that $0 < 1/n \ll \rho, \gamma \ll \nu \ll \tau \ll \eta < 1$. Suppose that G is a graph on n vertices and that $\mathcal{V}$ is a weak robust partition of G with parameters $\rho, \nu, \tau, \eta, k, \ell$. Suppose further that G contains a $\mathcal{V}$-tour $\mathcal{P}$ with parameter γ. Then G contains a Hamilton cycle.*

5. (4,0): Four robust expander components

sec40

The aim of this section is to prove the following lemma.

(4,0) **Lemma 5.1.** *Let $D, n \in \mathbb{N}$ and $0 < 1/n \ll \rho \ll \nu \ll \tau \ll 1$. Suppose that G is a 3-connected D-regular graph on n vertices with $D \geq n/4$. Suppose further that G has a robust partition $\mathcal{V}$ with parameters $\rho, \nu, \tau, 4, 0$. Then G contains a $\mathcal{V}$-tour with parameter $33/n$.*

We will find a $\mathcal{V}$-tour $\mathcal{P}$ as follows. Let $\mathcal{V} := \{V_1, \dots, V_4\}$. Suppose that there are $1 \leq i < j \leq 4$ such that $G[V_i, V_j]$ contains a large matching M. We can use 3-connectivity with the tripartition $\mathcal{V}' := \mathcal{V} \cup \{V_i \cup V_j\} \setminus \{V_i, V_j\}$ to obtain a path system $\mathcal{P}'$ such that $R_{\mathcal{V}'}(\mathcal{P}')$ is a $\mathcal{V}'$-tour. Then $\mathcal{P}'$ together with some suitable edges of M will form a $\mathcal{V}$-tour.

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[21] DK: previously had "A_j, B_j contain the same number of vertices of $\mathcal{P}$."

[22] I don't think there's any reason to define (A, B)-balanced (and certainly not $\mathrm{End}_{\mathcal{P}}(U), \mathrm{Int}_{\mathcal{P}}(U)$) since they are only needed to state the definition of a $\mathcal{V}$-tour.

[23] there we had a weak robust subpartition

[24] ν and τ are superfluous and only needed to define the WRSP

[25] there we had a weak robust subpartition

Suppose instead that for all $1 \le i < j \le 4$, every matching in $G[V_i, V_j]$ is small. In this case, we appeal to the result of Jackson, Li and Zhu [8] mentioned in the introduction: any longest cycle in G is dominating. Thus C visits all the V_i. Moreover, since there are very few edges between the V_i it follows that most of the edges of C lie within some V_i. If we remove all such edges, what remains is a $\mathcal{V}$-tour.

Let $\mathcal{V}'$ be a partition of $V(G)$ into three parts such that $\mathcal{V}$ is a refinement of $\mathcal{V}'$. Then, by Lemma 4.8, we can easily find a collection of paths $\mathcal{P}'$ such that $R_{\mathcal{V}'}(\mathcal{P}')$ is an Euler tour. The following result will enable us to 'extend' $\mathcal{P}'$ into $\mathcal{P}$ such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour.

plusmatching **Proposition 5.2.** *Let $\mathcal{U}$ be a partition of $V(G)$. Let $U, V \in \mathcal{U}$ and let $\mathcal{U}' := \mathcal{U} \cup \{U \cup V\} \setminus \{U, V\}$. Suppose that G contains a path system $\mathcal{P}'$ such that $R_{\mathcal{U}'}(\mathcal{P}')$ is an Euler tour. Suppose further that $G[U, V]$ contains a matching M of size at least $|V(\mathcal{P}') \cap (U \cup V)| + 2$. Then G contains a path system $\mathcal{P}$ with $E(\mathcal{P}) \supseteq E(\mathcal{P}')$ such that $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour and $|V(\mathcal{P}) \cap X| \le |V(\mathcal{P}') \cap X| + 2$ for all $X \in \mathcal{U}$.*

Proof. Note that there are at least two edges e, e' of M which are vertex-disjoint from $\mathcal{P}'$. Let $R' := R_{\mathcal{U}}(\mathcal{P}')$ and $R'' := R_{\mathcal{U}'}(\mathcal{P}')$. We have that $d_{R'}(U) + d_{R'}(V) = d_{R''}(U \cup V)$ is even since R'' is an Euler tour. Moreover, $d_{R'}(X) = d_{R''}(X)$ for all $X \in \mathcal{U}' \cap \mathcal{U}$.

If both $d_{R'}(U)$ and $d_{R'}(V)$ are odd, let $\mathcal{P} := \mathcal{P}' \cup \{e\}$.[26] Otherwise, both $d_{R'}(U)$ and $d_{R'}(V)$ are even (but one could be zero). In this case, let $\mathcal{P} := \mathcal{P}' \cup \{e, e'\}$.[27] It is straightforward to check that in both cases $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour. □

A subgraph H of a graph G is said to be *dominating* if $G \setminus V(H)$ is an independent set. In our proof of Lemma 5.1 we will use the following theorem of Jackson, Li and Zhu.

jlzdom **Theorem 5.3.** [8] *Let G be a 3-connected D-regular graph on n vertices with $D \ge n/4$. Then any longest cycle in C is dominating.*

Proof of Lemma 5.1*.* Let C be a longest cycle in G. Then Theorem 5.3 implies that C is dominating. We consider two cases according to the number of edges in C between classes of $\mathcal{V}$.

Case 1. *$e_C(U, V) \ge 12$ for some distinct $U, V \in \mathcal{V}$.*

Since C is a cycle we have that $\Delta(C[U, V]) \le 2$. König's theorem implies that $C[U, V]$ has a proper edge-colouring with at most two colours, and thus $C[U, V]$ contains a matching of size at least $e_C(U, V)/2 \ge 6$.

Let $\mathcal{V}' := \mathcal{V} \cup \{U \cup V\} \setminus \{U, V\}$. So $\mathcal{V}'$ is a tripartition of $V(G)$, and certainly $|V| \ge 3$ for each $V \in \mathcal{V}'$. Apply Lemma 4.8 to obtain a path system $\mathcal{P}'$ in G such that the consequences (i)–(iii) hold.[28] Then $R_{\mathcal{V}'}(\mathcal{P}')$ is an Euler tour and (iii) implies that $|V(\mathcal{P}') \cap X| \le 4$ for all $X \in \mathcal{V}'$.

Now Proposition 5.2 with $\mathcal{V}, \mathcal{V}'$ playing the roles of $\mathcal{U}, \mathcal{U}'$ implies that G contains a path system $\mathcal{P}$ such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, and $|V(\mathcal{P}) \cap X| \le 6$ for all $X \in \mathcal{V}$. So $\mathcal{P}$ is a $\mathcal{V}$-tour with $6/n$ playing the role of γ.

Case 2. *$e_C(U, V) \le 11$ for all distinct $U, V \in \mathcal{V}$.*

[26]Let $R := R_{\mathcal{U}}(\mathcal{P})$. Then $d_R(X) = d_{R'}(X)$ for all $X \in \mathcal{U}' \cap \mathcal{U}$. Moreover $d_R(U) = d_{R'}(U) + 1$, and similarly for V. Therefore every vertex in R has even positive degree. Furthermore, R'' is connected and so R is connected. Note that U might be isolated in R'. Therefore R is an Euler tour.

[27]Then $d_R(X) = d_{R'}(X)$ for all $X \in \mathcal{U}' \cap \mathcal{U}$. Moreover $d_R(U) = d_{R'}(U) + 2$, and similarly for V. The same reasoning as above implies that R is an Euler tour. We also have that $\mathrm{Int}_{\mathcal{P}}(X) = \mathrm{Int}_{\mathcal{P}'}(X)$ for all $X \in \mathcal{U}$, and $\Delta(R) \le \Delta(R) + 2$.

[28]NEW 'consequences of' everywhere...

Let $\mathcal{P}$ be the collection of disjoint paths with edge set $E(C)\setminus\bigcup_{V\in\mathcal{V}}E(C[V])$. For each $V\in\mathcal{V}$, let $\mathcal{P}_V:=\bigcup_{U\in\mathcal{V}\setminus\{V\}}\mathcal{P}[U,V]$. Then

eCV (5.1)
$$e\left(\mathcal{P}_V\right)=\sum_{U\in\mathcal{V}\setminus\{V\}}e_C(U,V)\leq 33.$$

Suppose that $|V(C)\cap V|<D-2\rho^{1/3}n$. Let $X:=V\setminus V(C)$. So X is an independent set in G. Moreover, (D7) implies that, for all but at most ρn vertices in $x\in V$, we have $d_V(x)\geq D-\rho n$. In particular, $|V|\geq D-\rho n$ and so $|X|\geq\rho^{1/3}n$. Thus there is some $x\in X$ such that $d_V(x)\geq D-\rho n$. Therefore x has a neighbour in X, a contradiction.

Thus $|V(C)\cap V|\geq D-2\rho^{1/3}n$ for all $V\in\mathcal{V}$. But
$$2|V(C)\cap V|=\sum_{v\in V}d_C(v)=2e_C(V)+e(\mathcal{P}_V)$$
and hence
$$e_C(V)=|V(C)\cap V|-\frac{1}{2}e(\mathcal{P}_V)\geq D-2\rho^{1/3}n-33/2>0.$$
Thus $E(C[V])\neq\emptyset$ for all $V\in\mathcal{V}$. It is straightforward to check that this implies that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour.[29] Finally, note that, for each $V\in\mathcal{V}$, (5.1) implies that we have $|V(\mathcal{P})\cap V|\leq e(\mathcal{P}_V)\leq 33$. So $\mathcal{P}$ is a $\mathcal{V}$-tour with parameter $33/n$. □

6. (0,2): Two bipartite robust expander components

sec02

The aim of this section is to prove the following lemma.

(0,2) **Lemma 6.1.** *Let $D,n\in\mathbb{N}$, let $0<1/n\ll\rho\ll\nu\ll\tau\ll\alpha<1$ and let $D\geq\alpha n$. Suppose that G is a 3-connected D-regular graph on n vertices and that $\mathcal{V}$ is a robust partition of G with parameters $\rho,\nu,\tau,0,2$. Then G contains a $\mathcal{V}$-tour with parameter $\rho^{1/3}$.*

We first give a brief outline of the argument.

6.1. **Sketch of the proof of Lemma 6.1.** Let $\mathcal{V}:=\{W_1,W_2\}$ be as above and let A_i,B_i be a bipartition of W_i such that $|A_i|\geq|B_i|$ and $G[W_i]$ is a bipartite robust expander component with bipartition A_i,B_i or B_i,A_i. (To be precise, $G[W_i]$ is a bipartite robust expander component with bipartition A_{W_i},B_{W_i}, where $\{A_{W_i},B_{W_i}\}=\{A_i,B_i\}$.)[30]

To prove Lemma 6.1, our aim is to find a 'balancing' path system $\mathcal{P}$ to which we can apply Lemma 4.9 and hence obtain a $\mathcal{V}$-tour. In other words, the path system has to 'compensate for' the differences in the sizes of the vertex classes A_i and B_i and has to 'join up' W_1 and W_2. (This also justifies why we do not specify whether $G[W_i]$ is a bipartite robust expander component with bipartition A_i,B_i or B_i,A_i.)[31]

One could try to first find a path system which balances W_1, and then add additional edges so that W_2 is also balanced; however these additional edges may cause W_1 to become unbalanced. So one must find a path system $\mathcal{P}$ which simultaneously balances both components.

This is not too difficult if both A_1 and A_2 contain sufficiently large matchings M_1 and M_2 (see Lemma 6.5). In this case, we use the 3-connectivity of G to modify $M_1\cup M_2$ to obtain $\mathcal{P}$.

So suppose that this is not the case. Then (see Lemmas 6.4 and 6.12) we show that we can choose $C_i\in\{A_i,B_i\}$ for each $i=1,2$ such that Vizing's and König's theorems on edge-colourings guarantee the following: $G[C_1],G[C_2],G[W_1,A_2]$ contain matchings $M_1,M_2,M_{1,2}$ respectively, such that

[29]Let $R:=R_{\mathcal{V}}(\mathcal{P})$. We have that $\mathcal{P}$ contains every edge of C which does not lie within some cluster of the partition. Now C meets every cluster of $\mathcal{V}$, so for any $V,V'\in\mathcal{V}$, there is a path $P\subseteq C$ between a vertex of V and a vertex of V'. Then $\mathcal{P}[V(P)]$ is a path system whose corresponding edges in R form a path between V and V'. Thus R is connected. Moreover each $V\in\mathcal{V}$ has even degree in R. Therefore R is an Euler tour.

[30]NEW29/5

[31]NEW29/5

the union $\mathcal{R}$ of these matchings balances both W_1 and W_2. However, two problems can arise: $\mathcal{R}$ may not connect W_1 and W_2 (it might contain no W_1W_2-path) and it might contain cycles.

Therefore the bulk of the proof of Lemma 6.1 is devoted to choosing M_1, M_2 and $M_{1,2}$ carefully to avoid these problems. Observe that since we use Vizing's and König's theorems to find matchings, we can actually find much larger matchings in $H \subseteq G$ when $\Delta(H)$ is small, and thus choosing a 'good' matching is easier in this case. So most of the difficulty in the proof arises from the presence of vertices of high degree.[32]

6.2. **Balanced subgraphs with respect to a partition.** Consider a graph G with vertex partition $\mathcal{V} := \{W_1, W_2\}$, where W_i has bipartition A_i, B_i for $i = 1, 2$. Write $\mathcal{V}^*$ for the ordered partition (A_1, B_1, A_2, B_2). Given $D \in \mathbb{N}$, we say that G is *D-balanced (with respect to $\mathcal{V}^*$)* if both of the following hold.

balancing

$$2e(A_1) - 2e(B_1) + e(A_1, W_2) - e(B_1, W_2) = D(|A_1| - |B_1|); \tag{6.1}$$
$$2e(A_2) - 2e(B_2) + e(A_2, W_1) - e(B_2, W_1) = D(|A_2| - |B_2|).$$

Proposition 4.7(i) easily implies that any D-regular graph with arbitrary ordered partition $\mathcal{V}^*$ is D-balanced.[33]

regbal

Proposition 6.2. *Suppose that G is a D-regular graph and let A_1, B_1, A_2, B_2 be a partition of $V(G)$. Then G is D-balanced with respect to (A_1, B_1, A_2, B_2).*

The next proposition shows that, to prove Lemma 6.1, it suffices to find a path system $\mathcal{P}$ which is 2-balanced with respect to $\mathcal{V}^*$, contains a W_1W_2-path, and does not have many edges.

sufficient

Proposition 6.3. *Let $n, D \in \mathbb{N}$ and $0 < 1/n \ll \rho \le \gamma \ll \nu \ll \tau \ll \alpha < 1$. Let G be a D-regular graph on n vertices with $D \ge \alpha n$. Suppose further that G has a robust partition $\mathcal{V} := \{W_1, W_2\}$ with parameters $\rho, \nu, \tau, 0, 2$. For each $i = 1, 2$, let A_i, B_i be the bipartition of W_i such that $|A_i| \ge |B_i|$ and $G[W_i]$ is a bipartite (ρ, ν, τ)-robust expander component with bipartition A_i, B_i or B_i, A_i.* [34] *Let $\mathcal{P}$ be a 2-balanced path system with respect to (A_1, B_1, A_2, B_2) in G. Suppose that $e(\mathcal{P}) \le \gamma n$ and that $\mathcal{P}$ contains at least one W_1W_2-path. Then G contains a $\mathcal{V}$-tour with parameter 18γ.*

Proof. Let p be the number of W_1W_2-paths in $\mathcal{P}$. Any W_1W_2-path in $\mathcal{P}$ contains an odd number of W_1W_2-edges. Since $\mathcal{P}$ is 2-balanced with respect to (A_1, B_1, A_2, B_2), we have that $e_{\mathcal{P}}(W_1, W_2) = e_{\mathcal{P}}(A_1, W_2) - e_{\mathcal{P}}(B_1, W_2) + 2e_{\mathcal{P}}(B_1, W_2)$ is even. Hence p is even. Since $p > 0$, we have that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour.

The hypothesis $e(\mathcal{P}) \le \gamma n$ implies that $|V(\mathcal{P}) \cap V| \le 2\gamma n$ for all $V \in \mathcal{V}$. Proposition 4.5 implies that $\mathcal{V}$ is a weak robust partition with parameters $2\gamma, \nu, \tau, \alpha^2/2, 0, 2$. Thus we can apply Lemma 4.9 with $\mathcal{V}, 0, 2, W_i, \{A_i, B_i\}, \mathcal{P}, 2\gamma$ playing the roles of $\mathcal{U}, k, \ell, W_j, \{A_j, B_j\}, \mathcal{P}, \rho$ to find a $\mathcal{V}$-tour $\mathcal{P}'$ with parameter 18γ.[35] □

The next lemma shows that we can find a D-balanced subgraph of G which only contains edges in some of the parts of G. (Recall the definition of $\lceil \cdot \rceil_\varepsilon$ from the end of Subsection 4.1.)

removeedges

Lemma 6.4. *Let $D \in \mathbb{N}$ be such that $D \ge 20$. Let G be a graph and let $\mathcal{V}^* := (A_1, B_1, A_2, B_2)$ be an ordered partition of $V(G)$ with $0 \le |A_i| - |B_i| \le D/2$ for $i = 1, 2$.*[36] *Suppose that $e_G(A_1, B_2) \le e_G(B_1, A_2)$ and $\Delta(G[A_i]) \le D/2$ for $i = 1, 2$.*[37] *Suppose further that G is D-balanced with respect to $\mathcal{V}^*$. Then one of the following holds:*

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[32]NEW 2nd paragraph onwards

[33]Proof of Prop 6.2: Let $W_i := A_i \cup B_i$. Apply Prop 4.7(i) with A_1, B_1, W_2 playing the roles of A, B, U to get the first D-balanced condition. Apply Prop 4.7(i) with A_2, B_2, W_1 playing the roles of A, B, U to get the second D-balanced condition.

[34]NEW29/5

[35]NEW29/5

[36]Deryk added this and $D \ge 20$, the latter ensures $\lceil e(A)/5 \rceil_{1/4} \ge \lceil 2e(A)/D \rceil$

[37]Added min deg condition for (ii).

(i) *for* $i = 1, 2$, $G[A_i]$ *contains a matching* M_i *of size* $|A_i| - |B_i| \leq \lceil e_G(A_i)/5 \rceil_{1/4}$*;*
(ii) *there exists a spanning subgraph* G' *of* G *which is* D*-balanced with respect to* $\mathcal{V}^*$ *and* $E(G') \subseteq E(G[C_1]) \cup E(G[C_2]) \cup E(G[A_1 \cup B_1, A_2])$*, where* $C_1 \in \{A_1, B_1\}$ *and* $C_2 \in \{A_2, B_2\}$*.*

Proof. Observe that the graph obtained by removing $E(G[A_i, B_i])$ from G for $i = 1, 2$ is D-balanced. So we may assume that $E(G[A_i, B_i]) = \emptyset$ for $i = 1, 2$. Consider each of the pairs

$$\{G[A_1], G[B_1]\}, \{G[A_2], G[B_2]\}, \{G[A_1, A_2], G[B_1, B_2]\}, \{G[A_1, B_2], G[B_1, A_2]\}$$

of induced subgraphs. For each such pair $\{J, J'\}$, remove $\min\{e_G(J), e_G(J')\}$ arbitrary edges from each of J, J' in G. Let H be the subgraph obtained from G in this way. Then H is D-balanced and for each pair $\{J, J'\}$, we have that $E(H[V(J)]) = \emptyset$ whenever $e_G(J) \leq e_G(J')$ (and vice versa). In particular, $e_H(A_1, B_2) = 0$. Suppose that we cannot take $G' := H$ so that (ii) holds. Then $H \subseteq G[C_1] \cup G[C_2] \cup G[B_1, A_2 \cup B_2]$ for some $C_1 \in \{A_1, B_1\}$ and $C_2 \in \{A_2, B_2\}$ with $e_H(B_1, B_2) \geq 1$. So $e_H(A_1, A_2) = 0$. Let $v_i := D(|A_i| - |B_i|) \geq 0$. Since H is D-balanced we have that $2e_H(A_1) - 2e_H(B_1) - e_H(B_1, A_2 \cup B_2) = v_1 \geq 0$. In particular, $e_H(A_1) \geq e_H(B_1)$. So $e_H(B_1) = 0$. Let $t := e_H(B_1, A_2)$. Thus

H*ineq1

$$2e_H(A_1) \geq v_1 + t + 1 \quad \text{and similarly} \tag{6.2}$$
$$2e_H(A_2) \geq v_2 - t + 1.$$

Suppose first that $t \geq v_2$. Then $2e_H(A_1) \geq v_1 + v_2 + 1$. Since G is D-balanced, summing the two equations in (6.1) implies that $v_1 + v_2$ is even. Let $H_{B_1A_2}$ consist of v_2 arbitrary edges in $H[B_1, A_2]$ and let H_{A_1} consist of $(v_1 + v_2)/2$ arbitrary edges in $H[A_1]$. In this case, we let $G' := H_{A_1} \cup H_{B_1A_2}$. So (ii) holds.

Suppose instead that $t < v_2$. First consider the case when $t = 0$.[38] Then (6.2) implies that $2e_G(A_i) \geq 2e_H(A_i) \geq v_i + 1$ for $i = 1, 2$. Since $\Delta(G[A_i]) \leq D/2$, Vizing's theorem implies that $G[A_i]$ contains a matching M_i of size[39]

$$\left\lceil \frac{e_G(A_i)}{D/2+1} \right\rceil \geq \left\lceil \frac{D(|A_i| - |B_i|)/2}{D/2+1} \right\rceil \geq |A_i| - |B_i| - \lfloor D/(D+2) \rfloor = |A_i| - |B_i|.$$

Note that the right hand side is at most $\lceil e(A_i)/5 \rceil_{1/4}$.[40] So (i) holds.

Therefore we may assume that $t > 0$. Recall that $v_1 \equiv v_2 \mod 2$. We will choose $H_{B_1A_2} \subseteq H[B_1, A_2]$ and $H_{A_i} \subseteq H[A_i]$ for $i = 1, 2$ by arbitrarily choosing edges according to the relative parities of v_1 and t, such that the following hold:

- if $v_1 + t$ is even then choose $e(H_{B_1A_2}) = t$, $2e(H_{A_1}) = v_1 + t$, $2e(H_{A_2}) = v_2 - t$;
- if $v_1 + t$ is odd then choose $e(H_{B_1A_2}) = t - 1$, $2e(H_{A_1}) = v_1 + t - 1$, $2e(H_{A_2}) = v_2 - t + 1$.

These choices are possible by (6.2). We let $G' := H_{A_1} \cup H_{A_2} \cup H_{B_1A_2}$. Observe that G' is D-balanced. So (ii) holds. □

Observe that the subgraph $M_1 \cup M_2$ of G guaranteed by Lemma 6.4(i) is a 2-balanced path system. The next lemma shows that, when G is 3-connected, one can modify such a path system into one which also contains paths between $A_1 \cup B_1$ and[41] $A_2 \cup B_2$.

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[38] Previously the proof was wrong because we neglected this case.

[39] Deryk changed this

[40] LATE CHANGE: This is clear if $e(A_i) \geq 10$, say. Note that $e(A_i) \geq D(|A_i| - |B_i|)/2$. So if $e(A_i) < 10$ we have $e(A_i) = 0$. In this case the assertion is also clear. (It is necessary to observe this since the assertion would not be true if, for example, $e(A_i) = 1$.)

[41] This lemma has been moved forward (before it was the last of the section). Then the reader can always assume that we are in case (ii) of Lemma 6.4 in the remainder of the section.

ensureconnected **Lemma 6.5.** *Let $n, D \in \mathbb{N}$ and $0 < 1/n \ll \gamma \ll 1$. Let G be a 3-connected D-regular graph on n vertices. Let W_1, W_2 be a partition of $V(G)$ and let A_i, B_i be a partition of W_i for $i = 1, 2$, where $|A_i| \geq |B_i|$. Suppose that there exist matchings M_1, M_2 in $G[A_1], G[A_2]$ respectively so that $|A_i| - |B_i| = e(M_i) \leq \lceil e(A_i)/5 \rceil_{1/4}$ and $e(M_i) \leq \gamma n$ for $i = 1, 2$. Then G contains a path system $\mathcal{P}$ which is 2-balanced with respect to (A_1, B_1, A_2, B_2) and contains a W_1W_2-path, and $e(\mathcal{P}) \leq 3\gamma n$.*

Proof. Proposition 6.2 implies that G is D-balanced with respect to (A_1, B_1, A_2, B_2). Suppose that there exist edges $e \in E(G[A_1, A_2])$ and $e' \in E(G[B_1, B_2])$. Then we can take $\mathcal{P} := M_1 \cup M_2 \cup \{e, e'\}$. We are similarly done if there exist edges $f \in E(G[A_1, B_2])$ and $f' \in E(G[B_1, A_2])$. If either of these two hold then we say that G contains a *balanced matching*. So we may assume that G does not contain a balanced matching. The 3-connectivity of G implies that there is a matching N of size at least three in $G[W_1, W_2]$. Since G does not contain a balanced matching, $e_N(C_1, C_2) \geq 2$ for some $C_i \in \{A_i, B_i\}$. So we can choose a matching N' of size two in $G[C_1, C_2]$. Let D_i be such that $\{C_i, D_i\} := \{A_i, B_i\}$. Note that $e_G(D_1, D_2) = 0$ or G would contain a balanced matching. Without loss of generality, we may assume that $e(M_1) \leq e(M_2)$.

Case 1. $e(M_2) > 0$.

Note that $1 \leq e(M_2) \leq e_G(A_2)/5 + 3/4$. Thus[42] $e_G(A_2) - e(M_2) \geq 4e_G(A_2)/5 - 3/4 > 0$. So we can always choose an edge $e_2 \in E(G[A_2]) \setminus E(M_2)$. If possible, let f_2 be the edge of M_2 spanned by $V(N') \cap A_2$. If there is no such edge, let f_2 be an arbitrary edge in M_2. Let

$$M_2' := \begin{cases} M_2 \setminus \{f_2\} & \text{if } C_2 = A_2 \\ M_2 \cup \{e_2\} & \text{if } C_2 = B_2. \end{cases}$$

Case 1.a. $e(M_1) > 0$.

Define e_1, f_1 and hence M_1' analogously to e_2, f_2, M_2'. It is straightforward to check that $\mathcal{P} := N' \cup M_1' \cup M_2'$ is as required in the lemma.

Case 1.b. $e(M_1) = 0$.

We have $|A_1| = |B_1|$. Without loss of generality we may suppose that $C_1 = A_1$ or we can swap A_1, B_1. So $e_G(A_1, W_2) \geq e_N(C_1, C_2) \geq 2$. Since G is D-balanced and $e_G(B_1, C_2) = e_G(B_1, W_2)$, this in turn implies that $2e_G(B_1) + e_G(B_1, C_2) \geq 2$. If $e_G(B_1) > 0$ let $e \in E(G[B_1])$ be arbitrary and define $\mathcal{P} := N' \cup M_2' \cup \{e\}$. Otherwise, there exists $e_{12} \in E(G[B_1, C_2])$. Let $e_{12}' \in E(N')$ be vertex-disjoint from e_{12}. If possible, let $f_2' \in E(M_2)$ be the edge spanning the endpoints of e_{12}, e_{12}' which lie in A_2; otherwise, let $f_2' \in E(M_2)$ be arbitrary. If $C_2 = A_2$, let $\mathcal{P} := M_2 \cup \{e_{12}, e_{12}'\} \setminus \{f_2'\}$. If $C_2 = B_2$, let $\mathcal{P} := M_2 \cup \{e_{12}, e_{12}'\}$. It is straightforward to check that in all cases $\mathcal{P}$ is as required in the lemma.[43]

Case 2. $e(M_2) = 0$.

So $e(M_1) = 0$ and $|A_i| = |B_i|$ for $i = 1, 2$. Without loss of generality, we may assume that $C_i := A_i$ (and hence $D_i := B_i$). Write $\{i, j\} = \{1, 2\}$. Since G is D-balanced we have that

$$2e_G(A_i) - 2e_G(B_i) + e_G(A_i, A_j) + e_G(A_i, B_j) - e_G(B_i, A_j) = 0.$$

So $2e_G(B_i) + e_G(B_i, A_j) \geq e_N(A_1, A_2) \geq 2$. Therefore either $e_G(B_i) > 0$ or $e_G(B_i, A_j) > 0$ (or both). So for $i = 1, 2$, either we can find $e_i \in E(G[B_i])$ or $e_{ij} \in E(G[B_i, A_j])$ (or both). Note that not both $e_G(B_1, A_2), e_G(A_1, B_2)$ can be positive since G does not contain a balanced matching.

Suppose that $e_G(B_1), e_G(B_2) > 0$. Let $\mathcal{P} := N' \cup \{e_1, e_2\}$ as required. If $e_G(B_1) = 0$ or $e_G(B_2) = 0$, then we may assume without loss of generality that $e_G(B_1) > 0$ and $e_G(B_2, A_1) > 0$. Let $e_{12}' \in N'$ be vertex-disjoint from e_{21}. Let $\mathcal{P} := \{e_1, e_{12}', e_{21}\}$. It is straightforward to check that in both cases $\mathcal{P}$ is as required in the lemma. □

[42] $\lceil a \rceil_\varepsilon \leq a + 1 - \varepsilon$

[43] Deryk changed the last 3 sentences

6.3. **Tools for finding matchings.** Given any bipartite graph G, König's theorem on edge-colourings guarantees that we can find a matching of size at least $\lceil e(G)/\Delta(G)\rceil$. The following lemma shows that, given any matching M in G, we can find a matching M' of at least this size such that $V(M) \subseteq V(M')$.

matchingextend

Lemma 6.6. *Let G be a bipartite graph with vertex classes V, W such that $\Delta(G) \leq \Delta$. Let M be a matching in G with $e(M) \leq \lceil e(G)/\Delta\rceil$. Then there exists a matching M' in G such that $e(M') = \lceil e(G)/\Delta\rceil$ and*[44] *$V(M) \subseteq V(M')$.*

Proof. Let M' be a matching in G such that $V(M) \subseteq V(M')$ and $e(M') \leq \lceil e(G)/\Delta\rceil$ is maximal with this property. Suppose that $e(M') < \lceil e(G)/\Delta\rceil$. Since, by König's theorem on edge-colourings, G contains a matching of size $\lceil e(G)/\Delta\rceil$, this means that M' is not a maximum matching. So, by Berge's lemma, G contains an augmenting path P for M', i.e. a path with endpoints not in $V(M')$ which alternates between edges in $E(M')$ and edges outside of $E(M')$. But then $P \setminus E(M')$ is a matching contradicting the maximality of $e(M')$. □

We now show that given a bipartite graph $G = (U, Z)$ and any partition V, W of Z, we can find a large matching in G which has the 'right' density in each of $G[U, V]$ and $G[U, W]$.

spreadmatching

Lemma 6.7. *Let G be a bipartite graph with vertex classes $U, V \cup W$, where V, W are disjoint. Suppose that $\Delta(G) \leq \Delta$. Let b_V, b_W be non-negative integers such that $b_V + b_W \leq \lceil e(G)/\Delta\rceil$, $b_V \leq \lceil e_G(U, V)/\Delta\rceil$ and $b_W \leq \lceil e_G(U, W)/\Delta\rceil$. Then G contains a matching M such that $e_M(U, V) = b_V$ and $e_M(U, W) = b_W$.*[45]

Proof. By increasing b_V, b_W if necessary, we may assume that $b_V + b_W = \lceil e(G)/\Delta\rceil$. Note that either $b_V = \lceil e_G(U, V)/\Delta\rceil$, or $b_W = \lceil e_G(U, W)/\Delta\rceil$, or both. Suppose without loss of generality that $b_V = \lceil e_G(U, V)/\Delta\rceil$. Choose a matching M' in G of size $\lceil e(G)/\Delta\rceil$. Let $m_V := e_{M'}(U, V)$ and let $m_W := e_{M'}(U, W)$. Let $k := b_V - m_V$. Then

$$m_W = \lceil e(G)/\Delta\rceil - m_V = b_V + b_W - m_V = b_W + k.$$

If $k = 0$ we are done, so suppose first that $k > 0$. Apply Lemma 6.6 to obtain a matching J_V in $G[U, V]$ such that $e(J_V) = b_V$ and $V(J_V) \supseteq V(M'[U, V])$. So $|(V(J_V) \setminus V(M'[U, V])) \cap U| = k$. Thus we can choose a submatching J_W of $M'[U, W]$ of size $m_W - k = b_W$ that is vertex-disjoint from J_V. Let $M := J_V \cup J_W$.

Otherwise, $k < 0$. Apply Lemma 6.6 to obtain a matching J_W in $G[U, W]$ such that $e(J_W) = b_W$ and $V(J_W) \supseteq V(M'[U, W])$. As above, we can choose a submatching J_V of $M'[U, V]$ of size b_V that is vertex-disjoint from J_W. Let $M := J_V \cup J_W$. □

6.4. **Acyclic unions of matchings.** The next lemma shows that, in a graph with low maximum degree, we can find a large matching that does not completely span a given set of vertices.[46]

sparsematching

Proposition 6.8. *Let $0 < 1/\Delta \ll \eta \ll 1$. Let G be a graph with $\Delta(G) \leq \eta\Delta$ and suppose that $e(G) \geq 2\eta\Delta$. Suppose that $K \subseteq V(G)$. Then there exists a matching M in G such that $e(M) = \lceil e(G)/\Delta\rceil$ and $M[K]$ is not a perfect matching.*

[44] DK: reformulated lemma and proof so that $e(M') = \lceil e(G)/\Delta\rceil$ instead of $e(M') \geq \lceil e(G)/\Delta\rceil$, since this is what we need later on

[45] Can't have ceilings for both. But that's okay. Fact: if have $b' + c'$ s.t. $b' + c' = \lceil b + c\rceil$ and $b' \leq \lceil b\rceil$, $c' \leq \lceil c\rceil$, then either $b' = \lceil b\rceil$ or $c' = \lceil c\rceil$, or both.

[46] Our aim is to find a 2-balanced path system $\mathcal{P}$ in G that consists of matchings in and between A_1, B_1, A_2, B_2. We require that $\mathcal{P}$ has a W_1W_2-path. Suppose that we have added a matching N between A_1 and W_2 to $\mathcal{P}$, and we now wish to add an additional matching M in $G[A_1]$. Then $M \cup N$ contains a W_1W_2-path unless $M[V(N) \cap A_1]$ is a perfect matching.

Proof. By Vizing's theorem, G contains a matching M' of size
$$\left\lceil \frac{e(G)}{\Delta(G)+1} \right\rceil \geq \left\lceil \frac{e(G)}{3\eta\Delta/2} \right\rceil \geq \left\lceil \frac{e(G)}{\Delta} \right\rceil + 1.$$
Delete edges so that M' has size $\lceil e(G)/\Delta \rceil + 1$. If M' contains an edge with both endpoints in K, remove this edge to obtain M. Otherwise, obtain M from M' by removing an arbitrary edge. □

Proposition 6.8 and the following observation will be used to guarantee that, given a matching M in $G[W_1, A_2]$, we can find a suitable matching N in $G[A_2]$ such that the path system $M \cup N$ contains a W_1A_2-path.

obvious **Fact 6.9.** *Let G be a graph with vertex partition U, V and let M be a non-empty matching between U and V. Let $K := V(M) \cap V$ and let M' be a matching in $G[V]$ such that $M'[K]$ is not a perfect matching. Then $M \cup M'$ is a path system containing a UV-path.*

Given a graph G with low maximum degree, vertex partition U, V and a non-empty matching M in $G[U, V]$, the next lemma shows that we can find matchings in $G[U], G[V]$ which extend M into a path system $\mathcal{P}$ containing a UV-path.

rsethreematchings **Lemma 6.10.** *Let $0 < 1/\Delta \ll \eta \ll 1$. Let G be a graph with partition U, V and suppose that $\Delta(G) \leq \eta\Delta$. Let M be a matching between U and V. Suppose further that $e_G(U) \leq e_G(V) \leq \eta\Delta^2$. Then there exist matchings M_U, M_V in $G[U], G[V]$ respectively such that*

- (i) *$\mathcal{P} := M \cup M_U \cup M_V$ is a path system;*
- (ii) *$e(M_U) \leq \lceil e_G(U)/\Delta \rceil$ with equality if $e_G(U) \geq \sqrt{\eta}\Delta$; and $e(M_V) \leq \lceil e_G(V)/\Delta \rceil$ with equality if $e_G(V) \geq \sqrt{\eta}\Delta$;*
- (iii) *if $M \neq \emptyset$, then $\mathcal{P}$ contains a UV-path.*

Proof. If $M = \emptyset$ then Vizing's theorem implies that we can find matchings M_U, M_V of size $\lceil e_G(U)/\Delta \rceil$, $\lceil e_G(V)/\Delta \rceil$ respectively. Then the consequences (i)–(iii) hold. So we may assume that $M \neq \emptyset$. If $e_G(U) \leq e_G(V) < \sqrt{\eta}\Delta$, then we are done by taking $M_U, M_V := \emptyset$. Suppose instead that $e_G(U) < \sqrt{\eta}\Delta \leq e_G(V)$. Apply Proposition 6.8 with $G[V], V(M) \cap V$ playing the roles of G, K to obtain a matching M_V in $G[V]$ such that $e(M_V) = \lceil e_G(V)/\Delta \rceil$ and $M_V[V(M) \cap V]$ is not a perfect matching. Fact 6.9 implies that we are done by taking $M_U = \emptyset$.

Therefore we may assume that $\sqrt{\eta}\Delta \leq e_G(U) \leq e_G(V)$. Apply Proposition 6.8 with $G[U], V(M) \cap U$ playing the roles of G, K to obtain a matching M_U in $G[U]$ of size $\lceil e_G(U)/\Delta \rceil$ such that $M_U[V(M) \cap U]$ is not a perfect matching. Let $\mathcal{P}_U$ be the path system with edge set $E(M) \cup E(M_U)$. So Fact 6.9 implies that $\mathcal{P}_U$ contains at least one UV-path P. Let $u_0 \in U$ and $v_0 \in V$ be the endpoints of P. Let Y be the set of all those vertices in V which are endpoints of a VV-path in $\mathcal{P}_U$. Now

|Y| (6.3) $$|Y| \leq 2e(M_U) = 2\lceil e_G(U)/\Delta \rceil \leq 2\lceil e_G(V)/\Delta \rceil.$$

Obtain G' from $G[V]$ by removing every edge incident with $Y \cup \{v_0\}$. So[47]
$$e(G') \geq e_G(V) - \eta\Delta(|Y|+1) \overset{(6.3)}{\geq} (1 - 4\sqrt{\eta})e_G(V) \geq e_G(V)/2.$$
So G' contains a matching of size
$$\lceil e(G')/(\eta\Delta+1) \rceil \geq \lceil e(G')/2\eta\Delta \rceil \geq \lceil e_G(V)/4\eta\Delta \rceil \geq \lceil e_G(V)/\Delta \rceil.$$
Let M_V be an arbitrary submatching of this matching of size $\lceil e_G(V)/\Delta \rceil$. Let $\mathcal{P} := M \cup M_U \cup M_V$.

Clearly (ii) holds. Observe that $\mathcal{P}$ has a UV-path, namely P. Hence (iii) holds. To show (i), it is enough to show that $\mathcal{P}$ is acyclic. Suppose not and let C be a cycle in $\mathcal{P}$. Now C contains at

[47]LATE CHANGE: $e(V) - \eta\Delta(|Y|+1) \geq e(V) - \eta\Delta(2\lceil e(V)/\Delta \rceil + 1) \geq e(V) - \eta\Delta(2e(V)/\Delta + 3) = e(V)(1 - 2\eta) - 3\eta\Delta \geq e(V)(1-2\eta) - 3\sqrt{\eta}e(V) \geq e(V)(1 - 4\sqrt{\eta})$.

least one edge $e \in E(M_V)$. Then both endpoints[48] of this edge belong to Y, and hence $e \notin E(G')$, a contradiction. □

The following is a version of Lemma 6.10 for sparse graphs which may have a small number of vertices with high degree.

threematchings

Lemma 6.11. *Let $0 < 1/\Delta \ll \rho \ll 1$. Let G be a graph with vertex partition U, V and suppose that $\Delta(G[U]), \Delta(G[V]) \leq \Delta$. Let M be a matching between U and V such that $e(M) \leq \rho\Delta$. Suppose further that $e_G(U), e_G(V) \leq \rho\Delta^2$. Then, for any integers $0 \leq a_U \leq \lceil e_G(U)/\Delta \rceil_{1/4}$ and $0 \leq a_V \leq \lceil e_G(V)/\Delta \rceil_{1/4}$, G contains a path system $\mathcal{P}$ such that*

(i) *$\mathcal{P}[U, V] = M$ and both of $\mathcal{P}[U]$,$\mathcal{P}[V]$ are matchings;*
(ii) *$e_{\mathcal{P}}(U) = a_U$, $e_{\mathcal{P}}(V) = a_V$;*
(iii) *if $M \neq \emptyset$, then $\mathcal{P}$ contains a UV-path.*

Proof. By removing edges in $G[U]$ and $G[V]$ we may assume without loss of generality that $a_U = \lceil e_G(U)/\Delta \rceil_{1/4}$ and $a_V = \lceil e_G(V)/\Delta \rceil_{1/4}$. Choose η with $\rho \ll \eta \ll 1$. Let $U' := \{u \in U : d_U(u) \geq \eta\Delta\}$ and define V' analogously. Then $2e_G(U) \geq \sum_{u \in U'} d_U(u) \geq |U'|\eta\Delta$ and similarly for V', so

U'
$$|U'|, |V'| \leq \sqrt{\rho}\Delta. \tag{6.4}$$

Let $U_0 := U \setminus U'$ and $V_0 := V \setminus V'$. Let H be the graph with vertex set $V(G)$ and edge set $E(G[U_0]) \cup E(G[V_0]) \cup M$. So $E_H(U) = E_G(U_0)$ and $E_H(V) = E_G(V_0)$.[49] Moreover, $\Delta(H) \leq 2\eta\Delta$. Note that

eG0
$$e_G(U_0) \geq e_G(U) - \Delta|U'| \quad \text{and} \quad e_G(V_0) \geq e_G(V) - \Delta|V'|. \tag{6.5}$$

Assume without loss of generality that $e_G(U_0) \leq e_G(V_0)$. Apply Lemma 6.10 with $H, M, U, V, 2\eta$ playing the roles of G, M, U, V, η to obtain matchings M_{U_0}, M_{V_0} in $H[U_0] = G[U_0], H[V_0] = G[V_0]$ respectively such that $\mathcal{P}_0 := M \cup M_{U_0} \cup M_{V_0}$ is a path system satisfying the consequences (i)–(iii) of Lemma 6.10. So $\mathcal{P}_0$ contains a UV-path if $M \neq \emptyset$. Moreover, $e(M_{U_0}) \leq \lceil e_G(U_0)/\Delta \rceil$ with equality if $e_G(U_0) \geq \sqrt{2\eta}\Delta$, and $e(M_{V_0}) \leq \lceil e_G(V_0)/\Delta \rceil$ with equality if $e_G(V_0) \geq \sqrt{2\eta}\Delta$. Thus

VP0
$$|V(\mathcal{P}_0)| \leq 2e(\mathcal{P}_0) \leq 2\left(e(M) + \lceil e_G(U)/\Delta \rceil + \lceil e_G(V)/\Delta \rceil\right) \leq \sqrt{\rho}\Delta. \tag{6.6}$$

For every $u \in U'$ and $v \in V'$ we have that

$$d_{U_0 \setminus V(\mathcal{P}_0)}(u), d_{V_0 \setminus V(\mathcal{P}_0)}(v) \overset{(6.6)}{\geq} \eta\Delta/2 \overset{(6.4)}{>} |U'|, |V'|.$$

[50] So for each $u \in U'$, we may choose a distinct neighbour $w_u \in U_0 \setminus V(\mathcal{P}_0)$ of u. Let $M_{U'} := \{uw_u : u \in U'\} \subseteq G[U', U_0 \setminus V(\mathcal{P}_0)]$. Define a matching $M_{V'}$ in $G[V', V_0 \setminus V(\mathcal{P}_0)]$ (which covers V') similarly.

Let $\mathcal{P} := \mathcal{P}_0 \cup M_{U'} \cup M_{V'}$. Note that $\mathcal{P}$ is a path system since $\mathcal{P}_0$ is. Certainly $\mathcal{P}[U, V] = \mathcal{P}_0[U, V] = M$, so (i) holds. Suppose that $e_G(U_0) \geq \sqrt{2\eta}\Delta$. Then

$$\begin{aligned} e_{\mathcal{P}}(U) = e(M_{U_0}) + e(M_{U'}) = \lceil e_G(U_0)/\Delta \rceil + |U'| &\overset{(6.5)}{\geq} \lceil e_G(U)/\Delta - |U'| \rceil + |U'| \\ = \lceil e_G(U)/\Delta \rceil \geq \lceil e_G(U)/\Delta \rceil_{1/4}. \end{aligned}$$

Suppose instead that $e_G(U_0) < \sqrt{2\eta}\Delta$. Then

$$e_{\mathcal{P}}(U) \geq |U'| \overset{(6.5)}{\geq} \lceil e_G(U)/\Delta - \sqrt{2\eta} \rceil \geq \lceil e_G(U)/\Delta \rceil_{1/4}$$

[48]Allan, at least one rather than both? Deryk:'at least one' would make sense too, but I'd leave it as it is

[49]edges of M may be incident to vertices in $U' \cup V'$ so we apply the sparse lemma to this H rather than $G[U_0] \cup G[V_0]$

[50]recall that $U' \cap U_0 = \emptyset$. Deryk changed calculation slightly

since $\sqrt{2\eta} < 1/4$. Analogous statements are true for $e_{\mathcal{P}}(V)$. So by removing edges in $e_{\mathcal{P}}(U), e_{\mathcal{P}}(V)$ if necessary, we may assume that (ii) holds. Note that $\mathcal{P}$ has a UV-path if $\mathcal{P}_0$ does (there is a one-to-one correspondence between the UV-paths in $\mathcal{P}$ and the UV-paths in $\mathcal{P}_0$).[51] □

6.5. **Rounding.** Given a small collection of reals which sum to an integer, the following lemma shows that we can suitably round these reals so that their sum is unchanged.[52] Lemmas 6.7 and 6.11 together enable us to find three matchings, one in each of $G[W_1], G[W_2]$ and $G[W_1, W_2]$, each of which is not too large, such that their union is a path system $\mathcal{P}$. Lemma 6.12 will allow us to choose the size of each matching correctly, so that $\mathcal{P}$ is 2-balanced.

rounding **Lemma 6.12.** *Let $0 < \varepsilon < 1/2$. Let $a_1, a_2, b, c \in \mathbb{R}$ with $b, c \geq 0$ and let $x_1, x_2 \in \mathbb{N}_0$. Suppose that*

$$2a_1 + b - c = 2x_1 \quad \text{and} \quad 2a_2 + b + c = 2x_2.$$

Then there exist integers a'_1, a'_2, b', c' such that

$$2a'_1 + b' - c' = 2x_1 \quad \text{and} \quad 2a'_2 + b' + c' = 2x_2,$$

where $0 \leq b' \leq \lceil b \rceil$, $0 \leq c' \leq \lceil c \rceil$, $b' + c' \leq \lceil b + c \rceil$; and for $i = 1, 2$, $|a'_i| \leq \lceil |a_i| \rceil_\varepsilon$; and finally $a'_i \geq 0$ if and only if $a_i \geq 0$.

Proof. Note that

rightsum
$$\lfloor 2a_1 \rfloor + \lceil b - c \rceil = 2x_1 \quad \text{and} \quad \lfloor 2a_2 \rfloor + \lceil b + c \rceil = 2x_2. \tag{6.7}$$

In particular, either $\lfloor 2a_1 \rfloor$, $\lceil b - c \rceil$ are both odd, or both even. The same is true for the pair $\lfloor 2a_2 \rfloor, \lceil b + c \rceil$. Let $A_i := \lfloor 2a_i \rfloor / 2$ for $i = 1, 2$. Let also

$$B := \frac{\lceil b + c \rceil + \lceil b - c \rceil}{2} \quad \text{and} \quad C := \frac{\lceil b + c \rceil - \lceil b - c \rceil}{2}.$$

Observe that $\{A_1, A_2, B, C\} \subseteq \mathbb{Z} \cup (\mathbb{Z} + 1/2)$. Let $i \in \{1, 2\}$. Suppose first that $a_i \geq 0$ (and so $A_i \geq 0$). If $a_i - \lfloor a_i \rfloor \leq \varepsilon$ then $2\lceil a_i \rceil_\varepsilon = 2\lfloor a_i \rfloor = \lfloor 2a_i \rfloor = 2A_i$. If $a_i - \lfloor a_i \rfloor > \varepsilon$ then $2\lceil a_i \rceil_\varepsilon = 2\lceil a_i \rceil \geq \lfloor 2a_i \rfloor = 2A_i$. Therefore $\lceil A_i \rceil \leq \lceil a_i \rceil_\varepsilon$. Suppose now that $a_i < 0$ (and so $A_i < 0$). If $a_i - \lfloor a_i \rfloor < 1 - \varepsilon$ then $2\lfloor a_i + \varepsilon \rfloor = 2\lfloor a_i \rfloor \leq \lfloor 2a_i \rfloor = 2A_i$. If $a_i - \lfloor a_i \rfloor \geq 1 - \varepsilon$ then $2\lfloor a_i + \varepsilon \rfloor = 2\lfloor a_i \rfloor + 2 = \lfloor 2a_i \rfloor + 1 = 2A_i + 1$ since $1 - \varepsilon \geq 1/2$. Since $-\lceil -a_i \rceil_\varepsilon = \lfloor a_i + \varepsilon \rfloor$, this shows that $-\lceil -a_i \rceil_\varepsilon \leq \lceil A_i \rceil$. Altogether this implies that

Ai
$$\begin{aligned} |A_i| &\leq \lceil |a_i| \rceil_\varepsilon \quad \text{when } A_i \in \mathbb{Z}, \quad \text{and} \\ |A_i + 1/2| &\leq \lceil |a_i| \rceil_\varepsilon \quad \text{when } A_i \in \mathbb{Z} + 1/2. \end{aligned} \tag{6.8}$$

We also have that

B+C
$$B + C = \lceil b + c \rceil \quad \text{and} \quad B - C = \lceil b - c \rceil. \tag{6.9}$$

Note that

Bbound
$$\begin{aligned} \lceil 2b \rceil = \lceil b + c + b - c \rceil \leq 2B \leq \lceil b + c + (b - c) \rceil + 1 = \lceil 2b \rceil + 1 \leq 2\lceil b \rceil + 1; \\ \lceil 2c \rceil - 1 = \lceil b + c - (b - c) \rceil - 1 \leq 2C \leq \lceil b + c - (b - c) \rceil = \lceil 2c \rceil \leq 2\lceil c \rceil. \end{aligned} \tag{6.10}$$

[51]The UV-paths in $\mathcal{P}$ are not precisely the UV-paths in $\mathcal{P}_0$. E.g. if uPv is a UV-path in $\mathcal{P}_0$ and $u \in U'$ (a vertex of large degree), then $w_u uPv$ is a (sub)path in $\mathcal{P}$.

[52]it is very important that we are able to round a_i to at most $\lceil a_i \rceil_\varepsilon$ (rather than just an integer which is at most $\lceil a_i \rceil$). Then, if $e_G(A_i)$ is very small, are not required to find any path-system edges in $G[A_i]$ (and indeed we cannot necessarily find any such edges).

It is straightforward to check that these equations (together with the definition of C) imply the following:

ineqs (6.11)
$$\begin{aligned} 0 \le B &\le \lceil b \rceil \quad \text{when } B \in \mathbb{Z} \\ 0 \le B - 1/2 &\le \lceil b \rceil \quad \text{when } B \in \mathbb{Z} + 1/2 \\ 0 \le C &\le \lceil c \rceil \quad \text{when } C \in \mathbb{Z} \\ 0 \le C - 1/2 < C + 1/2 &\le \lceil c \rceil \quad \text{when } C \in \mathbb{Z} + 1/2. \end{aligned}$$

Finally, note that (6.7) and (6.9) together imply that

hope (6.12)
$$2A_1 + B - C = 2x_1 \quad \text{and} \quad 2A_2 + B + C = 2x_2.$$

We choose a_1', a_2', b', c' as follows:

	a_1'	a_2'	b'	c'	
(i)	A_1	A_2	B	C	if $\lceil b+c \rceil$, $\lceil b-c \rceil$ both even;
(ii)	$A_1 + 1/2$	A_2	$B - 1/2$	$C + 1/2$	if $\lceil b+c \rceil$ even, $\lceil b-c \rceil$ odd;
(iii)	A_1	$A_2 + 1/2$	$B - 1/2$	$C - 1/2$	if $\lceil b+c \rceil$ odd, $\lceil b-c \rceil$ even;
(iv)	$A_1 + 1/2$	$A_2 + 1/2$	$B - 1$	C	if $b > 0$ and $\lceil b+c \rceil$, $\lceil b-c \rceil$ both odd;
(v)	$A_1 - 1/2$	$A_2 + 1/2$	B	$C - 1$	if $b = 0$ and $\lceil b+c \rceil$, $\lceil b-c \rceil$ both odd.

By the definition of A_i we have for each $i = 1, 2$ that $a_i' \ge 0$ if and only if $a_i \ge 0$. Then $\{a_1', a_2', b', c'\} \subseteq \mathbb{Z}$ and (6.12) implies that
$$2a_1' + b' - c' = 2x_1 \quad \text{and} \quad 2a_2' + b' + c' = 2x_2.$$
Moreover, $b' + c' \le B + C = \lceil b+c \rceil$. We claim that $0 \le b' \le \lceil b \rceil$ and $0 \le c' \le \lceil c \rceil$ and $|a_i'| \le \lceil |a_i| \rceil_\varepsilon$ for $i = 1, 2$ respectively in all cases (i)–(v). To see this, suppose first that we are in case (iv). Since $b > 0$, (6.10) implies that $B \ge \lceil 2b \rceil / 2 > 0$, so, since $B \in \mathbb{Z}$, $B - 1 \ge 0$ in this case.

Suppose now that we are in case (v). Then $\lceil c \rceil, \lceil -c \rceil = -\lfloor c \rfloor$ are both odd. Therefore $\lceil c \rceil, \lfloor c \rfloor$ are both odd so $\lceil c \rceil = \lfloor c \rfloor = c$. So $c \in \mathbb{N}_0$ is odd, $B = 0$ and $C = c$. Thus $C - 1 \ge 0$. Moreover $c = 2A_1 - 2x_1$, so $2A_1$ is odd and positive, which implies that $A_1 - 1/2 \ge 0$. Then (6.8) implies that $|A_1 - 1/2| \le \lceil |a_i| \rceil_\varepsilon$.[53]

In all cases (i)–(v), these last deductions together with (6.8)–(6.11) complete the proof of the lemma. □

6.6. **Proof of Lemma 6.1.** Before we can prove Lemma 6.1, we need one more preliminary result which[54] guarantees a path system $\mathcal{P}$ that can balance out the vertex class sizes of the bipartite graphs induced by the W_i. If $e_{\mathcal{P}}(W_1, W_2) = 0$, then we will use 3-connectivity (via Lemma 6.5) to modify $\mathcal{P}$ into a balanced path system which also links up the W_i.

2balanced **Lemma 6.13.** *Let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$ and let G be a D-regular graph on n vertices with $D \ge \alpha n$. Suppose that G has a robust partition $\mathcal{V} := \{W_1, W_2\}$ with parameters $\rho, \nu, \tau, 0, 2$. For each $i = 1, 2$, let A_i, B_i be the bipartition of W_i such that $|A_i| \ge |B_i|$ and $G[W_i]$ is a bipartite (ρ, ν, τ)-robust expander component with bipartition A_i, B_i or B_i, A_i.* [55] *Then*

- (i) *G contains a path system $\mathcal{P}$ which is 2-balanced with respect to (A_1, B_1, A_2, B_2) such that $e(\mathcal{P}) \le \sqrt{\rho} n$;*
- (ii) *if $e_{\mathcal{P}}(W_1, W_2) > 0$ then $\mathcal{P}$ contains a W_1W_2-path;*
- (iii) *for $i = 1, 2$, $\mathcal{P}[W_i]$ consists either of a matching in $G[A_i]$ of size at most $\lceil e_G(A_i)/5 \rceil_{1/4}$, or a matching in $G[B_i]$ of size at most $\lceil e_G(B_i)/5 \rceil_{1/4}$.*

[53]DK has changed the last 3 sentences slightly

[54]DK changed this para and replaced $\mathcal{P}[W_1, W_2] \ne \emptyset$ by $e_{\mathcal{P}}(W_1, W_2) > 0$ in the lemma below (and in the proof of Lemma 6.1

[55]NEW29/5

Proof. Write $\mathcal{V}^* := (A_1, B_1, A_2, B_2)$. Let $\Delta := D/2$ and note that

$$\Delta(G[A_i]), \Delta(G[B_i]), \Delta(G[W_1, W_2]) \leq \Delta$$

for $i = 1, 2$ by (D4) and (D5). Without loss of generality, we may suppose that $e_G(A_1, B_2) \leq e_G(B_1, A_2)$. Note that G is D-balanced with respect to $\mathcal{V}^*$ by Proposition 6.2. Apply Lemma 6.4 to G. Suppose that the consequence (i) of Lemma 6.4 holds. Then $G[A_i]$ contains a matching M_i of size $|A_i| - |B_i| \leq \lceil e_G(A_i)/5 \rceil_{1/4}$ for $i = 1, 2$.[56] Set $\mathcal{P} := M_1 \cup M_2$. So (iii) holds, (D3) and (C2) imply that (i) holds, and (ii) is vacuous.[57]

So we may assume that the consequence (ii) of Lemma 6.4 holds. Let H be a spanning subgraph of G which is D-balanced with respect to $\mathcal{V}^*$ such that $E(H) \subseteq E(G[C_1]) \cup E(G[C_2]) \cup E(G[W_1, A_2])$ for some $C_1 \in \{A_1, B_1\}$ and $C_2 \in \{A_2, B_2\}$. Observe that

eH (6.13)
$$e(H) \leq \sum_{i=1,2} \left(e_G(A_i, \overline{B_i}) + e_G(B_i, \overline{A_i}) \right) \stackrel{\text{(D3),(C3)}}{\leq} 2\rho n^2.$$

For each $H' \subseteq H$ and $i = 1, 2$, define

fdef (6.14)
$$f_i(H') = e_{H'}(A_i) - e_{H'}(B_i).$$

Now (6.1) implies that, for any $t \in \mathbb{N}_0$, H' is t-balanced if

fbal (6.15)
$$2f_i(H') + e_{H'}(A_i, W_j) - e_{H'}(B_i, W_j) = t(|A_i| - |B_i|)$$

for $\{i, j\} = \{1, 2\}$. Observe that $e_H(C_i) = e_H(W_i) = |f_i(H)|$. For $i = 1, 2$, let

ai (6.16)
$$a_i := f_i(H)/\Delta.$$

Then the D-balancedness of H and (6.15) imply that

$$2a_1 + \frac{e_H(A_1, A_2)}{\Delta} - \frac{e_H(B_1, A_2)}{\Delta} = 2(|A_1| - |B_1|)$$
$$\text{and} \quad 2a_2 + \frac{e_H(A_1, A_2)}{\Delta} + \frac{e_H(B_1, A_2)}{\Delta} = 2(|A_2| - |B_2|).$$

Apply Lemma 6.12 with $a_1, a_2, e_H(A_1, A_2)/\Delta, e_H(B_1, A_2)/\Delta, |A_1| - |B_1|, |A_2| - |B_2|, 1/4$ playing the roles of $a_1, a_2, b, c, x_1, x_2, \varepsilon$ to obtain integers a_1', a_2', b', c' with[58]

ai' (6.17)
$$|a_i'| \leq \lceil |a_i| \rceil_{1/4} = \lceil e_H(C_i)/\Delta \rceil_{1/4} \quad \text{for } i = 1, 2;$$

aiai' (6.18)
$$a_i' \geq 0 \quad \text{if and only if} \quad a_i \geq 0;$$

$0 \leq b' \leq \lceil e_H(A_1, A_2)/\Delta \rceil$; $0 \leq c' \leq \lceil e_H(B_1, A_2)/\Delta \rceil$ and

b'+c' (6.19)
$$b' + c' \leq \lceil e_H(W_1, A_2)/\Delta \rceil;$$

rounded1 (6.20)
$$2a_1' + b' - c' = 2(|A_1| - |B_1|) \quad \text{and} \quad 2a_2' + b' + c' = 2(|A_2| - |B_2|).$$

Apply Lemma 6.7 with $H[W_2, W_1], W_2, A_1, B_1$ playing the roles of G, U, V, W to obtain a matching M in $H[W_2, W_1]$ such that

wheresM (6.21)
$$e_M(A_1, A_2) = e_M(A_1, W_2) = b', \quad e_M(B_1, A_2) = e_M(B_1, W_2) = c'$$
$$\text{and} \quad e_M(W_1, B_2) = 0.$$

[56]DK added $\leq \lceil e_G(A_i)/5 \rceil_{1/4}$

[57]LATE CHANGE: justification for (i).

[58]DK added (6.18) and referred to it later on

Then (6.13)[59] and (6.19) imply that $e(M) = b' + c' \le \lceil e(H)/\Delta \rceil \le \sqrt{\rho}\Delta$.[60] By (6.13) and (6.17), we can apply Lemma 6.11 to H with $\sqrt{\rho}, M, \Delta, W_1, W_2, |a'_1|, |a'_2|$ playing the roles of $\rho, M, \Delta, U, V, a_U, a_V$ to obtain a path system $\mathcal{P}$ such that

Medges (6.22) $$\mathcal{P}[W_1, W_2] = M;$$

eP'Y (6.23) $$e_{\mathcal{P}}(W_i) = e_{\mathcal{P}}(C_i) = |a'_i| \quad \text{for } i = 1, 2;$$

$\mathcal{P}[C_i]$ is a matching for $i = 1, 2$, and if $M \neq \emptyset$, then $\mathcal{P}$ contains a W_1W_2-path. So (ii) holds. (Note that (6.23) follows from the fact that $H[W_i] = H[C_i]$.) Moreover, (6.17) and (6.23) imply that the matching $\mathcal{P}[C_i]$ has size at most $\lceil e_H(C_i)/\Delta \rceil_{1/4} \le \lceil e_G(C_i)/\Delta \rceil_{1/4} \le \lceil e_G(C_i)/5 \rceil_{1/4}$. So (iii) holds. Equations (6.14), (6.16), (6.18) and (6.23) imply that

sumai (6.24) $$f_i(\mathcal{P}) = a'_i.$$

Furthermore, by (6.21) and (6.22) we have

$$e_{\mathcal{P}}(A_1, W_2) - e_{\mathcal{P}}(B_1, W_2) = b' - c' \quad \text{and} \quad e_{\mathcal{P}}(W_1, A_2) - e_{\mathcal{P}}(W_1, B_2) = b' + c'.$$

Together with (6.15), (6.20) and (6.24), this implies that $\mathcal{P}$ is 2-balanced with respect to $\mathcal{V}^*$. Finally,

$$e(\mathcal{P}) = |a'_1| + |a'_2| + b' + c' \overset{(6.17),(6.19)}{\le} e(H)/\Delta + 3 \overset{(6.13)}{\le} \sqrt{\rho}n,$$

as required. □

Proof of Lemma 6.1. Let $\mathcal{V} := \{W_1, W_2\}$ and for $i = 1, 2$, let A_i, B_i be the partition of W_i guaranteed by (D3). If necessary, relabel A_i, B_i so that $|A_i| \ge |B_i|$.[61] Apply Lemma 6.13 to obtain a path system $\mathcal{P}$ which is 2-balanced with respect to (A_1, B_1, A_2, B_2) such that $e(\mathcal{P}) \le \sqrt{\rho}n$.

Suppose first that $e_{\mathcal{P}}(W_1, W_2) > 0$. Then $\mathcal{P}$ contains a W_1W_2-path by the consequence (ii) of Lemma 6.13. So we are done by Proposition 6.3. Therefore we may assume that $e_{\mathcal{P}}(W_1, W_2) = 0$. The consequence (iii) of Lemma 6.13 implies that, for each $i = 1, 2$, at least one of $\mathcal{P}[A_i], \mathcal{P}[B_i]$ is empty, and the other is a matching of size at most $\lceil e_G(B_i)/5 \rceil_{1/4}, \lceil e_G(A_i)/5 \rceil_{1/4}$ respectively.[62] The 2-balancedness of $\mathcal{P}$ implies that $e_{\mathcal{P}}(A_i) - e_{\mathcal{P}}(B_i) = |A_i| - |B_i| \ge 0$. So $\mathcal{P} = M_1 \cup M_2$ for some matchings $M_i \subseteq G[A_i]$. Apply Lemma 6.5 to obtain a path system $\mathcal{P}'$ which is 2-balanced with respect to (A_1, B_1, A_2, B_2) and contains a W_1W_2-path, and $e(\mathcal{P}) \le 3\sqrt{\rho}n$. Again, we are done by Proposition 6.3. □

7. (2,1) : Two robust expander components and one bipartite robust expander component

sec21

The aim of this section is to prove the following lemma.

(2,1) **Lemma 7.1.** *Let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll 1$. Let G be a 3-connected D-regular graph on n vertices where $D \ge n/4$. Let $\mathcal{X}$ be a robust partition of G with parameters $\rho, \nu, \tau, 2, 1$. Then G contains a Hamilton cycle.*

This — the final case — is the longest and most difficult. This is perhaps unsurprising given that the extremal example in Figure 1(i) has precisely this structure. Moreover, the presence of a bipartite robust expander component means that the path system we find to join the robust components needs to be balanced with respect to the bipartite component – the regularity of G is essential to achieve this. On the other hand, since we have to join up three components, the 3-connectivity of G is essential too. The main challenge is to find a path system which satisfies both requirements simultaneously, i.e. one that is both balanced and joins up the three components.

[59] DK referred to (6.13) instead of (C3)

[60] $\rho n^2/\Delta = 2\rho n^2/D \le 2\rho n/\alpha \le 2\rho D/\alpha^2 = 4\rho\Delta/\alpha^2 \le \sqrt{\rho}\Delta/2$

[61] NEW 30/5

[62] LATE CHANGE: swapped second A_i and B_i.

We need to invoke the degree bound $D \geq n/4$ for this. We begin by giving a brief outline of the argument.

7.1. **Sketch of the proof of Lemma 7.1.** Let $\mathcal{X} := \{V_1', V_2', W'\}$,[63] where $G[V_i']$ is a robust expander component for $i = 1, 2$, and $G[W']$ is a bipartite robust expander component. Let A', B' be a bipartition of W such that $|A'| \geq |B'|$ and $G[W]$ is a bipartite robust expander component with bipartition A', B' or B', A'. (To be precise, $G[W']$ is a bipartite robust expander component with bipartition $A_{W'}, B_{W'}$, where $\{A_{W'}, B_{W'}\} = \{A', B'\}$.)[64] To prove Lemma 7.1, Lemmas 4.9 and 4.10 imply that it is sufficient to find an $\mathcal{X}$-tour $\mathcal{P}$ such that

(X1) $\mathcal{P}$ contains few edges;
(X2) $R_{\mathcal{X}}(\mathcal{P})$ is an Euler tour;
(X3) $2e_{\mathcal{P}}(A') - 2e_{\mathcal{P}}(B') + e_{\mathcal{P}}(A', U') - e_{\mathcal{P}}(B', U') = 2(|A'| - |B'|)$ holds, where $U' := V_1' \cup V_2'$.

Note that (X1)–(X3) are independent of whether $G[W']$ is a bipartite robust expander component with bipartition A', B' or B', A'.[65] Note that $\mathcal{P}$-edges in $G[A', U'] \cup G[A']$ count 'positively' towards the goal of (X3), edges in $G[B', U'] \cup G[B']$ count 'negatively', and all other edges are 'neutral'. Therefore a natural approach to construct an $\mathcal{X}$-tour is to find two matchings $M_{A',U'}$ in $G[A', U']$ and $M_{A'}$ in $G[A']$ such that $\mathcal{P}_{\rm match} := M_{A',U'} \cup M_{A'}$ is an $\mathcal{X}$-tour (note that $\mathcal{P}_{\rm match}$ is always a path system). Unfortunately, this may be impossible. However, we can hope that there exists an $\mathcal{X}$-tour $\mathcal{P}$ *most* of whose edges lie in the union of two such matchings. In other words, we aim to construct matchings $M_{A'}$ and $M_{A',U'}$ which come as close as possible to satisfying (X1)–(X3).

Note that with the above approach, the requirement (X3) translates to $|M_{A',U'}| + 2|M_{A'}| = 2(|A'| - |B'|)$. By Proposition 4.7, *any* partition $\{U^*, A^*, B^*\}$ of $V(G)$ satisfies

ABU*
$$2e_G(A^*) - 2e_G(B^*) + e_G(A^*, U^*) - e_G(B^*, U^*) = D(|A^*| - |B^*|). \tag{7.1}$$

To find $M_{A',U'}$ and $M_{A'}$ we will use Vizing's theorem on edge colourings, which guarantees a matching of size $e(H)/(\Delta(H)+1)$ in a graph H, and König's theorem, which guarantees a matching of size $e(H)/\Delta(H)$ in a bipartite graph H. Suppose first that

maxdeg
$$\Delta(G[A', U']), \Delta(G[A']) \leq D/2. \tag{MaxDeg}$$

This then implies that we can find $M_{A',U'}$ and $M_{A'}$ such that

$$|M_{A',U'}| + 2|M_{A'}| \geq \frac{e_G(A', U')}{D/2} + \frac{2e_G(A')}{D/2+1}.$$

which is nearly at least $2(|A'| - |B'|)$ by (7.1). So by removing edges of $M_{A',U'}$ and $M_{A'}$ if necessary, we can ensure that $|M_{A',U'}|$ and $|M_{A'}|$ are very close to the correct sizes. Unfortunately $M_{A',U'} \cup M_{A'}$ may not satisfy (X2). In this case, we will modify $M_{A',U'} \cup M_{A'}$ to obtain the desired $\mathcal{P}$.

The above illustrates that (MaxDeg) is an important constraint, which we would like to achieve and apply. However, our robust partition $\mathcal{X}$ does not necessarily satisfy (MaxDeg): by (D4), $\Delta(G[A', U'])$ could be as large as $2D/3$, for example when $a \in A'$ satisfies $d_{V_1'}(a), d_{V_2'}(a), d_{B'}(a) = D/3$. For this reason, we will adjust the partition $\mathcal{X}$ slightly by moving a small number of vertices to obtain a weak robust partition $\mathcal{V} := \{V_1, V_2, W := A \cup B\}$ (where each part corresponds to its primed counterpart, and $|A| \geq |B|$) such that $\mathcal{V}$ *does* satisfy (MaxDeg). By Lemmas 4.10 and 4.9 it is still sufficient to find $\mathcal{P}$ with the properties above, with $\mathcal{V}$ replacing $\mathcal{X}$.

We prove Lemma 7.1 separately in each of the following four cases:

(i) $|A| - |B| \geq 2$ and $e_G(A, \overline{W})$ is at least a little larger than $3D/2$ (Subsection 7.5);
(ii) $|A| - |B| \geq 2$ and $e_G(A, \overline{W})$ is at most a little larger than $3D/2$ (Subsection 7.6);
(iii) $|A| - |B| = 1$ (Subsection 7.7);

[63]NEW entire proof sketch Sec 7.1
[64]NEW29/5
[65]NEW29/5

(iv) $|A| = |B|$ (Subsection 7.8).

The reason for these distinctions will be discussed at the end of Subsection 7.4. The full strength of the minimum degree bound $D \geq n/4$ is only used in the last two cases.

7.1.1. *A remark on a different approach.* Suppose for instance that we have $\Delta(G[A', U']), \Delta(G[A']) \leq D/100$ instead of (MaxDeg). Then we could find much larger matchings $M_{A',U'}$ and $M_{A'}$, and would have more freedom when choosing suitable edges from them to add to $\mathcal{P}$. By (D7), there are very few vertices in each component with many neighbours in other components, so moving these vertices would not reduce our expansion parameters by much. So one could hope to proceed as follows:

If there exists $a \in A'$ with, say, at least $D/100$ neighbours in A', move a to B'. If, for $i \in \{1, 2\}$, there exists $a \in A'$ with at least $D/200$ neighbours in V_i', move a to V_i'. If there exists $v \in V_i'$ with at least $D/100$ neighbours in A', move v to B'. After every step, we still have a weak robust partition. Continue until we have a weak robust partition for which one of the following holds:

(a) $\Delta(G[A', U']), \Delta(G[A']) \leq D/100$ and $|A'| - |B'| \geq 2$;
(b) $|A'| - |B'| \in \{0, 1\}$ and (MaxDeg) holds.

The idea would be that one could then replace cases (i) and (ii) above by the easier case (a).

However, such a process runs into difficulties as illustrated by the following example. Suppose that we have started the process above with partition $A'_{\text{old}}, B'_{\text{old}}, U'_{\text{old}}$ and have arrived at a partition A', B', U' which satisfies the following properties: $G[A']$ contains a triangle $a_1a_2a_3 \in A'$ such that $d(a_i, A') = 2$, $d(a_i, B') = D/2 - 2$ and $d(a_i, U') = D/2$ for each $i \leq 3$. Moreover, $|A'| = |B'| + 2$, $e_G(A') = e_G(B') + D/4$, $e_G(A', U') = 3D/2$ and $e_G(B', U') = 0$. (Note that (7.1) holds and $\Delta(G[A', U']) = D/2$.) Thus we aim to move one or two vertices from A'. If we move a_1 to B', then $\delta(G[A' \setminus \{a_1\}, B' \cup \{a_1\}]) = 2$ and so we no longer have a weak robust partition. If we move a_1 to U', then $d_G(a_2, U' \cup \{a_1\}) = D/2 + 1$, and so (MaxDeg) fails. A similar argument holds if we move two vertices from A' to other classes. If $\Delta(G[B']) \leq D/2$, then we could try to avoid this issue by first moving two or three of the a_i to U', and then swapping A' and B' to obtain a partition satisfying (b). However, since we might have moved several vertices $a \in A'_{\text{old}}$ (which might for instance satisfy $d(a, A'_{\text{old}}) = D/5$ and $d(a, B'_{\text{old}}) = 4D/5$) to B' at some earlier steps, we do not have any control on $\Delta(G[B'])$. Thus it is not clear that we can swap A' and B'. Moreover, one can modify the example to violate (MaxDeg) by a larger number, which is $o(n)$ say.

notation3

7.2. **Notation.** Throughout[66] the remainder of the paper, whenever we say that a graph G has vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$, we assume that $V(G)$ has a partition into parts V_1, V_2, W, each of size at least $|V(G)|/100 \geq 100$, that A and B are disjoint and that $|A| \geq |B|$. Moreover, we will say that G has a weak robust partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ (for some given parameters) if $\mathcal{V}$ satisfies the above properties and is a weak robust partition of G such that $G[V_1], G[V_2]$ are two robust expander components and $G[W]$ is a bipartite robust expander component, and the bipartition of W as specified by (D3′) is A, B or B, A.[67] We will use a similar notation when $\mathcal{V}$ is a robust partition of G.

Given $0 < \varepsilon < 1$ and $\Delta > 0$, consider any graph G with vertex partition U, A, B such that $\Delta(G[A]), \Delta(G[A, U]) \leq \Delta$. We say that[68]

character

$$\text{char}_{\Delta,\varepsilon}(G) := (\ell, m) \tag{7.2}$$

when $\ell := \lceil e_G(A)/\Delta \rceil_\varepsilon$ and m is the largest even integer less than or equal to $\lceil e_G(A, U)/\Delta \rceil_\varepsilon$. (Recall the definition of $\lceil \cdot \rceil_\varepsilon$ from the end of Subsection 4.1.) Given any path system $\mathcal{P}$ in G, we

[66]DK: rewrote this para. Formally we would need to be more careful, for example, we need that Lemma 7.11 also holds if the partition classes are slightly smaller than $n/5$, so that we can apply it later on to the partition $\mathcal{V}'$ obtained from $\mathcal{V}$ by moving a few vertices (eg in the proof of Lemma 7.13). But I'd suggest to gloss over this...

[67]NEW29/5

[68]DK had $\text{char}_{\Delta,\varepsilon}(G) = \text{char}_\varepsilon(G) := (\ell, m)$ before, but I don't think we ever use $\text{char}_\varepsilon(G)$

write

bal (7.3) $$\mathrm{bal}_{AB}(\mathcal{P}) := e_{\mathcal{P}}(A) - e_{\mathcal{P}}(B) + (e_{\mathcal{P}}(A,U) - e_{\mathcal{P}}(B,U))/2.$$

[69] When $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ is a vertex partition of G, we take $U := V_1 \cup V_2$ in the definitions of $\mathrm{char}_{\Delta,\varepsilon}$ and bal_{AB}.

It may be helpful to motivate these two crucial pieces of notation. We think of 'char' as being short for 'character'. The character of G encodes what sort of $\mathcal{V}$-tour $\mathcal{P}$ we can hope to find. Typically, when G has character (ℓ, m), a $\mathcal{V}$-tour will closely resemble the union of a matching of size ℓ in $G[A]$, and a matching of size m in $G[A,U]$. (Recall that, in a $\mathcal{V}$-tour $\mathcal{P}$, we have that $e_{\mathcal{P}}(W,U)$ is even.) The character of G together with Vizing's and König's theorems guarantee that we can find such matchings. The notion 'bal' is a measure of the 'balancedness' of a path system $\mathcal{P}$. One of our aims will be to find $\mathcal{P}$ with $\mathrm{bal}_{AB}(\mathcal{P}) = |A| - |B|$ (see (P2) below).[70]

Given $0 < \varepsilon < 1$, $\Delta > 0$ and a graph G with partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ and $\mathrm{char}_{\Delta,\varepsilon}(G) = (\ell, m)$, we will find a path system satisfying the following properties:[71]

(P1) $e(\mathcal{P}) \leq \ell + m + 6$;[72]
(P2) $\mathrm{bal}_{AB}(\mathcal{P}) = |A| - |B|$;
(P3) $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour.

reduction

7.3. **Preliminaries and a reduction.** In this subsection we show that, in order to prove Lemma 7.1, it is sufficient to prove Lemma 7.3 below. We then state some tools which will be used in the next subsections to do so. The following observation provides us with a convenient check for a path system $\mathcal{P}$ to be such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour.[73]

eulertour **Fact 7.2.** *Let G be a graph with vertex partition $\mathcal{V}$ into three parts.*[74] *Then, for a path system $\mathcal{P}$ in G,* (P3) *is equivalent to the following. For each $X \in \mathcal{V}$, $e_{\mathcal{P}}(X, \overline{X})$ is even and there exists $X' \in \mathcal{V} \setminus \{X\}$ such that $\mathcal{P}$ contains an XX'-path.*

The remainder of Section 7 is devoted to the proof of the following lemma, which states that G contains a path system satisfying (P1)–(P3) (when the partition $\mathcal{V}$ and the parameters involved are suitably defined).

aim **Lemma 7.3.** *Let $n, D \in \mathbb{N}$ and $\ell, m \in \mathbb{N}_0$. Let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \varepsilon \ll 1$. Let G be a 3-connected D-regular graph on n vertices where $D \geq n/4$. Suppose that G has a weak robust partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ with parameters $\rho, \nu, \tau, 1/16, 2, 1$ such that $|V_1|, |V_2| \geq D/2$ and $|A| \geq |B|$.*[75] *Suppose further that $\Delta(G[A, V_1 \cup V_2]) \leq D/2$, $d_{V_i}(x_i) \geq d_{V_j}(x_i)$ for all $x_i \in V_i$ and all $\{i,j\} = \{1,2\}$, and $d_A(a) \leq d_B(a)$ for all $a \in A$. Let $\mathrm{char}_{D/2,\varepsilon}(G) = (\ell, m)$. Then G contains a path system $\mathcal{P}$ satisfying* (P1)–(P3).

The following proposition gives bounds on ℓ and m when $\mathrm{char}_{\Delta,\varepsilon}(G) = (\ell, m)$.

ell **Proposition 7.4.** *Let $n, D \in \mathbb{N}$ and $\ell, m \in \mathbb{N}_0$. Let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \varepsilon, \eta \ll 1$ and suppose $D \geq n/4$. Let G be a graph on n vertices with weak robust partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ with parameters $\rho, \nu, \tau, \eta, 2, 1$. Suppose further that $\Delta(G[A]), \Delta(G[A, V_1 \cup V_2]) \leq D/2$ and that $\mathrm{char}_{D/2,\varepsilon}(G) = (\ell, m)$. Then $\ell, m \leq 12\rho n$.*

---

[69]we need $\mathrm{bal}_{AB}(\mathcal{P}) = |A| - |B|$ in order to apply Lemma 4.9 and modify $\mathcal{P}$ into a $\mathcal{V}$-tour.

[70]NEW paragraph

[71]Note that $\mathcal{P}$ satisfying (P1)–(P3) is *not* a $\mathcal{V}$-tour since we need to apply Lemma 4.9 first.

[72]we do not need to explicitly parametrise the robust partition.

[73]DK had "into at most three parts" before, but if $|\mathcal{V}| = 1$ then the fact is not necessarily true

[74]We were claiming that a graph (cf. reduced graph) is connected if and only if each each vertex has non-zero degree. This is only true for graphs with at most three vertices.

[75]DK: added $|V_1|, |V_2| \geq D/2$. This of course follows since $G[V_i]$ is a robust component. But now we don't have to refer to Lemma 4.2 whenever we want to use this fact. Added the same condition in the Claim of the proof of Lemma 7.1 below.

Proof. (D3′)[76] implies that $G[W]$ is ρ-close to bipartite with bipartition A, B. So $e_G(A)+e_G(A, V_1 \cup V_2) \leq \rho n^2$. Thus $\ell = \lceil 2e_G(A)/D \rceil_\varepsilon \leq 3\rho n^2/D \leq 12\rho n$. An almost identical calculation gives the same bound for m. □

We now show that, to prove Lemma 7.1, it suffices to prove Lemma 7.3.

Proof of Lemma 7.1 *(assuming Lemma* 7.3*).* Choose ε with $\tau \ll \varepsilon \ll 1$.[77] Let $\mathcal{X} = \{U_1, U_2, W' := A' \cup B'\}$ be a robust partition of G with parameters $\rho, \nu, \tau, 2, 1$, where $G[U_1], G[U_2]$ are (ρ, ν, τ)-robust expander components and $G[W']$ is a bipartite (ρ, ν, τ)-robust expander component with bipartition A', B' as guaranteed by (D3). We will alter $\mathcal{X}$ slightly so that it is a weak robust partition and that additionally the degree conditions of Lemma 7.3 hold.

Claim. *There exists a weak robust partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ of G with parameters $\rho^{1/3}, \nu/2, 2\tau, 1/16, 2, 1$ such that $|V_1|, |V_2| \geq D/2$, $|A| \geq |B|$, $\Delta(G[A, V_1 \cup V_2]) \leq D/2$, $d_{V_i}(x_i) \geq d_{V_j}(x_i)$ for all $x_i \in V_i$ and $\{i, j\} = \{1, 2\}$, and $d_A(a) \leq d_B(a)$ for all $a \in A$.*

To prove the claim, for $i = 1, 2$, let X_i be the collection of vertices $x \in U_i$ with $d_{\overline{U_i}}(x) > \rho n$. Then (D7) implies that $|X_i| \leq \rho n$. Let $Y_i := U_i \setminus X_i$. Then each $y \in Y_i$ satisfies

dYi (7.4) $$d_{Y_i}(y) = d(y) - d_{\overline{U_i} \cup X_i}(y) \geq d(y) - \rho n - |X_i| \geq d(y) - 2\rho n.$$

Let A_0 be the collection of vertices $a \in A'$ such that $d_{\overline{B'}}(a) \geq \sqrt{\rho} n$. Let $A_1 := A' \setminus A_0$. Define B_0, B_1 analogously. By (D3), $G[W']$ is ρ-close to bipartite with bipartition A', B'.[78] Therefore (C3) holds, from which one can easily derive that $|A_0|, |B_0| \leq 2\sqrt{\rho} n$.[79] Similarly as in (7.4), for each $a \in A_1$ and $b \in B_1$ we have

dA1 (7.5) $$d_{B_1}(a) \geq d(a) - 3\sqrt{\rho} n \quad \text{and} \quad d_{A_1}(b) \geq d(b) - 3\sqrt{\rho} n.$$

Let $V_0 := X_1 \cup X_2 \cup A_0 \cup B_0$. Then

V0 (7.6) $$|V_0| \leq 5\sqrt{\rho} n.$$

Among all partitions X_1', X_2', A_0', B_0' of V_0, choose one such that $e(A \cup B, V_1 \cup V_2)$ is minimised; and subject to $e(A \cup B, V_1 \cup V_2)$ being minimal we have that $e(V_1, V_2) + e(A) + e(B)$ is minimal,[80] where $V_i := Y_i \cup X_i'$, $A := A_1 \cup A_0'$ and $B := B_1 \cup B_0'$. It is easy to see that $d_{A \cup B}(w) \geq d_{V_1 \cup V_2}(w)$ for all $w \in A_0' \cup B_0'$; $d_{V_1 \cup V_2}(v) \geq d_{A \cup B}(v)$ for all $v \in X_1' \cup X_2'$; $d_{V_i}(v_i) \geq d_{V_j}(v_i)$ for all $v_i \in X_i'$ and $\{i, j\} = \{1, 2\}$; $d_A(a) \leq d_B(a)$ for all $a \in A_0'$; and $d_B(b) \leq d_A(b)$ for all $b \in B_0'$. If $v_i \in Y_i$, then (7.4) implies that $d_{V_i}(v_i) \geq d_{Y_i}(v_i) \geq d(v_i) - 2\rho n \geq d(v_i)/2$. So $d_{V_i}(v_i) \geq d_{A \cup B}(v_i), d_{V_j}(v_i)$ for $\{i, j\} = \{1, 2\}$. Similarly, (7.5) implies that, for all $w \in A_1 \cup B_1$ we have $d_{A \cup B}(w) \geq d_{V_1 \cup V_2}(w)$; for all $a \in A_1$ we have $d_A(a) \leq d_B(a)$ and for all $b \in B_1$ we have $d_B(b) \leq d_A(b)$. Observe that (7.4), (7.5) imply that $|V_i| \geq D - 2\rho n$ and $|A|, |B| \geq D - 3\sqrt{\rho} n$ respectively.[81] Then the degree conditions required in the claim hold, and they also hold with B, A playing the roles of A, B respectively.[82] We now prove that $\mathcal{V} := \{V_1, V_2, W := A \cup B\}$ is a weak robust partition with parameters $\rho^{1/3}, \nu/2, 2\tau, 1/16, 2, 1$. Property (D1′) is clear. We now prove (D2′). Observe that[83]

$$e(V_i, \overline{V_i}) \leq e(U_i, \overline{U_i}) + D|X_i| + D|X_i'| \leq (\rho + 6\sqrt{\rho}) n^2 \leq \rho^{1/3} n^2.$$

Therefore each V_i is a $\rho^{1/3}$-robust component of G. Note also that

$$|V_i \triangle U_i| \leq |V_0| \overset{(7.6)}{\leq} 5\sqrt{\rho} n \leq \nu |U_i|/2.$$

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[76] DK: new sentence

[77] Need to choose ε here since it is not defined in the statement of Lemma 7.1.

[78] DK: had W, A, B before

[79] If say $|A_0| > 2\sqrt{\rho} n$, then $e_G(A, \overline{B}) \geq |A_0| \sqrt{\rho} n/2 > \rho n^2$, contradicting (C3).

[80] this has been simplified

[81] LATE CHANGE: swapped sentences round because we need $|V_i| \geq D/2$ in addition.

[82] NEW 30/5

[83] DK: added $+D|X_i|$ to count $e(Y_i, \overline{V_i} \cap X_i)$ in next inequality

Lemma 4.1 implies that $G[V_i]$ is a $(\nu/2, 2\tau)$-robust expander. Therefore $G[V_i]$ is a $(\rho^{1/3}, \nu/2, 2\tau)$-robust expander component for $i = 1, 2$, so (D2′) holds. To prove (D3′), note that $|A \triangle A'| + |B \triangle B'| \leq 2|V_0| \leq \rho^{1/3} n/3$ where the final inequality follows from (7.6). Now Lemma 4.3 implies that $G[A \cup B]$ is a bipartite $(\rho^{1/3}, \nu/2, 2\tau)$-robust expander component of G with bipartition A, B. Thus (D3′) holds. Finally, (D4′) and (D5′) are clear from the degree conditions we have already obtained. Finally, if necessary, relabel A and B so that $|A| \geq |B|$. Then, as previously remarked, the degree conditions of the claim hold.[84] This completes the proof of the claim.

Given[85] the partition $\mathcal{V}$ of $V(G)$, let ℓ, m satisfy $\mathrm{char}_{D/2,\varepsilon}(G) = (\ell, m)$. Let $\mathcal{P}$ be a path system in G guaranteed by Lemma 7.3, i.e. $\mathcal{P}$ satisfies (P1)–(P3). Note[86] that $\mathcal{V}$ is also a weak robust partition with parameters $\rho^{1/3}, \nu/2, 2\tau, \varepsilon, 2, 1$. So (P1) and Proposition 7.4 with $\rho^{1/3}, \varepsilon$ playing[87] the roles of ρ, η imply that $e(\mathcal{P}) \leq 25\rho^{1/3} n$. Then, for each $X \in \mathcal{V}$ we have that $|V(\mathcal{P}) \cap X| \leq |V(\mathcal{P})| \leq 2e(\mathcal{P}) \leq 50\rho^{1/3} n \leq \rho^{1/4} n/9$. So Lemma 4.9 applied with $2, 1, W, \{A, B\}, \mathcal{P}, \rho^{1/4}/9$ playing the roles of $k, \ell, W_j, \{A_j, B_j\}, \mathcal{P}, \rho$[88] implies that G contains a path system $\mathcal{P}'$ that is a $\mathcal{V}$-tour with parameter $\rho^{1/4}$. Now Lemma 4.10 with $\mathcal{P}', \rho^{1/3}, \rho^{1/4}, \nu/2, 2\tau, 1/16, 2, 1$ playing the roles of $\mathcal{P}, \rho, \gamma, \nu, \tau, \eta, k, \ell$ implies that G contains a Hamilton cycle. □

tools

7.4. **Tools.** In this section we gather some useful tools which will be used repeatedly in the sections to come. We will often use the following lower bounds for $e_G(A), e_G(A, U)$ implied by $\mathrm{char}_{\Delta,\varepsilon}(G)$.[89]

charedges

Proposition 7.5. *Let $\Delta, \Delta' \in \mathbb{N}$ and $\ell, m \in \mathbb{N}_0$. Let $\Delta'/\Delta \leq \varepsilon < 1$. Suppose that G is a graph with vertex partition U, A, B such that $\Delta(G[A]), \Delta(G[A, U]) \leq \Delta$ and $\mathrm{char}_{\Delta,\varepsilon}(G) = (\ell, m)$. Then $e_G(A) \geq (\ell - 1)\Delta + \Delta'$ and $e_G(A, U) \geq (m - 1)\Delta + \Delta'$.*

Proof. We have that $\ell = \lceil e_G(A)/\Delta \rceil_\varepsilon = \lceil e_G(A)/\Delta - \varepsilon \rceil$ so $\ell - 1 < e_G(A)/\Delta - \varepsilon \leq (e_G(A) - \Delta')/\Delta$, as required. A near identical calculation proves the second assertion. □

The path system we require will contain edges in $G[A]$ and $G[V_1 \cup V_2, A]$, and will 'roughly look like' a matching within each of these subgraphs. The following lemma allows us to find a structure which in turn contains a large matching even if certain vertices need to be avoided.[90]

goodmatching2

Lemma 7.6. *Let $\Delta, \Delta' \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$ be such that $\ell/\Delta', \Delta'/\Delta, 1/\Delta' \ll 1$. Let G be a graph with $\Delta(G) \leq \Delta$, and let $e(G) \geq (\ell - 1)\Delta + \Delta'$. Then G contains one of the following:*[91]

- (i) *a matching M of size $\ell + 1$ and $uv \in E(G)$ with $u \notin V(M)$;*
- (ii) *ℓ vertices each with degree at least Δ'.*

Moreover, if $\ell \geq 1$ and $e(G) \geq \ell\Delta + 1$; or $\ell = 0$ and $e(G) \geq 2$, then (i) *holds.*

Proof. We will use induction on ℓ in order to show that either (i) or (ii) holds. The cases $\ell = 0, 1$ are trivial. Suppose now that $\ell \geq 2$.[92] Suppose first that $\Delta(G) \leq \Delta'$. Then, by Vizing's theorem, $E(G)$ can be properly coloured with at most $\Delta' + 1$ colours.[93] Therefore G contains a matching

[84] NEW 30/5

[85] DK: previously para started with "Let $\Delta := D/2$. Now (D3′) implies that $G[A \cup B]$ is $\rho^{1/3}$-close to bipartite, and therefore $e_G(A) + e_G(A, V_1 \cup V_2) \leq \rho^{1/3} n^2$."

[86] DK: new sentence

[87] DK: had 1/16 instead of ε before

[88] NEW29/5

[89] DK deleted $\Delta'/\Delta \ll 1$ in the next prop since we don't need it

[90] DK: replaced $1/\Delta \ll 1/\Delta' \ll 1$ with $\Delta'/\Delta, 1/\Delta' \ll 1$ in the prop and made similar changes in the statements of the other lemmas (since both Δ and Δ' will later be linear in n, we don't have that $1/\Delta \ll 1/\Delta'$. We need $1/\Delta' \ll 1$ in Lemma 7.6 since this doesn't follow from $\ell/\Delta' \ll 1$ in the case when $\ell = 0$.

[91] LATE CHANGE: new wording.

[92] We cannot only suppose $\ell \geq 1$ for the next part of the argument to work.

[93] Now $(\ell - 1)\Delta - \Delta' > (\ell + 1)(\Delta' + 1)$. This wasn't in a comment before.

of size

$$\left\lceil \frac{e(G)}{\Delta'+1} \right\rceil \geq \left\lceil \frac{(\ell-1)\Delta+\Delta'}{\Delta'+1} \right\rceil \geq \ell+2.$$

So (i) holds. Thus we may assume that there exists $x \in V(G)$ with $d(x) \geq \Delta'$. Let $G^- := G \setminus \{x\}$. Then $e(G^-) \geq e(G) - \Delta \geq (\ell-2)\Delta + \Delta'$. By induction, $e(G^-)$ contains either a matching M^- of size ℓ and $uv \in E(G^-)$ with $u \notin V(M^-)$, or $\ell - 1$ vertices of degree at least Δ'. In the first case, choose $y \in N(x) \setminus V(M^-)$ with $y \neq u$ and let $M := M^- \cup \{xy\}$. Then (i) holds. In the second case, x is our ℓth vertex of degree at least Δ' in G, so (ii) holds.

For the moreover part, suppose now that $\ell \geq 1$ and $e(G) \geq \ell\Delta + 1$. Suppose that (i) does not hold. Let $x_1, \ldots, x_\ell$ be ℓ distinct vertices of degree at least Δ'. Then $e(G \setminus \{x_1, \ldots, x_\ell\}) \geq e(G) - \Delta\ell \geq 1$. So G contains an edge e which is not incident to $\{x_1, \ldots, x_\ell\}$. We obtain a contradiction by considering $\{e, x_1z_1\} \cup \{x_1y_1, \ldots, x_\ell y_\ell\}$, where $z_1 \in N(x_1)$ avoids e and for $1 \leq i \leq \ell$ the vertices $y_i \in N(x_i)$ are distinct, and avoid e, z_1 and $x_1, \ldots, x_\ell$.

Finally, if $\ell = 0$, then any two edges of G satisfy (i). □

Given an even matching M in $G[A, V_1 \cup V_2]$ and a lower bound on $e_G(A)$, we would like to extend M into a path system $\mathcal{P}$ using edges from $G[A]$ so that $\mathrm{bal}_{AB}(\mathcal{P})$ is large. Lemma 7.6 gives us two useful structures in $G[A]$ from which we can choose suitable edges to add to M to form $\mathcal{P}$. The following proposition does this in the case when the consequence (i) of Lemma 7.6 holds.

casei **Proposition 7.7.** *Let G be a graph with vertex partition X, Y. Suppose that $G[Y]$ contains a matching M' of size $\ell+1$ and an edge uv with $u \notin V(M')$. Let M be a non-empty even matching of size m in $G[X,Y]$. Then G contains a path system $\mathcal{P}$ such that*[94]

(i) *$\mathcal{P}[X,Y] = M$ and $\mathcal{P} \subseteq M \cup M' \cup \{uv\}$;*
(ii) *$e_{\mathcal{P}}(Y) = \ell + 1$;*
(iii) *$\mathcal{P}$ contains at least two XY-paths.*

Proof. We will extend M by adding edges from $M' \cup \{uv\}$, so (i) automatically holds. Note that any path system $\mathcal{P}$ obtained in this way contains an even number of XY-paths. So it suffices to find such a $\mathcal{P}$ with at least one XY-path. If $M \cup M'$ contains an XY-path, then we are done by setting $\mathcal{P} := M \cup M'$. So suppose not. Then $M'[V(M) \cap Y]$ is a perfect matching M''. If $v \in V(M'')$, let f be the edge of M'' containing v. Otherwise, let $f \in E(M'')$ be arbitrary. We take $\mathcal{P} := M \cup M' \cup \{uv\} \setminus \{f\}$. Now both of the two edges in M which are incident to f lie in distinct XY-paths of $\mathcal{P}$, so (iii) holds. Clearly (ii) holds too. □

Following on from the previous proposition, we now consider how to extend M into $\mathcal{P}$ when instead the consequence (ii) of Lemma 7.6 holds in $G[A]$.

2paths **Proposition 7.8.** *Let $\Delta' \in \mathbb{N}$ and let $\ell, m, r \in \mathbb{N}_0$ with $\Delta' \geq 3\ell + m$. Let G be a graph with vertex partition X, Y and let M be a matching in $G[X,Y]$ of size m. Let $\{x_1, \ldots, x_\ell\} \subseteq Y$ such that $d_Y(x_i) \geq \Delta'$ and $|\{x_1, \ldots, x_\ell\} \setminus V(M)| \geq r$. Then there exists a path system $\mathcal{P} \subseteq G[X,Y] \cup G[Y]$ such that $e_{\mathcal{P}}(Y) = \ell + r$, $\mathcal{P}[X,Y] = M$ and every edge of M lies in a distinct XY-path in $\mathcal{P}$.*

Proof. Since $\Delta' \geq 3\ell + m$, $G[Y]$ contains a collection of ℓ vertex-disjoint paths $P_1, \ldots, P_\ell$ of length two with midpoints $x_1, \ldots, x_\ell$ respectively, such that $V(P_i) \cap V(M) \subseteq \{x_i\}$. For each $x_i \in V(M)$, delete one arbitrary edge from P_i. Let $\mathcal{P}$ consist of M together with $P_1, \ldots, P_\ell$. Then $\mathcal{P}$ is a path system, and every edge of M lies in a distinct XY-path. Moreover, $e_{\mathcal{P}}(Y) \geq 2\ell - (\ell - r) = \ell + r$. Delete additional edges from $\mathcal{P}[Y]$ if necessary. □

rounding2 **Proposition 7.9.** *Let $0 < \varepsilon < 1/3$. Let $a, b \in \mathbb{R}_{\geq 0}$ and let $x \in \mathbb{N}_0$. Suppose that $2a + b \geq 2x$. Let $a' := \lceil a \rceil_\varepsilon$ and let b' be the largest even integer of size at most $\lceil b \rceil_\varepsilon$. Then $a', b' \geq 0$ and $2a' + b' \geq 2x$.*

[94]DK: added "at least" in (iii)

Proof. Note that

$$2\lceil a\rceil_\varepsilon + \lceil b\rceil_\varepsilon = 2\lceil a-\varepsilon\rceil + \lceil b-\varepsilon\rceil \geq \lceil 2a-2\varepsilon+b-\varepsilon\rceil \geq \lceil 2x-3\varepsilon\rceil \geq 2x.$$

This implies the proposition.[95] □

`matchingsizes`

Proposition 7.10. *Let $D \in \mathbb{N}$ and let $0 < \varepsilon < 1/3$.*[96] *Let G be a D-regular graph and let U, A, B be a partition of $V(G)$ where $|A| \geq |B|$. Suppose that $\Delta(G[A,U]), \Delta(G[A]) \leq D/2$ and that $\mathrm{char}_{D/2,\varepsilon}(G) = (\ell, m)$. Then $\ell, m \geq 0$ and $\ell + m/2 \geq |A| - |B|$.*

Proof. The consequence (ii) of Proposition 4.7 implies that $4e(A)/D + 2e(A,U)/D \geq 2(|A|-|B|)$. Apply Proposition 7.9 with $2e(A)/D, 2e(A,U)/D, |A|-|B|$ playing the roles of a, b, x to obtain a', b'. Note that $a' = \ell$ and $b' = m$. □

We will first prove Lemma 7.3 in the case when $|A| - |B| \geq 2$. This constraint arises for the following reason. We will show that we can find a path system $\mathcal{P}$ such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, but $\mathcal{P}$ is 'overbalanced'. More precisely, $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$, which is at least as large as $|A|-|B|$ by Proposition 7.10. We would like to remove edges from $\mathcal{P}$ so that (P2) holds, and $R_{\mathcal{V}}(\mathcal{P})$ is still an Euler tour. However, there exist path systems $\mathcal{P}_0$ such that $\mathrm{bal}_{AB}(\mathcal{P}_0) = 2$, $R_{\mathcal{V}}(\mathcal{P}_0)$ is an Euler tour, but any $\mathcal{P}_0'$ with $E(\mathcal{P}_0') \subsetneq E(\mathcal{P}_0)$ is such that $R_{\mathcal{V}}(\mathcal{P}_0')$ is not an Euler tour. (For example, a matching of size two in $G[V_1, A]$ together with a matching of size two in $G[V_2, A]$, such that these edges are all vertex-disjoint.) So, if $|A| - |B| < 2$, we cannot guarantee, simply by removing edges, that we will ever be able to find $\mathcal{P}'$ with $\mathrm{bal}_{AB}(\mathcal{P}') = |A| - |B|$ without violating (P3).

We will split the case when $|A| - |B| \geq 2$ further into the subcases $m \geq 4$ and $m \leq 2$, i.e. when $e_G(A, V_1 \cup V_2)$ is at least a little larger than $3D/2$, and when it is not. We will call these the *dense* and *sparse* cases respectively.

`dense`

7.5. **The proof of Lemma 7.3 in the case when $|A| - |B| \geq 2$ and $m \geq 4$.** This subsection concerns the dense case when $m \geq 4$, i.e. when $e_G(A, V_1 \cup V_2)$ is at least slightly larger than $3D/2$.[97] Now $G[A, V_1 \cup V_2]$ contains a matching M of size m. We will add edges to M to obtain a path system $\mathcal{P}$ which satisfies (P1)–(P3). If $M[A, V_i]$ is an even non-empty matching for both $i = 1, 2$, then M satisfies (P3). In every other case we must modify M by adding and/or subtracting edges. We do this separately depending on the relative values of $e_M(A, V_1)$ and $e_M(A, V_2)$. We thus obtain a path system $\mathcal{P}_0$ which satisfies (P1) and (P3). Then we obtain $\mathcal{P}$ by adding edges to $\mathcal{P}_0$ from $G[A]$ so that (P2) is also satisfied. We must pay attention to the way in which these sets of edges interact to ensure that $\mathcal{P}$ still satisfies (P3).[98]

We begin with the subcase when $e_M(V_1, A), e_M(V_2, A)$ are both even and positive.[99]

`2,2`

Lemma 7.11. *Let $\Delta, \Delta' \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ and $m \in 2\mathbb{N}$ with $\Delta'/\Delta, m/\Delta', \ell/\Delta' \ll 1$. Let G be a graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Let M be a matching in $G[V_1 \cup V_2, A]$ of size m, and let $M_i := M[V_i, A]$ and $m_i := e(M_i)$. Suppose that $\{m_1, m_2\} \subseteq 2\mathbb{N}$. Let $e(A) \geq (\ell-1)\Delta + \Delta'$ and $\Delta(G[A]) \leq \Delta$. Then G contains a path system $\mathcal{P}$ such that $\mathcal{P} \subseteq G[A] \cup G[A, V_1 \cup V_2]$, $\mathcal{P}[A, V_1 \cup V_2] = M$, $e(\mathcal{P}) = \ell + m$, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$. Moreover, $\mathcal{P}$ contains at least one V_iA-path for each $i = 1, 2$.*

Proof. We will find $\mathcal{P}$ by adding suitable edges of $G[A]$ to M such that $\mathcal{P}$ contains at least one V_iA-path for each $i = 1, 2$.[100] Then by Fact 7.2 we have that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour. Apply

[95]If $\lceil b\rceil_\varepsilon$ is even, have $a' := \lceil a\rceil_\varepsilon$ and $b' := \lceil b\rceil_\varepsilon$. Otherwise, $2\lceil a\rceil_\varepsilon + \lceil b\rceil_\varepsilon$ is odd and at least $2x$, so $2\lceil a\rceil_\varepsilon + \lceil b\rceil_\varepsilon - 1 \geq 2x$. Note that, since $b \geq 0$, $\lceil b\rceil_\varepsilon = \lceil b-\varepsilon\rceil \geq \lceil b-1/3\rceil \geq \lceil -1/3\rceil = 0$. But $\lceil b\rceil_\varepsilon$ is odd so $\lceil b\rceil_\varepsilon \geq 1$. In this case we have $a' := \lceil a\rceil_\varepsilon$ and $b' := \lceil b\rceil_\varepsilon - 1$ (the latter assertion needs $\lceil b\rceil_\varepsilon \geq 1$).

[96]DK deleted $1/D \ll 1$

[97]If $e_G(A,U) \geq 3D/2 + 2\varepsilon$ then $\lceil e_G(A,U) - \varepsilon\rceil \geq \lceil 3+\varepsilon\rceil = 4$.

[98]I added a mini sketch..

[99]DK: added the moreover part of the next lemma - useful for Lemmas 7.13 and 7.14

[100]DK: reformulated first sentence

Lemma 7.6 to $G[A]$. Suppose first that the consequence (i) of Lemma 7.6 holds. Let M' be a matching of size $\ell+1$ in $G[A]$ and let $uv \in E(G[A])$ be such that $u \notin V(M')$. Then

bal22
$$\mathrm{bal}_{AB}(M \cup M') = \ell + m/2 + 1 \quad \text{and} \quad e(M \cup M') = \ell + m + 1. \tag{7.7}$$

If $M \cup M'$ contains a V_iA-path for both $i = 1,2$ we are done by setting $\mathcal{P} := M \cup M' \setminus \{e\}$ where $e \in M'$ is arbitrary. Suppose now that $M \cup M'$ contains a V_1A-path but no V_2A-path. Then $V(M_2) \cap A \subseteq V(M')$. Choose $e_2 \in E(M')$ with an endpoint in $V(M_2)$. Then $\mathcal{P} := M \cup M' \setminus \{e_2\}$ contains a V_iA-path for both $i = 1,2$, and (7.7) implies that $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$ and $e(\mathcal{P}) = \ell + m$, as required. The case when $M \cup M'$ contains a V_2A-path but no V_1A-path is identical.

So we may assume that $M \cup M'$ contains no V_iA-path for both $i = 1,2$. Suppose that there is $a_1a_2 \in E(M')$ with $a_i \in V(M_i)$. Then $\mathcal{P} := M \cup M' \setminus \{a_1a_2\}$ contains a V_iA-path with endpoint a_i for $i = 1,2$. Moreover, (7.7) implies that $\mathcal{P}$ satisfies the other conditions. Therefore we may assume that $M'_i := M'[V(M_i) \cap A]$ is a (non-empty) perfect matching for $i = 1,2$. Choose $f_i \in E(M'_i)$ for $i = 1,2$ such that $v \in V(f_1) \cup V(f_2)$ if possible. We set $\mathcal{P} := M \cup M' \cup \{uv\} \setminus \{f_1, f_2\}$. Note that every vertex in $V(f_i) \setminus \{v\}$ is the endpoint of a V_iA-path in $\mathcal{P}$. Then (7.7) implies that $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{AB}(M \cup M') + 1 - 2 = \ell + m/2$ and $e(\mathcal{P}) = \ell + m$, as required.

Suppose instead that the consequence (ii) of Lemma 7.6 holds and let $x_1, \dots, x_\ell$ be ℓ distinct vertices in A with $d_A(x_i) \geq \Delta'$ for all $1 \leq i \leq \ell$. Apply Proposition 7.8 with $G \setminus B, V_1 \cup V_2, A, M, x_i, 0$ playing the roles of G, X, Y, M, x_i, r to obtain a path system $\mathcal{P} \subseteq G[A] \cup G[A, V_1 \cup V_2]$ with $e_{\mathcal{P}}(A) = \ell$, $\mathcal{P}[A, V_1 \cup V_2] = M$ and such that every edge in M lies in a distinct AV_i-path in $\mathcal{P}$ for some $i \in \{1,2\}$. Therefore $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $e(\mathcal{P}) = \ell + m$, and since $V(\mathcal{P}) \cap B = \emptyset$ we have that $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$. □

We now consider the case when $e_M(V_1, A), e_M(V_2, A)$ are both odd and at least three.

3,3 **Lemma 7.12.** *Let $\Delta, \Delta' \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ and $m \in 2\mathbb{N}$ with $\Delta'/\Delta, m/\Delta', \ell/\Delta' \ll 1$. Let G be a graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Let $m < e_G(V_1 \cup V_2, A)$, $e_G(A) \geq (\ell-1)\Delta + \Delta'$ and $\Delta(G[A]) \leq \Delta$. Let M be a matching in $G[V_1 \cup V_2, A]$ of size m, and let $M_i := M[V_i, A]$, $m_i := e(M_i)$. Suppose $\{m_1, m_2\} \subseteq 2\mathbb{N}+1$. Then G contains a path system $\mathcal{P}$ such that $e(\mathcal{P}) \leq \ell + m$, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$.*

Proof. We will find $\mathcal{P}$ such that $e_{\mathcal{P}}(V_i, A) = e_{\mathcal{P}}(V_i, W)$ is even for $i = 1,2$, $e_{\mathcal{P}}(V_1, V_2) = 0$ and such that for each $X \in \mathcal{V}$, there exists $X' \in \mathcal{V} \setminus \{X\}$ such that $\mathcal{P}$ contains an XX'-path. Then by Fact 7.2 we have that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour.

Let us first suppose that $\ell = 0$. Since $m < e_G(V_1 \cup V_2, A)$, there exists an edge $e^+ \in G[V_1 \cup V_2, A] \setminus E(M)$. Suppose, without loss of generality, that $e^+ \in G[V_1, A]$. Let e^- be an arbitrary edge in M_2. Let $\mathcal{P} := M \cup \{e^+\} \setminus \{e^-\}$. Then $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}) = (m_1+1)/2 + (m_2-1)/2 = m/2$, as required.

Therefore we assume that $\ell \geq 1$. Apply Lemma 7.6 to $G[A]$. Suppose first that the consequence (i) of Lemma 7.6 holds. So $G[A]$ contains a matching M' of size $\ell+1$. Note that it suffices to find $e_i \in M_i$ for $i = 1,2$ such that $M \cup M' \setminus \{e_1, e_2\}$ contains a V_iA-path for $i = 1,2$. Then it is straightforward to check that we are done by setting $\mathcal{P} := M \cup M' \setminus \{e_1, e_2\}$.

We say that $xy \in E(G[A])$ is a *connecting edge* if $x \in V(M_1)$ and $y \in V(M_2)$. Suppose that M' contains no connecting edge. So $M \cup M'$ contains no V_1V_2-paths. But an even number of edges in M_i lie in V_iV_i-paths of $M \cup M'$. Since m_i is odd, there must be a V_iA-path P_i in $M \cup M'$ for $i = 1,2$. We are done by choosing $e_i \in E(M_i) \setminus E(P_i)$ arbitrarily.

Therefore we may assume that there exists a connecting edge $a_1a_2 \in M'$, with $a_i \in V(M_i)$. Suppose that there exists a second connecting edge $a'_1a'_2 \in M'$, with $a'_i \in V(M_i)$. Then we are done by choosing $e_1 \in M_1$ with endpoint a_1 and $e_2 \in M_2$ with endpoint a'_2. Therefore we may suppose that a_1a_2 is the only connecting edge in G. Let P be the V_1V_2-path containing a_1a_2. Let $\mathcal{P}' := (M \cup M') \setminus \{E(P)\}$. Then, for each $i = 1,2$, either $\mathcal{P}'$ contains a V_iA-path $P_{i,A}$, or a

V_iV_i-path $P_{i,i}$. In the first case, let e_i be an arbitrary edge of M_i that does not lie in $P_{i,A}$.[101] In the second case, let $e_i \in E(P_{i,i}) \cap E(M_i)$ be arbitrary.[102]

Suppose instead that the consequence (ii) of Lemma 7.6 holds in $G[A]$ and let $x_1, \dots, x_\ell$ be ℓ distinct vertices in A with $d_A(x_i) \geq \Delta'$ for all $1 \leq i \leq \ell$. Since $\ell \geq 1$, we can choose $e_1 \in M_1$ and $e_2 \in M_2$ so that $\{x_1, \dots, x_\ell\} \not\subseteq V(M \setminus \{e_1, e_2\})$. Apply Proposition 7.8 with $G \setminus B, V_1 \cup V_2, A, M \setminus \{e_1, e_2\}, x_i, 1$ playing the roles of G, X, Y, M, x_i, r to obtain a path system $\mathcal{P} \subseteq G[A] \cup G[A, V_1 \cup V_2]$ such that $e_{\mathcal{P}}(A) = \ell + 1$, $\mathcal{P}[A, V_1 \cup V_2] = M \setminus \{e_1, e_2\}$, and every edge in $M \setminus \{e_1, e_2\}$ lies in a distinct AV_i-path in $\mathcal{P}$ for some $i \in \{1, 2\}$. Then $e(\mathcal{P}) = \ell + m - 1$ and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + 1 + (m-2)/2 = \ell + m/2$. Since $\mathcal{P}[A, V_i]$ is an even matching for $i = 1, 2$ and $\mathcal{P}[V_1, V_2]$ is empty, we have that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and we are done. □

We now consider the case when $e_M(V_2, A)$ is odd and at least three, and $e_M(V_1, A) = 1$.

1,3 **Lemma 7.13.** *Let $\Delta, \Delta' \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ and $m \in 2\mathbb{N}$ with $\Delta'/\Delta, m/\Delta', \ell/\Delta' \ll 1$. Let G be a 3-connected graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Let $e_G(A) \geq (\ell - 1)\Delta + \Delta'$ and $\Delta(G[A]) \leq \Delta$. Let M_2 be a matching in $G[V_2, A]$ of size $m - 1$ where $3 \leq m - 1 < e_G(V_2, A)$ and let $e_1 \in G[V_1, A]$ be an edge not incident to M_2. Then G contains a path system $\mathcal{P}$ such that $e(\mathcal{P}) \leq \ell + m + 2$, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$.*

Proof. We will find a path system $\mathcal{P}$ such that, for each $X \in \mathcal{V}$, $e_{\mathcal{P}}(X, \overline{X})$ is even and there exists $X' \in \mathcal{V} \setminus \{X\}$ such that $\mathcal{P}$ contains an XX'-path. Then by Fact 7.2, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour. We will choose $\mathcal{P}$ such that $\mathcal{P}[V_1 \cup V_2, W]$ is obtained from $M_2 \cup \{e_1\}$ by adding/removing at most one edge.[103] Since G is 3-connected, G contains an edge v_1v with $v_1 \in V_1$ and $v \in V_2 \cup A \cup B$ such that vv_1 and e_1 are vertex-disjoint. We consider cases depending on the location of v.

Case 1. $v \in A$.

If possible, let e_2 be the edge of M_2 incident to v; otherwise, let e_2 be an arbitrary edge of M_2. Then we are done by applying Lemma 7.11 with $M_2 \cup \{e_1, v_1v\} \setminus \{e_2\}$ playing the role of M.

Case 2. $v \in V_2$.

If possible, choose $e_2 \in E(M_2)$ whose endpoint $v_2 \in V_2$ satisfies $v_2 = v$, otherwise let $e_2 \in E(M_2)$ be arbitrary. Set $V_1' := V_1 \cup \{v, v_2\}$ and $V_2' := V_2 \setminus \{v, v_2\}$. Observe that $e_{M_2 \cup \{e_1\}}(A, V_i') \in 2\mathbb{N}$ for $i = 1, 2$. Let $\mathcal{V}' := \{V_1', V_2', W\}$. Apply Lemma 7.11 with $G \setminus \{v_1\}, V_1', V_2', A, B, M_2 \cup \{e_1\}$ playing the roles of G, V_1, V_2, A, B, M to obtain a path system $\mathcal{P}'$ such that $\mathcal{P}' \subseteq G[A] \cup G[A, V_1' \cup V_2']$, $\mathcal{P}'[A, V_1' \cup V_2'] = M_2 \cup \{e_1\}$, $e(\mathcal{P}') = \ell + m$, $R_{\mathcal{V}'}(\mathcal{P}')$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}') = \ell + m/2$. Moreover, $\mathcal{P}'$ contains at least one $V_i'A$-path for each $i = 1, 2$. Let[104] P_i be such a path.

Let $\mathcal{P} := \mathcal{P}' \cup \{vv_1\}$. Then $e(\mathcal{P}) = \ell + m + 1$ and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$. Moreover, each of $e_{\mathcal{P}}(V_1, \overline{V_1}) = e_{\mathcal{P}'}(V_1', \overline{V_1'}) = 2$, $e_{\mathcal{P}}(V_2, \overline{V_2}) = e_{\mathcal{P}'}(V_2', \overline{V_2'}) + 2$ and $e_{\mathcal{P}}(W, \overline{W}) = e_{\mathcal{P}'}(W, \overline{W})$ is even. Now P_2 is a V_2A-path in $\mathcal{P}$. Similarly, if P_1 avoids e_2, then P_1 is a V_1A-path in $\mathcal{P}$. If P_1 contains e_2 and $v_2 = v$, then v_1vP_1 is a V_1A-path in $\mathcal{P}$. If $v_2 \neq v$ then v_1v is a V_1V_2-path in $\mathcal{P}$.[105] Therefore, by Fact 7.2, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, as required.

Case 3. $v \in B$.

Apply Lemma 7.6 to $G[A]$. Suppose first that the consequence (i) of Lemma 7.6 holds. Let M' be a matching of size $\ell + 1$ in $G[A]$ and let $uw \in E(G[A])$ with $u \notin V(M')$. Apply Proposition 7.7 with $G \setminus B, V_1 \cup V_2, A, M_2 \cup \{e_1\}, M', u, w$ playing the roles of G, X, Y, M, M', u, v to obtain a

[101]This is possible because $P_{i,A}$ contains at most one V_iA-edge and $e(M_i) \geq 3$.

[102]LATE CHANGE: Else e_i could have both endpoints in A.

[103]LATE CHANGE: The previous sentence was wrong.

[104]DK: before we had "Since $\mathcal{P}'[A, V_1 \cup V_2] = M_2 \cup \{e_1\}$ we have that, for each $i = 1, 2$, $\mathcal{P}'$ contains at least two distinct AV_i'-paths. Therefore, for each $i = 1, 2$, $\mathcal{P}'$ contains at least one AV_i'-path P_i with $V(P_i) \cap \{v, v_1, v_2\} = \emptyset$." But I don't see why $\mathcal{P}'$ contains at least two distinct AV_i'-paths.

[105]Note that v lies in a path of $\mathcal{P}'$ iff $v = v_2$.

path system $\mathcal{P}_0$ such that $\mathcal{P}_0[V_1 \cup V_2, A] = M_2 \cup \{e_1\}$; $e_{\mathcal{P}_0}(A) = \ell + 1$; and $\mathcal{P}_0$ contains at least two $(V_1 \cup V_2)A$-paths. But $\mathcal{P}_0$ contains at most one V_1A-path, and hence at least one V_2A-path P. Now the consequence (i) of Proposition 7.7 implies that $e_P(V_2, A) = 1$. So we can choose $e \in E(\mathcal{P}_0[V_2, A]) \setminus E(P)$. Let $\mathcal{P} := \mathcal{P}_0 \cup \{v_1v\} \setminus \{e\}$. Then $e_{\mathcal{P}}(X, \overline{X})$ is even for all $X \in \{V_1, V_2, W\}$ and $\mathcal{P}$ contains a V_1B-path and a V_2A-path. Moreover, $\mathrm{bal}_{AB}(\mathcal{P}) = e_{\mathcal{P}_0}(A) + e_{\mathcal{P}_0}(A, V_1 \cup V_2)/2 - 1 = \ell + m/2$, as required.

Suppose instead that the consequence (ii) of Lemma 7.6 holds. Then $G[A]$ contains ℓ distinct vertices $x_1, \dots, x_\ell$ such that $d_A(x_i) \geq \Delta'$ for all $1 \leq i \leq \ell$. Choose $e \in E(G[V_2, A]) \setminus E(M_2)$. If $\ell = 0$ then $\mathcal{P} := M_2 \cup \{e_1, v_1v, e\}$ is as required. Suppose now that $\ell = 1$. Let $w_1, y_1 \in N_A(x_1) \setminus V(M_2 \cup \{e_1\})$ be distinct. Suppose that $x_1 \notin V(e_1)$. If possible, choose e_2 to be the edge of M_2 that contains x_1; otherwise, let e_2 be an arbitrary edge of M_2. In this case we let $\mathcal{P} := M_2 \cup \{e_1, v_1v, w_1x_1y_1\} \setminus \{e_2\}$. Suppose now that $x_1 \in V(e_1)$. In this case we let $\mathcal{P} := M_2 \cup \{e_1, v_1v, e\} \cup \{x_1y_1\}$. In all cases, we have that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $e(\mathcal{P}) \leq \ell + m + 2$ and $\mathrm{bal}_{AB}(\mathcal{P}) = m/2 + 1$, as required.

Suppose finally that $\ell \geq 2$. Then we can choose $e_2 \in M_2$ so that $\{x_1, \dots, x_\ell\} \not\subseteq V(M_2 \cup \{e_1\} \setminus \{e_2\})$. Apply Proposition 7.8 with $G \setminus B, V_1 \cup V_2, A, M_2 \cup \{e_1\} \setminus \{e_2\}, x_i, 1$ playing the roles of G, X, Y, M, x_i, r to obtain a path system $\mathcal{P}_0$ in $G[A] \cup G[A, V_1 \cup V_2]$ such that $e_{\mathcal{P}_0}(A) = \ell + 1$, $\mathcal{P}_0[A, V_1 \cup V_2] = M_2 \cup \{e_1\} \setminus \{e_2\}$, and every edge in $M_2 \cup \{e_1\} \setminus \{e_2\}$ lies in a distinct AV_i-path in $\mathcal{P}_0$ for some $i \in \{1, 2\}$. Let $\mathcal{P} := \mathcal{P}_0 \cup \{v_1v\}$. Then $e(\mathcal{P}) = \ell + m + 1$ and

$$\mathrm{bal}_{AB}(\mathcal{P}) = e_{\mathcal{P}_0}(A) + e_{\mathcal{P}_0}(A, V_1 \cup V_2)/2 - 1/2 = \ell + 1 + (m-1)/2 - 1/2 = \ell + m/2.$$

Note finally that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour by Fact 7.2. □

We are now ready to prove a more general version of Lemmas 7.11–7.13 in which $G[A, V_1 \cup V_2]$ contains an arbitrary even matching of size at least four.

0,4 **Lemma 7.14.** *Let $\Delta, \Delta' \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ and $m \in 2\mathbb{N}$ with $\Delta'/\Delta, m/\Delta', \ell/\Delta' \ll 1$ and $m \geq 4$. Let $\Delta'/\Delta < \varepsilon < 1/3$. Let G be a 3-connected graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Suppose that $\Delta(G[A]), \Delta(G[A, V_1 \cup V_2]) \leq \Delta$ and $\mathrm{char}_{\Delta,\varepsilon}(G) = (\ell, m)$. Then G contains a path system $\mathcal{P}$ such that $e(\mathcal{P}) \leq \ell + m + 4$, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$.*

Proof. Write $U := V_1 \cup V_2$. Proposition 7.5 implies that

lmedges (7.8) $$e_G(A) \geq (\ell - 1)\Delta + \Delta' \quad \text{and} \quad e_G(A, U) \geq (m-1)\Delta + \Delta'.$$

Recall also that $m \leq \lceil e_G(A, U)/\Delta \rceil$ and m is even. Choose non-negative integers b_1, b_2 such that $b_i \leq \lceil e_G(A, V_i)/\Delta \rceil$ for $i = 1, 2$ and $b_1 + b_2 = m$. Apply Lemma 6.7 with $G[A, U], A, V_1, V_2$ playing the roles of G, U, V, W to obtain a matching M in $G[A, U]$ such that $e_M(A, V_i) = b_i$ for $i = 1, 2$. Without loss of generality we assume that $b_1 \leq b_2$. Suppose first that b_1, b_2 are both even and positive. Then we are done by applying Lemma 7.11. If b_1, b_2 are both odd and at least three, then we are done by applying Lemma 7.12.[106] Suppose that $b_1 = 1$. Then $\lceil e_G(A, V_2)/\Delta \rceil \geq b_2 = m - 1$ so $m - 1 < e_G(A, V_2)$. Therefore we can apply Lemma 7.13 with M playing the role of $M_2 \cup \{e_1\}$. So we can assume that $b_1 = 0$, and hence that $M \subseteq G[A, V_2]$. Suppose that $e_G(A, V_1) > 0$. Then there is an edge $e \in E(G[A, V_1])$ and $m - 1$ edges in M which are not incident with e. We are similarly done by applying Lemma 7.13. The only remaining case is when $e_G(A, V_1) = 0$. Now (7.8) implies that

nowhave (7.9) $$e_G(A, V_2) \geq (m-1)\Delta + \Delta'.$$

Since G is 3-connected, $G[V_1, \overline{V_1}]$ contains a matching of size three. So $G[V_1, V_2 \cup B]$ contains a matching of size three. Then at least one of $G[V_1, V_2]$, $G[V_1, B]$ contains a matching of size two.

Case 1. *$G[V_1, V_2]$ contains a matching M^* of size two.*

[106] $\lceil e(A, U)/\Delta \rceil < e(A, U)$ since $e(A, U) \geq 3\Delta + \Delta'$.

Choose two distinct edges $e_2, e_2' \in E(M)$ such that $|V(M^*) \cap \{v_2, v_2'\}|$ is as large as possible, where v_2, v_2' are the endvertices of e_2, e_2' in V_2.[107] Set $V_1' := V_1 \cup \{v_2, v_2'\}$ and $V_2' := V_2 \setminus \{v_2, v_2'\}$. Observe that $e_M(A, V_i') \in 2\mathbb{N}$ for $i = 1, 2$ since $m \geq 4$. Let $\mathcal{V}' := \{V_1', V_2', W\}$. Apply Lemma 7.11 with G, V_1', V_2', A, B, M playing[108] the roles of G, V_1, V_2, A, B, M to obtain a path system $\mathcal{P}'$ such that $\mathcal{P}' \subseteq G[A] \cup G[A, V_1' \cup V_2']$, $\mathcal{P}'[A, V_1' \cup V_2'] = M$, $e(\mathcal{P}') = \ell + m$, $R_{\mathcal{V}'}(\mathcal{P}')$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}') = \ell + m/2$.[109] Moreover, $\mathcal{P}'$ contains at least one $V_i'A$-path for each $i = 1, 2$. Let P_i be such a path. Then P_1 contains either e_2 or e_2'. Without loss of generality we may assume that P_1 contains e_2.

Let $\mathcal{P} := \mathcal{P}' \cup M^*$. Then $e(\mathcal{P}) = \ell + m + 2$ and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$. Moreover, each of $e_{\mathcal{P}}(V_1, \overline{V_1}) = e_{\mathcal{P}'}(V_1', \overline{V_1'}) = 2$, $e_{\mathcal{P}}(V_2, \overline{V_2}) = e_{\mathcal{P}'}(V_2', \overline{V_2'}) + 4$ and $e_{\mathcal{P}}(W, \overline{W}) = e_{\mathcal{P}'}(W, \overline{W})$ is even. Now P_2 is an V_2A-path in $\mathcal{P}$. If M^* contains an edge e which avoids both v_2, v_2' (and thus is vertex-disjoint from all edges in M), then e is a V_1V_2-path in $\mathcal{P}$. If there is no such edge e, then M^* contains an edge e' whose endvertex in V_2 is v_2. Then $e' \cup P_1$ is a V_1A-path in $\mathcal{P}$. Therefore, by Fact 7.2, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, as required.

Case 2. *$G[V_1, B]$ contains a matching M^* of size two.*

Apply Lemma 7.6 to $G[A]$. Suppose first that the consequence (i) of Lemma 7.6 holds. Then $G[A]$ contains a matching M' of size $\ell + 1$ and an edge uv with $u \notin V(M')$. Apply Proposition 7.7 with $G \setminus B, V_1 \cup V_2, A, M, M', u, v$ playing the roles of G, X, Y, M, M', u, v to obtain a path system $\mathcal{P}_0$ such that $\mathcal{P}_0[V_1 \cup V_2, A] = M$; $\mathcal{P}_0 \subseteq M \cup M' \cup \{uv\}$; $e_{\mathcal{P}_0}(A) = \ell + 1$; and $\mathcal{P}_0$ contains at least two V_2A-paths. Let $\mathcal{P} := \mathcal{P}_0 \cup M^*$. Then $\mathcal{P}$ contains at least two V_2A-paths and two V_1B-paths (namely the edges of M^*), so $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour. Moreover $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2$ and $e(\mathcal{P}) = \ell + m + 3$, as required.

Suppose now that the consequence (ii) of Lemma 7.6 holds in $G[A]$. Assume first that $\ell \geq 2$. Let $x_1, \dots, x_\ell$ be ℓ distinct vertices in A such that $d_A(x_i) \geq \Delta'$ for $1 \leq i \leq \ell$. Since $m \geq 4$, we can choose distinct $e_1, e_2 \in M$ such that $|\{x_1, \dots, x_\ell\} \setminus V(M \setminus \{e_1, e_2\})| \geq 2$. Then Proposition 7.8 applied with $G \setminus B, V_1 \cup V_2, A, M \setminus \{e_1, e_2\}, x_i, 2$ playing the roles of G, X, Y, M, x_i, r implies that there is a path system $\mathcal{P}' \subseteq G[A] \cup G[A, V_1 \cup V_2]$ such that $e_{\mathcal{P}'}(A) = \ell + 2$, $\mathcal{P}'[A, V_1 \cup V_2] = M \setminus \{e_1, e_2\}$, and such that every edge of $M \setminus \{e_1, e_2\}$ lies in a distinct AV_2-path. Let $\mathcal{P} := \mathcal{P}' \cup M^*$. Then $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $e(\mathcal{P}) = \ell + m + 2$, and

$$\mathrm{bal}_{AB}(\mathcal{P}) = e_{\mathcal{P}'}(A) + e_{\mathcal{P}'}(A, V_1 \cup V_2)/2 - 1 = \ell + 2 + (m-2)/2 - 1 = \ell + m/2.$$

Finally we consider the case when $\ell \leq 1$. Lemma 7.6 applied to $G[A, V_1 \cup V_2]$ and (7.9) imply that $G[A, V_1 \cup V_2]$ contains a matching M' of size m together with a matching M^+ of size two which is edge-disjoint from M', such that both edges in M^+ contain a vertex outside of $V(M')$. Since $e_G(A, V_1) = 0$ by our assumption, we have $M' \cup M^+ \subseteq G[A, V_2]$.[110] Suppose first that $\ell = 0$. In this case we let $\mathcal{P} := M' \cup M^+ \cup M^*$. It is clear that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $e(\mathcal{P}) = m + 4$ and $\mathrm{bal}_{AB}(\mathcal{P}) = m/2$, as required. The final case is when $\ell = 1$. Choose $e \in M^+$ and $e' \in M'$ such that $|V(e) \cap \{x_1\}| + |V(e') \cap \{x_1\}|$ is maximal.[111] So $\mathcal{P}' := M' \cup M^+ \setminus \{e, e'\}$ is a matching of size $m - 1$ together with an extra edge, and $x_1 \notin V(\mathcal{P}')$. In particular, $\mathcal{P}'$ contains a V_2A-path P_2. Since $m/\Delta' \ll 1$, we can choose distinct vertices w_1, y_1 in $N_A(x_1) \setminus V(\mathcal{P}')$. Let $\mathcal{P} := \mathcal{P}' \cup M^* \cup \{w_1x_1y_1\}$. Then P_2 is a V_2A-path in $\mathcal{P}$ and each edge of M^* is a V_1B-path in $\mathcal{P}$. So Fact 7.2 implies that $\mathcal{P}$ is an Euler tour. Moreover, $\mathrm{bal}_{AB}(\mathcal{P}) = m/2 + 1$, and $e(\mathcal{P}) = m + 4$, as required. □

The proof of Lemma 7.3 in the 'dense' case is now just a short step away.

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[107]DK: reformulated that sentence

[108]DK had $G \setminus E(M^*)$ instead of G, don't see why we have to delete M^*

[109]DK: changed the rest of Case 1, had "Since $\mathcal{P}'[A, V_1' \cup V_2'] = M$, we have that, for each $i = 1, 2$, $\mathcal{P}'$ contains at least two AV_i'-paths P_i, P_i'." before. I don't see why this is true.

[110]Deryk added new sentence

[111]DK: reformulated this sentence

Proof of Lemma 7.3 in the case when $|A| - |B| \geq 2$ and $m \geq 4$. Let $\Delta := D/2$. Observe that $d_A(a) \leq d_B(a)$ for all $a \in A$ implies that $\Delta(G[A]) \leq \Delta$. Proposition 7.10 implies that $\ell + m/2 \geq |A| - |B|$. Choose non-negative integers $\ell' \leq \ell$ and $m' \leq m$ such that m' is even, $\ell' + m'/2 = |A| - |B|$ and $m' \geq 4$. This is possible since $|A| - |B| \geq 2$. Let $\Delta' := \nu n$. Proposition 7.4 implies that $\ell', m' \leq 12\rho n$. Then $\Delta'/\Delta \ll 1$, $m'/\Delta' \ll 1$, $\ell'/\Delta' \ll 1$, $\Delta'/\Delta < \varepsilon$. Apply Lemma 7.14 with ℓ', m' playing the roles of ℓ, m to obtain a path system $\mathcal{P}$ such that $e(\mathcal{P}) \leq \ell' + m' + 4 \leq \ell + m + 4$, $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, and $\mathrm{bal}(\mathcal{P}) = \ell' + m'/2 = |A| - |B|$. So (P1)–(P3) hold. □

sparse

7.6. **The proof of Lemma 7.3 in the case when $|A| - |B| \geq 2$ and $m \leq 2$.** We now deal with the sparse case, i.e. when the largest even matching we can guarantee between A and $V_1 \cup V_2$ has size at most two. For this, we need to introduce some notation which will be used in all of the remaining cases.

7.6.1. *More notation and tools.* [112] In the previous case when $m \geq 4$, $G[A, V_1 \cup V_2]$ has a large matching which we used to suitably connect components. In the case $m \leq 2$ we cannot rely on this. So we use a 'basic connector'. Roughly speaking, a basic connector $\mathcal{P}$ is a path system with few edges such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour. So $\mathcal{P}$ satisfies (P1) and (P3), but not necessarily (P2) (i.e. we might have $\mathrm{bal}_{AB}(\mathcal{P}) \neq |A| - |B|$). We find a basic connector by Proposition 7.15 and then adjust it to satisfy (P2). Basic connectors will also be useful in the final two subsections, i.e. Sections 7.7 and 7.8, which concern the case when $|A| - |B| \leq 1$.

Given a path system $\mathcal{P}$, recall the definition of $F_{\mathcal{P}}(A)$ in (4.1). We say that $\mathcal{P}$ is a *basic connector*[113] (for $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$) if

(BC1) $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour;
(BC2) $e(\mathcal{P}) \leq 4$ and $|\mathrm{bal}_{AB}(\mathcal{P})| \leq 2$;
(BC3) $e_{\mathcal{P}}(A \cup B) = 0$;
(BC4) if $F_{\mathcal{P}}(A) = (a_1, a_2)$ then $\mathrm{bal}_{AB}(\mathcal{P}) \in \{a_1 + 2a_2 - 2, a_1 + 2a_2 - 1\}$ and $a_2 \leq 1$.

It can be shown that (BC1)–(BC3) imply (BC4) (cf. the proof of Proposition 7.15). Observe (BC3) implies that if $\mathcal{P}$ is a basic connector, then

BCeq

$$2\mathrm{bal}_{AB}(\mathcal{P}) = e_{\mathcal{P}}(A, V_1 \cup V_2) - e_{\mathcal{P}}(B, V_1 \cup V_2) = a_1 + 2a_2 - e_{\mathcal{P}}(B, V_1 \cup V_2). \quad (7.10)$$

Roughly speaking, the existence of a basic connector $\mathcal{P}$ follows from 3-connectivity. We would like to modify/extend $\mathcal{P}$ into a path system $\mathcal{P}'$ which balances the sizes of A, B, i.e. for which $\mathrm{bal}_{AB}(\mathcal{P}') = |A| - |B|$.[114] The following notion will be very useful for this. Given a graph G, disjoint $A_1, A_2 \subseteq V(G)$ and $t \in \mathbb{N}_0$, we say that[115]

$$\mathrm{acc}(G; A_1, A_2) \geq t$$

if G contains a path system $\mathcal{P}$ such that

(A1) $e(\mathcal{P}) = t$;
(A2) $d_{\mathcal{P}}(x_2) = 0$ for each $x_2 \in A_2$;
(A3) $d_{\mathcal{P}}(x_1) \leq 1$ for each $x_1 \in A_1$, and no path of $\mathcal{P}$ has both endpoints in A_1.

We say that such a $\mathcal{P}$ *accommodates* A_1, A_2, where 'acc' is chosen for 'accomodating'.[116]

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[112] ALLAN NEW

[113] when $G[V_1 \cup_2, A \cup B]$ is very sparse, we need to start with such an object (guaranteed by 3-connectivity) and adjust it as appropriate. When this graph is dense, we start with the large matching it contains and extend this into an appropriate path system.

[114] DK: changed this sentence

[115] Deryk introduced this def instead of $\mathrm{acc}(G; c_1, c_2)$ since there was a mistake in the proof of Prop 7.19 (and in the proof of Lemma 7.21)

[116] NEW sentence

In[117] a typical application of this notion, we have already constructed a path system $\mathcal{P}_0$. We let A_1 be the set of all those vertices in A which have degree one in $\mathcal{P}_0$ and A_2 be the set of all those vertices in A which have degree two in $\mathcal{P}_0$. Then, if $\mathrm{acc}(G[A]; A_1, A_2) \geq t$, we can find a path system $\mathcal{P}$ in $G[A]$ with t edges such that $\mathcal{P}_0 \cup \mathcal{P}$ is also a path system.

We now collect some tools which will be used to prove Lemma 7.3 in the case when $|A|-|B| \geq 2$ and $m \leq 2$. The next proposition uses Lemma 4.8 to show that G contains a basic connector.

BC **Proposition 7.15.** *Let G be a 3-connected graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Then G contains a basic connector $\mathcal{P}$.*

Proof. Apply Lemma 4.8 to G and $\mathcal{V}$ to obtain a path system $\mathcal{P}$ satisfying the conditions (i)–(iii). We claim that $\mathcal{P}$ is a basic connector. Write $F_{\mathcal{P}}(A) = (a_1, a_2)$ and $F_{\mathcal{P}}(B) = (b_1, b_2)$. In particular, (iii) implies that

degsum
$$a_1 + b_1 + 2(a_2 + b_2) \in \{2, 4\} \tag{7.11}$$

and $a_2+b_2 \leq 1$. Note that (BC1) and (BC3) are immediate from (ii) and (i) respectively. Moreover, (i) implies $e_{\mathcal{P}}(A \cup B) = 0$. So $e_{\mathcal{P}}(A, V_1 \cup V_2) = a_1 + 2a_2$ and $e_{\mathcal{P}}(B, V_1 \cup V_2) = b_1 + 2b_2$. So (7.11) implies that

$$2\mathrm{bal}_{AB}(\mathcal{P}) = a_1 + 2a_2 - b_1 - 2b_2 \in \{2a_1 + 4a_2 - 4, 2a_1 + 4a_2 - 2\}$$

and $|2\mathrm{bal}_{AB}(\mathcal{P})| \leq 4$, so (BC2) and (BC4) hold. □

By Proposition 7.15, we can find a basic connector $\mathcal{P}_0$ in G, which may not satisfy (P2). Our aim now is to find a suitable path system $\mathcal{P}_A$ in $G[A]$ so that $\mathcal{P}_0 \cup \mathcal{P}_A$ satisfies (P1)–(P3). Let A_i be the collection of all those vertices of A with degree i in $\mathcal{P}_0$. The next result shows[118] that it suffices to show that $\mathrm{acc}(G[A]; A_1, A_2) \geq |A| - |B| - \mathrm{bal}_{AB}(\mathcal{P}_0)$.[119]

addpaths **Proposition 7.16.** *Let G be a graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Let $\mathcal{P}_0$ be a basic connector in G and for $i = 1, 2$ let A_i be the collection of all those vertices of A with degree i in $\mathcal{P}_0$. Then, for any integer $0 \leq t \leq \mathrm{acc}(G[A]; A_1, A_2)$, we have that G contains a path system $\mathcal{P}$ such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{AB}(\mathcal{P}_0) + t$ and $e(\mathcal{P}) \leq t + 4$.*

Proof. Let $\mathcal{P}_A$ be a path system in $G[A]$ which accommodates A_1, A_2 such that $e(\mathcal{P}_A) = t$. Let $\mathcal{P} := \mathcal{P}_0 \cup \mathcal{P}_A$. Properties (A2) and (A3) imply that $\mathcal{P}$ is a path system. It is straightforward to check that (BC1) implies that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour. Moreover, $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{AB}(\mathcal{P}_0) + e(\mathcal{P}_A)$, as required. Finally, (BC2) gives the required bound on $e(\mathcal{P})$. □

7.6.2. *Building a basic connector from a matching.* The next lemma shows that in the case when $G[A, V_1 \cup V_2]$ contains a matching of size at least three, we can obtain a basic connector with additional useful properties.

3matching **Lemma 7.17.** *Let G be a 3-connected graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Suppose that $G[A, V_1 \cup V_2]$ contains a matching M of size three. Then one of the following holds:*

- (i) *G contains a basic connector $\mathcal{P}$ with $\mathrm{bal}_{AB}(\mathcal{P}) \geq 1$, and if $F_{\mathcal{P}}(A) = (a_1, a_2)$, then $a_1 \geq 2$;*
- (ii) *$e_G(A, V_i) = 0$ for some $i \in \{1, 2\}$, and for each $a \in A$, G contains matchings $M_{a,A}, M_{a,B}$ in $G[A \setminus \{a\}, V_j], G[B, V_i]$ respectively, where $j \in \{1, 2\} \setminus \{i\}$, each of which has size two. In particular, $\mathcal{P}_a := M_{a,A} \cup M_{a,B}$ is a basic connector with $\mathrm{bal}_{AB}(\mathcal{P}_a) = 0$, $a \notin V(\mathcal{P}_a)$ and $F_{\mathcal{P}}(A) = (2, 0)$.*[120]

[117] DK: changed this para

[118] I merged the two propositions that used to be here since the former was only ever used to prove the latter.

[119] Deryk changed this sentence and rewrote prop, replacing $\mathrm{acc}(G[A]; a_1, a_2)$ by $\mathrm{acc}(G[A]; A_1, A_2)$. We need this strengthening in the proof of Prop 7.20

[120] This statement has been strengthened, and the proof has changed accordingly (last few lines). DK: changed "moreover" to "in particular"

Proof. Without loss of generality we may assume that $e_M(A,V_2) \geq e_M(A,V_1)$. Suppose first that $e_G(A,V_1) > 0$. We claim that $G[A, V_1 \cup V_2]$ contains a matching M' of size three such that $e_{M'}(A,V_1) = 1$ and $e_{M'}(A,V_2) = 2$. To see this, we may assume that we cannot set $M' := M$, so $M \subseteq G[A,V_2]$. Let $e_1 \in E(G[A,V_1])$. Then $V(e_1) \cap V(M) \subseteq A$. If possible, let e' be the edge of M incident to e_1, otherwise let $e' \in E(M)$ be arbitrary. Let $M' := M \cup \{e_1\} \setminus \{e'\}$, proving the claim.

Since G is 3-connected, there exists $e \in E(G[V_1, \overline{V_1}])$ that is not incident with the unique edge $e_1 \in M'[A,V_1]$. Let x be the endpoint of e that does not lie in V_1. If $x \in V_2$ then we can choose $e_2 \in M'[A,V_2]$ which is not incident with e and then $\mathcal{P} := \{e, e_1, e_2\}$ is a path system with $\mathrm{bal}_{AB}(\mathcal{P}) = 1$ and $F_{\mathcal{P}}(A) = (2,0)$. It is easy to check that $\mathcal{P}$ is a basic connector, so (i) holds. If $x \in A \cup B$ then similarly $\mathcal{P} := M' \cup \{e\}$ satisfies (i).

Suppose now that $e_G(A,V_1) = 0$. Thus $e_M(A,V_2) = 3$. Since G is 3-connected, there is a matching M' of size three in $G[V_1, \overline{V_1}]$. Let $E(M') = \{e_1, e_2, e_3\}$ and let x_1, x_2, x_3 respectively be the endpoints of e_1, e_2, e_3 which do not lie in V_1. Note that $\{x_1, x_2, x_3\} \subseteq B \cup V_2$. Suppose first that $|V(M') \cap B| \leq 1$. Without loss of generality we assume that $\{x_1, x_2\} \subseteq V_2$. Let $e, e' \in E(M)$ be such that $\{x_1, x_2\} \not\subseteq V(\{e, e'\})$. Then $\mathcal{P} := \{e, e', e_1, e_2\}$ is such that $\mathrm{bal}_{AB}(\mathcal{P}) = 1$ and $F_{\mathcal{P}}(A) = (2,0)$. Moreover, $\mathcal{P}$ is a basic connector, so (i) holds. So without loss of generality we may assume that $|V(M') \cap B| \geq 2$ and $\{x_1, x_2\} \subseteq B$. Given an arbitrary $a \in A$, choose $e, e' \in E(M)$ such that $a \notin V(\{e, e'\})$. Let $M_{a,A} := \{e, e'\}$ and $M_{a,B} := \{e_1, e_2\}$. So (ii) holds.[121] □

We now show how this result implies that, whenever $G[A, V_1 \cup V_2]$ contains a matching of size two, we are again able to find a basic connector with additional useful properties (though not as useful as those in Lemma 7.17).

2matchingcor

Lemma 7.18. *Let G be a 3-connected graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Let M be a matching in $G[A, V_1 \cup V_2]$ of size two. Then G contains a basic connector $\mathcal{P}$ with $\mathrm{bal}_{AB}(\mathcal{P}) \geq 0$, and if $F_{\mathcal{P}}(A) = (a_1, a_2)$, then $a_1 \geq 1$.*

Proof. Write $U := V_1 \cup V_2$. Since G is 3-connected, $G[A \cup B, U]$ contains a matching M' of size three. We claim that $M \cup M'$ contains a matching M^* of size three such that at least two of the edges in M^* lie in $G[A,U]$. To see this, assume that $e_{M'}(A,U) \leq 1$ (or we could take $M^* := M'$). Assume further that there is no edge $e \in E(M')$ without an endpoint in $V(M)$ (or we could take $M^* := M \cup \{e\}$). Then, if we write $M := \{au, a'u'\}$ where $a, a' \in A$ and $u, u' \in U$, we have that M' consists of distinct edges $e_u, e_{u'}, e$ incident with u, u' and $\{a, a'\}$ respectively. Suppose that $a \in V(e)$. Then $e \in E(G[A,U])$ and so $e_u, e_{u'} \in E(G[B,U])$. Moreover, neither e nor e_u is incident with $a'u'$. We can set $M^* := \{a'u', e, e_u\}$. If instead $a' \in V(e)$, then we can set $M^* := \{au, e, e_{u'}\}$. This proves the claim.

If $M^* \subseteq G[A,U]$, we are done by Lemma 7.17. Otherwise, let bu be the unique edge in $M^*[B,U]$ with $u \in U$ and $b \in B$. Let $A' := A \cup \{b\}$ and $B' := B \setminus \{b\}$. Apply Lemma 7.17 with G, M^*, A', B' playing the roles of G, M, A, B. Suppose first that (i) holds. Then G contains a basic connector $\mathcal{P}$ with $\mathrm{bal}_{A'B'}(\mathcal{P}) \geq 1$. But $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{A'B'}(\mathcal{P}) - d_{\mathcal{P}}(b)$ if $b \in V(\mathcal{P})$ and $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{A'B'}(\mathcal{P})$ otherwise. If $d_{\mathcal{P}}(b) = 1$ then $\mathrm{bal}_{AB}(\mathcal{P}) \geq 0$, as required. Suppose that $d_{\mathcal{P}}(b) = 2$. Write $F_{\mathcal{P}}(A') := (a'_1, a'_2)$. Thus $a'_2 = 1$ by (BC4). Moreover, the consequence (i) of Lemma 7.17 implies that $a'_1 \geq 2$. Now $a'_1 + 2a'_2 \leq \mathrm{bal}_{A'B'}(\mathcal{P}) + 2 \leq 4$ by (BC2) and (BC4), so $(a'_1, a'_2) = (2,1)$ and $\mathrm{bal}_{A'B'}(\mathcal{P}) = 2$. Then $\mathrm{bal}_{AB}(\mathcal{P}) \geq 0$, as required. Let $F_{\mathcal{P}}(A) =: (a_1, a_2)$. As above, $(a_1, a_2) \in \{(a'_1 - 1, a'_2), (a'_1, a'_2 - 1), (a'_1, a'_2)\}$. So $a_1 \geq a'_1 - 1 \geq 1$ by the consequence (i) of Lemma 7.17. Suppose instead that the consequence (ii) of Lemma 7.17 holds. The 'in particular' part implies that G contains a basic connector $\mathcal{P}_b$ with $\mathrm{bal}_{A'B'}(\mathcal{P}_b) = 0$, $F_{\mathcal{P}_b}(A) = (2,0)$ and $b \notin V(\mathcal{P}_b)$. Then $\mathrm{bal}_{AB}(\mathcal{P}_b) = \mathrm{bal}_{A'B'}(\mathcal{P}_b)$, and $F_{\mathcal{P}_b}(A) = F_{\mathcal{P}_b}(A')$ as required. □

[121] Let $\mathcal{P}_a := M_1 \cup M_2$. Then $\mathrm{bal}_{AB}(\mathcal{P}_a) = 0$ and $a \notin V(\mathcal{P}_a)$ and $F_{\mathcal{P}_a} = (2,0)$. Moreover, $\mathcal{P}$ is a basic connector, so (ii) holds.

7.6.3. *Accommodating path systems.* The following proposition gives a lower bound for $\mathrm{acc}(G; A_1, A_2)$ whenever G contains several vertices of degree much larger than $|A_1| + |A_2|$ (i.e. when the consequence (ii) of Lemma 7.6 holds in G).[122]

build2paths

Proposition 7.19. *Let $\Delta' \in \mathbb{N}$ and let $\ell, a_1, a_2 \in \mathbb{N}_0$ be such that $\Delta' \geq 3\ell + a_1 + a_2$. Let G be a graph and let X be a collection of ℓ vertices in G such that $d_G(x) \geq \Delta'$ for all $x \in X$. Then for all disjoint $A_1, A_2 \subseteq V(G)$ with $|A_i| = a_i$ for $i = 1, 2$, we have*

$$\mathrm{acc}(G; A_1, A_2) \geq 2\ell - |X \cap A_1| - 2|X \cap A_2|.$$

Proof. Write $X := \{x_1, \dots, x_\ell\}$. Since $\Delta' \geq 3\ell + a_1 + a_2$ we can choose distinct vertices $w_1, \dots, w_\ell, y_1, \dots, y_\ell$ such that $\{w_i, y_i\} \subseteq N(x_i) \setminus (A_1 \cup A_2 \cup X)$. For each $1 \leq i \leq \ell$, define

Picases

$$P_i := \begin{cases} x_i y_i & \text{if } x_i \in A_1; \\ \emptyset & \text{if } x_i \in A_2; \\ w_i x_i y_i & \text{otherwise.} \end{cases} \tag{7.12}$$

Then $\mathcal{P} := \bigcup_{1 \leq i \leq \ell} P_i$ is a path system which accommodates A_1, A_2. Clearly

accbound2

$$\mathrm{acc}(G; A_1, A_2) \geq e(\mathcal{P}) = 2\ell - |X \cap A_1| - 2|X \cap A_2|, \tag{7.13}$$

as required. □

The following proposition shows that, if A contains a collection X of vertices of high degree and G contains a basic connector $\mathcal{P}_0$ which does not interact too much with X, then we can extend $\mathcal{P}_0$ such that it still induces an Euler tour but $\mathrm{bal}_{AB}(\mathcal{P}_0)$ has increased.

2paths2

Proposition 7.20. *Let $\Delta' \in \mathbb{N}$ and let $\ell, r \in \mathbb{N}_0$ be such that $\Delta' \geq 3\ell + 4$. Let G be a graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$ and let $\mathcal{P}_0$ be a basic connector in G. For $i = 1, 2$, let A_i be the collection of all those vertices in A with degree i in $\mathcal{P}_0$. Let $X := \{x_1, \dots, x_\ell\} \subseteq A$ where $d_A(x_i) \geq \Delta'$ for all $1 \leq i \leq \ell$. Suppose that $X \cap A_2 = \emptyset$ and $|X \setminus A_1| \geq r$. Then G contains a path system $\mathcal{P}$ such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{AB}(\mathcal{P}_0) + \ell + r$ and $e(\mathcal{P}) \leq \ell + r + 4$.*

Proof. Write $F_{\mathcal{P}_0}(A) := (a_1, a_2)$. So $|A_i| = a_i$ and hence $a_1 + a_2 = |V(\mathcal{P}_0) \cap A| \leq 4$ by (BC2) and (BC3). Therefore we can apply Proposition 7.19 to see that

$$\mathrm{acc}(G[A]; A_1, A_2) \geq 2\ell - |X \cap A_1| - 2|X \cap A_2| \geq 2\ell - (\ell - r) = \ell + r.$$

Then Proposition 7.16 implies that there exists a path system $\mathcal{P}$ as required. □

The following lemma gives lower bounds for $\mathrm{acc}(G[A]; A_1, A_2)$. Together with Proposition 7.16, this will enable us to see 'how far' we can extend a basic connector. We show that $\mathrm{acc}(G[A]; A_1, A_2)$ is 'sufficiently large' unless we are in one of two special cases.

accommodation

Lemma 7.21. *Let $k \in \{0, 1\}$, $\Delta, \Delta', \ell \in \mathbb{N}$ be such that $\ell + k \geq 2$. Suppose that $\Delta'/\Delta, \ell/\Delta' \ll 1$. Let G be a graph with vertex partition U, A and suppose that $e_G(A) \geq (\ell - 1)\Delta + \Delta'$ and $\Delta(G[A]), \Delta(G[A, U]) \leq \Delta$. Let $a_1, a_2 \in \mathbb{N}_0$ with $a_1 \geq k$ and $\Delta' \geq 3\ell + a_1 + a_2$. Let $A_1, A_2 \subseteq A$ be disjoint such that $|A_i| = a_i$ for $i = 1, 2$. Then one of the following holds.*

(I) $\mathrm{acc}(G[A]; A_1, A_2) \geq \ell - a_1 - 2a_2 + k + 2$;
(II) $k = 1$, $(a_1, a_2) = (1, 0)$ *and* $\mathrm{acc}(G[A]; A_1, A_2) \geq \ell + 1$;
(III) $k = 1$, $1 \leq \ell, a_1 + a_2 \leq 2$, $e_G(A) \leq \ell\Delta$ *and* $\mathrm{acc}(G[A]; A_1, A_2) \geq \ell - a_2$. *Moreover, let $X := \{x \in A : d_A(x) \geq \Delta'\}$. Then $|X| = \ell$ and all edges of $G[A]$ are incident with X.*

[122]Deryk replaced $\mathrm{acc}(G; a_1, a_2)$ by $\mathrm{acc}(G; A_1, A_2)$ in the next prop and the rest of the section (the next prop was wrong before).

Proof. Apply Lemma 7.6 to $G[A]$. Suppose first that (i) holds. Let M be a matching in $G[A]$ of size $\ell+1$ and let $uv \in E(G[A])$ be such that $u \notin V(M)$. Obtain M' from M by deleting all those edges with both endpoints in A_1 or at least one endpoint in A_2. Then M' accommodates A_1, A_2 by construction, so

acc1 (7.14)
$$\mathrm{acc}(G[A]; A_1, A_2) \geq e(M') \geq \ell + 1 - \lfloor a_1/2 \rfloor - a_2.$$

If $\lceil a_1/2 \rceil + a_2 \geq k+1$, then (7.14) implies that (I) holds.

So suppose instead that $\lceil a_1/2 \rceil + a_2 \leq k$. First consider the case $k=0$. Then $\lceil a_1/2 \rceil + a_2 = 0$ and hence $(a_1, a_2) = (0,0)$. Now $A_1 = A_2 = \emptyset$, so $M \cup \{uv\}$ is a path system which accommodates A_1, A_2, and $e(M \cup \{uv\}) = \ell + 2$, so (I) holds.

Now consider the case $k=1$. We have $\lceil a_1/2 \rceil + a_2 \leq 1$. But $a_1 \geq k \geq 1$ so $(a_1, a_2) = (1,0)$. Observe that $\mathrm{acc}(G[A]; A_1, A_2) \geq \ell + 1$ by (7.14). So (II) holds.

Suppose now that the consequence (i) of Lemma 7.6 does not hold in $G[A]$. Since $\ell \geq 1$, we have $e_G(A) \leq \ell\Delta$ by the final assertion in Lemma 7.6. Let $X := \{x \in A : d_A(x) \geq \Delta'\}$. Then $|X| \geq \ell$. Since the consequence (i) of Lemma 7.6 does not hold, we must have that $|X| = \ell$ and that all edges of $G[A]$ are incident with X.[123]

Apply Proposition 7.19 to see that

accheavy (7.15)
$$\begin{aligned}\mathrm{acc}(G[A]; A_1, A_2) &\geq 2\ell - |X \cap A_1| - 2|X \cap A_2| \geq 2\ell - \min\{a_1, \ell - a_2\} - 2a_2 \\ &= \ell - a_1 - 2a_2 + \max\{\ell, a_1 + a_2\} \geq \ell - a_2.\end{aligned}$$

[124] In particular, if $\max\{\ell, a_1 + a_2\} \geq k+2$, (7.15) implies that (I) holds. So we may suppose that $\max\{\ell, a_1 + a_2\} \leq k+1$. Recall that $k + \ell \geq 2$ and $a_1 \geq k$ in the hypothesis. Hence, we have $k=1$ and so $1 \leq \ell, a_1 + a_2 \leq 2$. So (III) holds. □

We are now ready to prove Lemma 7.3 in the case when $|A| - |B| \geq 2$ and $m \leq 2$. Roughly speaking, the approach is as follows. Proposition 7.15 implies that G contains a basic connector $\mathcal{P}_0$. When $m=2$, Lemmas 7.17 and 7.18 allow us to assume that $\mathrm{bal}_{AB}(\mathcal{P}_0)$ is non-negative. We would like to extend $\mathcal{P}_0$ to a path system $\mathcal{P}$ in such a way that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour and $\mathrm{bal}_{AB}(\mathcal{P}) = \ell + m/2 \geq |A| - |B|$. Proposition 7.16 implies that, in order to do this, it suffices to find a path system $\mathcal{P}_A$ in $G[A]$ which accommodates A_1, A_2 (where A_i is the collection of all those vertices in A with degree i in $\mathcal{P}_0$) and has enough edges. Now Lemma 7.21 implies that we can do this unless $m=2$, ℓ is small and $(|A_1|, |A_2|)$ takes one of a small number of special values. Some additional arguments are required in these cases.

Proof of Lemma 7.3 in the case when $|A| - |B| \geq 2$ and $m \leq 2$.[125] Let $k := m/2$. Since $m \in 2\mathbb{N}_0$ we have $k \in \{0,1\}$. Let $\Delta := D/2$, $\Delta' := \nu n$ and $U := V_1 \cup V_2$. Proposition 7.10 implies that

l+k (7.16)
$$\ell + k \geq |A| - |B| \geq 2.$$

Proposition 7.4 implies that $\ell, m \leq 12\rho n$. Then $\Delta'/\Delta, \ell/\Delta', m/\Delta' \ll 1$, $\Delta'/\Delta \ll \varepsilon$. Proposition 7.5 implies that

lm (7.17)
$$e_G(A) \geq (\ell-1)\Delta + \Delta' \quad \text{and} \quad e_G(A, U) \geq (m-1)\Delta + \Delta'.$$

By Proposition 7.15, G contains a basic connector $\mathcal{P}_0$. Further assume that $\mathrm{bal}_{AB}(\mathcal{P}_0)$ is maximal, and given $\mathrm{bal}_{AB}(\mathcal{P}_0)$, a_1 is maximal where $F_{\mathcal{P}_0}(A) := (a_1, a_2)$. Let

$$t := |A| - |B| - \mathrm{bal}_{AB}(\mathcal{P}_0).$$

Then (BC2) implies that $t \geq 0$. In fact we may assume that $t \geq 1$ as otherwise $\mathcal{P}_0$ satisfies (P1)–(P3). For $i = 1, 2$ let A_i be the set of all those vertices in A which have degree i in $\mathcal{P}_0$. So

[123] DK reformulated this sentence

[124] $\ell - a_1 - 2a_2 + \max\{\ell, a_1 + a_2\} \geq \ell - (a_1 + a_2) - a_2 + (a_1 + a_2)$

[125] DK changed quite a bit in this proof...

$|A_i| = a_i$. Proposition 7.16 implies that, to prove Lemma 7.3, it suffices to show that

$$\mathrm{acc}(G[A]; A_1, A_2) \geq t.$$

(To check (P1), note that (BC2) and (7.16) imply $t \leq |A| - |B| + 2 \leq \ell + k + 2 \leq \ell + m + 2$.)

Claim A.

(i) *Suppose that* $k = 1$. *Then* $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 0$, *and if* $\mathrm{bal}_{AB}(\mathcal{P}_0) = 0$ *then* $a_1 \geq 1$.
(ii) $a_1 \geq k$.[126]

To prove Claim A(i), note that if $k = 1$ (and so $m = 2$), then (7.17) and Lemma 7.6 imply that $G[A, U]$ contains a matching of size two. Together with Lemma 7.18 and our choice of $\mathcal{P}_0$ this in turn implies Claim A(i). Claim A(ii) clearly holds if $k = 0$, so assume $k = 1$. If $\mathrm{bal}_{AB}(\mathcal{P}_0) = 2$, then $a_1 \geq 1$ by (BC4). Together with Claim A(i) this shows that we may assume that $\mathrm{bal}_{AB}(\mathcal{P}_0) = 1$. By (BC4), we may further assume that $(a_1, a_2) = (0, 1)$. Then (7.10) implies that $e_{\mathcal{P}_0}(B, U) = 0$. But then $\mathcal{P}_0$ has no endpoints in $W = A \cup B$, contradicting (BC1). This completes the proof of Claim A.

Apply Lemma 7.21 with $G \setminus B, A, U, F_{\mathcal{P}_0}(A), \ell, k$ playing the roles of $G, A, U, (a_1, a_2), \ell, k$. Suppose first that (I) holds, so

$$\mathrm{acc}(G[A]; A_1, A_2) \geq \ell - a_1 - 2a_2 + k + 2 \overset{(BC4),(7.16)}{\geq} |A| - |B| - \mathrm{bal}_{AB}(\mathcal{P}_0) = t,$$

as required. Therefore we may assume that one of the consequences (II) or (III) of Lemma 7.21 holds. So $k = 1$ and therefore $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 0$ by Claim A(i). Suppose first that (II) holds. Then

$$\mathrm{acc}(G[A]; A_1, A_2) \geq \ell + 1 \overset{(7.16)}{\geq} |A| - |B| \geq t,$$

as required. Therefore we may assume that (III) holds. So $1 \leq \ell, a_1 + a_2 \leq 2$, $e_G(A) \leq \ell\Delta$ and $\mathrm{acc}(G[A]; A_1, A_2) \geq \ell - a_2$. Let $X := \{x \in A : d_A(x) \geq \Delta'\}$. Then the consequence (III) of Lemma 7.21 also implies that $|X| = \ell$ and all edges of $G[A]$ are incident with X.

We claim that we are done if $\mathrm{bal}_{AB}(\mathcal{P}_0) \neq a_2$. To see this, suppose first that $\mathrm{bal}_{AB}(\mathcal{P}_0) \leq a_2 - 1$. Since $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 0$ this implies that $a_2 = 1$ and $\mathrm{bal}_{AB}(\mathcal{P}_0) = 0$. But $a_1 \geq k \geq 1$ by Claim A(ii) and $a_1 + a_2 \leq 2$, so $a_1 = a_2 = 1$. This is a contradiction to (BC4). Suppose instead that $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq a_2 + 1$. Then

$$\mathrm{acc}(G[A]; A_1, A_2) \geq \ell - a_2 \geq \ell + 1 - \mathrm{bal}_{AB}(\mathcal{P}_0) = \ell + 1 - (|A| - |B|) + t \overset{(7.16)}{\geq} t.$$

Therefore we may assume that $\mathrm{bal}_{AB}(\mathcal{P}_0) = a_2$. In particular, this together with (BC4) implies that $\mathrm{bal}_{AB}(\mathcal{P}_0) \in \{0, 1\}$. We claim that we can further assume that

ellequal (7.18) $$\ell = |A| - |B| - 1.$$

Indeed, to see this, note that by (7.16), it suffices to show that we are done if $\ell \geq |A| - |B|$. But in this case we have $\mathrm{acc}(G[A]; A_1, A_2) \geq \ell - a_2 \geq |A| - |B| - a_2 = t$, as required.

We will now distinguish two cases.

Case 1. $G[A, U]$ *contains a matching of size three.*[127]

Recall that $\mathrm{bal}_{AB}(\mathcal{P}_0) \in \{0, 1\}$. So Lemma 7.17 and our choice of $\mathcal{P}_0$ imply that $a_1 \geq 2$. Since $a_1 + a_2 \leq 2$ we have that $(a_1, a_2) = (2, 0)$. Therefore $\mathrm{bal}_{AB}(\mathcal{P}_0) = a_2 = 0$. Now, by Lemma 7.17 and our choice of $\mathcal{P}_0$ we deduce that there is some $i \in \{1, 2\}$ such that for $j \in \{1, 2\} \setminus \{i\}$ and for

[126] KS: We do need to prove the claim like this. Lemma 7.18 implies there exists $\mathcal{P}$ with $\mathrm{bal}_{AB}(\mathcal{P}) \geq 0$ and $a_1 \geq 1$ (for this $\mathcal{P}$). So this could only give us some $\mathcal{P}$ with $\mathrm{bal}_{AB}(\mathcal{P}) = 0$ and $a_1 = 1$ (for this $\mathcal{P}$). By our choice of $\mathcal{P}_0$, we have that $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 0$. So we could have $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 1$. But then the existence of $\mathcal{P}$ tells us nothing about a_1 for $\mathcal{P}_0$, so this needs to be checked separately.

[127] KS: we can't assume that $\mathrm{bal}_{AB}(\mathcal{P}_0) \leq 0$, as we could have $(a_1, a_2) = (1, 1)$.

each $a \in A$, there are matchings $M_{a,A}, M_{a,B}$ in $G[A \setminus \{a\}, V_i], G[B, V_j]$ respectively, each of which has size two. Moreover, $\mathcal{P}_a := M_{a,A} \cup M_{a,B}$ is a basic connector with $\mathrm{bal}_{AB}(\mathcal{P}_a) = 0$.

Let $x \in X$ be arbitrary. (Recall that $|X| = \ell \geq 1$.) Apply Proposition 7.20 with $\mathcal{P}_x, V(M_{x,A}) \cap A, \emptyset, X, \ell, 1$ playing the roles of $\mathcal{P}_0, A_1, A_2, X, \ell, r$ to obtain a path system $\mathcal{P}$ in G such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{AB}(\mathcal{P}_x) + \ell + 1 = |A| - |B|$ (using (7.18)), and $e(\mathcal{P}) \leq \ell + 5$. Thus, $\mathcal{P}$ satisfies (P1)–(P3).

Case 2. *$G[A, U]$ does not contain a matching of size three.*

Together with König's theorem on edge-colourings this implies that $e_G(A, U) \leq 2\Delta$.

Claim B. *$X \cap V(\mathcal{P}_0) = \emptyset$.*

Since $e_G(A, U) \leq 2\Delta$, the consequence (ii) of Proposition 4.7 implies that

$$e_G(A) \geq \Delta(|A| - |B|) - e_G(A, U)/2 \overset{(7.18)}{\geq} \ell\Delta.$$

In fact, equality holds since $e_G(A) \leq \ell\Delta$ by the consequence (III) of Lemma 7.21. Since all edges of $G[A]$ are incident with X and $|X| = \ell$ it follows that $d_A(x) = \Delta = D/2$ for all $x \in X$. For all $x \in X$, $d_U(x) = D - d_A(x) - d_B(x) \leq D - 2d_A(x) = D - 2\Delta = 0$. The claim follows by (BC3).

Recall that we assume that $t \geq 1$. Observe that, since $\mathrm{bal}_{AB}(\mathcal{P}_0) \in \{0, 1\}$, the definition of t and (7.18) imply that $1 \leq t \leq |A| - |B| = \ell + 1$. Choose an arbitrary $X' \subseteq X$ with $|X'| = t - 1$. Apply Proposition 7.20 with $\mathcal{P}_0, X', t-1, 1$ playing the roles of $\mathcal{P}_0, X, \ell, r$ to obtain a path system $\mathcal{P}$ in G such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, $\mathrm{bal}_{AB}(\mathcal{P}) = \mathrm{bal}_{AB}(\mathcal{P}_0) + t = |A| - |B|$, and $e(\mathcal{P}) \leq \ell + 5$. Thus, $\mathcal{P}$ satisfies (P1)–(P3). □

+1

7.7. **The proof of Lemma 7.3 in the case when $|A| = |B| + 1$.** Note that the extremal example in Figure 1(i) satisfies the conditions of this case. Therefore the degree bound $D \geq n/4$ is essential here. We will follow a similar strategy as in Section 7.6. We first find a basic connector $\mathcal{P}_0$ and then modify it to obtain a path system $\mathcal{P}$ satisfying (P1)–(P3). To be more precise, $\mathcal{P}$ will satisfy $e(\mathcal{P}) \leq 6$ and $\mathrm{bal}_{AB}(\mathcal{P}) = 1$. Throughout this section, we will assume that the basic connector $\mathcal{P}_0$ is chosen so that $|\mathrm{bal}_{AB}(\mathcal{P}_0) - 1|$ is minimal. We will distinguish cases depending on the value of $\mathrm{bal}_{AB}(\mathcal{P}_0)$.

Let G be a D-regular graph with vertex partition A, B, U where $|A| = |B| + 1$. Then the consequence (i) of Proposition 4.7 implies that

balance1 (7.19)
$$2e_G(A) + e_G(A, U) = 2e_G(B) + e_G(B, U) + D.$$

We will need the following simple facts for the case when $|\mathrm{bal}_{AB}(\mathcal{P}_0)| = 2$.[128]

balmin **Proposition 7.22.** *Let G be a 3-connected graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Then the following holds:*

- (i) *if $\mathcal{P}_0$ is a basic connector in G with $\mathrm{bal}_{AB}(\mathcal{P}_0) = 2$, then $V(\mathcal{P}_0) \cap B = \emptyset$ and $\mathcal{P}_0[A, V_i]$ is a matching of size two for each $i = 1, 2$. In particular, $\mathcal{P}_0[A, V_1 \cup V_2]$ contains a matching of size three.*
- (ii) *if $e_G(B, U) > 0$ and G contains a basic connector $\mathcal{P}'_0$ with $\mathrm{bal}_{AB}(\mathcal{P}'_0) = 2$, then G also contains a basic connector $\mathcal{P}_0$ with $\mathrm{bal}_{AB}(\mathcal{P}_0) = 1$;*
- (iii) *if $e_G(A, U) > 0$ then G contains a basic connector $\mathcal{P}_0$ with $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq -1$;*
- (iv) *if $e_G(A, U), e_G(B, U) > 0$ then G contains a basic connector $\mathcal{P}_0$ with $|\mathrm{bal}_{AB}(\mathcal{P}_0)| \leq 1$.*

Proof. (i) follows immediately from (BC1)–(BC4). To prove (ii), note that by (i), for both $i = 1, 2$ there are matchings $M_i \subseteq G[A, V_i]$ of size two such that $\mathcal{P}'_0 = M_1 \cup M_2$. Let $e \in E(G[B, U])$ be arbitrary. Without loss of generality, suppose that $e \in E(G[B, V_1])$. If possible, let $e' \in E(M_1)$ be the edge incident with e; otherwise let $e' \in E(M_1)$ be arbitrary. Then $\mathcal{P}_0 := (\mathcal{P}'_0 \cup \{e\}) \setminus \{e'\}$

[128]DK: changed (ii) and its proof

is a basic connector with $\mathrm{bal}_{AB}(\mathcal{P}_0) = 1$, as required. (iii) and (iv) follow from Proposition 7.15 together with an argument similar to the one for (ii).[129] □

The next lemma concerns the case when $G[A, V_1 \cup V_2]$ contains a matching of size three. This extra condition ensures the existence of a basic connector with useful properties of which we can take advantage.[130]

3okay **Lemma 7.23.** *Let $n, D \in \mathbb{N}$ be such that $D \geq n/4$ and $1/n \ll 1$.*[131] *Let G be a 3-connected D-regular graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$, where $|V_i| \geq D/2$ for $i = 1, 2$. Suppose that $|A| = |B|+1$, that $\Delta(G[A, V_1 \cup V_2]) \leq D/2$ and that $G[A, V_1 \cup V_2]$ contains a matching of size three. Then G contains a path system $\mathcal{P}$ which satisfies* (P1)–(P3).

Proof. Let $U := V_1 \cup V_2$. Without loss of generality we may assume that $e_G(A, V_1) \leq e_G(A, V_2)$. We will obtain $\mathcal{P}$ by adding at most two edges to a basic connector $\mathcal{P}_0$. Therefore $e(\mathcal{P}) \leq 6$ so (P1) will hold. We may assume that there does not exist a basic connector $\mathcal{P}_0'$ with $\mathrm{bal}_{AB}(\mathcal{P}_0') = 1$ (otherwise we can take $\mathcal{P} := \mathcal{P}_0'$). Apply Lemma 7.17 to obtain a basic connector in G which satisfies (i) or (ii).

Case 1. *The consequence (i) of Lemma 7.17 holds.*

So G contains a basic connector $\mathcal{P}_0$ such that $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 1$ and, if $F_{\mathcal{P}_0}(A) = (a_1, a_2)$, then $a_1 \geq 2$. Thus $\mathrm{bal}_{AB}(\mathcal{P}_0) = 2$ by our assumption. The consequence (i) of Proposition 7.22 implies that $V(\mathcal{P}_0) \cap B = \emptyset$. Furthermore, the consequence (ii) of Proposition 7.22 implies that $e_G(B, U) = 0$. Suppose that $e_G(B) \geq 1$. For arbitrary $e \in E(G[B])$ we have that $\mathcal{P} := \mathcal{P}_0 \cup \{e\}$ satisfies (P1)–(P3). So we may assume that $e_G(B) = 0$. So (7.19) implies that

AUsum
$$2e_G(A) + e_G(A, U) = D. \tag{7.20}$$

Moreover, for each $b \in B$ we have that $N_G(b) \subseteq A$ and thus $|A| \geq D$. So $|B| \geq D - 1$ and since $D \geq n/4$ we have that $|U| \leq 2D+1$. We will only prove the case when $|V_1| = D - s$ for some $s \in \mathbb{N}_0$. (The same argument also works for $|V_2| = D - s$.) Recall that $s \leq D/2$ by assumption. Then every vertex in V_1 has at least $s + 1$ neighbours in $\overline{V_1}$. Since $e_G(B, U) = 0$ and $e_G(A, V_1) \leq e_G(A, V_2)$ we have that[132]

$$e_G(V_1, V_2) \geq e_G(V_1, \overline{V_1}) - e_G(A, V_1) \overset{(7.20)}{\geq} (s + 1)(D - s) - D/2 \geq D/2.$$

Suppose that $\mathcal{P}_0$ is a matching of size four in $G[A, U]$. Then, given any $e \in E(G[V_1, V_2])$, we can choose $e_i \in \mathcal{P}_0[A, V_i]$ such that e, e_1, e_2 is a matching of size three. Otherwise, the consequence (i) of Proposition 7.22 implies that $\mathcal{P}_0$ consists of vertex-disjoint paths u_1a_1, u_2a_2, v_1av_2, where $v_i, u_i \in V_i$ and $a, a_1, a_2 \in A$. Since $e_G(V_1, V_2) \geq 2$, we can pick $e \in E(G[V_1, V_2]) \setminus \{u_1u_2\}$. It is easy to see that we can similarly find $e_i \in E(\mathcal{P}_0[A, V_i])$ such that e, e_1, e_2 is a matching of size three. In both cases, $\mathcal{P} := \{e, e_1, e_2\}$ satisfies (P1)–(P3).

Case 2. *The consequence (ii) of Lemma 7.17 holds.*

Since $e_G(A, V_1) \leq e_G(A, V_2)$ this implies that $e_G(V_1, A) = 0$. Moreover, the consequence (ii) of Lemma 7.17 also implies that, for each $a \in A$, there are matchings $M_{a,A}, M_{a,B}$ in $G[A \setminus \{a\}, V_2], G[B, V_1]$ respectively, each of which has size two. In particular $e_G(B, U) \geq 2$.[133] Suppose that $e_G(A) > 0$. Let $aa' \in E(G[A])$. Then $\mathcal{P} := M_{a,A} \cup M_{a,B} \cup \{aa'\}$ satisfies (P1)–(P3). So we may assume that $e_G(A) = 0$. Then (7.19) implies that $e_G(A, V_2) = e_G(A, U) \geq D + e_G(B, U) \geq D + 2$.[134] The 'moreover' part of Lemma 7.6 with $G[A, V_2], D/2, 2$ playing the roles of G, Δ, ℓ implies that

[129] I don't think (iv) follows from the statements of (ii) and (iii)...
[130] DK: changed statement + proof of next lemma
[131] DK added $1/n \ll 1$ since we need $2/D = 1/\Delta \ll 1$ in order to apply Lemma 7.6
[132] for $0 \leq s \leq D/2$ the function $(s + 1)(D - s) - D/2 = -s^2 + sD + D/2$ is minimized if $s = 0$
[133] LATE CHANGE: new sentence.
[134] LATE CHANGE: New calculation, to save defining $\mathcal{P}_a$.

$G[A, V_2]$ contains a matching M_A of size three and an edge xy with $x \notin V(M_A)$. Let $a \in A$ be arbitrary. Then $\mathcal{P} := M_{a,B} \cup M_A \cup \{xy\}$ satisfies (P1)–(P3). □

The following proposition will be used to find edges in $G[A]$ which can be added to a basic connector $\mathcal{P}_0$ so that it is still a path system and $R_{\mathcal{V}}(\mathcal{P}_0)$ is still an Euler tour. For example, if $a \in A$ is such that $d_{\mathcal{P}_0}(a) = 2$, then we cannot add any edges in $G[A]$ which are incident with a. (Recall that the partition given in Lemma 7.3 satisfies $d_A(a) \leq d_B(a)$ for all $a \in A$.)

sumfact **Proposition 7.24.** *Let G be a D-regular graph with vertex partition A, B, U where $|A| = |B| + 1$. Let $a \in A$ be such that $d_A(a) \leq d_B(a)$. Then*

$$2e_G(A \setminus \{a\}) + e_G(A \setminus \{a\}, U) \geq e_G(B, U).$$

Proof. Note that[135]

$$\begin{aligned} 2e_G(A \setminus \{a\}) + e_G(A \setminus \{a\}, U) &= 2e_G(A) + e_G(A, U) - 2d_A(a) - d_U(a) \\ &\geq 2e_G(A) + e_G(A, U) - d_A(a) - d_B(a) - d_U(a) \\ &= 2e_G(A) + e_G(A, U) - D \overset{(7.19)}{\geq} e_G(B, U), \end{aligned}$$

as required. □

By Lemma 7.23, we may assume that $G[A, V_1 \cup V_2]$ contains no matching of size three. Then the consequence (i) of Proposition 7.22 allows us to assume that $\mathrm{bal}_{AB}(\mathcal{P}_0) \leq 0$ (or we are done). In the next lemma, we consider the case when $\mathrm{bal}_{AB}(\mathcal{P}_0) = 0$.

bal0 **Lemma 7.25.** *Let $D \in \mathbb{N}$. Let G be a 3-connected D-regular[136] graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Suppose that $|A| = |B| + 1$, $\Delta(G[A, V_1 \cup V_2]) \leq D/2$ and $d_A(a) \leq d_B(a)$ for all $a \in A$. Suppose further that $G[A, V_1 \cup V_2]$ does not contain a matching of size three. Let $\mathcal{P}_0$ be a basic connector in G with $\mathrm{bal}_{AB}(\mathcal{P}_0) = 0$. Then G contains a path system $\mathcal{P}$ which satisfies* (P1)–(P3).

Proof. Let $U := V_1 \cup V_2$. Since $G[A, U]$ does not contain a matching of size three, König's theorem on edge-colourings implies that

AUedges (7.21) $$e_G(A, U) \leq D.$$

Property (BC4) implies that $a_1 + 2a_2 \in \{1, 2\}$ and so $F_{\mathcal{P}_0}(A) \in \{(2, 0), (1, 0), (0, 1)\}$. We will distinguish cases based on the value of $F_{\mathcal{P}_0}(A)$.

Case 1. $F_{\mathcal{P}_0}(A) = (2, 0)$.

Then (7.10) implies that $e_{\mathcal{P}_0}(A, U) = e_{\mathcal{P}_0}(B, U) = 2$. Since $\mathcal{P}_0$ is an Euler tour and $e(\mathcal{P}_0) \leq 4$ by (BC1) and (BC2), there are distinct vertices $a, a' \in A$, a collection of distinct vertices $X := \{u, u', v, v'\} \subseteq U$ with $|X \cap V_i| = 2$ for $i = 1, 2$ and $b, b' \in B$ which are not necessarily distinct, such that $\mathcal{P}_0 := \{au, a'u', bv, b'v'\}$.[137]

Observe that we are done if there exists $e \in E(G[A]) \setminus \{aa'\}$ since then $\mathcal{P}_0 \cup \{e\}$ satisfies (P1)–(P3). So we may assume that $E(G[A]) \subseteq \{aa'\}$. Now

$$2 = e_{\mathcal{P}_0}(B, U) \leq e_G(B, U) \overset{(7.21)}{\leq} 2e_G(B) + e_G(B, U) + D - e_G(A, U) \overset{(7.19)}{=} 2e_G(A) \leq 2.$$

Therefore we have $e_G(B) = 0$, $e_G(A) = 1$, $e_G(A, U) = D$ and $e_G(B, U) = 2$, so $E(G[B, U]) = \{bv, b'v'\}$ and $E(G[A]) = \{aa'\}$.

[135] LATE CHANGE: Removed (i) from Prop.

[136] Deryk added D-regular

[137] The old proof didn't work because I was assuming that b, b' were distinct. So Case 1.b. below is essentially the extra part.

We will assume that either $\{u,u'\} \subseteq V_1$ and $\{v,v'\} \subseteq V_2$; or $\{u,v\} \subseteq V_1$ and $\{u',v'\} \subseteq V_2$ since the other cases are similar.

Case 1.a. $\{u,u'\} \subseteq V_1$ *and* $\{v,v'\} \subseteq V_2$.

Suppose that $e_G(V_1,V_2) \neq 0$. Let $v_1v_2 \in E(G[V_1,V_2])$ with $v_i \in V_i$. Choose $e_1 \in \mathcal{P}_0[A, V_1 \setminus \{v_1\}]$ and $e_2 \in \mathcal{P}_0[B, V_2 \setminus \{v_2\}]$. Then $\mathcal{P} := \{e_1, e_2, v_1v_2, aa'\}$ satisfies (P1)–(P3). Suppose that $e_G(A,V_2) \neq 0$. Let $a''x_2 \in E(G[A,V_2])$ with $a'' \in A$ and $x_2 \in V_2$. Choose $e_2 \in \mathcal{P}_0[B, V_2 \setminus \{x_2\}]$. Then $\mathcal{P} := \{au, a'u', a''x_2, e_2\}$ satisfies (P1)–(P3). Therefore $e_G(A \cup V_1, V_2) = 0$. So $E(G[V_2, \overline{V_2}]) = \{bv, b'v'\}$, contradicting the 3-connectivity of G.

Case 1.b. $\{u,v\} \subseteq V_1$ *and* $\{u',v'\} \subseteq V_2$.

We may assume that $b = b'$ since otherwise $\mathcal{P} := \mathcal{P}_0 \cup \{aa'\}$ satisfies (P1)–(P3). Since $G[A,U]$ does not contain a matching of size three, every edge in $G[A,U]$ is incident with at least one of a, a', u, u'. Suppose that there exists $a'' \in A \setminus \{a,a'\}$ such that $ua'' \in E(G)$. Then $\mathcal{P} := \mathcal{P}_0 \cup \{ua'', aa'\} \setminus \{ua\}$ satisfies (P1)–(P3). A similar deduction can be made with u' playing the role of u. Therefore every edge in $G[A,U]$ is incident with a or a'. Since $e_G(A,U) = D$ we have $d_U(a), d_U(a') = D/2$.

Suppose that $e_G(V_1,V_2) \neq 0$. Let $v_1v_2 \in E(G[V_1,V_2])$ with $v_i \in V_i$. If $v_1 \neq u$ and $v_2 \neq u'$ then $\mathcal{P} := \{au, a'u', v_1v_2\}$ satisfies (P1)–(P3). Therefore we may suppose, without loss of generality, that $v_1 = u$. Suppose that $v_2 \neq u'$. Then $\mathcal{P} := \{a'u', v_1v_2, bv, aa'\}$ satisfies (P1)–(P3). Therefore we may suppose that $v_2 = u$. Thus $uu' \in E(G)$. Since $d_U(a) \geq D/2$, we can choose $w \in N_U(a) \setminus \{v, v', u, u'\}$. Suppose that $w \in V_1$. Then $\mathcal{P} := \{aw, uu', aa', bv'\}$ satisfies (P1)–(P3). If $w \in V_2$ then $\mathcal{P} := \{aw, uu', aa', bv\}$ satisfies (P1)–(P3).

Thus we may assume that $e_G(V_1,V_2) = 0$. Choose $Y_a \in \{V_1,V_2\}$ such that $d_{Y_a}(a) \geq D/4$. Note that there is always such a Y_a. Define $Y_{a'}$ analogously. Suppose that $Y_{a'} = V_1$. Choose $w' \in N_{V_1}(a') \setminus \{u,v\}$. Then $\mathcal{P} := \mathcal{P}_0 \cup \{a'w'\} \setminus \{bv\}$ satisfies (P1)–(P3). We can argue similarly if $Y_a = V_2$.

Therefore we may assume that $Y_{a'} = V_2$ and $Y_a = V_1$. Suppose that $d_{V_1}(a') \neq 0$. Let $w' \in N_{V_1}(a')$. Since $d_{V_1}(a) \geq D/4$, we can choose $w \in N_{V_1}(a) \setminus \{w'\}$.[138] Then $\mathcal{P} := \mathcal{P}_0 \cup \{aw, a'w'\} \setminus \{au, bv\}$ satisfies (P1)–(P3).[139] So $d_{V_1}(a') = 0$. Since every edge of $G[A,U]$ is incident with a or a', we have that every edge in $G[A,V_1]$ is incident with a. We have shown that every edge in $G[V_1, \overline{V_1}]$ is incident with a or b, contradicting the 3-connectivity of G.

Case 2. $F_{\mathcal{P}_0}(A) = (1,0)$.

Then (7.10) implies that $e_G(B,U) \geq e_{\mathcal{P}_0}(B,U) = 1$. So (7.19) and (7.21) give $2e_G(A) = D + 2e_G(B) + e_G(B,U) - e_G(A,U) \geq 1$. Let $e \in E(G[A])$ be arbitrary. Then $\mathcal{P} := \mathcal{P}_0 \cup \{e\}$ satisfies (P1)–(P3).

Case 3. $F_{\mathcal{P}_0}(A) = (0,1)$.

Now (7.10) implies that $e_{\mathcal{P}_0}(B,U) = e_{\mathcal{P}_0}(A,U) = 2$. So (BC2) implies that $e_{\mathcal{P}_0}(V_1,V_2) = 0$ and that there exist distinct $v_i, u_i \in V_i$ for $i = 1,2$, and $b, b' \in B$ and $a \in A$ such that $\mathcal{P}_0 = \{v_1b, v_2b', u_1au_2\}$. Proposition 7.24 implies that $2e_G(A \setminus \{a\}) + e_G(A \setminus \{a\}, U) \geq 2$. Suppose first that $e_G(A \setminus \{a\}) \geq 1$. Choose $e \in E(G[A \setminus \{a\}])$. Then $\mathcal{P} := \mathcal{P}_0 \cup \{e\}$ satisfies (P1)–(P3). Therefore we may assume that $e_G(A \setminus \{a\}, U) \geq 2$. Suppose there exists $e' \in E(G[A \setminus \{a\}, U \setminus \{u_1,u_2\}])$. Without loss of generality, suppose that e' has an endpoint in V_1. Then $\mathcal{P} := \mathcal{P}_0 \cup \{e'\} \setminus \{v_1b\}$ satisfies (P1)–(P3). Therefore we may assume that G contains an edge $a'u_1$ where $a' \in A \setminus \{a\}$. Let $\mathcal{P}'_0 := \mathcal{P}_0 \cup \{a'u_1\} \setminus \{au_1\}$. Then $\mathcal{P}'_0$ is a basic connector with $\mathrm{bal}_{AB}(\mathcal{P}'_0) = 0$ and $F_{\mathcal{P}'_0}(A) = (2,0)$. So we are in Case 1. □

The next lemma concerns the case when $\mathrm{bal}_{AB}(\mathcal{P}_0) = -1$.

[138] DK replaced $w \in N_{V_1}(a) \setminus \{w', v\}$ by $w \in N_{V_1}(a) \setminus \{w'\}$

[139] DK replaced $\mathcal{P} := \mathcal{P}_0 \cup \{a'w'\} \setminus \{bv\}$ by $\mathcal{P} := \mathcal{P}_0 \cup \{aw, a'w'\} \setminus \{au, bv\}$

bal-1 **Lemma 7.26.** *Let $D \in \mathbb{N}$ where $D \geq 12$. Let G be a 3-connected D-regular graph with vertex partition $\mathcal{V} = \{V_1, V_2, W := A \cup B\}$. Suppose that $|A| = |B| + 1$, $\Delta(G[A, V_1 \cup V_2] \leq D/2$ and $d_A(a) \leq d_B(a)$ for all $a \in A$. Let $\mathcal{P}_0$ be a basic connector in G such that $|\mathrm{bal}_{AB}(\mathcal{P}_0) - 1|$ is minimal. Suppose that $\mathrm{bal}_{AB}(\mathcal{P}_0) = -1$. Then G contains a path system $\mathcal{P}$ which satisfies* (P1)–(P3).

Proof. Let $U := V_1 \cup V_2$. Observe that $G[A, U]$ does not contain a matching of size two since otherwise Lemma 7.18 would imply that $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 0$. Therefore $e_G(A, U) \leq D/2$, and so (7.19) implies that

D/4bound (7.22) $$e_G(A) \geq D/4.$$

Write $F_{\mathcal{P}_0}(A) := (a_1, a_2)$. Then (BC4) implies that $a_1 + 2a_2 \in \{0, 1\}$. So $(a_1, a_2) \in \{(0, 0), (1, 0)\}$. Suppose first that $(a_1, a_2) = (0, 0)$. Then by (7.22), we can choose distinct $e, e' \in E(G[A])$. In this case $\mathcal{P} := \mathcal{P}_0 \cup \{e, e'\}$ satisfies (P1)–(P3).

Now suppose that $(a_1, a_2) = (1, 0)$. Then (7.10) implies that

BUedges (7.23) $$e_G(B, U) \geq e_{\mathcal{P}_0}(B, U) = 3.$$

Let au be the single edge in $\mathcal{P}_0[A, U]$, where $a \in A$ and $u \in U$. Note that any edge in $E(G[A \setminus \{a\}, U])$ is incident with u since $G[A, U]$ contains no matching of size two. So $e_G(A \setminus \{a\}, U) = d_{A \setminus \{a\}}(u)$. Thus Proposition 7.24 and (7.23) imply that

3edges (7.24) $$2e_G(A \setminus \{a\}) + d_{A \setminus \{a\}}(u) \geq 3.$$

Suppose first that $d_A(a) \leq 1$. In this case, (7.22) implies that $e_G(A \setminus \{a\}) \geq D/4 - 1 \geq 2$. Let $e, e' \in E(G[A \setminus \{a\}])$ be distinct. Then $\mathcal{P} := \mathcal{P}_0 \cup \{e, e'\}$ satisfies (P1)–(P3).

Now suppose that $d_A(a) \geq 2$. Let $a', a'' \in N_A(a)$ be distinct. Suppose that $e_G(A \setminus \{a\}) \neq 0$. Then we can choose $e \in E(G[A \setminus \{a\}])$, and $\mathcal{P} := \mathcal{P}_0 \cup \{aa', e\}$ satisfies (P1)–(P3). Suppose instead that $e_G(A \setminus \{a\}) = 0$. Then $d_{A \setminus \{a\}}(u) \geq 3$ by (7.24), so there exists $a^* \in A \setminus \{a, a', a''\}$ such that $ua^* \in E(G[A, U])$.[140] We have that $\mathcal{P} := \mathcal{P}_0 \cup \{ua^*, a'aa''\} \setminus \{ua\}$ satisfies (P1)–(P3). □

We are now ready to combine the preceding lemmas to prove Lemma 7.3 fully in the case when $|A| = |B| + 1$.

Proof of Lemma 7.3 in the case when $|A| = |B| + 1$. Let $U := V_1 \cup V_2$. Suppose first that $G[A, U]$ contains a matching of size three. Then we are done by Lemma 7.23, so assume not. Proposition 7.15 implies that G contains a basic connector. Choose a basic connector $\mathcal{P}_0$ in G such that $|\mathrm{bal}_{AB}(\mathcal{P}_0) - 1|$ is minimal. Recall that (BC2) implies $|\mathrm{bal}_{AB}(\mathcal{P}_0)| \leq 2$. Since $G[A, U]$ does not contain a matching of size three, the consequence (i) of Proposition 7.22 implies that $\mathrm{bal}_{AB}(\mathcal{P}_0) \leq 1$. We may assume that $\mathrm{bal}_{AB}(\mathcal{P}_0) \leq 0$ or we are done. Lemmas 7.25 and 7.26 prove the lemma in the case when $\mathrm{bal}_{AB}(\mathcal{P}_0) = 0, -1$ respectively. So we may assume that $\mathrm{bal}_{AB}(\mathcal{P}_0) = -2$. Thus, by (7.10), we have $e_G(B, U) \geq 4$. Moreover, by the consequence (iii) of Proposition 7.22 we may assume that $e_G(A, U) = 0$. Now (7.19) implies $e_G(A) \geq D/2 + 2$. The 'moreover' part of Lemma 7.6 with $G[A], D/2, 1$ playing the roles of G, Δ, ℓ implies that $G[A]$ contains a matching M_A of size two and an edge aa' with $a \notin V(M_A)$. So $\mathcal{P} := \mathcal{P}_0 \cup M_A \cup \{aa'\}$ satisfies (P1)–(P3). □

equal

7.8. **The proof of Lemma 7.3 in the case when $|A| = |B|$.** In this subsection we consider the only remaining case of Lemma 7.3: when the bipartite vertex classes A and B have equal size. Our aim is to find a path system $\mathcal{P}$ such that $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour, and $\mathrm{bal}_{AB}(\mathcal{P}) = 0$. As in the previous section, we will appropriately modify a basic connector guaranteed by Proposition 7.15. The degree bound $D \geq n/4$ is used again here.

[140]Clearly $\mathcal{P}'_0 := \mathcal{P}_0 \cup \{ua^*\} \setminus \{ua\}$ is a basic connector with $\mathrm{bal}_{AB}(\mathcal{P}_0) = -1$ and $a \notin V(\mathcal{P}'_0)$.

Proof of Lemma 7.3 in the case when $|A| = |B|$. Let $U := V_1 \cup V_2$. The consequence (i) of Proposition 4.7 implies that

balance0
$$2e_G(A) + e_G(A, U) = 2e_G(B) + e_G(B, U). \tag{7.25}$$

Proposition 7.15 implies that G contains a basic connector. Choose a basic connector $\mathcal{P}_0$ in G such that $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$ is minimal. Write $F_{\mathcal{P}_0}(A) := (a_1, a_2)$.

Suppose first that $e_G(B, U) = 0$.[141] Then

$$2\mathrm{bal}_{AB}(\mathcal{P}_0) \overset{(7.10)}{=} a_1 + 2a_2 = e_{\mathcal{P}_0}(A, U) \leq e_G(A, U) \overset{(7.25)}{\leq} 2e_G(B).$$

(In particular, $\mathrm{bal}_{AB}(\mathcal{P}_0) \geq 0$.) Let $E' \subseteq E(G[B])$ be a collection of $\mathrm{bal}_{AB}(\mathcal{P}_0)$ distinct edges (so $|E'| \leq 2$ by (BC2)). Then $\mathcal{P} := \mathcal{P}_0 \cup E'$ satisfies (P1)–(P3). Thus we may assume that $e_G(B, U) \geq 1$ and a similar argument allows us to assume that $e_G(A, U) \geq 1$.

Together with the 3-connectivity of G, this implies that $G[W, U]$ contains a matching M of size two such that one edge is incident with A and one edge is incident with B.[142] The consequence (iv) of Proposition 7.22 and our choice of $\mathcal{P}_0$ together imply that $|\mathrm{bal}_{AB}(\mathcal{P}_0)| \leq 1$. Without loss of generality we suppose that $\mathrm{bal}_{AB}(\mathcal{P}_0) = -1$ (otherwise $\mathrm{bal}_{AB}(\mathcal{P}_0) = 1$ and we could swap A and B, or $\mathrm{bal}_{AB}(\mathcal{P}_0) = 0$ and we are done by taking $\mathcal{P} := \mathcal{P}_0$). Then (BC4) implies that $(a_1, a_2) \in \{(0, 0), (1, 0)\}$. If $e_G(A) \geq 1$ then, for any $e \in E(G[A])$ we have that $\mathcal{P} := \mathcal{P}_0 \cup \{e\}$ satisfies (P1)–(P3). So we may assume that

eGA
$$e_G(A) = 0. \tag{7.26}$$

Claim 1. *$G[A, U]$ does not contain a matching of size two.*

To prove the claim, suppose not. We will show that if $G[A, U]$ contains a matching of size two, then the minimality of $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$ will be contradicted. First consider the case when $(a_1, a_2) = (1, 0)$. So $e_{\mathcal{P}_0}(A, U) = 1$ and therefore $e_{\mathcal{P}_0}(B, U) = 3$ by (7.10). But (BC2) implies that $e(\mathcal{P}_0) \leq 4$, so $e_{\mathcal{P}_0}(V_1, V_2) = 0$. Now by (BC1) we have that $|V(\mathcal{P}_0) \cap V_i| = 2$ for $i = 1, 2$, and $d_{\mathcal{P}_0}(v) = 1$ for all $v \in V(\mathcal{P}_0) \cap V_i$. In particular, $e_{\mathcal{P}_0}(V_i, B) > 0$ for both $i = 1, 2$. Let e be the single edge in $\mathcal{P}_0[A, U]$. Without loss of generality, we may assume that $G[A, U]$ contains an edge e' which is vertex-disjoint from e. (Otherwise, $G[A, U]$ contains a matching $av, a'v'$ such that $e = av'$. Then $\mathcal{P}_0' := \mathcal{P}_0 \cup \{a'v'\} \setminus \{e\}$ is a basic connector with $\mathrm{bal}_{AB}(\mathcal{P}_0') = \mathrm{bal}_{AB}(\mathcal{P}_0)$ and $a'v'$ is the single edge in $\mathcal{P}_0'[A, U]$; and av is an edge which is vertex-disjoint from $a'v'$.)[143] Suppose first that e' has an endpoint in V_1. If possible, choose $f \in E(\mathcal{P}_0[V_1, B])$ which is incident with e'; otherwise let $f \in E(\mathcal{P}_0[V_1, B])$ be arbitrary. Then $\mathcal{P} := \mathcal{P}_0 \cup \{e'\} \setminus \{f\}$ contradicts the minimality of $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$. The case when e' has an endpoint in V_2 is similar.

Suppose now that $(a_1, a_2) = (0, 0)$. Then $e_{\mathcal{P}_0}(A, U) = 0$ and hence $e_{\mathcal{P}_0}(B, U) = 2$. Moreover, $\mathcal{P}_0[B, U]$ is a matching e, e' since $\mathcal{P}_0$ is an Euler tour by (BC1). Now $d_{R_{\mathcal{V}}(\mathcal{P}_0)}(V_i) \geq 2$ for $i = 1, 2$, so $e_{\mathcal{P}_0}(V_1, V_2) \geq 1$. But (BC2) implies that $e(\mathcal{P}_0) \leq 4$, so $e_{\mathcal{P}_0}(V_1, V_2) \leq 2$. Suppose that $e_{\mathcal{P}_0}(V_1, V_2) = 1$ and let $f \in E(\mathcal{P}_0[V_1, V_2])$. Then $\mathcal{P}_0 = \{e, e', f\}$ is a matching of size three. Moreover $e_{\mathcal{P}_0}(B, V_i) = 1$ for $i = 1, 2$. If there exists $e_A \in E(G[A, U] \setminus V(f))$ then we can replace one of e, e' by e_A to contradict the minimality of $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$. Therefore there is a matching $\{e_A, e_A'\} \subseteq E(G[A, U])$ such that both e_A, e_A' are incident to $V(f)$. Then they are vertex-disjoint from $\{e, e'\}$, so $\mathcal{P} := \{e, e', e_A, e_A'\}$ contradicts the minimality of $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$. Suppose now that $e_{\mathcal{P}_0}(V_1, V_2) = 2$. Then $\mathcal{P}_0[B, U] \subseteq G[B, V_i]$ for some $i = 1, 2$. Without loss of generality we assume that $i = 2$. Suppose that there exists $e_A \in E(G[A, V_1])$. Choose $f \in E(\mathcal{P}_0[V_1, V_2])$ that is not incident to e_A. Choose $e_B \in E(\mathcal{P}_0[B, V_2])$ that is not incident to f. Then $\mathcal{P} := \{e_A, f, e_B\}$ contradicts the minimality of $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$. Therefore we may assume that there is a matching

[141] DK changed this para

[142] $G[W, U]$ contains a matching of size three by 3-connectivity. If it is contained in $G[A, U]$ then replace one edge by the edge in $G[B, U]$. and vice versa.

[143] this bracket is more detailed.

$M_A \subseteq G[A, V_2]$ of size two. There is at least one V_1V_2-path in $\mathcal{P}_0$ (which consists of a single edge f'). Choose $e \in M_A$ which is not incident to f'. If possible, let e_B be the edge of $\mathcal{P}_0[B, V_2]$ which is incident to e; otherwise let $e_B \in E(\mathcal{P}_0[B, V_2])$ be arbitrary. Then $\mathcal{P} := \mathcal{P}_0 \cup \{e\} \setminus \{e_B\}$ contradicts the minimality of $|\mathrm{bal}_{AB}(\mathcal{P}_0)|$. This completes the proof of the claim.

Therefore $e_G(A, U) \leq D/2$ since $\Delta(G[A, U)] \leq D/2$. So (7.25) and (7.26) together imply that

eWU
$$e_G(W, U) = e_G(B, U) - e_G(A, U) + 2e_G(A, U) \leq D. \tag{7.27}$$

Suppose first that $|A| = |B| = D - k$ for some $k \in \mathbb{N}$. Then (7.26) implies that, for all $a \in A$, we have $d_U(a) = D - d_A(a) - d_B(a) \geq D - |B| = k$. So $e_G(A, U) \geq k|A| = k(D - k) \geq D - 1$, a contradiction. So $|A| = |B| \geq D$ and hence $|U| = n - |A| - |B| \leq n - 2D \leq 2D$ since $D \geq n/4$.

Claim 2. *There exists a matching M' of size three in $G[V_1, V_2]$.*

To prove the claim, assume without loss of generality that $|V_1| \leq |V_2|$. Then there exists $s \in \mathbb{N}_0$ such that $|V_1| = D - s$. Recall from our assumption in Lemma 7.3 that[144] $|V_1| \geq D/2$. Suppose first that $s \geq 2$. Then

$$\begin{aligned} e_G(V_1, V_2) &\geq D|V_1| - e_G(U, W) - 2\binom{|V_1|}{2} \overset{(7.27)}{\geq} |V_1|(D - |V_1| + 1) - D \\ &\geq \min\{D^2/4 - D/2, 2D - 6\} \geq D + 1. \end{aligned} \tag{7.28}$$

Recall that $d_{V_i}(x_i) \geq d_{V_j}(x_i)$ for all $x_i \in V_i$ and $\{i, j\} = \{1, 2\}$. So $\Delta(G[V_1, V_2]) \leq D/2$. Therefore we are done by König's theorem on edge-colourings.

Thus we may assume that $s \in \{0, 1\}$. Let $H := G[V_1, V_2]$. Suppose that H contains no matching of size three. By König's theorem on vertex covers, H contains a vertex cover $\{v_i, v_j\}$ where $v_i \in V_i$, $v_j \in V_j$ and i, j are not necessarily distinct. So $e(H) \leq d_H(v_i) + d_H(v_j)$. Note that the complement $\overline{G}$ of G satisfies[145]

missingedges
$$\begin{aligned} e_{\overline{G}}(V_1) + e_{\overline{G}}(V_2) &\geq d_{\overline{G}[V_i]}(v_i) + d_{\overline{G}[V_j]}(v_j) - 1 = |V_i| - d_{V_i}(v_i) + |V_j| - d_{V_j}(v_j) - 3 \\ &\geq D - d_{V_i}(v_i) + D - d_{V_j}(v_j) - 5 \geq d_H(v_i) + d_H(v_j) - 5 \\ &\geq e(H) - 5. \end{aligned} \tag{7.29}$$

Therefore by counting the degrees in G of the vertices in U, we have that

$$\begin{aligned} e_G(U, W) &= \sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v) - 2e(H) - 2e_G(V_1) - 2e_G(V_2) \\ &= D(|V_1| + |V_2|) - 2e(H) - 2\left(\binom{|V_1|}{2} - e_{\overline{G}}(V_1) + \binom{|V_2|}{2} - e_{\overline{G}}(V_2)\right) \\ &\overset{(7.29)}{\geq} D(|V_1| + |V_2|) - 10 - 2\binom{|V_1|}{2} - 2\binom{|V_2|}{2} \\ &= |V_1|(D - |V_1|) + |V_2|(D - |V_2|) + |V_1| + |V_2| - 10 \geq 2D - 14, \end{aligned}$$

a contradiction to (7.27).[146] This proves the claim.

Recall that M is a matching of size two in $G[W, U]$ with one edge incident to A and one edge incident to B. Assume without loss of generality that $e_M(V_2, W) \geq e_M(V_1, W)$. There exists $e \in E(M')$ which is vertex-disjoint from M. Suppose first that $e_M(V_2, W) = 2$. Let $e' \in E(M') \setminus \{e\}$ be arbitrary. Then $\mathcal{P} := M \cup \{e, e'\}$ satisfies (P1)–(P3). Suppose instead that $e_M(V_2, W) = e_M(V_1, W) = 1$. Then $\mathcal{P} := M \cup \{e\}$ satisfies (P1)–(P3). This completes the proof of Lemma 7.3 in all cases. □

[144] DK rewrote this sentence

[145] -1: if $v_iv_j \in E(\overline{G})$ and $i = j$.

[146] $|V_1|(D - |V_1|) + |V_2|(D - |V_2|) = 0$ if $s = 0$ and is at least $(D - 1) - (D + 1) = -2$ when $s = 1$. Moreover, $|V_1| + |V_2| \geq 2D - 2$

8. The proof of Theorem 1.1

proof

We are now ready to prove Theorem 1.1. It is a consequence of Theorem 4.6 and Lemma 4.10 (both proved in [10]), as well as Lemmas 5.1, 6.1 and 7.1.

Proof of Theorem 1.1. Choose a non-decreasing function $g : (0,1) \to (0,1)$ with $g(x) \leq x$ for all $x \in (0,1)$ such that the requirements of Proposition 4.5 and Lemmas 4.10, 5.1, 6.1, 7.1 (each applied, where relevant, with $1/32, 1/4$ playing the roles of η, α) are satisfied whenever $n, \rho, \gamma, \nu, \tau$ satisfy

hierarchy

$$1/n \leq g(\rho), g(\gamma); \quad \rho, \gamma \leq g(\nu); \quad \nu \leq g(\tau); \quad \tau \leq g(1/32). \tag{8.1}$$

Choose τ, τ' so that

$$0 \leq \tau' \leq \tau \leq g(1/32), 40^{-3} \quad \text{and} \quad \tau' \leq g(\tau).$$

Define a function $g' : (0,1) \to (0,1)$ by $g'(x) = (g(x))^3$.[147] Apply Theorem 4.6 with $g', \tau', 1/20$ playing the roles of g, τ, ε to obtain an integer n_0.[148] Let G be a 3-connected D-regular graph on $n \geq n_0$ vertices where $D \geq n/4$. We may assume that the consequence (ii) of Theorem 4.6 holds or we are done. Thus there exist ρ, ν with $1/n_0 \leq \rho \leq \nu \leq \tau'$, $1/n_0 \leq g'(\rho)$ and $\rho \leq g'(\nu)$; and $(k, \ell) \in \{(4,0), (2,1), (0,2)\}$ such that G has a robust partition $\mathcal{V}$ with parameters $\rho, \nu, \tau', k, \ell$ (and thus also a robust partition with parameters ρ, ν, τ, k, ℓ).

Let $\gamma := \rho^{1/3}$. Note that $n, \rho, \gamma, \nu, \tau$ satisfy (8.1).[149] Apply Lemmas 5.1, 6.1 in the cases when (k, ℓ) equals $(4,0), (0,2)$ respectively to obtain a $\mathcal{V}$-tour of G with parameter γ.[150] Proposition 4.5 implies that $\mathcal{V}$ is a weak robust partition with parameters $\rho, \nu, \tau, 1/32, k, \ell$. Then Lemma 4.10 implies that G contains a Hamilton cycle. Apply Lemma 7.1 in the case when $(k, \ell) = (2, 1)$ to obtain a Hamilton cycle in G. This completes the proof of the theorem. □

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[147] Note that $g'(x) < g(x) \leq g(x^{1/3})$ for all $x \in (0,1)$, where the second inequality holds since g is non-decreasing. Note also that $2\tau'^{1/3} \leq \varepsilon$.

[148] Have $\gamma = \rho^{1/3}$ in Lemma 6.1 so use g^3 to ensure that here $\gamma = \rho^{1/3} \leq g(\nu)$.

[149] Then $1/n \leq 1/n_0 \leq g'(\rho) \leq g(\rho^{1/3}) = g(\gamma)$, and $\gamma = \rho^{1/3} \leq (g')^{1/3}(\nu) = g(\nu)$

[150] (Note that a $\mathcal{V}$-tour with parameter γ is a $\mathcal{V}$-tour with parameter γ' whenever $\gamma \leq \gamma'$.)

[151] NEW and updated refs

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