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DOI:
10.1017/S0308210515000712

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Document Version
Peer reviewed version

Citation for published version (Harvard):
Meyer, JC & Needham, DJ 2016, 'Aspects of Hadamard well-posedness for classes of non-Lipschitz semilinear parabolic partial differential equations', Proceedings of the Royal Society of Edinburgh: Section A (Mathematics), vol. 146, no. 4, pp. 777-832. https://doi.org/10.1017/S0308210515000712

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Aspects of Hadamard well-posedness for classes of non-Lipschitz semi-linear parabolic partial differential equations

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(MS received 'Received date'; 'Accepted date')

In this paper we study classical solutions of the Cauchy problem for a class of non-Lipschitz semi-linear parabolic partial differential equations in one spatial dimension with sufficiently smooth initial data. When the nonlinearity is Lipschitz continuous results concerning existence, uniqueness and continuous dependence on initial data are well established (see, for example, the texts of Friedman [7], and Smoller [32] and in the context of the present paper, see also Meyer [19]), as are the associated results concerning Hadamard well-possedness. In this paper we first consider the situation when the nonlinearity is Hölder continuous, and then when the nonlinearity is upper Lipschitz continuous. Finally we consider the situation when the nonlinearity is both Hölder continuous and upper Lipschitz continuous. In each case the focus is placed upon the question of existence, uniqueness and continuous dependence on initial data, and thus upon aspects of Hadamard well-posedness.

1. Introduction and Motivation

This paper addresses two classes of non-Lipschitz semi-linear parabolic Cauchy problems. Specifically, we consider the problem of finding a continuous and bounded function $u:\mathbb{R}\times[0,T]\to\mathbb{R}$ (for some $T>0$), for which u_t, u_x and u_{xx} exist and are continuous in $\mathbb{R}\times(0,T]$, and which satisfies,

$$u_t - u_{xx} = f(u) \quad \forall (x,t)\in\mathbb{R}\times(0,T],$$

$$u(x,0) = u_0(x) \quad \forall x\in\mathbb{R}.$$

Here $f:\mathbb{R}\to\mathbb{R}$ is the nonlinear term and $u_0:\mathbb{R}\to\mathbb{R}$ is the initial data. We refer to this initial value problem as (B-R-D-C). A function u which satisfies all of the above conditions is referred to as a solution to (B-R-D-C) on $\mathbb{R}\times[0,T]$ with initial data u_0. In the situation when the nonlinear term $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, extensive results for (B-R-D-C), concerning existence, uniqueness and continuous dependence on initial data u_0, and consequently on Hadamard well-posedness, are well established for (B-R-D-C) (see, for example, Friedman [7], Fife [6], Rothe [27], Smoller [32], Samarvskii et al [29], Volpert et al [33], Leach and Needham [12], and in the context of the present paper, see Meyer [19], Chapter 6). It is the purpose of this paper to consider the corresponding questions for (B-R-D-C) in detail for two broader classes of nonlinear term $f:\mathbb{R}\to\mathbb{R}$. In particular we will consider the situation when $f:\mathbb{R}\to\mathbb{R}$ is Hölder continuous and when $f:\mathbb{R}\to\mathbb{R}$ is upper

Lipschitz continuous, after which these cases may be combined to consider the situation when $f : \mathbb{R} \to \mathbb{R}$ is both Hölder and upper Lipschitz continuous. The relevance and motivation for considering nonlinear terms $f : \mathbb{R} \to \mathbb{R}$ in these classes is detailed in the specific studies of Aguirre and Escobedo [1], Grundy and Peletier [9], Herrero and Velázquez [11], Meyer and Needham [21], Needham [23], McCabe, Leach and Needham [15], [16], [17] and [18] and discussed in detail in Meyer [19], (Chapter 1).

The paper is structured as follows. In Section 2 we introduce the fundamental notation, concepts and definitions used throughout the paper. In Section 3 we obtain an equivalence lemma for (B-R-D-C) with Hölder continuous nonlinearity f which shows that solutions $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ of (B-R-D-C) on $\mathbb{R} \times [0,T]$, with sufficiently smooth u_0 are equivalent to continuous and bounded functions $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ which satisfy the integral equation

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau$$

for all $(x,t) \in \mathbb{R} \times [0,T]$. This is achieved through the study of the regularity of the function $\phi : \mathbb{R} \times [0,T] \to \mathbb{R}$ given by

$$\phi(x,t) = \int_0^t \int_{-\infty}^{\infty} F(x + 2\sqrt{t-\tau}\lambda, \tau) e^{-\lambda^2} ds d\tau$$

for all $(x,t) \in \mathbb{R} \times [0,T]$, where $F : \mathbb{R} \times [0,T] \to \mathbb{R}$ is bounded and Hölder continuous in it's first argument uniformly in it's second argument. Specifically we obtain bounds on $\phi_t(x,t)$, $\phi_x(x,t)$ and $\phi_{xx}(x,t)$ for all $(x,t) \in \mathbb{R} \times (0,T]$. New Schauder-type estimates on solutions to (B-R-D-C) then follows directly.

In Section 4 we briefly reference well established results concerning (B-R-D-C) when the nonlinearity f is Lipschitz continuous. Namely, we reference existence, uniqueness and continuous dependence (on a set of sufficiently smooth initial data) results. The results established in Section 3 (in particular the new Schauder estimates) and those classical results referenced at the outset of Section 4 will play a key role in the remainder of the paper. In the substance of Section 4 we obtain our main results for (B-R-D-C) when the nonlinearity f is Hölder continuous. The aim of this section is to establish generic theory for (B-R-D-C) when f is Hölder continuous, which generalises the particular studies of Aguirre and Escobedo [1] and Needham [23] relating to (B-R-D-C) with specific Hölder continuous nonlinearities. We do not, in general, expect generic uniqueness when f is Hölder continuous, as a simple counter example demonstrates (see, for example, Meyer [19], Example 8.28). However, we establish a local existence result for (B-R-D-C) which takes the following form: For sufficiently smooth u_0, there exists $T > 0$ (dependent on f and u_0) such that there exist functions $\underline{u}, \overline{u} : \mathbb{R} \times [0,T] \to \mathbb{R}$ which are solutions to (B-R-D-C) on $\mathbb{R} \times [0,T]$ with initial data u_0. Moreover, if $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is any other solution to (B-R-D-C) on $\mathbb{R} \times [0,T]$ with initial data u_0, then,

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t) \quad \forall (x,t) \in \mathbb{R} \times [0,T].$$

This is achieved by considering the solutions to a sequence of approximating problems to (B-R-D-C) which have Lipschitz continuous nonlinearities. A solution to each of the sequence of approximating problems is guaranteed to exist uniformly locally via the classical results recalled at the outset of Section 4. It is then shown that the limit of this sequence of approximating solutions is a local solution to (B-R-D-C). The maximal and minimal aspect of the solutions is a result of the choice of approximating problems, specifically, the maximal solution is the limit of a sequence of non-increasing super-solutions to (B-R-D-C), whilst the minimal solution is the limit of a sequence of non-decreasing sub-solutions to (B-R-D-C). This local result is extended to a global result under the additional condition of a priori bounds on solutions to (B-R-D-C). From the construction of $\underline{u}$ and $\overline{u}$ in the proof of the local existence result above, we also obtain a conditional comparison theorem, which states that any supersolution $\overline{u}' : \mathbb{R} \times [0,T] \to \mathbb{R}$ and subsolution $\underline{u}' : \mathbb{R} \times [0,T] \to \mathbb{R}$ to (B-R-D-C) with initial data u_0 on $\mathbb{R} \times [0,T]$ satisfies,

$$\overline{u}'(x,t) \geq \underline{u}(x,t), \quad \underline{u}'(x,t) \leq \overline{u}'(x,t) \quad \forall (x,t) \in \mathbb{R} \times [0,T].$$

It follows that when uniqueness holds for the solution to (B-R-D-C) for a particular u_0, then this conditional comparison theorem becomes a classical comparison theorem. In addition, we establish qualitative structural properties of maximal and minimal solutions to (B-R-D-C). These properties are then employed, in conjunction with the construction of $\underline{u}$ and $\overline{u}$ in the proof of the local existence result, to establish a conditional continuous dependence result for (B-R-D-C). The conditional part of this result is on the uniqueness of solutions to (B-R-D-C), the existence of $\lim_{x\to\pm\infty} u_0(x)$, and the local uniqueness of solutions to the initial value problems for the ordinary differential equation, given by,

$$v_t = f(v), \quad v(0) = \lim_{x\to\pm\infty} u_0(x).$$

In Section 5 we consider (B-R-D-C) when the nonlinearity f is upper Lipschitz continuous. To begin this section, we state a uniqueness result which follows from a comparison theorem for (B-R-D-C) with upper Lipschitz nonlinearity f given in Meyer and Needham [20]. This comparison theorem is then used in conjunction with Gronwall's inequality to establish a conditional continuous dependence result for solutions to (B-R-D-C) on a set of sufficiently smooth initial data. The conditional part of this result is on the existence of solutions to (B-R-D-C) when the nonlinearity f is upper Lipschitz continuous; specifically, we suppose that for all initial data in the set of sufficiently smooth data, there exists a solution to (B-R-D-C) on $\mathbb{R}\times[0,T]$ for some $T > 0$. Under additional technical conditions, this continuous dependence result is extended to a global continuous dependence result.

Section 6 brings together the results in Section 4 and Section 5. Specifically, we consider (B-R-D-C) with nonlinearity f, which is both Hölder and upper Lipschitz continuous. A local existence and uniqueness result follows immediately from the results in Section 4 and Section 5. Additionally, since the conditions of the continuous dependence results in Section 5 are achieved by a priori bounded (B-R-D-C) with nonlinearity which is Hölder continuous, it follows that (B-R-D-C) with nonlinearity f, which is both Hölder and upper Lipschitz continuous, is continuously dependent on a set of sufficiently smooth initial data. These results amount to a well-posedness statement for a priori bounded (B-R-D-C) with sufficiently smooth initial data u_0 and nonlinearity f, which is both Hölder and upper Lipschitz continuous.

Finally, in Section 6 we present examples of the applications of the theory developed in the paper, and we discuss the extensions and limitations of this work, in particular in relation to broadening the class of initial data for which the theory applies.

2. Notation and Definitions

We begin by introducing the regions in which the forthcoming initial value problems will be defined. With $T > 0$, $\delta \in [0, T)$ and $X > 0$ we introduce,

$$D_T = \mathbb{R} \times (0, T], \quad \bar{D}_T = \mathbb{R} \times [0, T], \quad \partial D = \mathbb{R} \times \{0\},$$

$$\bar{D}_T^{\delta} = \mathbb{R} \times [\delta, T], \quad \bar{D}_T^{\delta, X} = [-X, X] \times [\delta, T].$$

The content of the paper concerns the study of classical solutions $u : \bar{D}_T \to \mathbb{R}$ to the following semi-linear parabolic Cauchy problem;

$$u_t = u_{xx} + f(u) \quad \forall (x, t) \in D_T, \tag{2.1}$$

$$u(x, 0) = u_0(x) \quad \forall x \in \mathbb{R}, \tag{2.2}$$

$$u(x, t) \text{ is uniformly bounded as } |x| \to \infty \text{ for } t \in [0, T]. \tag{2.3}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is the nonlinearity and $u_0 : \mathbb{R} \to \mathbb{R}$ is the initial data. The initial data is contained in the set of functions $u_0 : \mathbb{R} \to \mathbb{R}$ which are bounded, continuous, with bounded and continuous derivative and bounded and piecewise continuous second derivative, which is denoted as

$$\mathrm{BPC}^2(\mathbb{R}).$$

The partial differential equation (PDE) (2.1) is generally referred to as a *reaction-diffusion equation*, and the initial value problem given by (2.1)-(2.3) will be referred to throughout the paper as the *bounded, reaction-diffusion Cauchy problem*, abbreviated to (B-R-D-C). Moreover, throughout the paper, we adopt the following classical definition of solution to (B-R-D-C), namely,

Definition 2.1. A solution to (B-R-D-C) is a function $u : \bar{D}_T \to \mathbb{R}$ which is continuous and bounded on $\bar{D}_T$ and for which u_t, u_x and u_{xx} exist and are continuous on D_T. Moreover $u : \bar{D}_T \to \mathbb{R}$ must satisfy each of (2.1)-(2.3).

The questions addressed in this paper concern the global well-posedness of (B-R-D-C) in the sense of Hadamard [14]. In particular, for a given $f : \mathbb{R} \to \mathbb{R}$, we seek to establish,

(P1) (Existence) For each $u_0 \in \mathcal{A} \subset \mathrm{BPC}^2(\mathbb{R})$, there exists a solution $u : \bar{D}_T \to \mathbb{R}$ to (B-R-D-C) on $\bar{D}_T$ for each $T > 0$,

(P2) (Uniqueness) Whenever $u : \bar{D}_T \to \mathbb{R}$ and $v : \bar{D}_T \to \mathbb{R}$ are solutions to (B-R-D-C) on $\bar{D}_T$ for the same $u_0 \in \mathcal{A} \subset \mathrm{BPC}^2(\mathbb{R})$, then $u = v$ on $\bar{D}_T$ for each $T > 0$,

(P3) (Continuous Dependence) Given that (P1) and (P2) are satisfied for (B-R-D-C), then given any $u_0' \in \mathcal{A} \subset \text{BPC}^2(\mathbb{R})$ and $\epsilon > 0$, there exists a $\delta > 0$ (which may depend on u_0', T and ϵ) such that for all $u_0 \in \mathcal{A} \subset \text{BPC}^2(\mathbb{R})$, then,

$$\sup_{x\in\mathbb{R}} |u_0(x) - u_0'(x)| < \delta \implies \sup_{(x,t)\in\bar{D}_T} |u'(x,t) - u(x,t)| < \epsilon$$

where $u : \bar{D}_T \to \mathbb{R}$ and $u' : \bar{D}_T \to \mathbb{R}$ are the solutions to (B-R-D-C) corresponding respectively to $u_0, u_0' \in \mathcal{A} \subset \text{BPC}^2(\mathbb{R})$. This must hold for each $T > 0$.

When the above three properties (P1)-(P3) are satisfied by (B-R-D-C), then (B-R-D-C) is said to be *globally well-posed* on $\mathcal{A}$. Moreover, when (P1)-(P3) are satisfied by (B-R-D-C) and the constant δ in (P3) depends only on u_0' and ϵ (that is, being independent of T), then (B-R-D-C) is said to be *uniformly globally well-posed* on $\mathcal{A}$. When one or more of the properties (P1)-(P3) are not satisfied, then (B-R-D-C) is said to be *ill-posed* on $\mathcal{A}$. In addition to well-posedness, we shall address some fundamental qualitative features of solutions to (B-R-D-C).

In conjunction with solutions, we introduce two concepts which will be used throughout the paper. Firstly,

Definition 2.2. Let $\overline{u}$, $\underline{u} : \bar{D}_T \to \mathbb{R}$ be continuous on $\bar{D}_T$ and such that $\underline{u}_t$, $\underline{u}_x$, $\underline{u}_{xx}$, $\overline{u}_t$, $\overline{u}_x$, $\overline{u}_{xx}$ exist and are continuous on D_T. Suppose further that

$$N[\overline{u}] \equiv \overline{u}_t - \overline{u}_{xx} - f(\overline{u}) \geq 0 \quad \text{on } D_T,$$

$$N[\underline{u}] \equiv \underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \leq 0 \quad \text{on } D_T,$$

$$\underline{u}(x,0) \leq u_0(x) \leq \overline{u}(x,0) \quad \forall \ x \in \mathbb{R},$$

$$\underline{u} \text{ and } \overline{u} \text{ are uniformly bounded as } |x| \to \infty \text{ for t } \in [0,T].$$

Then on $\bar{D}_T$, $\underline{u}$ is called a *regular subsolution* (R-S-B) and $\overline{u}$ is called a *regular supersolution* (R-S-P) to (B-R-D-C).

In addition, we require the concept of (B-R-D-C) being a priori bounded. This is formalised in the following definition:

Definition 2.3. Suppose that, for (B-R-D-C), we can exhibit a constant $l_T > 0$ for each $0 \leq T \leq T^*$ (and some $T^* > 0$) which depends only upon T and $||u_0||_B$, and which is non-decreasing in $0 \leq T \leq T^*$. Suppose, furthermore, that if $u : \bar{D}_T \to \mathbb{R}$ is any solution to (B-R-D-C) on $\bar{D}_T$, then it can be demonstrated that,

$$\sup_{(x,t)\in\bar{D}_T} |u(x,t)| \leq l_T,$$

for each $0 \leq T \leq T^*$. We say that (B-R-D-C) is *a priori bounded* on $\bar{D}_T$ for each $0 \leq T \leq T^*$, with bound l_T.

In (B-R-D-C), the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is referred to as the reaction function, and throughout the paper we will restrict attention to those reaction functions f from one or more of the following classes of functions. Firstly we write $f \in L$ whenever $f : \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous* on every closed bounded interval. Similarly, we write $f \in H_\alpha$ $(0 < \alpha \le 1)$ whenever $f : \mathbb{R} \to \mathbb{R}$ is *Hölder continuous of degree* α on every closed bounded interval (note that $H_1 = L$). The third class of functions which will be considered in the paper satisfies a one sided Lipschitz condition, as follows,

Definition 2.4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper Lipschitz continuous* when f is continuous, and for any closed bounded interval $E \subset \mathbb{R}$, there exists a constant $k_E > 0$ such that for all $x, y \in E$, with $y \ge x$,

$$f(y) - f(x) \le k_E(y - x).$$

The set of all such functions $f : \mathbb{R} \to \mathbb{R}$ will be denoted by L_u.

Further discussion of the class of functions L_u can be found in Meyer [19]. In addition to the above, when f depends upon a parameter, we introduce the following two classes of functions.

Definition 2.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following condition: For any pair of closed bounded intervals $U, A \subset \mathbb{R}$, there exist constants $k_U > 0$ and $k_A > 0$ such that for all $(u_1, \alpha_1), (u_2, \alpha_2) \in U \times A$,

$$|f(u_1, \alpha_1) - f(u_2, \alpha_2)| \le k_U|u_1 - u_2| + k_A|\alpha_1 - \alpha_2|.$$

The set of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfy the preceding condition is denoted by L'.

together with,

Definition 2.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy the following conditions: For any pair of closed bounded intervals $U, A \subset \mathbb{R}$, then there exists constants $k_U > 0$ and $k_A > 0$ such that,

(a) for all $u_1, u_2 \in U$ with $u_1 > u_2$, then

$$f(u_1, \alpha) - f(u_2, \alpha) \le k_U(u_1 - u_2) \quad \forall \alpha \in A.$$

(b) for all $\alpha_1, \alpha_2 \in A$, then

$$|f(u, \alpha_1) - f(u, \alpha_2)| \le k_A|\alpha_1 - \alpha_2| \quad \forall u \in U.$$

The set of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfy the preceding conditions is denoted by L'_u.

We note that Definition 2.5 implies that $f(u, \cdot), f(\cdot, \alpha) \in L$, uniformly on compact intervals in u and α respectively, whilst Definition 2.6 implies that $f(\cdot, \alpha) \in L_u$ and $f(u, \cdot) \in L$, uniformly on compact intervals in α and u respectively.

3. Equivalence Lemma, Integral Equation and Schauder Estimates

The main content of this section is an equivalence lemma which establishes that solutions to (B-R-D-C) with $f \in H_\alpha$ are equivalent to solutions of an associated integral equation. This formulation then allows novel Schauder estimates for solutions of (B-R-D-C) with $f \in H_\alpha$ to be developed which play a crucial role in Section 4.

3.1. Convolution functions

To begin, let $F : \bar{D}_T \to \mathbb{R}$ be continuous and bounded. Thus, there exists a constant $M_T > 0$ such that

$$|F(x,t)| \leq M_T \quad \forall (x,t) \in \bar{D}_T. \tag{3.1}$$

We introduce the convolution function $\phi : \bar{D}_T \to \mathbb{R}$ as

$$\phi(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} F(x + 2\sqrt{t-\tau}\ w, \tau) e^{-w^2} dw d\tau \quad \forall (x,t) \in \bar{D}_T. \tag{3.2}$$

It is readily established that ϕ is well-defined on $\bar{D}_T$. Also

$$\phi(x,0) = 0 \quad \forall x \in \mathbb{R}. \tag{3.3}$$

In addition, $\phi : \bar{D}_T \to \mathbb{R}$ is continuous and bounded with

$$|\phi(x,t)| \leq M_T T \quad \forall (x,t) \in \bar{D}_T. \tag{3.4}$$

We next define, on $\bar{D}_T^\delta$ $(0 < \delta < T)$, the sequence of functions $\phi_n : \bar{D}_T^\delta \to \mathbb{R}$ for $n = N_\delta,\ N_\delta + 1, ...$, with $N_\delta = [\delta^{-1}] + 1$, as

$$\phi_n(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} F(x + 2\sqrt{t-\tau}\ w, \tau) e^{-w^2} dw d\tau \quad \forall (x,t) \in \bar{D}_T^\delta. \tag{3.5}$$

The function ϕ_n $(n = N_\delta,\ N_\delta + 1, ...)$ has the following properties:

(a) ϕ_n is continuous on $\bar{D}_T^\delta$.

(b) ϕ_n is bounded on $\bar{D}_T^\delta$, with $|\phi_n(x,t)| \leq M_T T$ $\forall (x,t) \in \bar{D}_T^\delta$.

(c) $\phi_n(x,t) \to \phi(x,t)$ as $n \to \infty$ uniformly $\forall (x,t) \in \bar{D}_T^\delta$.

We now observe that by a simple substitution $(s = x + 2\sqrt{t-\tau}\ w)$, we may write

$$\phi_n(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{F(s,\tau)}{(t-\tau)^{1/2}} e^{-\frac{(s-x)^2}{4(t-\tau)}} ds d\tau \quad \forall (x,t) \in \bar{D}_T^\delta. \tag{3.6}$$

It follows from (3.6), via standard results on uniform convergence of integrals [2], that ϕ_{nx}, ϕ_{nxx} and ϕ_{nt} exist and are continuous on $\bar{D}_T^\delta$, with,

$$\phi_{nx}(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{F(x + 2\sqrt{t-\tau}\ w, \tau)}{(t-\tau)^{1/2}} w e^{-w^2} dw d\tau \quad \forall (x,t) \in \bar{D}_T^\delta, \quad (3.7)$$

$$\phi_{nxx}(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{F(x + 2\sqrt{t-\tau}\ w, \tau)}{(t-\tau)} (w^2 - 1/2) e^{-w^2} dw d\tau \ \forall (x,t) \in \bar{D}_T^\delta, \quad (3.8)$$

$$\phi_{nt}(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{F(x + 2\sqrt{t-\tau}\ w, \tau)}{(t-\tau)} (w^2 - 1/2) e^{-w^2} dw d\tau + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} F(x + 2\sqrt{1/n}\ w, t - 1/n) e^{-w^2} dw \quad \forall (x,t) \in \bar{D}_T^\delta. \quad (3.9)$$

We observe from (3.7) that,

$$\begin{aligned} |\phi_{nx}(x,t)| &\le \frac{M_T}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{1}{(t-\tau)^{1/2}} |w| e^{-w^2} dw d\tau \\ &= \frac{M_T}{\sqrt{\pi}} \left[-2(t-\tau)^{1/2} \right]_0^{t-1/n} \\ &= \frac{2M_T}{\sqrt{\pi}} \left(t^{1/2} - (1/n)^{1/2} \right) \le \frac{2M_T}{\sqrt{\pi}} \left(T^{1/2} + 1 \right) \quad \forall (x,t) \in \bar{D}_T^\delta, \end{aligned} \quad (3.10)$$

and so ϕ_{nx} is bounded on $\bar{D}_T^\delta$, uniformly in n. We now have:

Lemma 3.1. *$\phi : \bar{D}_T \to \mathbb{R}$ in (3.2) is such that ϕ_x exists and is continuous and bounded on D_T, with*

$$|\phi_x(x,t)| \le \frac{2M_T}{\sqrt{\pi}} \left(T^{1/2} + 1 \right) \quad \forall (x,t) \in D_T.$$

Proof. First we recall that ϕ_n and ϕ_{nx} are continuous and bounded on $\bar{D}_T^\delta$ and that $\phi_n \to \phi$ as $n \to \infty$ uniformly on $\bar{D}_T^\delta$. Now let $n \ge m \ge N_\delta$ and $(x,t) \in \bar{D}_T^\delta$, then,

$$\begin{aligned} |\phi_{nx}(x,t) - \phi_{mx}(x,t)| &= \left| \frac{1}{\sqrt{\pi}} \int_{t-1/m}^{t-1/n} \int_{-\infty}^{\infty} \frac{F(x + 2\sqrt{t-\tau}\ w, \tau)}{(t-\tau)^{1/2}} w e^{-w^2} dw d\tau \right| \\ &\le \frac{M_T}{\sqrt{\pi}} \int_{t-1/m}^{t-1/n} \int_{-\infty}^{\infty} \frac{1}{(t-\tau)^{1/2}} |w| e^{-w^2} dw d\tau \\ &= \frac{2M_T}{\sqrt{\pi}} \left(\left(\frac{1}{m} \right)^{1/2} - \left(\frac{1}{n} \right)^{1/2} \right) \le \frac{2M_T}{\sqrt{\pi}} (1/m + 1/n) \end{aligned}$$

for all $(x,t) \in \bar{D}_T^\delta$. It follows that $\{\phi_{nx}\}$ is uniformly convergent on $\bar{D}_T^\delta$ as $n \to \infty$, via the Cauchy condition [28], and moreover via Theorem 7.17 in [28], that ϕ_x exists, is continuous and bounded on $\bar{D}_T^\delta$, with $\phi_{nx} \to \phi_x$ as $n \to \infty$ uniformly on $\bar{D}_T^\delta$. Moreover, using (3.10), we have

$$|\phi_x(x,t)| \le \frac{2M_T}{\sqrt{\pi}} \left(T^{1/2} + 1 \right) \quad \forall (x,t) \in \bar{D}_T^\delta. \quad (3.11)$$

Now all of the above holds for any fixed $0 < \delta < T$, and so it follows that ϕ_x exists, is continuous and bounded on D_T, with,

$$|\phi_x(x,t)| \le \frac{2M_T}{\sqrt{\pi}}\left(T^{1/2}+1\right) \quad \forall (x,t) \in D_T. \tag{3.12}$$

The proof is complete. □

We now restrict $F : \bar{D}_T \to \mathbb{R}$ to satisfy the additional condition:

(H) $F : \bar{D}_T \to \mathbb{R}$ is continuous, bounded and uniformly Hölder continuous of degree $0 < \alpha \le 1$ with respect to $x \in \mathbb{R}$, uniformly for $t \in [0,T]$. That is, there exists a constant $k_T > 0$ (independent of $t \in [0,T]$) such that,

$$|F(y,t) - F(x,t)| \le k_T |y-x|^\alpha \quad \forall (y,t),(x,t) \in \bar{D}_T.$$

We observe that,

$$\begin{aligned}
|\phi_{nxx}(x,t)| &\le \left| \frac{1}{\sqrt{\pi}} \int_0^{t-1/n}\int_{-\infty}^{\infty} \frac{F(x+2\sqrt{t-\tau}\ w,\tau) - F(x,\tau)}{(t-\tau)} (w^2 - 1/2)e^{-w^2} dw d\tau \right| \\
&\quad + \left| \frac{1}{\sqrt{\pi}} \int_0^{t-1/n}\int_{-\infty}^{\infty} \frac{F(x,\tau)}{(t-\tau)} (w^2-1/2)e^{-w^2} dw d\tau \right| \quad (=0) \\
&\le \frac{k_T}{\sqrt{\pi}} \int_0^{t-1/n}\int_{-\infty}^{\infty} \frac{2^\alpha}{(t-\tau)^{1-\alpha/2}} |w|^\alpha |w^2-1/2| e^{-w^2} dw d\tau \\
&= \frac{2^\alpha k_T}{\sqrt{\pi}} I_\alpha \left[\frac{-2}{\alpha}(t-\tau)^{\alpha/2} \right]_0^{t-1/n} \\
&= \frac{2^{\alpha+1} k_T}{\alpha\sqrt{\pi}} I_\alpha \left(t^{\alpha/2} - (1/n)^{\alpha/2} \right) \le \frac{2^{\alpha+1} k_T}{\alpha\sqrt{\pi}} I_\alpha \left(1 + T^{\alpha/2} \right)
\end{aligned} \tag{3.13}$$

for all $(x,t) \in \bar{D}_T^\delta$, where a direct integration establishes the term $(=0)$ above, and with

$$I_\alpha = \int_{-\infty}^{\infty} |w|^\alpha |w^2 - 1/2| e^{-w^2} dw > 0. \tag{3.14}$$

Similarly,

$$|\phi_{nt}(x,t)| \le \frac{2^{\alpha+1} k_T}{\alpha\sqrt{\pi}} I_\alpha \left(1 + T^{\alpha/2}\right) + M_T \quad \forall (x,t) \in \bar{D}_T^\delta. \tag{3.15}$$

Thus, under condition (H), both ϕ_{nt} and ϕ_{nxx} are continuous and bounded (uniformly in n) on $\bar{D}_T^\delta$, for each $n = N_\delta,\ N_\delta + 1,\ \dots$.

We next observe the following:

(I) With $n \geq m \geq N_\delta$ and $(x,t) \in \bar{D}_T^\delta$,

$$\begin{aligned}
&\left|\int_{t-1/m}^{t-1/n}\int_{-\infty}^{\infty}\frac{F(x+2\sqrt{t-\tau}\ w,\tau)}{(t-\tau)}(w^2-1/2)e^{-w^2}dwd\tau\right| \\
&\quad\leq \int_{t-1/m}^{t-1/n}\int_{-\infty}^{\infty}\frac{\left|F(x+2\sqrt{t-\tau}\ w,\tau)-F(x,\tau)\right|}{(t-\tau)}|w^2-1/2|e^{-w^2}dwd\tau \\
&\qquad+\left|\int_{t-1/m}^{t-1/n}\int_{-\infty}^{\infty}\frac{F(x,\tau)}{(t-\tau)}(w^2-1/2)e^{-w^2}dwd\tau\right| \quad (=0) \\
&\quad\leq k_T\int_{t-1/m}^{t-1/n}\int_{-\infty}^{\infty}\frac{2^\alpha}{(t-\tau)^{1-\alpha/2}}|w|^\alpha|w^2-1/2|e^{-w^2}dwd\tau \\
&\quad= k_T 2^\alpha I_\alpha\left[-\frac{2}{\alpha}(t-\tau)^{\alpha/2}\right]_{t-1/m}^{t-1/n} \\
&\quad= \frac{2^{\alpha+1}k_T I_\alpha}{\alpha}\left((1/m)^{\alpha/2}-(1/n)^{\alpha/2}\right) \quad \forall (x,t)\in\bar{D}_T^\delta. \qquad (3.16)
\end{aligned}$$

(II) Let $n \geq N_\delta$. Given any $\epsilon > 0$, there exists $\sigma > 0$ (depending upon ϵ, δ, X, T) such that for all $(x_0,t_0),(x_1,t_1) \in \bar{D}_T^{\delta-1/N_\delta,X}$ with $|(x_1-x_0,t_1-t_0)| < \sigma$, then

$$|F(x_1,t_1)-F(x_0,t_0)| < \epsilon/2,$$

since F is continuous and therefore uniformly continuous on $\bar{D}_T^{\delta-1/N_\delta,X}$. Now let $(x,t) \in \bar{D}_T^{\delta,X}$, then,

$$\begin{aligned}
&\left|\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}F(x+2\sqrt{1/n}\ w,t-1/n)e^{-w^2}dw - F(x,t)\right| \\
&\quad\leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}|F(x+2\sqrt{1/n}\ w,t-1/n)-F(x,t-1/n)|e^{-w^2}dw \\
&\qquad+|F(x,t-1/n)-F(x,t)| \\
&\quad\leq \frac{k_T}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{2^\alpha}{n^{\alpha/2}}|w|^\alpha e^{-w^2}dw + |F(x,t-1/n)-F(x,t)| \\
&\quad\leq \frac{2^\alpha k_T J_\alpha}{\sqrt{\pi}}\frac{1}{n^{\alpha/2}} + |F(x,t-1/n)-F(x,t)| \qquad (3.17)
\end{aligned}$$

where

$$J_\alpha = \int_{-\infty}^{\infty}|w|^\alpha e^{-w^2}dw > 0. \qquad (3.18)$$

Now since $(x,t) \in \bar{D}_T^{\delta,X}$ and $n \geq N_\delta$, then

$$(x,t-1/n),(x,t) \in \bar{D}_T^{\delta-1/N_\delta,X}.$$

Take $n > 1/\sigma + 1$, and so,

$$|(x,t-1/n)-(x,t)| < \sigma \quad \forall (x,t) \in \bar{D}_T^{\delta,X},$$

and so,

$$|F(x,t-1/n) - F(x,t)| < \epsilon/2 \quad \forall (x,t) \in \bar{D}_T^{\delta,X}. \tag{3.19}$$

Therefore, given any $\epsilon > 0$, then for all

$$n > \max\left\{1/\sigma + 1, \left(\frac{2^{\alpha+1}k_T J_\alpha}{\sqrt{\pi}\epsilon}\right)^{2/\alpha} + 1\right\},$$

we have,

$$\left|\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} F(x + 2\sqrt{1/n}\ w, t - 1/n)e^{-w^2}dw - F(x,t)\right| < \epsilon/2 + \epsilon/2 = \epsilon \quad \forall (x,t) \in \bar{D}_T^{\delta,X}.$$

Thus,

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} F(x + 2\sqrt{1/n}\ w, t - 1/n)e^{-w^2}dw \to F(x,t) \tag{3.20}$$

as $n \to \infty$ uniformly on $\bar{D}_T^{\delta,X}$ (any $\delta, X > 0$). We now have:

Lemma 3.2. *$\phi : \bar{D}_T \to \mathbb{R}$ is such that ϕ_t and ϕ_{xx} exist, are continuous and bounded on D_T, with,*

$$|\phi_{xx}(x,t)| \le \frac{2^{\alpha+1}k_T I_\alpha}{\alpha\sqrt{\pi}}(1 + T^{\alpha/2}), \tag{3.21}$$

$$|\phi_t(x,t)| \le \frac{2^{\alpha+1}k_T I_\alpha}{\alpha\sqrt{\pi}}(1 + T^{\alpha/2}) + M_T. \tag{3.22}$$

Moreover,

$$\phi_t(x,t) = \phi_{xx}(x,t) + F(x,t) \quad \forall (x,t) \in D_T. \tag{3.23}$$

Proof. First we recall that ϕ_n and ϕ_{nx} are continuous and bounded uniformly in n on $\bar{D}_T^\delta$ and that $\phi_n \to \phi$ and $\phi_{nx} \to \phi_x$ as $n \to \infty$ uniformly on $\bar{D}_T^\delta$. Moreover, ϕ_{nxx} is continuous and bounded uniformly in n on $\bar{D}_T^\delta$. Now let $n \ge m \ge N_\delta$ and $(x,t) \in \bar{D}_T^\delta$, then it follows from (3.16) that

$$|\phi_{nxx}(x,t) - \phi_{mxx}(x,t)| \le \frac{2^{\alpha+1}k_T I_\alpha}{\alpha\sqrt{\pi}}\left((1/m)^{\alpha/2} + (1/n)^{\alpha/2}\right) \quad \forall (x,t) \in \bar{D}_T^\delta.$$

It follows that $\{\phi_{nxx}\}$ is uniformly convergent on $\bar{D}_T^\delta$ as $n \to \infty$, via the Cauchy condition [28], and moreover via Theorem 7.17 in [28], that ϕ_{xx} exists, is continuous and is bounded on $\bar{D}_T^\delta$, with

$$\phi_{nxx} \to \phi_{xx} \text{ as } n \to \infty \text{ uniformly on } \bar{D}_T^\delta. \tag{3.24}$$

It follows from (3.24) and (3.13) that

$$|\phi_{xx}(x,t)| \le \frac{2^{\alpha+1}k_T I_\alpha}{\alpha\sqrt{\pi}}\left(1 + T^{\alpha/2}\right) \quad \forall (x,t) \in \bar{D}_T^\delta. \tag{3.25}$$

Again recall that ϕ_n and ϕ_{nt} are continuous and bounded uniformly in n on $\bar{D}_T^\delta$ and $\phi_n \to \phi$ as $n \to \infty$ uniformly on $\bar{D}_T^\delta$. It now follows from (3.9) together with

(3.16) and (3.20) that $\{\phi_{nt}\}$ is uniformly convergent on $\bar{D}_T^{\delta,X}$ (any $X > 0$) as $n \to \infty$, and so, moreover, that ϕ_t exists, and is continuous and bounded on $\bar{D}_T^{\delta,X}$, with,

$$\phi_{nt} \to \phi_t \text{ as } n \to \infty \text{ uniformly on } \bar{D}_T^{\delta,X}. \tag{3.26}$$

Moreover, using (3.15), we have,

$$|\phi_t(x,t)| \le \frac{2^{\alpha+1} k_T I_\alpha}{\alpha\sqrt{\pi}} \left(1 + T^{\alpha/2}\right) + M_T \quad \forall (x,t) \in \bar{D}_T^{\delta,X}. \tag{3.27}$$

Now, all of the above holds for any $X > 0$. Thus, ϕ_t exists, is continuous and bounded on $\bar{D}_T^{\delta}$, with,

$$|\phi_t(x,t)| \le \frac{2^{\alpha+1} k_T I_\alpha}{\alpha\sqrt{\pi}} \left(1 + T^{\alpha/2}\right) + M_T \quad \forall (x,t) \in \bar{D}_T^{\delta}. \tag{3.28}$$

We next observe that since all of the above holds for all $0 < \delta < T$, then ϕ_t and ϕ_{xx} exist and are continuous on D_T whilst (3.25) and (3.28) establish that ϕ_t and ϕ_{xx} are bounded on D_T with both (3.25) and (3.28) continuing to hold on D_T.

Finally, to obtain (3.23), let $(x,t) \in \bar{D}_T^{\delta,X}$, then it follows from (3.8), (3.9), (3.16) and (3.20) that

$$(\phi_{nt}(x,t) - \phi_{nxx}(x,t)) \to F(x,t) \text{ as } n \to \infty \text{ uniformly on } \bar{D}_T^{\delta,X}. \tag{3.29}$$

Also from (3.24) and (3.26), we have

$$(\phi_{nt}(x,t) - \phi_{nxx}(x,t)) \to \phi_t(x,t) - \phi_{xx}(x,t) \text{ as } n \to \infty \text{ uniformly on } \bar{D}_T^{\delta,X}. \tag{3.30}$$

Uniqueness of limits, together with (3.29) and (3.30), then gives,

$$\phi_t(x,t) - \phi_{xx}(x,t) = F(x,t) \quad \forall (x,t) \in \bar{D}_T^{\delta,X}. \tag{3.31}$$

However, (3.31) holds for any $X > 0$ and $0 < \delta < T$ and so continues to hold on D_T. The proof is complete. □

The results of this subsection will be essential at a later stage. We now introduce appropriate function spaces.

3.2. Function Spaces

Associated with (B-R-D-C), we introduce the Banach Spaces

$$(B_A^T, \|\cdot\|_A) \text{ and } (B_B, \|\cdot\|_B),$$

where,

$$B_A^T = \{u : \bar{D}_T \to \mathbb{R} : u \text{ is continuous and bounded on } \bar{D}_T\}, \tag{3.32}$$

$$B_B = \{v : \mathbb{R} \to \mathbb{R} : v \text{ is continuous and bounded on } \mathbb{R}\}, \tag{3.33}$$

and

$$\|u\|_A = \sup_{(x,t)\in\bar{D}_T} |u(x,t)| \quad \forall u \in B_A^T, \quad \|v\|_B = \sup_{x\in\mathbb{R}} |v(x)| \quad \forall v \in B_B.$$

Remark 3.3.

(i) It follows immediately from Definitions (3.32) and (3.33) that when $u(\cdot,\cdot) \in B_A^T$ then $u(\cdot,t) \in B_B$ for each $t \in [0,T]$.

(ii) Whenever $u : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C), then $u \in B_A^T$.

The following elementary lemma will be useful at a later stage,

Lemma 3.4. *Let $u \in B_A^T$. Then $H : [0,T] \to \mathbb{R}^+ \cup \{0\}$, defined by*

$$H(t) = \|u(\cdot,t)\|_B \quad \forall t \in [0,T],$$

is such that $H \in L^1([0,T])$.

Proof. First, observe that $H : [0,T] \to \mathbb{R}^+ \cup \{0\}$ is well-defined, via Remark 3.3 (i). Also, $H : [0,T] \to \mathbb{R}^+ \cup \{0\}$ is bounded, with $0 \le H(t) \le \|u\|_A$. Next introduce the sequence of functions $\{H_n : [0,T] \to \mathbb{R}^+ \cup \{0\}\}_{n\in\mathbb{N}}$ such that

$$H_n(t) = \sup_{x\in[-n,n]} |u(x,t)| \quad \forall t \in [0,T]. \tag{3.34}$$

Since $u \in B_A^T$, then it follows from (3.34) that $H_n \in C([0,T]) \subset L^1([0,T])$, with,

$$0 \le H_n(t) \le \|u\|_A \quad \forall t \in [0,T]. \tag{3.35}$$

In addition,

$$0 \le H_1(t) \le H_2(t) \le \ \dots\ \le H_n(t) \le \ \dots\ \le \|u\|_A \quad \forall t \in [0,T]. \tag{3.36}$$

Moreover, it follows from (3.36) that,

$$H_n(t) \to H(t) \text{ as } n \to \infty \tag{3.37}$$

for each $t \in [0,T]$. It is an immediate consequence of (3.35), (3.36), (3.37) and the monotone convergence theorem ([35], Theorem 2, p.96) that $H \in L^1([0,T])$. □

Next, we establish a generalisation of Gronwall's inequality [8], which will also be useful at a later stage,

Proposition 3.5. *Let $\phi : [0,T] \to \mathbb{R}$ be such that $\phi \in L^1([0,T])$ and $\phi(t) \ge 0$ for all $t \in [0,T]$. Suppose that,*

$$\phi(t) \le a + bt + k\int_0^t \phi(s)ds \quad \forall t \in [0,T],$$

with $a \ge 0$, $b \ge 0$, $k > 0$ constants. Then,

$$\phi(t) \le (a + bt)e^{kt} \quad \forall t \in [0,T].$$

Proof. It follows from the first inequality above that,

$$\phi(t)e^{-kt} + (-k)e^{-kt}\int_0^t \phi(s)ds \leq (a+bt)e^{-kt} \quad \forall t \in [0,T]. \tag{3.38}$$

Since $\phi \in L^1([0,T])$, it then follows from [35] (Proposition 2, p103), that, after an integration, (3.38) becomes,

$$e^{-kt}\int_0^t \phi(s)ds \leq \int_0^t (a+bs)e^{-ks}ds \leq \frac{1}{k}(a+bt)(1-e^{-kt}) \quad \forall t \in [0,T],$$

from which we obtain,

$$\int_0^t \phi(s)ds \leq \frac{1}{k}(a+bt)(e^{kt}-1) \quad \forall t \in [0,T]. \tag{3.39}$$

It then follows, via (3.39), that,

$$\phi(t) \leq a+bt+k\int_0^t \phi(s)ds \leq a+bt+(a+bt)(e^{kt}-1) = (a+bt)e^{kt} \quad \forall t \in [0,T],$$

as required. □

Now let $f : \mathbb{R} \to \mathbb{R}$ be such that $f \in H_\alpha$ for some $\alpha \in (0,1]$, $u \in B_A^T$ and $\hat{u} \in B_B$. We introduce the following functions $v, w : \bar{D}_T \to \mathbb{R}$, defined by,

$$v(x,t) = \int_{-\infty}^{\infty} \hat{u}(x+2\sqrt{t}\lambda)e^{-\lambda^2}d\lambda \quad \forall (x,t) \in \bar{D}_T, \tag{3.40}$$

$$w(x,t) = \int_0^t\int_{-\infty}^{\infty} f(u(x+2\sqrt{t-\tau}\lambda,\tau))e^{-\lambda^2}d\lambda d\tau \quad \forall (x,t) \in \bar{D}_T. \tag{3.41}$$

It is straightforward to establish (see, Meyer [19], Chapter 5, Section 2) that $v, w \in B_A^T$ and

$$v(x,0) = \sqrt{\pi}\hat{u}(x), \quad w(x,0) = 0 \quad \forall x \in \mathbb{R}. \tag{3.42}$$

The expression (3.40) for v may be re-written via simple substitution. For $(x,t) \in D_T$, we make the substitution $s = x + 2\sqrt{t}\lambda$ in (3.40), after which we obtain,

$$v(x,t) = \frac{1}{2\sqrt{t}}\int_{-\infty}^{\infty} \hat{u}(s)e^{-\frac{(x-s)^2}{4t}}ds \quad \forall (x,t) \in D_T. \tag{3.43}$$

We next introduce $F : \bar{D}_T \to \mathbb{R}$ such that,

$$F(x,t) = f(u(x,t)) \quad \forall (x,t) \in \bar{D}_T. \tag{3.44}$$

Now, since $f \in H_\alpha$ and $u \in B_A^T$, then F is bounded and continuous on $\bar{D}_T$. It then follows from Section 3.1, that we may write

$$w(x,t) = \lim_{n\to\infty}\int_0^{t-1/n}\int_{-\infty}^{\infty} \frac{f(u(s,\tau))}{2\sqrt{t-\tau}}e^{-\frac{(x-s)^2}{4(t-\tau)}}ds\ d\tau \quad \forall (x,t) \in D_T. \tag{3.45}$$

We can now state:

Lemma 3.6. *The functions $v, w \in B_A^T$ are such that v_t, v_x, v_{xx} and w_x, all exist and are continuous on D_T. Moreover, the derivatives are given by,*

$$v_x(x,t) = \frac{1}{t^{\frac{1}{2}}} \int_{-\infty}^{\infty} \hat{u}(x + 2\sqrt{t}w) w e^{-w^2} dw \quad \forall (x,t) \in D_T,$$

$$v_t(x,t) = v_{xx}(x,t) = \frac{1}{t} \int_{-\infty}^{\infty} \hat{u}(x + 2\sqrt{t}w)(w^2 - 1/2) e^{-w^2} dw \quad \forall (x,t) \in D_T,$$

$$w_x(x,t) = \lim_{n\to\infty} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}w, \tau))}{(t-\tau)^{\frac{1}{2}}} w e^{-w^2} dw d\tau \quad \forall (x,t) \in D_T.$$

Suppose also that u_x exists and is bounded on D_T, then w_t and w_{xx} also exist and are continuous on D_T, with,

$$w_{xx}(x,t) = \lim_{n\to\infty} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}w, \tau))}{(t-\tau)} (w^2 - 1/2) e^{-w^2} dw d\tau \quad \forall (x,t) \in D_T,$$

$$w_t(x,t) = \lim_{n\to\infty} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}w, \tau))}{(t-\tau)} (w^2 - 1/2) e^{-w^2} dw d\tau + \sqrt{\pi} f(u(x,t)) \quad \forall (x,t) \in D_T. \tag{3.46}$$

Proof. We first consider v. We introduce the function

$$\phi : [-a,a] \times [t_0,T] \times (-\infty,\infty) \to \mathbb{R},$$

given by,

$$\phi(x,t,s) = \frac{\hat{u}(s)}{2\sqrt{t}} e^{-\frac{(x-s)^2}{4t}} \tag{3.47}$$

for $(x,t,s) \in [-a,a] \times [t_0,T] \times (-\infty,\infty)$ (for any $a > 0$ and $0 < t_0 < T$). Then

$$v(x,t) = \int_{-\infty}^{\infty} \phi(x,t,s) ds$$

on $[-a,a] \times [t_0,T]$. Now, an examination of (3.47) shows that ϕ_t, ϕ_x and ϕ_{xx} all exist and are continuous on $[-a,a] \times [t_0,T] \times (-\infty,\infty)$, whilst the improper integrals

$$\int_{-\infty}^{\infty} \phi_x(x,t,s) ds, \quad \int_{-\infty}^{\infty} \phi_t(x,t,s) ds, \quad \int_{-\infty}^{\infty} \phi_{xx}(x,t,s) ds$$

are uniformly convergent for all $(x,t) \in [-a,a] \times [t_0,T]$. It follows that v_t, v_x and v_{xx} all exist and are continuous on $[-a,a] \times [t_0,T]$, for any $a > 0$ and $0 < t_0 < T$. Thus v_t, v_x and v_{xx} all exist and are continuous on D_T. Moreover,

$$v_x = \int_{-\infty}^{\infty} \phi_x(x,t,s) ds, \quad v_t = \int_{-\infty}^{\infty} \phi_t(x,t,s) ds, \quad v_{xx} = \int_{-\infty}^{\infty} \phi_{xx}(x,t,s) ds. \tag{3.48}$$

The given derivatives are now obtained by replacing ϕ_t, ϕ_x and ϕ_{xx} in the above, followed by the substitution $s = x + 2\sqrt{t}w$.

We now consider w. First we recall that $f \in H_\alpha$ and $u \in B_A^T$ so that $f(u)$ is bounded and continuous on $\bar{D}_T$. It then follows, via Lemma 3.1, that w_x exists and is continuous on D_T, and the derivative formula follows via (3.7). Next, when $u \in B_A^T$ is such that u_x exists and is bounded on D_T, it follows with $f \in H_\alpha$, that $f(u)$ satisfies condition (H) (in Section 3.1) on $\bar{D}_T$ (via an application of the mean value theorem). It then follows from Lemma 3.2 that w_t and w_{xx} exist and are continuous on D_T. The derivative formulae follow from (3.8), (3.9) and (3.20). □

3.3. Equivalence Lemma

We are now in a position to establish an equivalence lemma for (B-R-D-C) when $f \in H_\alpha$. This relates solutions of an associated *integral equation* to solutions of (B-R-D-C). We have,

Lemma 3.7. *Let $f \in H_\alpha$ for some $\alpha \in (0,1]$ and $u_0 \in BPC^2(\mathbb{R})$. Then, the following statements are equivalent:*

(a) $u \in B_A^T$ and $u : \bar{D}_T \to \mathbb{R}$ satisfies the integral equation

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau \quad \forall (x,t) \in \bar{D}_T.$$

(b) $u : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C) on $\bar{D}_T$.

Proof. To begin, we prove (a)⇒(b). Suppose (a) holds for $u : \bar{D}_T \to \mathbb{R}$ with $u \in B_A^T$. In particular, via (3.42), we have,

$$u(x,0) = u_0(x) \ \forall \ x \in \mathbb{R}, \tag{3.49}$$

whilst,

$$u(x,t) \text{ is uniformly bounded as } |x| \to \infty \text{ for } t \in [0,T] \tag{3.50}$$

as $u \in B_A^T$. Now we have, via Lemma 3.6, Lemma 3.1 and (a) that u_x exists and is bounded on D_T. It then follows, again via Lemma 3.6, that u_t, u_x and u_{xx} all exist and are continuous on D_T. Finally using the derivative formula given in Lemma 3.6, a direct substitution shows that,

$$u_t - u_{xx} - f(u) = 0 \text{ on } D_T. \tag{3.51}$$

Together, (3.49), (3.50) and (3.51) imply that $u : \bar{D}_T \to \mathbb{R}$ is a solution of (B-R-D-C) on $\bar{D}_T$.

We now prove (b)⇒(a). Let $u : \bar{D}_T \to \mathbb{R}$ be a solution of (B-R-D-C) on $\bar{D}_T$. Then $u \in B_A^T$ and $f \circ u : \bar{D}_T \to \mathbb{R}$ is bounded and continuous. It then follows (see Meyer, [19], Theorem 4.9) that $u : \bar{D}_T \to \mathbb{R}$ satisfies the integral equation in (a). □

At this stage it is also convenient to state an associated lemma for (B-R-D-C) when $f \in L_u$. Namely,

Lemma 3.8. *Let $f \in L_u$ and $u_0 \in BPC^2(\mathbb{R})$, and let $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then*

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau \quad \forall (x,t) \in \bar{D}_T.$$

Proof. Let $u : \bar{D}_T \to \mathbb{R}$ be a solution of (B-R-D-C) on $\bar{D}_T$. Then, since $f \in L_u$, we conclude that $f \circ u : \bar{D}_T \to \mathbb{R}$ is bounded and continuous. It then follows that $u : \bar{D}_T \to \mathbb{R}$ satisfies the above integral equation (see Meyer [19], Theorem 4.9). □

3.4. Derivative Estimates

We now move on to establishing derivative bounds for solutions to (B-R-D-C) on $\bar{D}_T$ when $f \in H_\alpha$. These provide a new generalisation of the classical Schauder derivative bounds for solutions to (B-R-D-C) on $\bar{D}_T$, when $f \in L$ (see, Schauder [30], [31], Friedman [7]). We first need the following:

Lemma 3.9. *Let $f \in H_\alpha$ for some $\alpha \in (0,1]$ and $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then,*

$$|u_x(x,t)| \leq \frac{2M_T}{\sqrt{\pi}}(1 + T^{\frac{1}{2}}) + M_0' \quad \forall (x,t) \in D_T,$$

where $M_0' > 0$ is an upper bound for $|u_0'| : \mathbb{R} \to \mathbb{R}$ and $M_T > 0$ is an upper bound for $|f \circ u| : \bar{D}_T \to \mathbb{R}$.

Proof. Let $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then, via Lemma 3.7 and Lemma 3.6, for any $(x,t) \in D_T$,

$$\begin{aligned} u_x(x,t) &= \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \right)_x \\ &\quad + \left(\frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau \right)_x \\ &= \frac{1}{\sqrt{\pi} t^{\frac{1}{2}}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) \lambda e^{-\lambda^2} d\lambda \\ &\quad + \lim_{n \to \infty} \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)^{\frac{1}{2}}} \lambda e^{-\lambda^2} d\lambda d\tau \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0'(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \\ &\quad + \lim_{n \to \infty} \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)^{\frac{1}{2}}} \lambda e^{-\lambda^2} d\lambda d\tau \end{aligned}$$

following an integration by parts. It follows that, for any $(x,t) \in D_T$,

$$\begin{aligned} |u_x(x,t)| &\leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |u_0'(x + 2\sqrt{t}\lambda)| e^{-\lambda^2} d\lambda \\ &\quad + \lim_{n \to \infty} \frac{1}{\sqrt{\pi}} \left| \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)^{\frac{1}{2}}} \lambda e^{-\lambda^2} d\lambda d\tau \right| \\ &\leq M_0' + \lim_{n \to \infty} \frac{1}{\sqrt{\pi}} \left| \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)^{\frac{1}{2}}} \lambda e^{-\lambda^2} d\lambda d\tau \right|. \end{aligned} \tag{3.52}$$

Now, for any $(x,t) \in D_T$ and $0 < 1/n < min\{1,t\}$,

$$\begin{aligned}
&\left| \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x+2\sqrt{t-\tau}\lambda,\tau))}{(t-\tau)^{\frac{1}{2}}} \lambda e^{-\lambda^2} d\lambda d\tau \right| \\
&\quad \leq \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{|f(u(x+2\sqrt{t-\tau}\lambda,\tau))|}{(t-\tau)^{\frac{1}{2}}} |\lambda| e^{-\lambda^2} d\lambda d\tau \\
&\quad \leq M_T \left(\int_0^{t-1/n} \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau \right) \left(\int_{-\infty}^{\infty} |\lambda| e^{-\lambda^2} d\lambda \right) \\
&\quad = 2M_T (t^{\frac{1}{2}} - (1/n)^{\frac{1}{2}}) \leq 2M_T (T^{\frac{1}{2}} + 1).
\end{aligned} \tag{3.53}$$

It follows from (3.52) and (3.53), that,

$$|u_x(x,t)| \leq M_0' + \frac{2M_T}{\sqrt{\pi}}(1 + T^{\frac{1}{2}}) \quad \forall (x,t) \in D_T,$$

as required. □

Remark 3.10. Observe that the proof of Lemma 3.9 only requires that the solution $u : \bar{D}_T \to \mathbb{R}$ satisfies an integral equation as in Lemma 3.7 or Lemma 3.8. Therefore Lemma 3.9 can also be established for $f \in L_u$. However, since subsequent applications of this derivative estimate only concern $f \in H_\alpha$, it is stated as above.

We next have,

Lemma 3.11. *Let $f \in H_\alpha$ for some $\alpha \in (0,1]$ and let $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then $f \circ u : \bar{D}_T \to \mathbb{R}$ satisfies,*

$$|f(u(y,t)) - f(u(x,t))| \leq k_T |y - x|^\alpha \quad \forall (x,t),(y,t) \in \bar{D}_T,$$

where,

$$k_T = k_E \left(\frac{2M_T}{\sqrt{\pi}}(1 + T^{\frac{1}{2}}) + M_0' \right)^\alpha$$

and $k_E > 0$ is a Hölder constant for $f : \mathbb{R} \to \mathbb{R}$ on the closed bounded interval $[-U_T, U_T]$, with $U_T > 0$ being an upper bound for $|u| : \bar{D}_T \to \mathbb{R}$.

Proof. Let $(x,t),\ (y,t) \in D_T$, then $u(x,t),\ u(y,t) \in [-U_T, U_T]$, and so, since $f \in H_\alpha$, then,

$$|f(u(y,t)) - f(u(x,t))| \leq k_E |u(y,t) - u(x,t)|^\alpha \tag{3.54}$$

where $k_E > 0$ is a Hölder constant for $f : \mathbb{R} \to \mathbb{R}$ on the closed bounded interval $[-U_T, U_T]$. However it follows from the mean value theorem together with Lemma 3.9, that,

$$|u(y,t) - u(x,t)| \leq \left(\frac{2M_T}{\sqrt{\pi}}(1 + T^{\frac{1}{2}}) + M_0' \right) |y - x|. \tag{3.55}$$

Combining (3.54) and (3.55) we obtain

$$|f(u(y,t)) - f(u(x,t))| \leq k_T |y - x|^\alpha \ \ \forall \ (x,y),(y,t) \in D_T, \tag{3.56}$$

with,

$$k_T = k_E \left(\frac{2M_T}{\sqrt{\pi}} (1 + T^{\frac{1}{2}}) + M_0' \right)^{\alpha}.$$

Now, for fixed $x, y \in \mathbb{R}$, the left-hand side of (3.56) is continuous for $t \in [0, T]$, whilst the right-hand side of (3.56) is independent of t. It follows that the inequality (3.56) extends from D_T onto $\bar{D}_T$, and the proof is complete. □

We are now in a position to state,

Lemma 3.12. *Let $f \in H_\alpha$ for some $\alpha \in (0,1]$ and $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then,*

$$|u_{xx}(x,t)| \le \frac{2^{\alpha+1} I_\alpha}{\alpha\sqrt{\pi}} k_T (1 + T^{\frac{1}{2}\alpha}) + M_0'' \quad \forall (x,t) \in D_T,$$

$$|u_t(x,t)| \le \frac{2^{\alpha+1} I_\alpha}{\alpha\sqrt{\pi}} k_T (1 + T^{\frac{1}{2}\alpha}) + M_0'' + M_T \quad \forall (x,t) \in D_T,$$

where $M_0'' > 0$ is an upper bound for $|u_0''| : \mathbb{R} \to \mathbb{R}$ and I_α is given by (3.14).

Proof. Let $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then u_x exists and is bounded on D_T, via Lemma 3.9. It then follows, via Lemma 3.6 and Lemma 3.7, for any $(x,t) \in D_T$,

$$\begin{aligned}
u_{xx}(x,t) &= \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \right)_{xx} \\
&\quad + \left(\frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau \right)_{xx} \\
&= \frac{1}{\sqrt{\pi} t} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda)(\lambda^2 - 1/2) e^{-\lambda^2} d\lambda \\
&\quad + \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)} (\lambda^2 - 1/2) e^{-\lambda^2} d\lambda d\tau \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0''(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \\
&\quad + \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)} (\lambda^2 - 1/2) e^{-\lambda^2} d\lambda d\tau
\end{aligned}$$

following an integration by parts. Thus, for any $(x,t) \in D_T$,

$$\begin{aligned}
|u_{xx}(x,t)| &\le \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |u_0''(x + 2\sqrt{t}\lambda)| e^{-\lambda^2} d\lambda \\
&\quad + \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \left| \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)} (\lambda^2 - 1/2) e^{-\lambda^2} d\lambda d\tau \right| \\
&\le M_0'' + \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \left| \int_0^{t-1/n} \int_{-\infty}^{\infty} \frac{f(u(x + 2\sqrt{t-\tau}\lambda, \tau))}{(t-\tau)} (\lambda^2 - 1/2) e^{-\lambda^2} d\lambda d\tau \right|
\end{aligned} \tag{3.57}$$

Now, for any $(x,t) \in D_T$ and $0 < 1/n < min\{1,t\}$,

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}}\left|\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{f(u(x+2\sqrt{t-\tau}\lambda,\tau))}{(t-\tau)}(\lambda^2-1/2)e^{-\lambda^2}d\lambda d\tau\right| \\
&\quad\leq \frac{1}{\sqrt{\pi}}\left|\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{f(u(x+2\sqrt{t-\tau}\lambda,\tau))-f(u(x,\tau))}{(t-\tau)}(\lambda^2-1/2)e^{-\lambda^2}d\lambda d\tau\right| \\
&\qquad+\frac{1}{\sqrt{\pi}}\left|\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{f(u(x,\tau))(\lambda^2-1/2)}{(t-\tau)}e^{-\lambda^2}d\lambda d\tau\right| \qquad (3.58)
\end{aligned}
$$

via the triangle inequality. However, the second term on the right-hand side of (3.58) vanishes, and so,

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}}\left|\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{f(u(x+2\sqrt{t-\tau}\lambda,\tau))}{(t-\tau)}(\lambda^2-1/2)e^{-\lambda^2}d\lambda d\tau\right| \\
&\quad\leq \frac{1}{\sqrt{\pi}}\left|\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{f(u(x+2\sqrt{t-\tau}\lambda,\tau))-f(u(x,\tau))}{(t-\tau)}(\lambda^2-1/2)e^{-\lambda^2}d\lambda d\tau\right| \\
&\quad\leq \frac{1}{\sqrt{\pi}}\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{|f(u(x+2\sqrt{t-\tau}\lambda,\tau))-f(u(x,\tau))|}{(t-\tau)}|\lambda^2-1/2|e^{-\lambda^2}d\lambda d\tau \\
&\quad\leq \frac{2^\alpha k_T}{\sqrt{\pi}}\int_0^{t-1/n}\int_{-\infty}^{\infty}\frac{|\lambda|^\alpha|\lambda^2-1/2|}{(t-\tau)^{1-\alpha/2}}e^{-\lambda^2}d\lambda d\tau \quad \text{(via Lemma 3.11)} \qquad (3.59) \\
&\quad\leq \frac{2^\alpha I_\alpha k_T}{\sqrt{\pi}}\int_0^{t-1/n}\frac{1}{(t-\tau)^{1-\alpha/2}}d\tau \\
&\quad\leq \frac{2^{\alpha+1}I_\alpha k_T}{\alpha\sqrt{\pi}}(t^{\alpha/2}-1/n^{\alpha/2}) \leq \frac{2^{\alpha+1}I_\alpha k_T}{\alpha\sqrt{\pi}}(1+T^{\alpha/2}), \qquad (3.60)
\end{aligned}
$$

where I_α is given by (3.14). It follows from (3.57) and (3.60) that,

$$
|u_{xx}(x,t)| \leq M_0'' + \frac{2^{\alpha+1}I_\alpha k_T}{\alpha\sqrt{\pi}}(1+T^{\frac{\alpha}{2}}) \quad \forall (x,t)\in D_T, \qquad (3.61)
$$

as required. Now, since $u : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C) on $\bar{D}_T$, then,

$$
u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \quad \forall (x,t) \in D_T, \qquad (3.62)
$$

via Definition 2.1. Thus, via the triangle inequality and (3.61),

$$
|u_t(x,t)| \leq |u_{xx}(x,t)| + |f(u(x,t))| \leq M_T + M_0'' + \frac{2^{\alpha+1}I_\alpha k_T(1+T^{\frac{\alpha}{2}})}{\alpha\sqrt{\pi}}
$$

for all $(x,t) \in D_T$, as required. □

An immediate consequence of this is,

Corollary 3.13. *Let $f \in H_\alpha$ for some $\alpha \in (0,1]$ and $u : \bar{D}_T \to \mathbb{R}$ be a solution to (B-R-D-C) on $\bar{D}_T$. Then u is uniformly continuous on $\bar{D}_T$.*

Proof. It follows from Lemma 3.9 and Lemma 3.12 together with the mean value theorem, that,

$$|u(x_2,t_2) - u(x_1,t_1)| \leq M(|x_2 - x_1| + |t_2 - t_1|) \quad \forall (x_2,t_2),(x_1,t_1) \in D_T,$$

with $M > 0$ being the maximum of the derivative bounds for u_x and u_t on D_T. Since u is continuous on $\bar{D}_T$, it follows that,

$$|u(x_2,t_2) - u(x_1,t_1)| \leq M(|x_2 - x_1| + |t_2 - t_1|) \quad \forall (x_2,t_2),(x_1,t_1) \in \bar{D}_T,$$

and the result follows. □

Remark 3.14. For fixed $\alpha \in (0,1]$ and $T > 0$, Lemma 3.9 and Lemma 3.12 establish that for any solution $u : \bar{D}_T \to \mathbb{R}$ of (B-R-D-C) on $\bar{D}_T$, then $|u_t|$, $|u_x|$ and $|u_{xx}|$ are bounded on D_T, with bounds which only depend upon α, T, $\|u_0'\|_B$, $\|u_0''\|_B$, $\|f \circ u\|_A$, $\|u\|_A$ and a Hölder constant for f on $[-U_T, U_T]$ with U_T being an upper bound for $\|u\|_A$. Such results are often referred to as *Schauder Estimates* for (B-R-D-C) (after J. Schauder [30], [31] of whose results for elliptic problems were extended to parabolic problems by A. Friedman, [7]). For additional information concerning the development of these type of results, see [10] and [26]. We note that the bounds obtained in this subsection for $f \in H_\alpha$ and $\alpha \in (0,1]$ become singular as $\alpha \to 0^+$, and replicate the classical results when $\alpha = 1$.

4. Hölder Continuous Theory

In this section, we first recall well established results concerning (B-R-D-C) (see, for example Needham [24], Smoller [32] and in the context of this paper, see Meyer [19]) when $f \in L$. Namely that (B-R-D-C) is globally well posed on $\mathrm{BPC}^2(\mathbb{R})$ when $f \in L$ and (B-R-D-C) is a priori bounded for any $T > 0$. Throughout the rest of this section we will refer to classical results when $f \in L$ via reference to those versions detailed in Meyer [19].

We begin by establishing a local existence result for maximal and minimal solutions (see Definition 4.1 and Remark 4.4) to (B-R-D-C) when $f \in H_\alpha$, with global existence obtained under the condition of a priori bounds. However, unlike the corresponding classical result for (B-R-D-C) when $f \in L$ (see [19], Theorem 6.2), uniqueness is not obtained generically for (B-R-D-C) when $f \in H_\alpha$. Additionally, from the construction of the local existence result for $f \in H_\alpha$, a conditional comparison theorem is obtained for (B-R-D-C) when $f \in H_\alpha$. We then establish qualitative properties of maximal and minimial solutions to (B-R-D-C) for $f \in H_\alpha$ under certain conditions on f and $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. To conclude the section, we employ the local existence result, along with the qualitative properties of solutions to (B-R-D-C) to establish a conditional conditnuous dependence result for (B-R-D-C) with $f \in H_\alpha$. We first introduce the notion of maximal and minimal solutions to (B-R-D-C).

Definition 4.1. Let $f \in H_\alpha$ for some $\alpha \in (0,1)$ and $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. Let

$$\mathcal{S} = \left\{ u : \bar{D}_T \to \mathbb{R} : u \text{ is a solution to the given (B-R-D-C) on } \bar{D}_T \right\}.$$

Then $\overline{u} : \bar{D}_T \to \mathbb{R}$ is said to be a *maximal solution* to the given (B-R-D-C) when $\overline{u} \in \mathcal{S}$ and for all $u \in \mathcal{S}$, $\overline{u}(x,t) \geq u(x,t)$ for all $(x,t) \in \bar{D}_T$. Correspondingly, $\underline{u} : \bar{D}_T \to \mathbb{R}$ is said to be a *minimal solution* to the given (B-R-D-C) when $\underline{u} \in \mathcal{S}$ and for all $u \in \mathcal{S}$, $\underline{u}(x,t) \leq u(x,t)$ for all $(x,t) \in \bar{D}_T$.

Remark 4.2. For a given (B-R-D-C), when $\underline{u} = \overline{u}$ on $\bar{D}_T$, then (B-R-D-C) has a unique solution on $\bar{D}_T$.

We can now state the first main results of this section.

Theorem 4.3. *Consider (B-R-D-C) with $f \in H_\alpha$ for some $\alpha \in (0,1)$, and $u_0 \in BPC^2(\mathbb{R})$. Then there exists a minimal and a maximal solution to (B-R-D-C) on $\bar{D}_\delta$, with*

$$\delta = \min\left\{\frac{(m_0 + a')}{c'}, \frac{(m_0 - b')}{c'}\right\} \geq \frac{1}{c'},$$

where $m_0 = \|u_0\|_B + 1$, $a' = \inf_{x\in\mathbb{R}} u_0(x)$, $b' = \sup_{x\in\mathbb{R}} u_0(x)$ and,

$$c' = \max\left\{\left|\inf_{y\in[-m_0,m_0]}\{f(y)\} - 1\right|, \left|\sup_{y\in[-m_0,m_0]}\{f(y)\} + 1\right|\right\}.$$

In addition, with $\underline{u} : \bar{D}_\delta \to \mathbb{R}$ and $\overline{u} : \bar{D}_\delta \to \mathbb{R}$ being the minimal and maximum solutions respectively, then

$$\max\{\|\underline{u}\|_A, \|\overline{u}\|_A\} \leq m_0.$$

In what follows we develop a constructional proof of Theorem 4.3, and, in doing so, we establish Proposition 4.17. As a consequence of this we have the following elementary observations concerning Theorem 4.3:

Remark 4.4. Let $\overline{u}, \underline{u} : \bar{D}_\delta \to \mathbb{R}$ be the maximal and minimal solutions to (B-R-D-C) as given in Theorem 4.3. Then $\overline{u}$ and $\underline{u}$ are, respectively, maximal and minimal solutions to (B-R-D-C) on $\bar{D}_{\delta'}$, for any $0 < \delta' \leq \delta$, and on $\bar{D}^{\delta_1}_{\delta_2}$, for any $0 \leq \delta_1 < \delta_2 \leq \delta$, in the sense that, for any solution $u : \bar{D}_\delta \to \mathbb{R}$ of (B-R-D-C), then $\underline{u} \leq u \leq \overline{u}$ on $\bar{D}^{\delta_1}_{\delta_2}$. Now, let $\overline{u}^c : \bar{D}_T \to \mathbb{R}$ be a function obtained by repeated application of Theorem 4.3 and glueing together the associated maximal solution and its domain at each stage. Then $\overline{u}^c : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C), and is a maximal solution to (B-R-D-C) on $\bar{D}_T$. Similarly let $\underline{u}^c : \bar{D}_T \to \mathbb{R}$ be a function obtained by repeated application of Theorem 4.3 and glueing together the associated minimal solution and its domain at each stage. Then $\underline{u}^c : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C), and is a minimal solution to (B-R-D-C) on $\bar{D}_T$. In what follows, we will refer to $\overline{u}^c : \bar{D}_T \to \mathbb{R}$ (when it exists on $\bar{D}_T$) as a *constructed maximal solution* to (B-R-D-C) on $\bar{D}_T$. Similarly, we will refer to $\underline{u}^c : \bar{D}_T \to \mathbb{R}$ (when it exists on $\bar{D}_T$) as a *constructed minimal solution* to (B-R-D-C) on $\bar{D}_T$. Note that a constructed maximal (minimal) solution to (B-R-D-C) on $\bar{D}_T$ is a maximal (minimal) solution to (B-R-D-C) on $\bar{D}_T$. However the converse does not necessarily follow; a maximal (minimal) solution to (B-R-D-C) on $\bar{D}_T$ need not be a constructed maximal (minimal) solution to (B-R-D-C) on $\bar{D}_T$; for example, a maximal (minimal) solution may exist on $\bar{D}_T$, whilst the *constructed* maximal (minimal) solution may have undergone blow-up *before* $t = T$.

Immediate consequences of the above are,

Corollary 4.5. *Let $f \in H_\alpha$ for some $\alpha \in (0,1)$ and $u_0 \in BPC^2(\mathbb{R})$. Then there exists a global constructed maximal (minimal) solution to (B-R-D-C) on $\bar{D}_\infty$ or there exists T_u $(T_l) > 0$ such that (B-R-D-C) has a constructed maximal (minimal) solution on $\bar{D}_{T_u} \backslash (\mathbb{R} \times \{T_u\})$ $(\bar{D}_{T_l} \backslash (\mathbb{R} \times \{T_l\}))$ which cannot be continued onto $\bar{D}_{T_u}$ $(\bar{D}_{T_l})$.*

Proof. This follows directly from repeated application of Theorem 4.3 to (B-R-D-C) and Remark 4.4. □

Corollary 4.6. *Let $f \in H_\alpha$ for some $\alpha \in (0,1)$ and $u_0 \in BPC^2(\mathbb{R})$. Let*

$$\overline{u}^c(\underline{u}^c) : \bar{D}_{T^*} \backslash (\mathbb{R} \times \{T^*\}) \to \mathbb{R}$$

be a constructed maximal (minimal) solution to (B-R-D-C) which cannot be continued onto $\bar{D}_{T^*}$. Then $\|\overline{u}^c(\cdot,t)\|_B$ $(\|\underline{u}^c(\cdot,t)\|_B)$ is unbounded as $t \to T^{*-}$.

Proof. This follows similar steps to the proof of the corresponding classical result when $f \in L$ (see [19], Theorem 6.10), via Theorem 4.3 and Remark 4.4. □

We now begin to establish Theorem 4.3, and must first prove a density result, namely,

Proposition 4.7. *Consider $f \in H_\alpha$ with $\alpha \in (0,1)$. Let $k_H > 0$ be a Hölder constant for f on the closed bounded interval $E \subset \mathbb{R}$. Then, on E, given any $\epsilon > 0$, there exists a Lipschitz continuous function $g : E \to \mathbb{R}$ such that,*

$$|f(x) - g(x)| < \epsilon \quad \forall x \in E,$$

where g is also a Hölder continuous function of degree α on E with Hölder constant $3k_H$.

Proof. Let $E \subset \mathbb{R}$ be a closed bounded interval, and $k_H > 0$ be a Hölder constant for f on E. Now, given any $\epsilon > 0$, set δ as follows,

$$\delta = \left(\frac{\epsilon}{2k_H}\right)^{1/\alpha}. \tag{4.1}$$

Then, for all $x, y \in E$, with $|x - y| < \delta$, we have,

$$|f(y) - f(x)| < \frac{\epsilon}{2}. \tag{4.2}$$

We may write $E = [a,b] \subset \mathbb{R}$. Now take $N \in \mathbb{N}$ with $N > \frac{(b-a)}{\delta}$ and divide the interval E into uniform sub-intervals X_n $(n = 1, ..., N)$, defined by

$$X_n = [x_{n-1}, x_n], \text{ where } x_0 = a,\ x_N = b,\ x_n = x_{n-1} + \frac{(b-a)}{N} \tag{4.3}$$

for each $1 \le n \le N$. Next define $l_n : X_n \to \mathbb{R}$, for $1 \le n \le N$, as

$$l_n(x) = \left(\frac{f(x_n)(x - x_{n-1}) + f(x_{n-1})(x_n - x)}{x_n - x_{n-1}}\right) \quad \forall x \in X_n \tag{4.4}$$

and define $g : E \to \mathbb{R}$ such that on each interval $X_n \subset E$, $g(x) = l_n(x)$ for all $x \in X_n$. Note that g defined by (4.3) and (4.4) is Lipschitz continuous on E with Lipschitz constant given by

$$k_E^l = \max_{1 \le n \le N} \left| \frac{f(x_n) - f(x_{n-1})}{(x_n - x_{n-1})} \right|.$$

Let $x \in E$, then there exists n such that $x \in X_n$ for some $n = 1, 2, ...N$ and so,

$$\begin{aligned} |f(x) - g(x)| &\le |f(x) - f(x_n)| + |f(x_n) - g(x)| \\ &= |f(x) - f(x_n)| + |g(x_n) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

via (4.1), (4.2), (4.3) and (4.4). It remains to shown that g is also Hölder continuous of degree $0 < \alpha < 1$ on E with Hölder constant $3k_H$. Observe that since $g(x_n) = f(x_n)$ for each $n = 0, 1, 2, .., N$, then on each interval X_n, we have,

$$\left| \frac{dg}{dx} \right| = \left| \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \right| \le \left| \frac{(x_n - x_{n-1})^{\alpha} k_H}{x_n - x_{n-1}} \right| = |x_n - x_{n-1}|^{\alpha - 1} k_H \quad \forall x \in X_n. \quad (4.5)$$

It follows from the mean value theorem with (4.5), that for any $x, y \in X_n$,

$$|g(x) - g(y)| \le |x_n - x_{n-1}|^{\alpha-1} k_H |x - y| = k_H \left| \frac{x - y}{x_n - x_{n-1}} \right|^{1-\alpha} |x - y|^{\alpha} \le k_H |x - y|^{\alpha}. \quad (4.6)$$

Now for $x \in X_n$ and $y \in X_m$ where, without loss of generality, $m > n$, then via (4.6) and (4.4),

$$\begin{aligned} |g(x) - g(y)| &\le |g(x) - g(x_n)| + |f(x_n) - f(x_{m-1})| + |g(x_{m-1}) - g(y)| \\ &\le k_H |x - x_n|^{\alpha} + k_H |x_n - x_{m-1}|^{\alpha} + k_H |x_{m-1} - y|^{\alpha} \le 3k_H |x - y|^{\alpha}, \quad (4.7) \end{aligned}$$

since $x \le x_n \le x_{m-1} \le y$. Inequalities (4.6) and (4.7) establish that g is Hölder continuous of degree α on E with Hölder constant $3k_H$, as required. □

Remark 4.8. Let $f \in H_\alpha$ for some $\alpha \in (0, 1)$ and $E = [a, b]$ for $b > a$. If $g : E \to \mathbb{R}$ is constructed as in Proposition 4.7, then $f(a) = g(a)$ and $f(b) = g(b)$.

Next we proceed to construct two sequences of functions which will subsequently be shown to converge to the minimal and maximal solutions to (B-R-D-C).

Proposition 4.9. *Let $f \in H_\alpha$ for some $\alpha \in (0, 1)$, and $E = [a, b]$ be a closed bounded interval. Let $k_H > 0$ be a Hölder constant for f on $[a, b]$. Then there exist sequences $\{\overline{f}_n\}_{n \in \mathbb{N}}$ and $\{\underline{f}_n\}_{n \in \mathbb{N}}$, such that for each $n \in \mathbb{N}$ the functions $\overline{f}_n, \underline{f}_n : \mathbb{R} \to \mathbb{R}$ satisfy,*

(a) $\overline{f}_n$ and $\underline{f}_n$ are Lipschitz continuous on every closed bounded interval $E' \subset \mathbb{R}$.

(b) $\overline{f}_n$ and $\underline{f}_n$ are Hölder continuous of degree α on every closed bounded interval $E' \subset \mathbb{R}$, with Hölder constant independent of $n \in \mathbb{N}$.

(c) $\overline{f}_n(u) \to f(u)$ and $\underline{f}_n(u) \to f(u)$ as $n \to \infty$ uniformly for all $u \in E$.

(d) $\underline{f}_n(u) \le f(u) \le \overline{f}_n(u)$ for all $u \in E$ and for each $n \in \mathbb{N}$.

(e) $\overline{f}_{n+1}(u) \leq \overline{f}_n(u)$ *and* $\underline{f}_{n+1}(u) \geq \underline{f}_n(u)$ *for all* $u \in \mathbb{R}$ *and for each* $n \in \mathbb{N}$.

Proof. The Lipschitz density result in Proposition 4.7 guarantees that there exists a sequence of Lipschitz continuous functions $g_n : [a,b] \to \mathbb{R}$ (each of which is also Hölder continuous on $[a,b]$ of degree $0 < \alpha < 1$, with Hölder constant $3k_H$ on $[a,b]$) such that $g_n(a) = f(a)$, $g_n(b) = f(b)$ and which satisfy,

$$\sup_{u \in E}\{|f - g_n|(u)\} \leq 1/2^n, \tag{4.8}$$

for each $n \in \mathbb{N}$. Now define $\overline{f}_n, \underline{f}_n : \mathbb{R} \to \mathbb{R}$, for each $n \in \mathbb{N}$, to be,

$$\overline{f}_n(u) = \begin{cases} g_{n+2}(u) + \frac{1}{2^n} & ;\ u \in [a,b] \\ g_{n+2}(a) + \frac{1}{2^n} & ;\ u \in (-\infty, a) \\ g_{n+2}(b) + \frac{1}{2^n} & ;\ u \in (b, \infty), \end{cases} \qquad \underline{f}_n(u) = \begin{cases} g_{n+2}(u) - \frac{1}{2^n} & ;\ u \in [a,b] \\ g_{n+2}(a) - \frac{1}{2^n} & ;\ u \in (-\infty, a) \\ g_{n+2}(b) - \frac{1}{2^n} & ;\ u \in (b, \infty). \end{cases} \tag{4.9}$$

We now give the proof for $\{\underline{f}_n\}$, with the proof for $\{\overline{f}_n\}$ following similarly. Statements (a) and (b) follow immediately from (4.9). Next we observe that,

$$|\underline{f}_n - f|(u) \leq |g_{n+2} - f|(u) + 1/2^n \leq 5/2^{n+2}, \tag{4.10}$$

for all $u \in [a,b]$ and $n \in \mathbb{N}$, and so $\underline{f}_n(u) \to f(u)$ as $n \to \infty$ uniformly for $u \in [a,b]$, which establishes statement (c). Also observe that for any $u \in [a,b]$ and $n \in \mathbb{N}$, we have,

$$\underline{f}_n(u) = g_{n+2}(u) - 1/2^n \leq (f(u) + 1/2^{n+2}) - 1/2^n \leq f(u) - 3/2^{n+2} \leq f(u), \tag{4.11}$$

from which statement (d) follows. It remains to establish that the sequence $\{\underline{f}_n\}_{n\in\mathbb{N}}$ is non-decreasing on $\mathbb{R}$. Observe via (4.8) and (4.9), that for any $n \in \mathbb{N}$,

$$\underline{f}_{n+1}(u) \geq \left(f(u) - \frac{1}{2^{n+3}}\right) - \frac{1}{2^{n+1}} = f(u) - \frac{5}{2^{n+3}}, \tag{4.12}$$

$$\underline{f}_n(u) \leq \left(f(u) + \frac{1}{2^{n+2}}\right) - \frac{1}{2^n} = f(u) - \frac{6}{2^{n+3}}, \tag{4.13}$$

for all $u \in [a,b]$. Combining (4.12) and (4.13) gives,

$$\underline{f}_{n+1}(u) - \underline{f}_n(u) \geq \frac{1}{2^{n+3}} > 0, \tag{4.14}$$

for all $u \in [a,b]$. In addition it follows from (4.9) that

$$\underline{f}_{n+1}(u) - \underline{f}_n(u) = \frac{1}{2^{n+1}} > 0, \tag{4.15}$$

for all $u \in (-\infty, a) \cup (b, \infty)$. Statement (e) follows from (4.14) and (4.15). This completes the proof for $\{\underline{f}_n\}$. □

Remark 4.10. In developing the proof of Theorem 4.3, for the given $f \in H_\alpha$ and $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ associated with (B-R-D-C), we will use the corresponding sequences $\{\underline{f}_n\}_{n\in\mathbb{N}}$ and $\{\overline{f}_n\}_{n\in\mathbb{N}}$ as constructed in Proposition 4.9, with the interval $[a,b] = [-m_0, m_0]$ where $m_0 = ||u_0||_B + 1$.

We now consider the sequences of (B-R-D-C) problems with reaction functions $f = \overline{f}_n$ and $f = \underline{f}_n$ as in (4.9), and initial data $u_0 \in \text{BPC}^2(\mathbb{R})$. Henceforth, these sequences of problems will be referred to as (B-R-D-C)_n^u and (B-R-D-C)_n^l respectively, for each $n \in \mathbb{N}$ (here superscripts u and l indicate upper and lower respectively). We now investigate the problems (B-R-D-C)_n^u and (B-R-D-C)_n^l.

Proposition 4.11. *For each $n \in \mathbb{N}$, any solution $\overline{u}_n, \underline{u}_n : \bar{D}_T \to \mathbb{R}$ to the problems (B-R-D-C)_n^u and (B-R-D-C)_n^l respectively, satisfies the inequalities,*

$$-c't + a' \leq \underline{u}_n(x,t) \leq \overline{u}_n(x,t) \leq c't + b',$$

for all $(x,t) \in \bar{D}_T$, and any $T > 0$, where,

$$c' = \max\left\{\left|\inf_{y\in[-m_0,m_0]}\{f(y)\} - 1\right|, \left|\sup_{y\in[-m_0,m_0]}\{f(y)\} + 1\right|\right\},$$

$$a' = \inf_{x\in\mathbb{R}} u_0(x), \quad b' = \sup_{x\in\mathbb{R}} u_0(x).$$

Proof. For convenience, we define $\overline{v}, \underline{v} : \bar{D}_T \to \mathbb{R}$ to be;

$$\underline{v}(x,t) = a' - c't, \quad \overline{v}(x,t) = b' + c't,$$

for all $(x,t) \in \bar{D}_T$. We now make a straightforward application of the classical Comparison Theorem (see [19], Theorem 6.1), in which we take $\underline{v}$ and $\underline{u}_n$, $\underline{u}_n$ and $\overline{u}_n$, and $\overline{u}_n$ and $\overline{v}$ as regular subsolutions and regular supersolutions to (B-R-D-C)_n^l, (B-R-D-C)_n^u and (B-R-D-C)_n^u respectively, which follows on observing,

$$\underline{v}_t - \underline{v}_{xx} + c' \leq 0, \quad \underline{u}_{nt} - \underline{u}_{nxx} + c' = \underline{f}_n(\underline{u}_n) + c' \geq 0, \tag{4.16}$$

$$\underline{u}_{nt} - \underline{u}_{nxx} - \overline{f}_n(\underline{u}_n) = \underline{f}_n(\underline{u}_n) - \overline{f}_n(\underline{u}_n) \leq 0, \quad \overline{u}_{nt} - \overline{u}_{nxx} - \overline{f}_n(\overline{u}_n) = 0 \leq 0, \tag{4.17}$$

$$\overline{u}_{nt} - \overline{u}_{nxx} - c' = \overline{f}_n(\overline{u}_n) - c' \leq 0, \quad \overline{v}_t - \overline{v}_{xx} - c' \geq 0, \tag{4.18}$$

on D_T, whilst,

$$\underline{v}(x,0) \leq \underline{u}_n(x,0) \leq \overline{u}_n(x,0) \leq \overline{v}(x,0), \tag{4.19}$$

for all $x \in \mathbb{R}$. Now applying the classical Comparison Theorem to each previously stated pair of regular subsolutions and regular supersolutions gives,

$$a' - c't \leq \underline{u}_n(x,t) \leq \overline{u}_n(x,t) \leq b' + c't,$$

for all $(x,t) \in \bar{D}_T$, as required. □

Remark 4.12. Proposition 4.11 ensures that, with $\delta > 0$ as given in Theorem 4.3,

$$-m_0 \leq a' - c't \leq \underline{u}_n(x,t) \leq \overline{u}_n(x,t) \leq b' + c't \leq m_0$$

for all $(x,t) \in \bar{D}_\delta$. Hence (B-R-D-C)_n^u and (B-R-D-C)_n^l are a priori bounded on $\bar{D}_\delta$, for each $n \in \mathbb{N}$, with a priori bounds independent of $n \in \mathbb{N}$.

We now have,

Proposition 4.13. *The problems (B-R-D-C)$_n^u$ and (B-R-D-C)$_n^l$ ($n \in \mathbb{N}$) have unique solutions $\overline{u}_n : \bar{D}_\delta \to \mathbb{R}$ and $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$ respectively. Moreover the inequalities in Proposition 4.11 and Remark 4.12 hold on $\bar{D}_\delta$.*

Proof. It follows from Remark 4.12 that each of (B-R-D-C)$_n^l$ and (B-R-D-C)$_n^u$ is a priori bounded on $\bar{D}_\delta$ for each $n \in \mathbb{N}$. Furthermore, Proposition 4.9 ensures $\overline{f}_n, \underline{f}_n \in L$ for each $n \in \mathbb{N}$. It then follows from the classical Global Existence Theorem (see [19], Theorem 6.4) that (B-R-D-C)$_n^u$ and (B-R-D-C)$_n^l$ have unique solutions on $\bar{D}_\delta$ for each $n \in \mathbb{N}$. These solutions must satisfy the inequalities in Proposition 4.11 and Remark 4.12 on $\bar{D}_\delta$. □

Now that both of the sequences of functions $\{\underline{u}_n\}_{n\in\mathbb{N}}$ and $\{\overline{u}_n\}_{n\in\mathbb{N}}$ have been constructed, it remains to show that they converge to the respective minimal and maximal solutions of the original (B-R-D-C). The remainder of the theory will be presented only for the minimal solution with the theory for the maximal solution following exactly the same steps. We next establish derivative estimates on $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$. In particular, we have

Proposition 4.14. *Let $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$ be the (unique) solution to (B-R-D-C)$_n^l$ ($n \in \mathbb{N}$). Then, on D_δ, we have,*

$$|\underline{u}_{nx}(x,t)| \le \frac{2c'}{\sqrt{\pi}}(1+\delta^{1/2}) + M_0', \quad |\underline{u}_{nt}(x,t)| \le \frac{2^{(\alpha+1)} I_\alpha}{\alpha\sqrt{\pi}} k_\delta (1+\delta^{\alpha/2}) + c' + M_0''$$

for all $(x,t) \in D_\delta$. Here, $k_H > 0$ is a Hölder constant for $f \in H_\alpha$ on $[-m_0, m_0]$, I_α is given by (3.14) *and*

$$M_0' = \sup_{x\in\mathbb{R}} |u_0'(x)|, \quad M_0'' = \sup_{x\in\mathbb{R}} |u_0''(x)|, \quad k_\delta = 3k_H \left(\frac{2c'}{\sqrt{\pi}}(1+\delta^{1/2}) + M_0' \right)^\alpha .$$

Proof. This follows directly from Lemma 3.9 and Lemma 3.12, on recalling that $\underline{f}_n : \mathbb{R} \to \mathbb{R}$ is Hölder continuous of degree α on $[-m_0, m_0] \subset \mathbb{R}$, with Hölder constant $3k_H$. □

Remark 4.15. We observe that all bounds in Proposition 4.14 are independent of $n \in \mathbb{N}$.

Before examining the limit of the sequence $\{\underline{u}_n\}_{n\in\mathbb{N}}$, two further results are required. The first is used to show that the sequence $\{\underline{u}_n\}_{n\in\mathbb{N}}$ is non-decreasing. The second is used to establish part of a comparison theorem. This can be achieved similarly for the sequence $\{\overline{u}_n\}_{n\in\mathbb{N}}$.

Proposition 4.16. *Let $\underline{u}_n$, $\underline{u}_{n+1} : \bar{D}_\delta \to \mathbb{R}$ be the unique solutions to (B-R-D-C)$_n^l$ and (B-R-D-C)$_{n+1}^l$ respectively. Then for each $n \in \mathbb{N}$,*

$$\underline{u}_{n+1}(x,t) \ge \underline{u}_n(x,t) \quad \forall\ (x,t) \in \bar{D}_\delta.$$

Proof. Recall from Proposition 4.9 that $\underline{f}_n : \mathbb{R} \to \mathbb{R}$ is such that $f \in L$ for any $n \in \mathbb{N}$, and,

$$\underline{f}_{n+1}(u) \geq \underline{f}_n(u)$$

for all $u \in \mathbb{R}$. The result then follows via a simple application of the classical Comparison Theorem (see [19], Theorem 6.1). □

Proposition 4.17. *Let $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$ be the unique solution to (B-R-D-C)^l_n on $\bar{D}_\delta$ and $v : \bar{D}_\delta \to \mathbb{R}$ be continuous, bounded and have continuous derivatives v_t, v_x and v_{xx} on D_δ, and such that,*

$$v_t - v_{xx} - f(v) \geq 0$$

for all $(x,t) \in D_\delta$. Suppose in addition, that,

$$v(x,0) \geq u_0(x)$$

for all $x \in \mathbb{R}$. Then for all $(x,t) \in \bar{D}_\delta$,

$$\underline{u}_n(x,t) \leq v(x,t).$$

Proof. To begin fix $n \in \mathbb{N}$. Since v is bounded on $\bar{D}_\delta$, there exists $M > 0$ such that,

$$|v(x,t)| \leq M \quad \forall (x,t) \in \bar{D}_\delta.$$

When $M \leq m_0$, then

$$v_t - v_{xx} - \underline{f}_n(v) \geq f(v) - \underline{f}_n(v) \geq 0,$$

$$\underline{u}_{nt} - \underline{u}_{nxx} - \underline{f}_n(\underline{u}_n) = 0 \leq 0$$

for all $(x,t) \in D_\delta$, via Proposition 4.9, whilst,

$$v(x,0) \geq u_0(x) = \underline{u}_n(x,0) \quad \forall x \in \mathbb{R}. \tag{4.20}$$

Upon taking v and $\underline{u}_n$ as a regular supersolution and regular subsolution respectively, an application of the classical Comparison Theorem (see [19], Theorem 6.1) gives,

$$\underline{u}_n(x,t) \leq v(x,t) \quad \forall (x,t) \in \bar{D}_\delta. \tag{4.21}$$

When $M > m_0$ define $\underline{f}'_n : \mathbb{R} \to \mathbb{R}$ by

$$\underline{f}'_n(u) = \begin{cases} \underline{f}_n(u) & ; \; u \in [-m_0, m_0] \\ g^-_{n+2}(u) - 1/2^n & ; \; u \in [-M, -m_0] \\ g^+_{n+2}(u) - 1/2^n & ; \; u \in [m_0, M] \\ g^-_{n+2}(-M) - 1/2^n & ; \; u \in (-\infty, -M) \\ g^+_{n+2}(M) - 1/2^n & ; \; u \in (M, \infty) \end{cases} \tag{4.22}$$

where $g^-_n : [-M, -m_0] \to \mathbb{R}$ and $g^+_n : [m_0, M] \to \mathbb{R}$ are constructed as in Proposition 4.7, and hence are Lipschitz continuous on $[-M, -m_0]$ and $[m_0, M]$ respectively, and,

$$\max\left\{ \sup_{u \in [-M,-m_0]} |g^-_n(u) - f(u)|, \sup_{u \in [m_0, M]} |g^+_n(u) - f(u)| \right\} < 1/2^n.$$

Moreover, via Remark 4.8 and arguments contained in the proof of Proposition 4.9, $\underline{f}_n' \in L$ and $\underline{f}_n'(u) \le f(u)$ for all $u \in [-M, M]$. Now, taking v and $\underline{u}_n$ to be a regular supersolution and regular subsolution respectively, which follows from (4.20) and the inequalities,

$$v_t - v_{xx} - \underline{f}_n'(v) \ge f(v) - \underline{f}_n'(v) \ge 0,$$

$$\underline{u}_{nt} - \underline{u}_{nxx} - \underline{f}_n'(\underline{u}) = \underline{f}_n(\underline{u}_n) - \underline{f}_n'(\underline{u}_n) = 0 \le 0$$

for all $(x,t) \in D_\delta$, we apply the classical Comparison Theorem to v and $\underline{u}_n$ which gives

$$\underline{u}_n(x,t) \le v(x,t) \quad \forall (x,t) \in \bar{D}_\delta. \tag{4.23}$$

The result follows from (4.21) and (4.23), as required. □

Remark 4.18. Note that in Proposition 4.17, any solution $u : \bar{D}_\delta \to \mathbb{R}$ to (B-R-D-C) on $\bar{D}_\delta$ satisfies the conditions on v. Therefore, for all $n \in \mathbb{N}$,

$$\underline{u}_n(x,t) \le u(x,t) \quad \forall (x,t) \in \bar{D}_\delta.$$

Proposition 4.17 and Remark 4.18 guarantee that any limit function of $\{\underline{u}_n\}_{n\in\mathbb{N}}$ is less than or equal to any solution of (B-R-D-C) on $\bar{D}_\delta$. Therefore, if a limit function of $\{\underline{u}_n\}_{n\in\mathbb{N}}$ is itself a solution to (B-R-D-C), then it must be a minimal solution. We now proceed to establish that the sequence $\{\underline{u}_n\}_{n\in\mathbb{N}}$ does indeed have a limit in $||\cdot||_A$, and that the limit function provides a solution to (B-R-D-C) on $\bar{D}_\delta$.

For each $(x,t) \in \bar{D}_\delta$, we consider the real sequence, $\{\underline{u}_n(x,t)\}_{n\in\mathbb{N}}$. It follows from Proposition 4.16 and Remark 4.12, that this real sequence is non-decreasing and bounded above, and hence is convergent for each $(x,t) \in \bar{D}_\delta$. Thus we may introduce the function $u^* : \bar{D}_\delta \to \mathbb{R}$ given by,

$$u^*(x,t) = \lim_{n\to\infty} \underline{u}_n(x,t) \quad \forall\ (x,t) \in \bar{D}_\delta, \tag{4.24}$$

where we note that

$$\underline{u}_n \to u^* \text{ as } n \to \infty \textit{ pointwise} \text{ on } \bar{D}_\delta. \tag{4.25}$$

We also have from Remark 4.12 and (4.24), that,

$$-m_0 \le u^*(x,t) \le m_0 \quad \forall\ (x,t) \in \bar{D}_\delta. \tag{4.26}$$

Next we have,

Lemma 4.19. *The sequence of functions $\{\underline{u}_n\}_{n\in\mathbb{N}}$ has a subsequence $\{\underline{u}_{n_j}\}_{j\in\mathbb{N}}$ such that*

$$\underline{u}_{n_j} \to u^* \textit{ as } j \to \infty \textit{ uniformly on } \bar{D}_\delta^{0,X},$$

for every $X > 0$. Moreover $u^* : \bar{D}_\delta \to \mathbb{R}$ is continuous on $\bar{D}_\delta$.

Proof. Consider the sequence of functions $\{\underline{u}_n\}_{n\in\mathbb{N}}$ in $\bar{D}_\delta$. Then each function $\underline{u}_n$, $n \in \mathbb{N}$, is continuous on $\bar{D}_\delta$ as it is a solution to $(\text{B-R-D-C})_n^l$ on $\bar{D}_\delta$. Also, we have, for each $n \in \mathbb{N}$,

$$|\underline{u}_n(x,t)| \le m_0 \quad \forall (x,t) \in \bar{D}_\delta, \tag{4.27}$$

via Remark 4.12. Now $\underline{u}_{nt}$ and $\underline{u}_{nx}$ exist and are continuous on D_δ and so it follows from the mean value theorem that for any $(x_0,t_0),\ (x_1,t_1)\in D_\delta$, then,

$$|\underline{u}_n(x_1,t_1)-\underline{u}_n(x_0,t_0)| = |\underline{u}_{nt}(\xi,\eta)(t_1-t_0)+\underline{u}_{nx}(\xi,\eta)(x_1-x_0)| \tag{4.28}$$

with $(\xi,\eta)\in D_\delta$ lying on the straight line joining (x_0,t_0) to (x_1,t_1). It follows from (4.28) and Proposition 4.14, that,

$$\begin{aligned}|\underline{u}_n(x_1,t_1)-\underline{u}_n(x_0,t_0)| &\le |\underline{u}_{nt}(\xi,\eta)||t_1-t_0|+|\underline{u}_{nx}(\xi,\eta)||x_1-x_0|\\ &\le \max\left\{\frac{2c'}{\sqrt{\pi}}(1+\delta^{1/2})+M_0',\ \frac{2^{\alpha+1}I_\alpha}{\alpha\sqrt{\pi}}k_\delta(1+\delta^{\alpha/2})+c'+M_0''\right\}\\ &\quad\times(|t_1-t_0|+|x_1-x_0|)\\ &\le \max\left\{\frac{2c'}{\sqrt{\pi}}(1+\delta^{1/2})+M_0',\ \frac{2^{\alpha+1}I_\alpha}{\alpha\sqrt{\pi}}k_\delta(1+\delta^{\alpha/2})+c'+M_0''\right\}\\ &\quad\times\sqrt{2}\,(|(x_1,t_1)-(x_0,t_0)|)\,. \end{aligned}\tag{4.29}$$

Since (4.29) holds for all $(x_1,t_1),\ (x_0,t_0)\in D_\delta$, and $\underline{u}_n$ is continuous on $\bar{D}_\delta$, then it follows that (4.29) holds for all $(x_1,t_1),\ (x_0,t_0)$ on $\bar{D}_\delta$. It is then an immediate consequence of (4.29) that the sequence of functions $\{\underline{u}_n\}_{n\in\mathbb{N}}$ are uniformly equicontinuous on $\bar{D}_\delta$. Moreover, it follows from (4.27) that $\{\underline{u}_n\}_{n\in\mathbb{N}}$ are uniformly bounded (by m_0) on $\bar{D}_\delta$. It then follows immediately from the *Ascoli-Arzéla compactness criterion* (see, for example, [28], p.154-158) that there exists a subsequence $\{\underline{u}_{n_j}\}_{j\in\mathbb{N}}$ and a continuous function $u_c:\bar{D}_\delta\to\mathbb{R}$ such that,

$$\underline{u}_{n_j}\to u_c \text{ as } j\to\infty \text{ uniformly on } \bar{D}_\delta^{0,X}, \tag{4.30}$$

for any $X>0$. From (4.30), we have that for each $(x,t)\in\bar{D}_\delta$, the real sequence $\{\underline{u}_{n_j}(x,t)\}_{n_j\in\mathbb{N}}$, is such that,

$$\underline{u}_{n_j}(x,t)\to u_c(x,t) \text{ as } j\to\infty. \tag{4.31}$$

It also follows from (4.25) (convergence of subsequences of convergent real sequences) that,

$$\underline{u}_{n_j}(x,t)\to u^*(x,t) \text{ as } j\to\infty. \tag{4.32}$$

It follows from (4.31) and (4.32) (uniqueness of limits of convergent real sequences) that,

$$u^*(x,t)=u_c(x,t)\quad\forall\ (x,t)\in\bar{D}_\delta$$

and so $u^*:\bar{D}_\delta\to\mathbb{R}$ is continuous and via (4.30), $\underline{u}_{n_j}\to u^*$ as $j\to\infty$ uniformly on $\bar{D}_\delta^{0,X}$, for any $X>0$, as required. □

As a consequence we have:

Corollary 4.20. *For any $X>0$, $\underline{u}_n\to u^*$ as $n\to\infty$ uniformly on $\bar{D}_\delta^{0,X}$.*

Proof. Follows directly from Lemma 4.19 and Proposition 4.16. □

We now have,

Proposition 4.21. *Let $u : \bar{D}_\delta \to \mathbb{R}$ be any solution to (B-R-D-C). Then,*

$$u^*(x,t) \le u(x,t) \quad \forall (x,t) \in \bar{D}_\delta.$$

Proof. It follows from Proposition 4.17 that for each $n \in \mathbb{N}$,

$$\underline{u}_n(x,t) \le u(x,t) \quad \forall (x,t) \in \bar{D}_\delta.$$

It then follows from (4.25) that,

$$u^*(x,t) \le u(x,t) \quad \forall (x,t) \in \bar{D}_\delta,$$

as required. □

Remark 4.22. $u^* : \bar{D}_\delta \to \mathbb{R}$ is continuous and from (4.26), $|u^*(x,t)| \le m_0$, for all $(x,t) \in \bar{D}_\delta$, so u^* is bounded on $\bar{D}_\delta$. It follows that $u^* \in B_A^\delta$ and $\|u^*\|_A \le m_0$.

With Remark 4.22 it remains to establish that $u^* : \bar{D}_\delta \to \mathbb{R}$ satisfies the appropriate integral equation in Lemma 3.7. To begin, we introduce the function $v : \bar{D}_\delta \to \mathbb{R}$, as follows,

$$v(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \tag{4.33}$$

for all $(x,t) \in \bar{D}_\delta$. We note that v is well-defined and $v \in B_A^\delta$. Moreover, since the initial data $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ to each problem $(\text{B-R-D-C})_n^l$ is the same for each $n \in \mathbb{N}$, it remains only to consider the functions $\underline{w}_n : \bar{D}_\delta \to \mathbb{R}$ $(n \in \mathbb{N})$ and $w : \bar{D}_\delta \to \mathbb{R}$ defined as follows,

$$\underline{w}_n(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \underline{f}_n \left(\underline{u}_n(x + 2\sqrt{t-\tau}\lambda, \tau) \right) e^{-\lambda^2} d\lambda d\tau,$$

$$w(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f \left(u^*(x + 2\sqrt{t-\tau}\lambda, \tau) \right) e^{-\lambda^2} d\lambda d\tau \tag{4.34}$$

for all $(x,t) \in \bar{D}_\delta$. We note that these functions are well-defined, since $\underline{u}_n \in B_A^\delta$ $(n \in \mathbb{N})$ (as it is a solution to $(\text{B-R-D-C})_n^l$) and $u^* \in B_A^\delta$ (via Remark 4.22). Moreover $w, \underline{w}_n \in B_A^\delta$ $(n \in \mathbb{N})$. We also observe that, $\underline{f}_n(\underline{u}_n), f(u^*) \in B_A^\delta$, and

$$\|\underline{f}_n(\underline{u}_n)\|_A \le c', \quad \|f(u^*)\|_A \le c' \tag{4.35}$$

for all $n \in \mathbb{N}$, via Remark 4.22. We now have,

Lemma 4.23. *For each $(x,t) \in \bar{D}_\delta$, the real sequence $\{\underline{w}_n(x,t)\}_{n\in\mathbb{N}}$ is convergent, and,*

$$\lim_{n\to\infty} \underline{w}_n(x,t) = w(x,t).$$

Proof. Given any $\epsilon > 0$, take

$$\lambda_\epsilon = \max\left\{ \frac{8c'(1+\delta)}{\sqrt{\pi}\epsilon},\ 1 \right\}. \tag{4.36}$$

Now fix $(x,t) \in \bar{D}_\delta$, then,

$$
\begin{aligned}
|\underline{w}_n(x,t) - w(x,t)| \le{}& \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} |\underline{f}_n\left(\underline{u}_n(x+2\sqrt{t-\tau}\lambda,\tau)\right) \\
&\qquad - f\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau \\
\le{}& \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\lambda_\epsilon}^{\lambda_\epsilon} |\underline{f}_n\left(\underline{u}_n(x+2\sqrt{t-\tau}\lambda,\tau)\right) \\
&\qquad - f\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau \\
&+ \frac{1}{\sqrt{\pi}} \int_0^t \int_{\lambda_\epsilon}^{\infty} |\underline{f}_n\left(\underline{u}_n(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau \\
&+ \frac{1}{\sqrt{\pi}} \int_0^t \int_{\lambda_\epsilon}^{\infty} |f\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau \\
&+ \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{-\lambda_\epsilon} |\underline{f}_n\left(\underline{u}_n(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau \\
&+ \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{-\lambda_\epsilon} |f\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau
\end{aligned}
$$

and so,

$$
\begin{aligned}
|\underline{w}_n(x,t) - w(x,t)| <{}& \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\lambda_\epsilon}^{\lambda_\epsilon} |\underline{f}_n\left(\underline{u}_n(x+2\sqrt{t-\tau}\lambda,\tau)\right) \\
&\qquad - f\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right)| e^{-\lambda^2} d\lambda d\tau + \frac{\epsilon}{2}, \quad (4.37)
\end{aligned}
$$

on using (4.35) and (4.36). Now, via Corollary 4.20, Proposition 4.9 and Proposition 4.11, it follows that there exists $N_\epsilon \in \mathbb{N}$, independent of $(\lambda,\ \tau) \in [-\lambda_\epsilon, \lambda_\epsilon] \times [0,t]$ such that for all $n \ge N_\epsilon$, then,

$$
\begin{aligned}
&|\underline{f}_n\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right) - f\left(u^*(x+2\sqrt{t-\tau}\lambda,\tau)\right)| < \frac{\epsilon}{4\delta}, \\
&|\underline{u}_n\left(x+2\sqrt{t-\tau}\lambda,\tau\right) - u^*\left(x+2\sqrt{t-\tau}\lambda,\tau\right)| \le \left(\frac{\epsilon}{12k_H\delta}\right)^{1/\alpha}
\end{aligned}
$$

for all $(\lambda,\ \tau) \in [-\lambda_\epsilon, \lambda_\epsilon] \times [0,t]$ with $k_H > 0$ being a Hölder constant for $f \in H_\alpha$ on $[-m_0, m_0]$. It then follows from (4.37) that, for all $n \ge N_\epsilon$ (which may depend on

$(x,t) \in \bar{D}_\delta$), then via Proposition 4.9,

$$
\begin{aligned}
&|\underline{w}_n(x,t) - w(x,t)| \\
&\quad < \frac{1}{\sqrt{\pi}} \int_0^\delta \int_{-\lambda_\epsilon}^{\lambda_\epsilon} \Big(|\underline{f}_n \left(\underline{u}_n(x + 2\sqrt{t-\tau}\lambda, \tau)\right) - \underline{f}_n \left(u^*(x + 2\sqrt{t-\tau}\lambda, \tau)\right)| + \\
&\quad\quad |\underline{f}_n \left(u^*(x + 2\sqrt{t-\tau}\lambda, \tau)\right) - f\left(u^*(x + 2\sqrt{t-\tau}\lambda, \tau)\right)| \Big) e^{-\lambda^2} d\lambda d\tau + \frac{\epsilon}{2} \\
&\quad = \frac{1}{\sqrt{\pi}} \int_0^\delta \int_{-\lambda_\epsilon}^{\lambda_\epsilon} \left(3k_H |\underline{u}_n - u^*|^\alpha (x + 2\sqrt{t-\tau}\lambda, \tau) + \frac{\epsilon}{4\delta} \right) e^{-\lambda^2} d\lambda d\tau + \frac{\epsilon}{2} \\
&\quad \le \frac{1}{\sqrt{\pi}} \int_0^\delta \int_{-\lambda_\epsilon}^{\lambda_\epsilon} \left(\frac{\epsilon}{4\delta} + \frac{\epsilon}{4\delta} \right) e^{-\lambda^2} d\lambda d\tau + \frac{\epsilon}{2} \\
&\quad \le \frac{\epsilon}{2\delta\sqrt{\pi}} \int_0^\delta \int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda d\tau + \frac{\epsilon}{2} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Therefore for each $(x,t) \in \bar{D}_\delta$, the real sequence $\{\underline{w}_n(x,t)\}_{n\in\mathbb{N}}$ is convergent and

$$
\lim_{n\to\infty} \underline{w}_n(x,t) = w(x,t),
$$

as required. □

We now have,

Lemma 4.24. *The function $u^* : \bar{D}_\delta \to \mathbb{R}$ is such that, $u^* \in B_A^\delta$, and,*

$$
\begin{aligned}
u^*(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u^*(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau \quad \forall (x,t) \in \bar{D}_\delta.
\end{aligned}
$$

Proof. For each $n \in \mathbb{N}$, then by construction $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$ is a solution to (B-R-D-C)$_n^l$ on $\bar{D}_\delta$. Since, for each $n \in \mathbb{N}$, (B-R-D-C)$_n^l$ has $\underline{f}_n \in H_\alpha$, it follows from Lemma 3.7 that $\underline{u}_n \in B_A^\delta$ and

$$
\begin{aligned}
\underline{u}_n(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \underline{f}_n \left(\underline{u}_n(x + 2\sqrt{t-\tau}\lambda, \tau)\right) e^{-\lambda^2} d\lambda d\tau \\
&= v(x,t) + \underline{w}_n(x,t) \qquad (4.38)
\end{aligned}
$$

for all $(x,t) \in \bar{D}_\delta$. Now fix $(x,t) \in \bar{D}_\delta$. It then follows from (4.38), (4.25) and Lemma 4.23 that,

$$
u^*(x,t) = v(x,t) + w(x,t) \quad \forall (x,t) \in \bar{D}_\delta,
$$

which, via (4.33) and (4.34) becomes,

$$
\begin{aligned}
u^*(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f\left(u^*(x + 2\sqrt{t-\tau}\lambda, \tau)\right) e^{-\lambda^2} d\lambda d\tau
\end{aligned}
$$

for all $(x,t) \in \bar{D}_\delta$. In addition, via Remark 4.22, $u^* \in B_A^\delta$. The proof is complete. □

It now follows immediately from Lemma 4.24 and Lemma 3.7 that $u^* : \bar{D}_\delta \to \mathbb{R}$ provides a solution to (B-R-D-C) on $\bar{D}_\delta$. That $u^* : \bar{D}_\delta \to \mathbb{R}$ is a minimal solution to (B-R-D-C) on $\bar{D}_\delta$ follows from Proposition 4.21 and the bound follows from Remark 4.22. The proof of Theorem 4.3 is complete. A global existence theorem can now be established, namely,

Theorem 4.25. *Consider (B-R-D-C) with $f \in H_\alpha$ for some $\alpha \in (0,1)$. When (B-R-D-C) is a priori bounded on $\bar{D}_T$ for any $0 \le T \le T'$, then (B-R-D-C) has a constructed minimal and a constructed maximal solution on $\bar{D}_{T'}$.*

Proof. The proof is a direct application of Theorem 4.3, with the a priori bounds allowing $[0,T']$ to be covered in a finite number of steps. The main details are standard (see [19], Theorem 6.4), after which the maximal and minimal properties follow from Remark 4.4. □

Following Proposition 4.17 we also have the following comparison-type result.

Proposition 4.26. *Let $f \in H_\alpha$ for some $\alpha \in (0,1)$, and let $\underline{w}, \overline{w} : \bar{D}_T \to \mathbb{R}$ be a regular subsolution and a regular supersolution to (B-R-D-C), respectively. Let $\underline{u}^c, \overline{u}^c : \bar{D}_T \to \mathbb{R}$ be constructed minimal and maximal solutions to (B-R-D-C), then*

$$\underline{u}^c(x,t) \le \overline{w}(x,t) \quad \text{and} \quad \overline{u}^c(x,t) \ge \underline{w}(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

Proof. We give a proof for the first inequality. The second inequality follows the same argument, with obvious modifications. Now, $\underline{u}^c : \bar{D}_T \to \mathbb{R}$ is a constructed minimal solution to (B-R-D-C). It follows, via Remark 4.4, and the construction of $\underline{u}^c$, that Proposition 4.17 holds on each constructional subdomain of $\bar{D}_T$ in turn. The result then follows. □

Remark 4.27. We observe that when uniqueness holds for (B-R-D-C) on $\bar{D}_T$, then $\underline{u}^c = \overline{u}^c$ on $\bar{D}_T$ and Proposition 4.26 becomes a full Comparison Theorem for (B-R-D-C).

The issue we have not addressed this far is uniqueness, and we may anticipate that general uniqueness, where $f \in H_\alpha$ for $\alpha \in (0,1)$, is false, via the following simple example.

Example 4.28. Consider the (B-R-D-C) problem where $f : \mathbb{R} \to \mathbb{R}$ is given by,

$$f(u) = \begin{cases} u^p & ; \ u > 0 \\ 0 & ; \ u \le 0, \end{cases}$$

for some $p \in (0,1)$ and $u_0 : \mathbb{R} \to \mathbb{R}$ is such that $u_0(x) = 0$ for all $x \in \mathbb{R}$. Simple calculations show that $f \in H_p \backslash L_u$ and $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. Now define $u_1, u_2 : \bar{D}_T \to \mathbb{R}$ for any $T > 0$ to be,

$$u_1(x,t) = 0 \quad \forall (x,t) \in \bar{D}_T,$$

$$u_2(x,t) = \begin{cases} 0 & ; \ (x,t) \in \mathbb{R} \times [0,t_s] \\ ((1-p)(t-t_s))^{1/(1-p)} & ; \ \mathbb{R} \times (t_s, T] \end{cases}$$

for any $0 \le t_s < T$. It is readily verified that u_1 and u_2 are distinct solutions to (B-R-D-C).

We next consider a further pathological example to illustrate the breadth of Theorem 4.3, where the reaction function is non-Lipschitz on every closed bounded interval.

Example 4.29. Consider (B-R-D-C) with reaction function $f_{\alpha,b} : \mathbb{R} \to \mathbb{R}$ given by

$$f_{\alpha,b}(u) = \sum_{n=0}^{\infty} b^{-n\alpha} \cos(b^n u) \tag{4.39}$$

for all $u \in \mathbb{R}$, where $b > 1$ and $\alpha \in (0,1)$. This function was used by Weierstrass [34], to exhibit the existence of a real valued function which is everywhere continuous, but non-differentiable almost everywhere. As a consequence of Rademachers Theorem [13] (p.100), this function is not Lipschitz continuous on any closed bounded interval. However, for any $\alpha' \in (0,\alpha)$,

$$\begin{aligned}|f_{\alpha,b}(u) - f_{\alpha,b}(v)| &\leq \sum_{n=0}^{\infty} b^{-n\alpha} |\cos(b^n u) - \cos(b^n v)| \\ &\leq 2\sum_{n=0}^{\infty} b^{-n\alpha} |b^n u - b^n v|^{\alpha'} \\ &= 2\sum_{n=0}^{\infty} b^{n(\alpha'-\alpha)} |u-v|^{\alpha'} \\ &= \frac{2}{(1-b^{(\alpha'-\alpha)})} |u-v|^{\alpha'}\end{aligned}$$

for any $u, v \in \mathbb{R}$. Hence $f_{\alpha,b} \in H_{\alpha'}$ with Hölder constant $\frac{2}{(1-b^{(\alpha'-\alpha)})}$ on any closed bounded interval. Moreover, f is bounded on $\mathbb{R}$ with

$$|f_{\alpha,b}(u)| \leq \frac{1}{(1-b^{-\alpha})} \quad \forall u \in \mathbb{R}. \tag{4.40}$$

Now let $u : \bar{D}_T \to \mathbb{R}$ be any solution to (B-R-D-C), and let $w_+ : \bar{D}_T \to \mathbb{R}$ and $w_- : \bar{D}_T \to \mathbb{R}$ be such that

$$w_+(x,t) = \frac{t}{(1-b^{-\alpha})} + \sup_{\lambda\in\mathbb{R}} u_0(\lambda), \quad w_-(x,t) = \frac{-t}{(1-b^{-\alpha})} + \inf_{\lambda\in\mathbb{R}} u_0(\lambda)$$

for all $(x,t) \in \bar{D}_T$. Then,

$$u_t - u_{xx} - \frac{1}{(1-b^{-\alpha})} = f_{\alpha,b}(u) - \frac{1}{(1-b^{-\alpha})} \leq 0, \quad w_{+t} - w_{+xx} - \frac{1}{(1-b^{-\alpha})} = 0 \geq 0$$

for all $(x,t) \in D_T$. It follows via the classical Comparison Theorem (see [19], Theorem 6.1), that

$$u(x,t) \leq w_+(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

Similarly, we establish that

$$u(x,t) \geq w_-(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

Thus,

$$\frac{-T}{(1-b^{-\alpha})} - \|u_0\|_B \le u(x,t) \le \frac{T}{(1-b^{-\alpha})} + \|u_0\|_B \quad \forall (x,t) \in \bar{D}_T,$$

and so

$$\|u\|_A \le \frac{T}{(1-b^{-\alpha})} + \|u_0\|_B.$$

We conclude that (B-R-D-C) is a priori bounded on $\bar{D}_T$ for any $T > 0$. Thus (B-R-D-C) has a global constructed minimal solution $\underline{u}^c : \bar{D}_\infty \to \mathbb{R}$ and a global constructed maximal solution $\overline{u}^c : \bar{D}_\infty \to \mathbb{R}$, via Theorem 4.25, and,

$$\frac{-t}{(1-b^{-\alpha})} + \inf_{\lambda\in\mathbb{R}} u_0(\lambda) \le \underline{u}^c(x,t) \le \overline{u}^c(x,t) \le \frac{t}{(1-b^{-\alpha})} + \sup_{\lambda\in\mathbb{R}} u_0(\lambda) \quad \forall (x,t) \in \bar{D}_\infty.$$

We remark that the approach adopted here in the proof of Theorem 4.3 was primarily motivated by the specific problems in [23], [1] and [21]. However, the methodology is remarkably similar to that developed in the context of ordinary differential equations in Carathéodory [4]. Carathéodory's approach has been used in [5] (p.45) to establish an analogous result to Theorem 4.3 for the ordinary differential equation problem

$$u_t = f(u,t), \quad u(0) = u_0$$

on $t \in [0,T]$ with $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous function in both variables. The methodology is similar in the sense that successive approximations are made and the Ascoli-Arzela compactness theorem is used to establish the existence of a limit. In addition, global existence results for second order parabolic partial differential equations (similar to Theorem 4.25), under various hypotheses, are available in [25] and [3]. The results in [25] are obtained by similar monotonicity methods, whereas the results in [3] are obtained by examining the limit of a sequence of Dirichlet problems with expanding domains together with the theory developed in [31] and [22] to guarantee existence and regularity of solutions to the approximating Dirichlet problems (under the assumption of the existence of global supersolutions and subsolutions).

We complete this section by establishing structural qualitative features of maximal and minimal solutions to (B-R-D-C). The first three results do not require that the associated maximal and minimal solution be constructed, whereas the fourth and fifth results require that the associated maximal and minimal solution is constructed. Once qualitative features have been established, we provide a conditional continuous dependence result for solutions to (B-R-D-C). To begin, we have,

Proposition 4.30. *Consider (B-R-D-C) with $f \in H_\alpha$ for some $\alpha \in (0,1)$, such that $u_0 : \mathbb{R} \to \mathbb{R}$ is constant. Suppose that $\overline{u}, \underline{u} : \bar{D}_T \to \mathbb{R}$ are a maximal solution and a minimal solution to (B-R-D-C) on $\bar{D}_T$, respectively. Then $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are independent of $x \in \mathbb{R}$ for each $t \in [0,T]$.*

Proof. First, for each $k \in \mathbb{R}$, consider $w^{(k)} : \bar{D}_T \to \mathbb{R}$ given by,

$$w^{(k)}(x,t) = \overline{u}(x+k,t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.41}$$

It follows immediately, that for each $k \in \mathbb{R}$, $w^{(k)}$ is a solution to (B-R-D-C) on $\bar{D}_T$. Now, since $\overline{u} : \bar{D}_T \to \mathbb{R}$ is a maximal solution to (B-R-D-C) on $\bar{D}_T$, it follows from Definition 4.1 that

$$w^{(k)}(x,t) \leq \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_T, \tag{4.42}$$

and so,

$$\overline{u}(x+k,t) \leq \overline{u}(x,t) \quad \forall (x,t,k) \in \bar{D}_T \times \mathbb{R}. \tag{4.43}$$

Now, take $(x_1,t),(x_2,t) \in \bar{D}_T$, and set $x = x_2$ and $k = x_1 - x_2$ in (4.43), to obtain

$$\overline{u}(x_1,t) \leq \overline{u}(x_2,t). \tag{4.44}$$

Next set $x = x_1$ and $k = x_2 - x_1$ in (4.43) to obtain

$$\overline{u}(x_2,t) \leq \overline{u}(x_1,t). \tag{4.45}$$

It follows from (4.44) and (4.45) that $\overline{u}(x_1,t) = \overline{u}(x_2,t)$, as required. The result for $\underline{u} : \bar{D}_T \to \mathbb{R}$ follows by a symmetrical argument. □

Remark 4.31. It follows from Proposition 4.30 that for (B-R-D-C) with $f \in H_\alpha$ for some $\alpha \in (0,1)$, and $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ such that u_0 is constant ($u_0 = c$), then a maximal solution $\overline{u} : \bar{D}_T \to \mathbb{R}$ and a minimal solution $\underline{u} : \bar{D}_T \to \mathbb{R}$ to (B-R-D-C) on $\bar{D}_T$ will be given by a maximal solution $\overline{U} : [0,T] \to \mathbb{R}$ and a minimal solution $\underline{U} : [0,T] \to \mathbb{R}$ to the following initial value problem,

$$u_t = f(u) \quad \forall t \in (0,T], \quad u(0) = c.$$

The existence of $\overline{U}, \underline{U} : [0,T] \to \mathbb{R}$ are guaranteed by Proposition 4.30.

We also have,

Proposition 4.32. *Consider (B-R-D-C) with $f \in H_\alpha$ for some $\alpha \in (0,1)$ and $u_0 \in BPC^2(\mathbb{R})$ such that*

$$u_0(x) = u_0(-x) \quad \forall x \in \mathbb{R}.$$

Suppose that $\overline{u}, \underline{u} : \bar{D}_T \to \mathbb{R}$ are a maximal solution and a minimal solution to (B-R-D-C) on $\bar{D}_T$, respectively. Then,

$$\overline{u}(x,t) = \overline{u}(-x,t) \quad \textit{and} \quad \underline{u}(x,t) = \underline{u}(-x,t) \quad \forall (x,t) \in \bar{D}_T.$$

Proof. First introduce $w : \bar{D}_T \to \mathbb{R}$ as

$$w(x,t) = \overline{u}(-x,t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.46}$$

Then, it follows that

$$w(x,0) = u_0(x) \quad \forall x \in \mathbb{R}. \tag{4.47}$$

Additionally, it follows from (4.46) that

$$w_t - w_{xx} - f(w) = 0 \quad \forall (x,t) \in D_T. \tag{4.48}$$

Therefore, via (4.47) and (4.48), $w : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C) on $\bar{D}_T$. Thus, via (4.46) and Definition 4.1, we have

$$\overline{u}(-x,t) = w(x,t) \le \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.49}$$

Upon considering $(x,t) = (\pm s, t)$ in (4.49), it follows that

$$\overline{u}(s,t) = \overline{u}(-s,t) \quad \forall (s,t) \in \bar{D}_T,$$

as required. The result for $\underline{u}$ follows a symmetrical argument. □

Additionally, we have,

Proposition 4.33. *Consider (B-R-D-C) with* $f \in H_\alpha$ *for some* $\alpha \in (0,1)$ *and* $u_0 \in BPC^2(\mathbb{R})$ *such that for some* $X > 0$*,*

$$u_0(x) = u_0(x + X) \quad \forall x \in \mathbb{R}.$$

Suppose that $\overline{u}, \underline{u} : \bar{D}_T \to \mathbb{R}$ *are a maximal solution and a minimal solution to (B-R-D-C) on* $\bar{D}_T$*, respectively. Then,*

$$\overline{u}(x,t) = \overline{u}(x + X, t) \quad \text{and} \quad \underline{u}(x,t) = \underline{u}(x + X, t) \quad \forall (x,t) \in \bar{D}_T.$$

Proof. First, consider $w : \bar{D}_T \to \mathbb{R}$ given by

$$w(x,t) = \overline{u}(x + X, t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.50}$$

It follows immediately, that $w : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C) on $\bar{D}_T$. Now, since $\overline{u}$ is a maximal solution to (B-R-D-C) on $\bar{D}_T$, it follows from Definition 4.1 that,

$$w(x,t) \le \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_T,$$

and so,

$$\overline{u}(x + X, t) \le \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.51}$$

Similarly, we may establish that,

$$\overline{u}(x - X, t) \le \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.52}$$

Now, put $(x,t) = (s,t)$ in (4.51) and $(x,t) = (s + X, t)$ in (4.52), from which it follows that,

$$\overline{u}(x + X, t) = \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_T,$$

as required. The result for $\underline{u}$ follows a symmetrical argument. □

Remark 4.34. We note here, that if we consider (B-R-D-C) with $f \in H_\alpha$, for some $\alpha \in (0,1)$, and initial data $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ such that

$$u_0(x) = -u_0(-x) \quad \forall x \in \mathbb{R},$$

then the method of proof adopted in the above propositions fails to establish a similar conclusion for a maximal or a minimal solution to (B-R-D-C) on $\bar{D}_T$.

We now consider a result concerning constructed maximal and constructed minimal solutions.

Proposition 4.35. *Consider (B-R-D-C) with* $f \in H_\alpha$ *for some* $\alpha \in (0,1)$*, and* $u_0 \in BPC^2(\mathbb{R})$ *such that* u_0 *is non-decreasing (non-increasing). Suppose that* $\overline{u}^c, \underline{u}^c : \bar{D}_T \to \mathbb{R}$ *are a constructed maximal solution and a constructed minimal solution to (B-R-D-C) on* $\bar{D}_T$*, respectively. Then* $\overline{u}^c(x,t)$ *and* $\underline{u}^c(x,t)$ *are non-decreasing (non-increasing) with* $x \in \mathbb{R}$ *for each* $t \in [0,T]$*.*

Proof. We give a proof in the non-decreasing case, with the non-increasing case following similarly. First, for each $k \geq 0$, introduce $\underline{w}^{(k)} : \bar{D}_T \to \mathbb{R}$ such that

$$\underline{w}^{(k)}(x,t) = \overline{u}^c(x-k,t) \quad \forall (x,t,k) \in \bar{D}_T \times \bar{\mathbb{R}}^+. \tag{4.53}$$

It follows that $\underline{w}^{(k)} : \bar{D}_T \to \mathbb{R}$ is a regular subsolution to (B-R-D-C) on $\bar{D}_T$. An application of Proposition 4.26 then gives,

$$\overline{u}^c(x-k,t) = \underline{w}^{(k)}(x,t) \leq \overline{u}^c(x,t) \quad \forall (x,t,k) \in \bar{D}_T \times \bar{\mathbb{R}}^+. \tag{4.54}$$

Now take $(x_1,t),(x_2,t) \in \bar{D}_T$, with $x_2 \geq x_1$, and set $x = x_2$ with $k = x_2 - x_1$ in (4.54), to obtain

$$\overline{u}^c(x_1,t) \leq \overline{u}^c(x_2,t) \quad \forall x_2 \geq x_1,\ t \in [0,T].$$

Therefore, $\overline{u}^c(x,t)$ is non-decreasing with $x \in \mathbb{R}$ for each $t \in [0,T]$. The result for $\underline{u}^c$ follows a symmetrical argument. □

We now consider the behavior of solutions to (B-R-D-C) as $|x| \to \infty$.

Proposition 4.36. *Consider (B-R-D-C) with* $f \in H_\alpha$ *for some* $\alpha \in (0,1)$*, and* $u_0 \in BPC^2(\mathbb{R})$ *such that*

$$\lim_{x \to \pm\infty} u_0(x) = u_0^{\pm}.$$

Let $\overline{u}^c, \underline{u}^c : \bar{D}_\delta \to \mathbb{R}$ *be the constructed maximal solution and the constructed minimal solution to (B-R-D-C) on* $\bar{D}_\delta$*, as given by Theorem 4.3, respectively. Then,*

$$\limsup_{x \to \pm\infty} \overline{u}^c(x,t) \leq \overline{U}^{\pm}(t), \quad \liminf_{x \to \pm\infty} \underline{u}^c(x,t) \geq \underline{U}^{\pm}(t)$$

uniformly for $t \in [0,\delta]$*, where* $\overline{U}^{\pm} : [0,\delta] \to \mathbb{R}$ *is the maximal solution and* $\underline{U}^{\pm} : [0,\delta] \to \mathbb{R}$ *is the minimal solution respectively, to the initial value problem,*

$$U_t^{\pm} = f(U^{\pm}) \quad \forall t \in (0,\delta], \quad U^{\pm}(0) = u_0^{\pm}. \tag{4.55}$$

Proof. First note that the existence of $\overline{U}^{\pm}, \underline{U}^{\pm} : [0,\delta] \to \mathbb{R}$ is guaranteed by [5, Theorem 2.3, p.10]. We consider the case $x \to +\infty$, with the case $x \to -\infty$ following similarly. To begin, let $\delta > 0$ and $\underline{u}^c : \bar{D}_\delta \to \mathbb{R}$ be as in Theorem 4.3. Moreover, let $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$ be the solution to the problem (B-R-D-C)$)_n^l$ for each $n \in \mathbb{N}$, as employed in the construction of Theorem 4.3, via Proposition 4.13. It follows immediately from Remark 4.12 and Remark 4.18 that

$$-m_0 \leq \underline{u}_n(x,t) \leq \underline{u}^c(x,t) \quad \forall (x,t) \in \bar{D}_\delta, \tag{4.56}$$

with m_0 as in Theorem 4.3. Moreover, via (4.56),

$$-m_0 \le \liminf_{x\to\infty} \underline{u}_n(x,t) \le \liminf_{x\to\infty} \underline{u}^c(x,t) \tag{4.57}$$

uniformly for $t \in [0,\delta]$ and $n \in \mathbb{N}$. Now for $n \in \mathbb{N}$, since $\underline{u}_n : \bar{D}_\delta \to \mathbb{R}$ is bounded and $\underline{f}_n \in L$, it follows from [33, Theorem 5.2, p. 239] that

$$\underline{u}_n(x,t) \to \underline{U}_n^+(t), \tag{4.58}$$

as $x \to +\infty$ uniformly for $t \in [0,\delta]$, where $\underline{U}_n^+ : [0,\delta] \to \mathbb{R}$ is the unique classical solution (see [5, Theorem 2.3, p.10]) to the problem;

$$\underline{U}_{nt}^+ = \underline{f}_n(\underline{U}_n^+) \quad \forall t \in (0,\delta], \quad \underline{U}_n^+(0) = u_0^+ - \frac{1}{2n}. \tag{4.59}$$

Observe that $\underline{U}_n^+ : [0,\delta] \to \mathbb{R}$ is continuous and, via Remark 4.12 and (4.58), is bounded uniformly for $(t,n) \in [0,\delta] \times \mathbb{N}$ with

$$|\underline{U}_n^+(t)| \le m_0 \quad \forall t \in [0,\delta]. \tag{4.60}$$

Additionally, it follows immediately from Remark 4.10 and (4.59) that

$$|\underline{U}_n^+(t_1) - \underline{U}_n^+(t_2)| \le \sup_{t\in(0,\delta]} |\underline{U}_{nt}^+(t)||t_1 - t_2| \le c'|t_1 - t_2| \quad \forall t_1, t_2 \in [0,\delta], \tag{4.61}$$

with c' independent of $n \in \mathbb{N}$, and as in Theorem 4.3. Thus, it follows from (4.60) that the sequence of continuous functions $\{\underline{U}_n^+\}_{n\in\mathbb{N}}$ is uniformly bounded and, via (4.61), uniformly equicontinuous. It then follows immediately from the *Ascoli-Arzéla compactness criterion* ([28][p.154-158]) that there exists a subsequence $\{\underline{U}_{n_j}^+\}_{j\in\mathbb{N}}$ ($1 \le n_1 < n_2 < n_3 < \ldots$ and $n_j \to \infty$ as $j \to \infty$) and a continuous function $U : [0,\delta] \to \mathbb{R}$ such that

$$\underline{U}_{n_j}^+(t) \to U(t) \text{ as } j \to \infty \text{ uniformly for } t \in [0,\delta], \tag{4.62}$$

and with $U : [0,T] \to \mathbb{R}$ satisfying the bound (4.60) above. Now, let $\epsilon > 0$. Then, via (4.62) and Remark 4.10, there exists $N \in \mathbb{N}$ such that for all $n_j \ge N$,

$$\frac{1}{2n_j} < \frac{\epsilon}{3}, \tag{4.63}$$

$$|\underline{f}_{n_j}(u) - f(u)| < \frac{\epsilon}{3\delta} \quad \forall u \in [-m_0, m_0], \tag{4.64}$$

$$|\underline{U}_{n_j}^+(\tau) - U(\tau)| < \left(\frac{\epsilon}{3\delta k_H}\right)^{1/\alpha} \quad \forall \tau \in [0,\delta], \tag{4.65}$$

where k_H is a Hölder constant for $f \in H_\alpha$ on $[-m_0, m_0]$. It now follows from (4.59), (4.63), (4.64) and (4.65) that, with $n_j \geq N$,

$$\begin{aligned}
&\left| \underline{U}^+_{n_j}(t) - u_0^+ - \int_0^t f(U(\tau))d\tau \right| \\
&\quad = \left| \left(u_0^+ - \frac{1}{2n_j} + \int_0^t \underline{f}_{n_j}(\underline{U}^+_{n_j}(\tau))d\tau \right) - u_0^+ - \int_0^t f(U(\tau))d\tau \right| \\
&\quad \leq \int_0^t |\underline{f}_{n_j}(\underline{U}^+_{n_j}(\tau)) - f(U(\tau))| d\tau + \frac{1}{2n_j} \\
&\quad < \int_0^t \left(\frac{\epsilon}{3\delta} + \left| f(\underline{U}^+_{n_j}(\tau)) - f(U(\tau)) \right| \right) d\tau + \frac{\epsilon}{3} \\
&\quad \leq \frac{2\epsilon}{3} + \int_0^t k_H |\underline{U}^+_{n_j}(\tau) - U(\tau)|^\alpha d\tau < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

for all $t \in [0, \delta]$. Thus, it follows that

$$\underline{U}^+_{n_j}(t) \to u_0^+ + \int_0^t f(U(\tau))d\tau, \tag{4.66}$$

as $j \to \infty$, uniformly for $t \in [0, \delta]$. Therefore, it follows from (4.62), (4.66) and the uniqueness of limits of real sequences, that

$$U(t) = u_0^+ + \int_0^t f(U(\tau))d\tau \quad \forall t \in [0, \delta], \tag{4.67}$$

and therefore, since U is continuous on $[0, \delta]$, that U is a classical solution of the initial value problem,

$$U'(t) = f(U) \quad \forall t \in (0, \delta], \quad U(0) = u_0^+. \tag{4.68}$$

Now, suppose that $V : [0, \delta] \to \mathbb{R}$ is a solution to (4.68), and set $M_v = \sup_{t \in [0,\delta]} |V(t)|$ and $M = \max\{M_v, m_0 + 1\}$. Upon taking $\underline{u} : \bar{D}_\delta \to \mathbb{R}$ as $\underline{u}(x, t) = \underline{U}^+_{n_j}(t)$ and $\overline{u} : \bar{D}_\delta \to \mathbb{R}$ as $\overline{u}(x, t) = V(t)$ as a regular subsolution and a regular supersolution to (B-R-D-C) with $f = \underline{f}'_{n_j}$ given by (4.22) (with M as above) and $u_0 = u_0^+$, then an application of the classical Comparison Theorem gives

$$\underline{U}^+_{n_j}(t) \leq V(t) \quad \forall t \in [0, \delta],$$

and so,

$$U(t) \leq V(t) \quad \forall t \in [0, \delta].$$

Therefore, it follows that $U : [0, \delta] \to \mathbb{R}$ is a minimal solution of (4.68), and so $U = \underline{U}^+$ on $\bar{D}_\delta$. Now, via (4.57) and (4.58), we have,

$$\underline{U}^+_{n_j}(t) = \lim_{x \to \infty} \underline{u}_{n_j}(x, t) = \liminf_{x \to \infty} \underline{u}_{n_j}(x, t) \leq \liminf_{x \to \infty} \underline{u}^c(x, t)$$

uniformly for $t \in [0, \delta]$ and $n_j \in \mathbb{N}$. It follows, via (4.62), that,

$$\liminf_{x \to \infty} \underline{u}^c(x, t) \geq U(t) = \underline{U}^+(t)$$

uniformly for $t \in [0, \delta]$, as required. The corresponding result as $x \to -\infty$ follows similarly. A symmetrical argument establishes the results for $\overline{u}^c(x, t)$ as $x \to \pm\infty$. □

Remark 4.37. When the initial value problems,

$$U_t^\pm = f(U^\pm) \quad \forall t \in (0,T], \quad U^\pm(0) = u_0^\pm, \tag{4.69}$$

(corresponding to global versions of (4.55)) in Proposition 4.36 have unique solutions $U^\pm$ for $0 \le T \le T'$, then the maximal solutions $\overline{U}^\pm : [0,\delta] \to \mathbb{R}$ and the minimal solutions $\underline{U}^\pm : [0,\delta] \to \mathbb{R}$ to the initial value problems in Proposition 4.36 will be equal to $U^\pm : [0,\delta] \to \mathbb{R}$. It then follows from Proposition 4.36 and Definition 4.1 that

$$U^\pm(t) \le \liminf_{x\to\pm\infty} \underline{u}^c(x,t) \le \limsup_{x\to\pm\infty} \underline{u}^c(x,t) \le \liminf_{x\to\pm\infty} \overline{u}^c(x,t) \le \limsup_{x\to\pm\infty} \overline{u}^c(x,t) \le U^\pm(t)$$

uniformly for $t \in [0,\delta]$. Thus, the following limits exist,

$$\lim_{x\to\pm\infty} \underline{u}^c(x,t) = \lim_{x\to\pm\infty} \overline{u}^c(x,t) = U^\pm(t) \tag{4.70}$$

uniformly for $t \in [0,\delta]$. Suppose now that $\overline{u}^c, \underline{u}^c : \bar{D}_{T'} \to \mathbb{R}$ are the constructed maximal solution and the constructed minimal solution to (B-R-D-C) with f and u_0 on $\bar{D}_{T'}$, respectively. Since solutions to (4.69) are unique on $[0,T]$ for any $0 \le T \le T'$, then it follows that $[0,\delta]$ in (4.70) can be replaced by $[0,T']$.

Finally, we provide the following conditional continuous dependence result.

Theorem 4.38. *Consider (B-R-D-C) with $f \in H_\alpha$ for some $\alpha \in (0,1)$ and $u_0 \in BPC^2(\mathbb{R})$. For $u_0^* \in BPC^2(\mathbb{R})$ that satisfies*

$$\lim_{x\to\pm\infty} u_0^*(x) = u_0^\pm,$$

suppose that $u^* : \bar{D}_{T'} \to \mathbb{R}$ is the unique solution to (B-R-D-C) on $\bar{D}_T$ for any $0 < T \le T'$. Moreover, suppose that the initial value problem

$$u_t = f(u) \quad \forall t \in (0,T'], \quad u(0) = c, \tag{4.71}$$

has solutions $U^\pm : [0,T'] \to \mathbb{R}$ for $c = u_0^+$ and $c = u_0^-$, which are unique on $[0,T]$ for any $0 \le T \le T'$. Then, for any $\epsilon > 0$, there exists $\delta^* > 0$ such that any solution $u : \bar{D}_{T'} \to \mathbb{R}$ to (B-R-D-C) with initial data $u_0 \in BPC^2(\mathbb{R})$ that satisfies $\|u_0 - u_0^*\|_B < \delta^*$ (of which there is at least one), also satisfies $\|u - u^*\|_A < \epsilon$.

Proof. Since $u^* : \bar{D}_{T'} \to \mathbb{R}$ is the unique solution to (B-R-D-C) on $\bar{D}_T$ for any $0 < T \le T'$, it follows that $u^* : \bar{D}_{T'} \to \mathbb{R}$ is a constructed solution, say in N_c applications of Theorem 4.3. It then follows from Proposition 4.36 and Remark 4.37 that

$$\lim_{x\to\pm\infty} u^*(x,t) = U^\pm(t) \tag{4.72}$$

uniformly for $t \in [0,T']$.

We now consider the first application of Theorem 4.3 in this construction procedure for $u^* : \bar{D}_{T'} \to \mathbb{R}$ and suppose that $\epsilon_1 > 0$. Let $\underline{u}_n, \overline{u}_n : \bar{D}_\delta \to \mathbb{R}$ be the unique solutions to (B-R-D-C)_n^l and (B-R-D-C)_n^u on $\bar{D}_\delta$, as employed in the proof of Theorem 4.3, respectively, with δ as in Theorem 4.3. Then, for any $X > 0$, via Corollary 4.20 and a symmetric argument, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$,

$$\max\{|\underline{u}_n - u^*|, \ |\overline{u}_n - u^*|\} < \frac{\epsilon_1}{2} \quad \text{on } \bar{D}_\delta^{0,X}. \tag{4.73}$$

We now proceed with the argument for large positive x. It follows from Remark 4.18 and Proposition 4.16 (with a symmetrical argument) that

$$\underline{u}_n(x,t) \le \underline{u}_{n+1}(x,t) \le u^*(x,t) \le \overline{u}_{n+1}(x,t) \le \overline{u}(x,t) \quad \forall (x,t) \in \bar{D}_\delta. \tag{4.74}$$

Additionally, since (B-R-D-C)^l_n and (B-R-D-C)^u_n have $\underline{f}_n, \overline{f}_n \in L$, then it follows from [33, Theorem 5.2, p. 239] that

$$\lim_{x\to\infty} \underline{u}_n(x,t) = \underline{U}^+_n(t), \quad \lim_{x\to\infty} \overline{u}_n(x,t) = \overline{U}^+_n(t) \tag{4.75}$$

uniformly for $t \in [0,\delta]$, where $\underline{U}^+_n, \overline{U}^+_n : [0,\delta] \to \mathbb{R}$ are respectively, the unique solutions to

$$\underline{U}^+_{nt} = \underline{f}_n(\underline{U}^+_n) \quad \forall t \in (0,\delta], \quad \underline{U}^+_n(0) = u^+_0 - \frac{1}{2n}, \tag{4.76}$$

$$\overline{U}^+_{nt} = \overline{f}_n(\overline{U}^+_n) \quad \forall t \in (0,\delta], \quad \overline{U}^+_n(0) = u^+_0 + \frac{1}{2n}. \tag{4.77}$$

Now, since $U^+ : [0,T'] \to \mathbb{R}$ is unique on $[0,\delta]$, it follows, as in the proof of Proposition 4.36, that there exists a subsequence $\{n_j\}_{j\in\mathbb{N}}$ of $1,2,3...$, such that

$$\underline{U}^+_{n_j} \to U^+ \text{ and } \overline{U}^+_{n_j} \to U^+ \text{ uniformly as } n_j \to \infty \text{ on } [0,\delta]. \tag{4.78}$$

Thus, it follows from (4.78) and (4.74) that there exists $N'_1 \in \mathbb{N}$ such that for all $n \ge N'_1$,

$$\max\{|(\underline{U}^+_n - U^+)(t)|,\ |(\overline{U}^+_n - U^+)(t)|\} < \frac{\epsilon_1}{2} \quad \forall t \in [0,\delta]. \tag{4.79}$$

Now, it follows from (4.72), (4.74), (4.75) and (4.79) that there exists $X_1 > 0$ such that for all $n \ge N'_1$,

$$-\frac{\epsilon_1}{2} < (\underline{u}_n - u^*)(x,t) \le (\overline{u}_n - u^*)(x,t) < \frac{\epsilon_1}{2} \quad \forall (x,t) \in [X_1,\infty) \times [0,\delta]. \tag{4.80}$$

Via a symmetrical argument, it follows that there exists $X'_1 > 0$ and $N''_1 \in \mathbb{N}$ such that for all $n \ge N''_1$,

$$-\epsilon_1 < (\underline{u}_n - u^*)(x,t) \le (\overline{u}_n - u^*)(x,t) < \epsilon_1 \quad \forall (x,t) \in \bar{D}_\delta \backslash \bar{D}^{0,X'_1}_\delta. \tag{4.81}$$

Therefore, it follows from (4.73) with $X = X'_1$ and (4.81) that there exists $N_1{*} \in \mathbb{N}$ such that for all $n \ge N_1{*}$, we have

$$-\epsilon_1 < (\underline{u}_n - u^*)(x,t) \le (\overline{u}_n - u^*)(x,t) < \epsilon_1 \quad \forall (x,t) \in \bar{D}_\delta. \tag{4.82}$$

Now, set $\delta_1 = 1/(2N_1{*})$ and suppose that $u : \bar{D}_\delta \to \mathbb{R}$ is a solution to (B-R-D-C) with f and $u_0 \in \text{BPC}^2(\mathbb{R})$ on $\bar{D}_\delta$, such that $\|u_0 - u^*_0\|_B < \delta_1$. Upon setting $M = \max\{\|u\|_A, m_0 + 1\}$, and taking $\underline{u} : \bar{D}_\delta \to \mathbb{R}$ as $\underline{u}(x,t) = \underline{u}_{N_1*}(x,t)$ and $\overline{u} : \bar{D}_\delta \to \mathbb{R}$ as $\overline{u}(x,t) = u(x,t)$ as a regular subsolution and a regular supersolution to (B-R-D-C) with $\underline{f}'_{N_1*}$ given by (4.22) (with M above) and u_0, it follows from the classical Comparison Theorem that

$$\underline{u}_{N_1*}(x,t) \le u(x,t) \quad \forall (x,t) \in \bar{D}_\delta. \tag{4.83}$$

A symmetrical argument then establishes, with (4.83) that

$$\underline{u}_{N_1*}(x,t) \le u(x,t) \le \overline{u}_{N_1*}(x,t) \quad \forall (x,t) \in \bar{D}_\delta. \tag{4.84}$$

It follows from (4.84) that (B-R-D-C) with f and u_0 is a priori bounded on $\bar{D}_\delta$, and hence, that there exists a constructed maximal (minimal) solution $\overline{u}^c(\underline{u}^c) : \bar{D}_\delta \to \mathbb{R}$ to (B-R-D-C) with f and u_0, and additionally, via (4.82), that

$$\|u - u^*\|_A < \epsilon_1. \tag{4.85}$$

Therefore, we have exhibited that for any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that (B-R-D-C) with f and u_0 such that $\|u_0 - u_0^*\|_B < \delta_1$, has a constructed maximal (minimal) solution on $\bar{D}_\delta$ and for any solution u to (B-R-D-C) with f and u_0 on $\bar{D}_\delta$, then $\|u - u^*\|_A < \epsilon_1$.

We now construct the solution $u^* : \bar{D}_{T'} \to \mathbb{R}$ to (B-R-D-C) with f and u_0^* in the supposed N_c steps, generating for any $\epsilon_i \in \mathbb{R}$, the pair $(\delta_i, \epsilon_i) \in \mathbb{R}^2$ for each $i = 1...N_c$, as produced in the above construction. By setting

$$\epsilon_{N_c} = \epsilon \text{ and } \epsilon_i = \min\{\delta_{i+1}, \epsilon\} \text{ for } i = 1...N_c - 1,$$

we obtain $\delta_1 := \delta^*$, such that, for all $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ such that $\|u_0 - u_0^*\|_B < \delta^*$, there exist a constructed maximal (minimal) solution $\overline{u}^c(\underline{u}^c) : \bar{D}_{T'} \to \mathbb{R}$ to (B-R-D-C) with f and u_0, and moreover, that $\|u - u^*\|_A < \epsilon$, as required. □

Remark 4.39. In Theorem 4.38, there is no guarantee that the solution to (B-R-D-C) with f and u_0 such that $\|u_0 - u^*\|_B < \delta^*$ is unique. The result is an extension of a corresponding result for the initial value problem for a first order ordinary differential equation, as detailed in Chapter 2, Section 4 of [5].

5. Upper Lipschitz Continuous Theory

In this section we exhibit a uniqueness result and a conditional continuous dependence result regarding the data $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. These are used to establish a conditional global well-posedness result for $f \in L_u$. Under an additional technical condition this result becomes a conditional uniform global well-posedness result. Although the classical Comparison Theorem for (B-R-D-C) when $f \in L$ is no longer applicable, we have established a corresponding Comparison Theorem for (B-R-D-C) when $f \in L_u$ in [20] (Theorem 4.4). Throughout this section, we will refer to this Comparison Theorem as (ULC). As an immediate consequence of (ULC) we are able to establish uniqueness for (B-R-D-C) when $f \in L_u$.

Theorem 5.1. *Let $f \in L_u$, then (B-R-D-C) has at most one solution on $\bar{D}_T$ for any $T > 0$.*

Proof. Follows directly from (ULC). □

Remark 5.2. Although we have established that (B-R-D-C) has at most one solution on $\bar{D}_T$ when $f \in L_u$, we are yet to establish whether such a solution exists.

We next establish a conditional continuous dependence result, both on initial data u_0 and reaction function parameter α. We have,

Theorem 5.3. *Let $f \in L_u^{'}$, and let $u_1, u_2 : \bar{D}_T \to \mathbb{R}$ be (unique) solutions to (B-R-D-C) on $\bar{D}_T$ corresponding to $u_0 = u_0^1 : \mathbb{R} \to \mathbb{R}$ and $u_0 = u_0^2 : \mathbb{R} \to \mathbb{R}$, where $u_0^1, u_0^2 \in BPC^2(\mathbb{R})$, and $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively. Let M_U and M_A be positive constants such that*

$$\max\{\|u_1\|_A, \|u_2\|_A\} \le M_U, \quad \max\{|\alpha_1|, |\alpha_2|\} \le M_A.$$

Suppose further that $f = f(u,\alpha)$ is non-decreasing with respect to $\alpha \in [-M_A, M_A]$ for each $u \in [-M_U, M_U]$, and $\alpha_2 \ge \alpha_1$,

$$u_0^2(x) - u_0^1(x) \ge 0 \quad \forall x \in \mathbb{R}, \tag{5.1}$$

then

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_B \le \left(\|u_0^2 - u_0^1\|_B + k_A(\alpha_2 - \alpha_1)t\right) e^{k_U t} \quad \forall t \in [0,T],$$

where $k_A > 0$ is a Lipschitz constant for $f(u,\alpha)$ with respect to $\alpha \in [-M_A, M_A]$ uniformly for $u \in [-M_U, M_U]$, and, k_U is an upper Lipschitz constant for $f(u,\alpha)$ with respect to $u \in [-M_U, M_U]$ uniformly for $\alpha \in [-M_A, M_A]$.

Proof. Under the above conditions on $f(u,\alpha)$ for $(u,\alpha) \in [-M_U, M_U] \times [-M_A, M_A]$, it is straightforward to verify that $u_1 : \bar{D}_T \to \mathbb{R}$ is a regular subsolution and $u_2 : \bar{D}_T \to \mathbb{R}$ is a regular supersolution to that (B-R-D-C) with $\alpha = \alpha_1$ and $u_0 = u_0^1$. It then follows from (ULC) that,

$$u_1(x,t) \le u_2(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

Now, via the conditions on $f(u,\alpha)$ and Lemma 3.8 we have,

$$\begin{aligned}
0 &\le (u_2 - u_1)(x,t) \\
&\le \|u_0^2 - u_0^1\|_B + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (f(u_2,\alpha_2) - f(u_1,\alpha_1))(x + 2\sqrt{t-\tau}\lambda, \tau) e^{-\lambda^2} d\lambda d\tau \\
&= \|u_0^2 - u_0^1\|_B + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (f(u_2,\alpha_2) - f(u_1,\alpha_2))(x + 2\sqrt{t-\tau}\lambda, \tau) e^{-\lambda^2} d\lambda d\tau \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (f(u_1,\alpha_2) - f(u_1,\alpha_1))(x + 2\sqrt{t-\tau}\lambda, \tau) e^{-\lambda^2} d\lambda d\tau \\
&\le \|u_0^2 - u_0^1\|_B + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} k_U (u_2 - u_1)(x + 2\sqrt{t-\tau}\lambda, \tau) e^{-\lambda^2} d\lambda d\tau \\
&\quad + k_A(\alpha_2 - \alpha_1)t \\
&\le \|u_0^2 - u_0^1\|_B + k_A(\alpha_2 - \alpha_1)t + k_U \int_0^t \|(u_2 - u_1)(\cdot,\tau)\|_B d\tau \quad \forall (x,t) \in \bar{D}_T.
\end{aligned} \tag{5.2}$$

Since the right hand side of (5.2) is independent of x, then we have,

$$\|(u_2 - u_1)(\cdot,t)\|_B \le \|u_0^2 - u_0^1\|_B + k_A(\alpha_2 - \alpha_1)t + k_U \int_0^t \|(u_2 - u_1)(\cdot,\tau)\|_B d\tau \tag{5.3}$$

for all $t \in [0,T]$. As $\|(u_2 - u_1)(\cdot,t)\|_B \in L^1([0,T])$ (via Lemma 3.4), an application of Proposition 3.5 to (5.3), gives,

$$\|(u_2 - u_1)(\cdot,t)\|_B \le \left(\|u_0^2 - u_0^1\|_B + k_A(\alpha_2 - \alpha_1)t\right) e^{k_U t} \quad \forall t \in [0,T], \tag{5.4}$$

as required. □

A corollary to this result, which removes the ordering on the initial data, is,

Corollary 5.4. *Let $f \in L_u^{'}$ and satisfy all of the conditions given in Theorem 5.3. Let $u_1 : \bar{D}_T \to \mathbb{R}$ and $u_2 : \bar{D}_T \to \mathbb{R}$ be as described in Theorem 5.3 with the exception of condition* (5.1)*. In addition, let $u_3 : \bar{D}_T \to \mathbb{R}$ be a (unique) solution to (B-R-D-C) corresponding to $u_0 = u_0^3 : \mathbb{R} \to \mathbb{R}$ and $\alpha = \alpha_2$. Let M_A and M_U be positive constants such that,*

$$\max\{\|u_1\|_A, \|u_2\|_A, \|u_3\|_A\} \le M_U, \quad \max\{|\alpha_1|, |\alpha_2|\} \le M_A.$$

Suppose that $\alpha_2 \ge \alpha_1$ and for $i = 1, 2$

$$\|u_0^3 - u_0^i\|_B \le \delta \quad \text{and} \quad u_0^3(x) \ge u_0^i(x) \quad \forall x \in \mathbb{R}$$

with $\delta \ge 0$. Then,

$$\max_{i,j=1,2,3} \|u_i(\cdot,t) - u_j(\cdot,t)\|_B \le (2\delta + tk_A(\alpha_2 - \alpha_1))\, e^{k_U t} \quad \forall t \in [0,T],$$

where $k_A > 0$ and $k_U > 0$ are as defined in Theorem 5.3.

Proof. We may apply Theorem 5.3 to obtain

$$\|u_3(\cdot,t) - u_1(\cdot,t)\|_B \le \left(\|u_0^3 - u_0^1\|_B + tk_A(\alpha_2 - \alpha_1)\right) e^{k_U t} \quad \forall t \in [0,T], \tag{5.5}$$

$$\|u_3(\cdot,t) - u_2(\cdot,t)\|_B \le \left(\|u_0^3 - u_0^2\|_B\right) e^{k_U t} \quad \forall t \in [0,T]. \tag{5.6}$$

Now,

$$\begin{aligned} \|u_2(\cdot,t) - u_1(\cdot,t)\|_B &\le \|u_3(\cdot,t) - u_1(\cdot,t)\|_B + \|u_3(\cdot,t) - u_2(\cdot,t)\|_B \\ &\le \left(\|u_0^3 - u_0^1\|_B + \|u_0^3 - u_0^2\|_B\right) e^{k_U t} + tk_A(\alpha_2 - \alpha_1) e^{k_U t} \end{aligned} \tag{5.7}$$

for all $t \in [0,T]$ via (5.5), (5.6) and the triangle inequality. However,

$$\max\{\|u_0^3 - u_0^1\|_B, \|u_0^3 - u_0^2\|_B\} \le \delta$$

and so, it follows from (5.7) that,

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_B \le (2\delta + tk_A(\alpha_2 - \alpha_1))\, e^{k_U t} \quad \forall t \in [0,T]. \tag{5.8}$$

The result follows from (5.5), (5.6) and (5.8). □

We now have,

Theorem 5.5. *Let $f \in L_u$ and suppose that (B-R-D-C) has a (unique) solution $u : \bar{D}_T \to \mathbb{R}$ for every $u_0 \in BPC^2(\mathbb{R})$. Let $u_0^* \in BPC^2(\mathbb{R})$ have the corresponding (unique) solution $u^* : \bar{D}_T \to \mathbb{R}$. Then given any $\epsilon > 0$, and any $u_0 \in BPC^2(\mathbb{R})$ such that,*

$$\|u_0 - u_0^*\|_B < \min\left\{\frac{1}{2}, \frac{1}{3}\epsilon e^{-k_U T}\right\},$$

it follows that,

$$\|u - u^*\|_A < \epsilon.$$

Here $k_U > 0$ is an upper Lipschitz constant for $f \in L_u$ on the interval $[-M_U, M_U]$, with $M_U > 0$ depending upon u_0^* and T.

Proof. Let $u^* : \bar{D}_T \to \mathbb{R}$ be the (unique) solution to (B-R-D-C) on $\bar{D}_T$ with $u_0 = u_0^* \in \text{BPC}^2(\mathbb{R})$, and $u_\delta^* : \bar{D}_T \to \mathbb{R}$ be the (unique) solution to (B-R-D-C) on $\bar{D}_T$ with $u_0 = u_0^* + \delta \in \text{BPC}^2(\mathbb{R})$, with $0 < \delta \le 1/2$. In addition, let $u^\pm : \bar{D}_T \to \mathbb{R}$ be the unique solutions to (B-R-D-C) on $\bar{D}_T$ with

$$u_0 = \inf_{x\in\mathbb{R}} u_0^*(x) - 1 \in \text{BPC}^2(\mathbb{R}) \text{ and } u_0 = \sup_{x\in\mathbb{R}} u_0^*(x) + 1 \in \text{BPC}^2(\mathbb{R}),$$

respectively. It follows from Theorem 5.1 and the translation invariance of the reaction-diffusion equation in (B-R-D-C), that there exist $U_+, U_- \in C^1([0,T])$ such that $u^+(x,t) = U_+(t)$ and $u^-(x,t) = U_-(t)$ for all $(x,t) \in \bar{D}_T$. Now let $u_0 \in \text{BPC}^2(\mathbb{R})$ such that,

$$\|u_0 - u_0^*\|_B < \delta, \tag{5.9}$$

with corresponding solution $u : \bar{D}_T \to \mathbb{R}$. It then follows from (ULC) with (5.9), that,

$$U_-(t) \le u^*(x,t) \le U_+(t), \;\; U_-(t) \le u_\delta^*(x,t) \le U_+(t), \;\; U_-(t) \le u(x,t) \le U_+(t), \tag{5.10}$$

for all $(x,t) \in \bar{D}_T$. Thus,

$$\|u^*\|_A, \; \|u_\delta^*\|_A, \; \|u\|_A \le M_U, \tag{5.11}$$

where $M_U > 0$ is given by

$$M_U = \max\left\{ \sup_{t\in[0,T]} |U_-(t)|, \sup_{t\in[0,T]} |U_+(t)| \right\}.$$

An application of Theorem 5.3 now gives,

$$\|u_\delta^*(\cdot,t) - u^*(\cdot,t)\|_B \le \delta e^{k_U t}, \quad \|u(\cdot,t) - u_\delta^*(\cdot,t)\|_B \le 2\delta e^{k_U t}, \tag{5.12}$$

for all $t \in [0,T]$ with $k_U > 0$ being an upper Lipschitz constant for $f \in L_u$ on $[-M_U, M_U]$. It follows from (5.12) and the triangle inequality that,

$$\|u(\cdot,t) - u^*(\cdot,t)\|_B \le \|u(\cdot,t) - u_\delta^*(\cdot,t)\|_B + \|u_\delta^*(\cdot,t) - u^*(\cdot,t)\|_B \le 3\delta e^{k_U t} \quad \forall t \in [0,T]. \tag{5.13}$$

Now set $\delta = \min\left\{\frac{1}{2}, \frac{1}{3}\epsilon e^{-k_U T}\right\}$ and the result follows from (5.9) and (5.13). □

We now introduce the following sets, where $I \subset \mathbb{R}$ is a closed bounded interval:

$$\text{BPC}_+^2(\mathbb{R}) = \{u_0 \in \text{BPC}^2(\mathbb{R}) : u_0(x) \ge 0 \;\; \forall x \in \mathbb{R}\},$$

$$A_I(\mathbb{R}) = \left\{u_0 \in \text{BPC}^2(\mathbb{R}) : u_0(x) \in I \;\; \forall x \in \mathbb{R}\right\},$$

$$A_{I+}(\mathbb{R}) = \left\{u_0 \in \text{BPC}_+^2(\mathbb{R}) : u_0(x) \in I \;\; \forall x \in \mathbb{R}\right\}.$$

We now have the following corollary concerning these alternative sets of initial data.

Corollary 5.6. *Under the conditions of Theorem 5.5, with $BPC^2(\mathbb{R})$ replaced by $BPC_+^2(\mathbb{R})$ throughout, then the same conclusion holds. Similarly, for any closed interval $I \subset \mathbb{R}$, $BPC^2(\mathbb{R})$ may be replaced by either of $A_I(\mathbb{R})$ or $A_{I+}(\mathbb{R})$, with the same conclusion holding in Theorem 5.5.*

Proof. For $\mathrm{BPC}_+^2(\mathbb{R})$, the proof follows the same steps as the proof of Theorem 5.5 upon replacing the initial data for $u^- : \bar{D}_T \to \mathbb{R}$ from $u_0 = \inf_{x\in\mathbb{R}}\{u_0^*(x)\} - 1$ to $u_0 = 0$ for all $x \in \mathbb{R}$, since the former is not guaranteed to be in the set $\mathrm{BPC}_+^2(\mathbb{R})$. The proof is similar for $A_I(\mathbb{R})$ and $A_{I^+}(\mathbb{R})$. □

We now have the following conditional global well-posedness result.

Corollary 5.7. *Let $f \in L_u$ and suppose that (B-R-D-C) has a solution $u : \bar{D}_T \to \mathbb{R}$ for every $u_0 \in BPC^2(\mathbb{R})$ and any $T > 0$. Then (B-R-D-C) is globally well-posed on $BPC^2(\mathbb{R})$. An equivalent statement holds with $BPC^2(\mathbb{R})$ replaced by $BPC_+^2(\mathbb{R})$, $A_I(\mathbb{R})$ or $A_{I^+}(\mathbb{R})$.*

Proof. For any of the initial data sets concerned, (P1) is satisfied according to the conditions of the corollary and (P2) follows from Theorem 5.1. For $\mathrm{BPC}^2(\mathbb{R})$, (P3) follows from Theorem 5.5 and for $\mathrm{BPC}_+^2(\mathbb{R})$, $A_I(\mathbb{R})$ and $A_{I^+}(\mathbb{R})$, (P3) follows from Corollary 5.6. The proof is complete. □

With an additional technical condition on solutions of (B-R-D-C) with $f \in L_u$, we can improve Corollary 5.7 to obtain a conditional uniform global well-posedness result, namely,

Theorem 5.8. *Let $f \in L_u$ and suppose that (B-R-D-C) has a (unique) solution $u : \bar{D}_\infty \to \mathbb{R}$ for every $u_0 \in BPC^2(\mathbb{R})$. Let $u_0^* \in BPC^2(\mathbb{R})$ have the corresponding (unique) solution $u^* : \bar{D}_\infty \to \mathbb{R}$. Moreover, suppose that there exists $T' \geq 0$, such that for any $u_0 \in BPC^2(\mathbb{R})$, the corresponding solution $u : \bar{D}_\infty \to \mathbb{R}$ satisfies,*

$$u(x,t) \in E \subset \mathbb{R} \quad \forall (x,t) \in \bar{D}_\infty^{T'},$$

with $E \subseteq \mathbb{R}$ being an interval, such that $f : E \to \mathbb{R}$ is non-increasing. Then given any $\epsilon > 0$, there exists $\delta > 0$, depending only upon T', u_0^*, f and ϵ, such that for any $u_0 \in BPC^2(\mathbb{R})$ that satisfies $\|u_0 - u_0^*\|_B < \delta$, it follows that for any $T > 0$,

$$\|(u - u^*)(\cdot,t)\|_B < \epsilon \quad \forall t \in [0,T].$$

Proof. Without loss of generality let $T' \geq 1$. Let $u^* : \bar{D}_\infty \to \mathbb{R}$ be the unique solution to (B-R-D-C) with $u_0 = u_0^* \in \mathrm{BPC}^2(\mathbb{R})$. In addition let $u_\pm^* : \bar{D}_{T'} \to \mathbb{R}$ be the unique solutions to (B-R-D-C) with $u_0 = \inf_{x\in\mathbb{R}} u_0^*(x) - 1 \in \mathrm{BPC}^2(\mathbb{R})$ and $u_0 = \sup_{x\in\mathbb{R}} u_0^*(x) + 1 \in \mathrm{BPC}^2(\mathbb{R})$, respectively. It follows from Theorem 5.1 and the translational invariance of the reaction-diffusion equation in (B-R-D-C), that there exist $U_+, U_- \in C^1([0,T'])$ such that $u_+^*(x,t) = U_+(t)$ and $u_-^*(x,t) = U_-(t)$ for all $(x,t) \in \bar{D}_{T'}$. Now let

$$M_U = \sup_{t\in[0,T']} \{\max\{|U_-(t)|, |U_+(t)|\}\}$$

and set $k_U > 0$ to be an upper Lipschitz constant for $f \in L_u$ on $[-M_U, M_U]$. Then, given $\epsilon > 0$, via Theorem 5.5, there exists $\delta' > 0$, depending on T', u_0^*, f and ϵ, such that, for all $u_0 \in \mathrm{BPC}^2(\mathbb{R})$, with corresponding solution $u : \bar{D}_\infty \to \mathbb{R}$, which satisfy $\|u_0 - u_0^*\|_B \leq \delta'$, we have

$$\|(u - u^*)(\cdot,t)\|_B < \frac{\epsilon}{4(1 + 2T'k_U)} \quad \forall t \in [0,T']. \tag{5.14}$$

Now, set

$$\delta = \min\left\{\delta', \frac{1}{8}\epsilon, 1\right\} \tag{5.15}$$

and henceforth consider $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ such that $\|u_0 - u_0^*\|_B < \delta$. Next, let $u_\delta^* : \bar{D}_\infty \to \mathbb{R}$ be the unique solution to (B-R-D-C) with initial data $u_0 = u_0^* + \delta \in \mathrm{BPC}^2(\mathbb{R})$. Then, via (ULC),

$$\max\{u^*(x,t), u(x,t)\} \le u_\delta^*(x,t) \quad \forall (x,t) \in \bar{D}_\infty. \tag{5.16}$$

Thus, it follows from (5.16) and Lemma 3.8 that

$$\begin{aligned} 0 &\le u_\delta^*(x,t) - u(x,t) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(u_0^*(x + 2\sqrt{t}\lambda) + \delta - u_0(x + 2\sqrt{t}\lambda)\right) e^{-\lambda^2} d\lambda \\ &\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \left(f(u_\delta^*(x + 2\sqrt{t-\tau}\lambda, \tau)) - f(u(x + 2\sqrt{t-\tau}\lambda, \tau))\right) e^{-\lambda^2} d\lambda d\tau \\ &\le 2\delta + \frac{1}{\sqrt{\pi}} \int_0^{T'} \int_{-\infty}^{\infty} \left(f(u_\delta^*(x + 2\sqrt{t-\tau}\lambda, \tau)) - f(u(x + 2\sqrt{t-\tau}\lambda, \tau))\right) e^{-\lambda^2} d\lambda d\tau \\ &\quad + \frac{1}{\sqrt{\pi}} \int_{T'}^t \int_{-\infty}^{\infty} \left(f(u_\delta^*(x + 2\sqrt{t-\tau}\lambda, \tau)) - f(u(x + 2\sqrt{t-\tau}\lambda, \tau))\right) e^{-\lambda^2} d\lambda d\tau \end{aligned} \tag{5.17}$$

for all $(x,t) \in \bar{D}_T^{T'}$ and any $T > T'$. In addition, it follows from (5.15) and (ULC) that

$$U_-(t) \le u^*(x,t) \le U_+(t), \quad U_-(t) \le u_\delta^*(x,t) \le U_+(t), \quad U_-(t) \le u(x,t) \le U_+(t)$$

for all $(x,t) \in \bar{D}_{T'}$. Thus, for $B_A^{T'}$, we conclude that,

$$\max\{\|u^*\|_A, \|u_\delta^*\|_A, \|u\|_A\} \le M_U. \tag{5.18}$$

Therefore, it follows from (5.16), (5.14) and Lemma 3.4 that

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \int_0^{T'} \int_{-\infty}^{\infty} \left(f(u_\delta^*(x + 2\sqrt{t-\tau}\lambda, \tau)) - f(u(x + 2\sqrt{t-\tau}\lambda, \tau))\right) e^{-\lambda^2} d\lambda d\tau \\ &\qquad \le \frac{1}{\sqrt{\pi}} \int_0^{T'} \int_{-\infty}^{\infty} k_U \left(u_\delta^*(x + 2\sqrt{t-\tau}\lambda, \tau) - u(x + 2\sqrt{t-\tau}\lambda, \tau)\right) e^{-\lambda^2} d\lambda d\tau \\ &\qquad \le \int_0^{T'} k_U \|(u_\delta^* - u)(\cdot, \tau)\|_B d\tau < \int_0^{T'} \frac{2k_U \epsilon}{4(1 + 2T' k_U)} d\tau < \frac{1}{4}\epsilon \end{aligned} \tag{5.19}$$

for all $(x,t) \in \bar{D}_T^{T'}$. Additionally, since $u_\delta^*(x,t), u(x,t) \in E$ for all $(x,t) \in \bar{D}_T^{T'}$, then it follows that

$$\frac{1}{\sqrt{\pi}} \int_{T'}^t \int_{-\infty}^{\infty} \left(f(u_\delta^*(x + 2\sqrt{t-\tau}\lambda, \tau)) - f(u(x + 2\sqrt{t-\tau}\lambda, \tau))\right) e^{-\lambda^2} d\lambda d\tau \le 0 \tag{5.20}$$

for all $(x,t) \in \bar{D}_T^{T'}$, via (5.16) and observing that $f \in L_u$ is non-increasing on E. Thus, it follows from (5.17), (5.19), (5.20) and (5.15) that

$$0 \le u_\delta^*(x,t) - u(x,t) < 2\delta + \frac{1}{4}\epsilon \le \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \frac{1}{2}\epsilon \tag{5.21}$$

for all $(x,t) \in \bar{D}_T^{T'}$. Since the right hand side of (5.21) is independent of x, then we have

$$\|(u_\delta^* - u)(\cdot,t)\|_B < \frac{1}{2}\epsilon \tag{5.22}$$

for all $t \in [T', T]$. Moreover, since (5.22) holds for any $T \ge T'$, it follows that (5.22) holds for $t \in [T', \infty)$. Thus, we conclude from (5.15), (5.14) and (5.22) that

$$\|(u_\delta^* - u)(\cdot,t)\|_B < \frac{\epsilon}{2} \tag{5.23}$$

for all $t \in [0,\infty)$. Thus, it follows from (5.23) that

$$\|(u^* - u)(\cdot,t)\|_B \le \|(u^* - u_\delta^*)(\cdot,t)\|_B + \|(u_\delta^* - u)(\cdot,t)\|_B < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $t \in [0,\infty)$. The result then follows for δ given by (5.15), as required. □

Corollary 5.9. *In Theorem 5.8, the initial data set $BPC^2(\mathbb{R})$ can be replaced by either $BPC_+^2(\mathbb{R})$, $A_I(\mathbb{R})$ or $A_{I+}(\mathbb{R})$ with the same conclusion holding.*

Proof. For $\mathrm{BPC}_+^2(\mathbb{R})$, the result follows on replacing $\mathrm{BPC}^2(\mathbb{R})$ by $\mathrm{BPC}_+^2(\mathbb{R})$ in the proof of Theorem 5.8. The proof is similar for $A_I(\mathbb{R})$ and $A_{I+}(\mathbb{R})$. □

Corollary 5.10. *Let $f \in L_u$ and suppose that (B-R-D-C) has a solution $u : \bar{D}_\infty \to \mathbb{R}$ for every $u_0 \in BPC^2(\mathbb{R})$. Moreover, suppose that there exists a $T' \ge 0$, such that for any $u_0 \in BPC^2(\mathbb{R})$, the corresponding solution $u : \bar{D}_\infty \to \mathbb{R}$ satisfies,*

$$u(x,t) \in E \subset \mathbb{R} \quad \forall (x,t) \in \bar{D}_\infty^{T'},$$

with $E \subseteq \mathbb{R}$ being an interval, such that $f : E \to \mathbb{R}$ is non-increasing. Then, (B-R-D-C) is uniformly globally well-posed on $BPC^2(\mathbb{R})$. An equivalent statement holds with $BPC^2(\mathbb{R})$ replaced by $BPC_+^2(\mathbb{R})$, $A_I(\mathbb{R})$ or $A_{I+}(\mathbb{R})$.

Proof. For any $u_0 \in \mathrm{BPC}^2(\mathbb{R})$, $\mathrm{BPC}_+^2(\mathbb{R})$, $A_I(\mathbb{R})$ or $A_{I+}(\mathbb{R})$, (P1) is satisfied according to the conditions of the corollary and (P2) follows from Theorem 5.1. For $\mathrm{BPC}^2(\mathbb{R})$, (P3) follows from Theorem 5.8 and for $\mathrm{BPC}_+^2(\mathbb{R})$, $A_I(\mathbb{R})$ and $A_{I+}(\mathbb{R})$, (P3) follows from Corollary 5.9. The proof is complete. □

We now have the following example,

Example 5.11. Consider the (B-R-D-C) with reaction function $f \in L_u$, given by,

$$f(u) = \begin{cases} u^p\,(u - 1/2)\,(1-u)^q & ;\ u \in [0,1] \\ 0 & ;\ u \in \mathbb{R}\backslash[0,1] \end{cases}$$

with $p, q \in (0,1)$. We can immediately state,

(i) Suppose (B-R-D-C) has a solution $u : \bar{D}_\infty \to \mathbb{R}$ for all $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. Then (B-R-D-C) is globally well-posed on $\mathrm{BPC}^2(\mathbb{R})$ via Corollary 5.7.

(ii) Suppose (B-R-D-C) has a solution $u : \bar{D}_\infty \to \mathbb{R}$ for all $u_0 \in A_I(\mathbb{R})$ with I being a closed bounded interval such that $I \subset (-\infty, 1/2)$ or $I \subset (1/2, \infty)$. Then (B-R-D-C) is uniformly globally well-posed on $A_I(\mathbb{R})$ via Corollary 5.10 upon taking $E = [a, u_{min}]$, where $a = \min\{0, \min I\}$ and $u_{min} \in (0, 1/2)$ with $f(u_{min}) = \inf_{u\in\mathbb{R}} f(u)$, and $E = [u_{max}, b]$, where $u_{max} \in (1/2, 1)$ with $f(u_{max}) = \sup_{u\in\mathbb{R}} f(u)$ and $b = \max\{1, \max I\}$, respectively.

Remark 5.12. The development of the theory in this section was motivated by the observation in [12] that a related problem to (B-R-D-C) has uniqueness for $f \in L_u$ together with an associated comparison theorem, which suggests that development of the theory when $f \in L_u$ would be fruitful. It should be noted that non-increasing functions $f \in L_u$ have been considered in related problems (see Theorem 5, [7] (p.201)).

6. Hölder Upper Lipschitz Continuous Theory

In this section, we bring together the results in Section 4 and Section 5. These results are complimentary, in that, combined they establish a global well-posedness result for (B-R-D-C) under the additional condition of a priori bounds. To begin, we have,

Theorem 6.1. *Consider (B-R-D-C) with $f \in H_\alpha \cap L_u$ for some $\alpha \in (0,1)$ and initial data $u_0 \in BPC^2(\mathbb{R})$. Then there exists a unique solution to (B-R-D-C) on $\bar{D}_\delta$, with $\delta > 0$ as in Theorem 4.3.*

Proof. Follows directly from Theorem 4.3 and Theorem 5.1. □

Similarly, we also have,

Theorem 6.2. *Consider (B-R-D-C) with $f \in H_\alpha \cap L_u$ for some $\alpha \in (0,1)$ with initial data $u_0 \in BPC^2(\mathbb{R})$. When (B-R-D-C) is a priori bounded on $\bar{D}_T$ for any $T > 0$, then (B-R-D-C) is globally well-posed on $BPC^2(\mathbb{R})$.*

Proof. It follows from Theorem 4.25 and Theorem 5.1 that for any $u_0 \in \mathrm{BPC}^2(\mathbb{R})$, there exists a unique solution $u : \bar{D}_\infty \to \mathbb{R}$ to (B-R-D-C) corresponding to the initial data u_0, and hence, (P1) and (P2) are satisfied. (P3) then follows from Corollary 5.7. □

With the additional technical condition introduced in the previous section, we also have,

Theorem 6.3. *Consider (B-R-D-C) with $f \in H_\alpha \cap L_u$ for some $\alpha \in (0,1)$ with initial data $u_0 \in BPC^2(\mathbb{R})$. Suppose that (B-R-D-C) is a priori bounded for any $T > 0$, and suppose that there exists $T' \geq 0$ such that for any $u_0 \in BPC^2(\mathbb{R})$, the corresponding (unique) solution $u : \bar{D}_\infty \to \mathbb{R}$ to (B-R-D-C) satisfies*

$$u(x,t) \in E \subseteq \mathbb{R} \quad \forall (x,t) \in \bar{D}_\infty^{T'},$$

with $E \subseteq \mathbb{R}$ being an interval, such that $f : E \to \mathbb{R}$ is non-increasing. Then (B-R-D-C) is uniformly globally well-posed on $BPC^2(\mathbb{R})$.

Proof. Follows directly from Theorem 6.2 and Theorem 5.8. □

Remark 6.4. The replacement of $\mathrm{BPC}^2(\mathbb{R})$ with $\mathrm{BPC}^2_+(\mathbb{R})$, $A_I(\mathbb{R})$ or $A_{I+}(\mathbb{R})$ in Theorem 6.2 and Theorem 6.3, is allowable (via Corollary 5.9).

We now exhibit applications of the above theory with the following examples:

Example 6.5. Consider (B-R-D-C) with $f \in H_\alpha$, such that f is strictly decreasing on $\mathbb{R}$, with initial data $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. Since $f : \mathbb{R} \to \mathbb{R}$ is strictly decreasing, it follows that $f \in L_u$. Let $u : \bar{D}_T \to \mathbb{R}$ be a solution to this (B-R-D-C). There are now three cases to consider:

(i) $f : \mathbb{R} \to \mathbb{R}$ is strictly positive. Define $\underline{u}, \overline{u} : \bar{D}_T \to \mathbb{R}$ to be

$$\underline{u}(x,t) = \inf_{x\in\mathbb{R}}\{u_0(x)\}, \quad \overline{u}(x,t) = \sup_{x\in\mathbb{R}}\{u_0(x)\} + f\left(\inf_{x\in\mathbb{R}}\{u_0(x)\}\right)t \quad \forall (x,t) \in \bar{D}_T.$$

It follows that $\underline{u}(x,t) \le \overline{u}(x,t)$ for all $(x,t) \in \bar{D}_T$ and hence,

$$\underline{u}_t - \underline{u}_{xx} - f(\underline{u}) = -f(\underline{u}) < 0,$$

$$\overline{u}_t - \overline{u}_{xx} - f(\overline{u}) > f\left(\inf_{x\in\mathbb{R}}\{u_0(x)\}\right) - f\left(\sup_{x\in\mathbb{R}}\{u_0(x)\}\right) \ge 0 \quad \forall (x,t) \in D_T.$$

Finally, since $\underline{u}(x,0) \le u_0(x) \le \overline{u}(x,0)$, via (ULC), we have

$$\inf_{x\in\mathbb{R}}\{u_0(x)\} \le u(x,t) \le \sup_{x\in\mathbb{R}}\{u_0(x)\} + f\left(\inf_{x\in\mathbb{R}}\{u_0(x)\}\right)t \quad \forall (x,t) \in \bar{D}_T.$$

(ii) $f : \mathbb{R} \to \mathbb{R}$ is strictly negative. Similarly we obtain,

$$\inf_{x\in\mathbb{R}}\{u_0(x)\} + f\left(\sup_{x\in\mathbb{R}}\{u_0(x)\}\right)t \le u(x,t) \le \sup_{x\in\mathbb{R}}\{u_0(x)\} \quad \forall (x,t) \in \bar{D}_T.$$

(iii) There exists a unique $u^* \in \mathbb{R}$ such that $f(u^*) = 0$. Define $\underline{u}, \overline{u} : \bar{D}_T \to \mathbb{R}$ to be,

$$\underline{u}(x,t) = \min\left\{\inf_{x\in\mathbb{R}}\{u_0(x)\},\ u^*\right\} \quad \forall (x,t) \in \bar{D}_T,$$

$$\overline{u}(x,t) = \max\left\{\sup_{x\in\mathbb{R}}\{u_0(x)\},\ u^*\right\} \quad \forall (x,t) \in \bar{D}_T.$$

It follows similarly that

$$\min\left\{\inf_{x\in\mathbb{R}}\{u_0(x)\},\ u^*\right\} \le u(x,t) \le \max\left\{\sup_{x\in\mathbb{R}}\{u_0(x)\},\ u^*\right\} \quad \forall (x,t) \in \bar{D}_T.$$

Thus in all three cases (B-R-D-C) is a priori bounded on $\bar{D}_T$ for any $T > 0$, and hence, via Theorem 6.2, (B-R-D-C) is globally well-posed on $\mathrm{BPC}^2(\mathbb{R})$. Additionally, since f is strictly decreasing, via Theorem 6.3 (with $E = \mathbb{R}$) it follows that (B-R-D-C) is uniformly globally well-posed on $\mathrm{BPC}^2(\mathbb{R})$. It should be mentioned that with additional considerations, one can obtain a similar result for the problem given above with "strictly decreasing on $\mathbb{R}$" replaced by "non-increasing on $\mathbb{R}$", where additional cases have to be considered.

Example 6.6. Consider the (B-R-D-C) problem given in Example 5.11 and observe that not only is $f \in L_u$, but $f \in H_\alpha \cap L_u$ where $\alpha = \min\{p, q\}$. Thus, we can now extend the conclusions in Example 5.11, specifically:

(i) (B-R-D-C) is a priori bounded on $\bar{D}_T$ for any $T \geq 0$, via a simple application of (ULC).

(ii) (B-R-D-C) is globally well-posed on $\mathrm{BPC}^2(\mathbb{R})$ via Theorem 6.2.

(iii) Let I be a closed bounded interval such that $I \subset (-\infty, 1/2)$ or $I \subset (1/2, \infty)$. Then (B-R-D-C) is uniformly globally well-posed on $A_I(\mathbb{R})$ via an application of (ULC) and Remark 6.4.

Example 6.7. Consider (B-R-D-C) with $f \in H_q \cap L_u$, given by

$$f(u) = \begin{cases} u^p(1-u)^q & ; u \in [0,1] \\ 0 & ; u \in \mathbb{R}\backslash[0,1], \end{cases}$$

with $p \in [1, \infty)$ and $q \in (0,1)$ with initial data $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. Immediately, we have,

(i) (B-R-D-C) is a priori bounded on $\bar{D}_T$ for any $T \geq 0$, via a simple application of (ULC).

(ii) (B-R-D-C) is globally well-posed on $\mathrm{BPC}^2(\mathbb{R})$ via Theorem 6.2.

(iii) Let I be a closed bounded interval such that $I \subset (0, \infty)$. Then (B-R-D-C) is uniformly globally well-posed on $A_I(\mathbb{R})$ via an application of (ULC) and Remark 6.4.

7. Discussion

In this paper we have established that for $f \in H_\alpha$ and $u_0 \in \mathrm{BPC}^2(\mathbb{R})$, there exists a solution to (B-R-D-C) on $\bar{D}_T$ for some $T > 0$. However, if $f \notin H_\alpha$, we are currently unaware if there exists a solution to (B-R-D-C) on $\bar{D}_T$ for some $T > 0$. For example, consider the (B-R-D-C) with $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(u) = \begin{cases} 0 & ; u \leq 0 \\ -\left(\frac{-1}{\log(u)}\right)^{1/2} & ; 0 < u < 1/2 \\ -\left(\frac{-1}{\log(1/2)}\right)^{1/2} & ; u \geq 1/2, \end{cases} \tag{7.1}$$

with initial data $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ given by

$$u_0(x) = \sin(x) e^{-x^2} \quad \forall x \in \mathbb{R}. \tag{7.2}$$

We observe that $f \notin H_\alpha$ for any $\alpha \in (0,1]$. The principal hindrance to a means of obtaining an existence result for this (B-R-D-C) is the lack of an equivalence lemma (of the form of Lemma 3.7) for $f \notin H_\alpha$. Specifically, since f given by (7.1) is continuous, it follows (see [19], Theorem 4.9) that if $u : \bar{D}_T \to \mathbb{R}$ is a solution to (B-R-D-C) with f as in (7.1) and initial data u_0 given by (7.2), then u satisfies,

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2\sqrt{t}\lambda) e^{-\lambda^2} d\lambda + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f(u(x + 2\sqrt{t-\tau}\lambda, \tau)) e^{-\lambda^2} d\lambda d\tau \tag{7.3}$$

for all $(x,t) \in \bar{D}_T$. However, if we consider $u : \bar{D}_T \to \mathbb{R}$ such that $u \in B_A^T$ and u satisfies (7.3), then we cannot guarantee that u is a solution to (B-R-D-C), since u_t and u_{xx} may not exist on D_T. It is worth noting that $f : \mathbb{R} \to \mathbb{R}$ given by (7.1) is non-increasing on $\mathbb{R}$, and hence, that $f \in L_u$ and the theory developed in Section 5 can be applied to this (B-R-D-C), if a solution is shown to exist.

It should also be noted that much of the theory in Sections 3, 4 and 5 can be obtained for the more general initial data sets of the following form,

$$\mathcal{U}_0 = \{u_0 : \mathbb{R} \to \mathbb{R} : u_0 \text{ is bounded and continuous on } \mathbb{R}\}.$$

Although these sets have not been considered here, the methodology contained in this paper can be adapted to encompass sets of this type. Specifically, the Equivalence Lemma follows similarly and the derivative estimates of Section 3 follow upon replacing condition (H) on $F(x,t)$ with a non-uniform Hölder condition in x on $F(x,t)$, such that the Hölder constant is allowed to blow up as $t \to 0^+$. This leads to derivative bounds for solutions of (B-R-D-C), which blow up as $t \to 0^+$, as expected. The majority of the theory in Sections 4 and 5 follows similarly, but in numerous instances, technical adjustments must be made (see [19], Chapter 10, Section 1).

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