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THE METHOD OF LAYER POTENTIALS IN L^p AND ENDPOINT SPACES FOR ELLIPTIC OPERATORS WITH L^∞ COEFFICIENTS

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Abstract. We consider layer potentials associated to elliptic operators $Lu = -\operatorname{div}(A\nabla u)$ acting in the upper half-space $\mathbb{R}^{n+1}_+$ for $n \geq 2$, or more generally, in a Lipschitz graph domain, where the coefficient matrix A is L^∞ and t-independent, and solutions of $Lu = 0$ satisfy interior estimates of De Giorgi/Nash/Moser type. A "Calderón-Zygmund" theory is developed for the boundedness of layer potentials, whereby sharp L^p and endpoint space bounds are deduced from L^2 bounds. Appropriate versions of the classical "jump-relation" formulae are also derived. The method of layer potentials is then used to establish well-posedness of boundary value problems for L with data in L^p and endpoint spaces.

Contents

1. Introduction

Consider a second order, divergence form elliptic operator

$$L = -\operatorname{div} A(x)\nabla \quad \text{in } \mathbb{R}^{n+1} := \{X = (x,t) : x \in \mathbb{R}^n,\ t \in \mathbb{R}\}, \tag{1.1}$$

where the gradient and divergence act in all $n+1$ variables, and A is an $(n+1)\times(n+1)$ matrix of L^∞, t-independent, complex coefficients, satisfying the uniform ellipticity condition

$$\Lambda^{-1}|\xi|^2 \leq \operatorname{Re}\langle A(x)\xi, \xi\rangle := \operatorname{Re}\sum_{i,j=1}^{n+1} A_{ij}(x)\,\xi_j\,\bar{\xi}_i, \quad \|A\|_\infty \leq \Lambda, \tag{1.2}$$

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for some $\Lambda \in (0,\infty)$, for all $\xi \in \mathbb{C}^{n+1}$, and for almost every $x \in \mathbb{R}^n$. The operator L is interpreted in the usual weak sense via the accretive sesquilinear form associated with (1.2). In particular, we say that u is a solution of $Lu = 0$, or simply $Lu = 0$, in an open set $\Omega \subset \mathbb{R}^{n+1}$, if $u \in L^2_{1,\mathrm{loc}}(\Omega)$ and $\int_{\mathbb{R}^{n+1}} A\nabla u \cdot \nabla\Phi = 0$ for all $\Phi \in C_0^\infty(\Omega)$.

Throughout the paper, we shall impose the following " **standard assumptions**":

(1) The operator $L = -\operatorname{div} A\nabla$ is of the type defined in (1.1) and (1.2) above, with t-independent coefficient matrix $A(x,t) = A(x)$.
(2) Solutions of $Lu = 0$ in open subsets of $\mathbb{R}^{n+1}$ satisfy the De Giorgi/Nash/Moser (DG/N/M) estimates defined in (1.3) and (1.4) below.

The paper has two principal aims. First, we prove sharp L^p and endpoint space bounds for layer potentials associated to any operator L that, along with its Hermitian adjoint $L^* = -\operatorname{div} A^*\nabla$, where A^* is the conjugate transpose of the matrix A, satisfies the standard assumptions. These results are of "Calderón-Zygmund" type, in the sense that the L^p and endpoint space bounds are deduced from L^2 bounds. Second, we use the layer potential method to obtain well-posedness results for boundary value problems for certain classes of such operators L. The precise definitions of the layer potentials, and a brief historical summary of previous work (including the known L^2 bounds), is given below.

Let us now discuss some preliminaries in order to state the main results of the paper. For any measurable subset E of $\mathbb{R}^d$ (typically $d = n+1$ or $d = n$), the symbol 1_E denotes the characteristic function of E defined on $\mathbb{R}^d$, the symbol $|E|$ denotes the d-dimensional Lebesgue measure of E, and $\fint_E f := |E|^{-1}\int_E f$ denotes the integral mean of a measurable function f on E. For notational convenience, capital letters will often be used to denote points in $\mathbb{R}^{n+1}$, e.g., $X = (x,t)$, $Y = (y,s)$, where $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$. In this notation, let $B(X,r) := \{Y \in \mathbb{R}^{n+1} : |X-Y| < r\}$ and $\Delta(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ denote, respectively, balls of radius $r > 0$ in $\mathbb{R}^{n+1}$ and in $\mathbb{R}^n$. The letter Q will be used to denote a generic cube in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes and side length $\ell(Q)$. The convention is adopted whereby C denotes a finite positive constant that may change from one line to the next but depends only on the relevant preceding hypotheses. We will often write C_p to emphasize when such a constant depends on a specific parameter p. We may also write $a \lesssim b$ to denote $a \leq Cb$, and $a \approx b$ to indicate that $a \lesssim b \lesssim a$, for some $a, b \in \mathbb{R}$.

De Giorgi/Nash/Moser (DG/N/M) estimates. We say that a locally square integrable function u is *locally Hölder continuous*, or equivalently, satisfies *De Giorgi/Nash (DG/N) estimates* in an open set $\Omega \subset \mathbb{R}^{n+1}$, if there is a positive constant $C_0 < \infty$, and an exponent $\alpha \in (0,1]$, such that for any ball $B = B(X,R)$ whose concentric double $2B := B(X,2R)$ is contained in Ω, we have

$$|u(Y) - u(Z)| \leq C_0 \left(\frac{|Y-Z|}{R}\right)^{\alpha} \left(\fint_{2B} |u|^2\right)^{1/2}, \tag{1.3}$$

whenever $Y, Z \in B$. Since $u = u - \fint_B u + \fint_B u$, it is clear that any function u satisfying (1.3) also satisfies Moser's *local boundedness* estimate (see [48])

$$\sup_{Y\in B} |u(Y)| \leq \widetilde{C}_0 \left(\fint_{2B} |u|^2\right)^{1/2}. \tag{1.4}$$

Moreover, as is well-known, (1.4) self improves to

$$\sup_{Y\in B} |u(Y)| \leq C_r \left(\fint_{2B} |u|^r\right)^{1/r}, \qquad \forall\, r \in (0,\infty). \tag{1.5}$$

Remark 1.1. It is well-known (see [19, 48, 49]) that when the coefficient matrix A is real, solutions of $Lu = 0$ satisfy the DG/N/M estimates (1.3) and (1.4), and the relevant constants depend quantitatively on ellipticity and dimension only (for this result, the matrix A need not be t-independent). Moreover, estimate (1.3), which implies (1.4), is stable under small complex perturbations of the coefficients in the L^∞ norm (see, e.g., [23, Chapter VI] or [2]). Therefore, the standard assumption (2) above holds automatically for small complex perturbations of real symmetric elliptic coefficients. We also note that in the t-independent setting considered here, the DG/N/M estimates always hold when the ambient dimension $n+1$ is equal to 3 (see [1, Section 11]).

We shall refer to the following quantities collectively as the "**standard constants**": the dimension n in (1.1), the ellipticity parameter Λ in (1.2), and the constants C_0 and α in the DG/N/M estimates (1.3) and (1.4).

In the presence of DG/N/M estimates for L and L^*, by [27], both L and L^* have fundamental solutions $E : \{(X,Y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : X \neq Y\} \to \mathbb{C}$ and $E^*(X,Y) := \overline{E(Y,X)}$, respectively, satisfying $E(X,\cdot)$, $E(\cdot,X) \in L^2_{1,\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \{X\})$ and

$$L_{x,t}\, E(x,t,y,s) = \delta_{(y,s)}, \qquad L^*_{y,s}\, E^*(y,s,x,t) = L^*_{y,s}\, \overline{E(x,t,y,s)} = \delta_{(x,t)}, \tag{1.6}$$

where δ_X denotes the Dirac mass in $\mathbb{R}^{n+1}$ at the point X. In particular, this means that

$$\int_{\mathbb{R}^{n+1}} A(x)\nabla_{x,t}E(x,t,y,s) \cdot \nabla\Phi(x,t)\, dxdt = \Phi(y,s), \qquad (y,s) \in \mathbb{R}^{n+1}, \tag{1.7}$$

for all $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$. Moreover, by the t-independence of our coefficients,

$$E(x,t,y,s) = E(x,t-s,y,0), \qquad (x,t),\ (y,s) \in \mathbb{R}^{n+1},$$

As is customary, we then define the single and double layer potential operators, associated to L, in the upper and lower half-spaces, $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$, by

$$\begin{aligned} \mathcal{S}^\pm f(x,t) &:= \int_{\mathbb{R}^n} E(x,t,y,0)\, f(y)\, dy, \qquad (x,t) \in \mathbb{R}^{n+1}_\pm, \\ \mathcal{D}^\pm f(x,t) &:= \int_{\mathbb{R}^n} \overline{\Big(\partial_{\nu^*} E^*(\cdot,\cdot,x,t)\Big)}(y,0)\, f(y)\, dy, \qquad (x,t) \in \mathbb{R}^{n+1}_\pm, \end{aligned} \tag{1.8}$$

where f is any suitable function defined on $\mathbb{R}^n$, and ∂_{ν^*} denotes the outer co-normal derivative (with respect to $\mathbb{R}^{n+1}_+$) associated to the adjoint matrix A^*, i.e.,

$$\Big(\partial_{\nu^*} E^*(\cdot,\cdot,x,t)\Big)(y,0) := -e_{n+1} \cdot A^*(y) \Big(\nabla_{y,s} E^*(y,s,x,t)\Big)\Big|_{s=0}, \tag{1.9}$$

where $e_{n+1} := (0,\dots,0,1)$ is the standard unit basis vector in the t direction. Similarly, using the notational convention that $t = x_{n+1}$, we define the outer co-normal derivative of u with respect to A by

$$\partial_\nu u := -e_{n+1} \cdot A\nabla u = -\sum_{j=1}^{n+1} A_{n+1,j}\, \partial_{x_j}.$$

When we are working in a particular half-space (usually the upper one, by convention), for simplicity of notation, we shall often drop the superscript and write, e.g., $\mathcal{S}$, $\mathcal{D}$ in lieu of $\mathcal{S}^+$, $\mathcal{D}^+$. At times, it may be necessary to identify the operator L to which the layer potentials are associated (when this is not clear from context), in which case we shall write $\mathcal{S}_L$, $\mathcal{D}_L$, and so on.

We note at this point that the single layer potential $\mathcal{S}f(x,t)$, as well as its boundary trace

$$S f(x) := \int_{\mathbb{R}^n} E(x,0,y,0)\, f(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{1.10}$$

are well-defined as absolutely convergent integrals for every $t \in \mathbb{R}$, for a.e. $x \in \mathbb{R}^n$, and for all $f \in L^p(\mathbb{R}^n)$, $1 < p < n$, by virtue of the estimate

$$\int_{\mathbb{R}^n} |E(x,t,y,0)|\,|f(y)|\, dy \;\lesssim\; I_1(|f|)(x)\,,$$

which follows from (3.1) below (here, I_1 denotes the classical Riesz potential of order 1). Moreover, by (3.1) and (3.2), the single layer potential has a well-defined realisation as an element of $L^1_{\mathrm{loc}}(\mathbb{R}^n)/\mathbb{C}$ for all $f \in L^p(\mathbb{R}^n)$, $n \le p < n+\varepsilon$, with $\varepsilon > 0$ depending upon the DG/N exponent (see, e.g., [20] or [52, IV.1.2.2] for a similar argument). The double layer potential $\mathcal{D}f(\cdot,t)$ is also well-defined as an absolutely convergent integral for all $f \in L^p(\mathbb{R}^n)$, $2-\varepsilon < p < \infty$, and for $t \neq 0$, by virtue of the estimate

$$\int_{\mathbb{R}^n} \left|\Big(\nabla E(x,t,\cdot,\cdot)\Big)(y,0)\right|^q dy \;\lesssim\; |t|^{n(1-q)}\,, \qquad x \in \mathbb{R}^n, \quad 1 < q < 2+\varepsilon\,, \tag{1.11}$$

which follows from [1, Lemmata 2.5 and 2.8, and Proposition 2.1], for some $\varepsilon > 0$ depending only on dimension and ellipticity.

We shall also define, in Section 2 below, boundary singular integrals

$$\begin{aligned}
K f(x) &:= \text{“p.v.”} \int_{\mathbb{R}^n} \overline{\Big(\partial_{\nu^*} E^*(\cdot,\cdot,x,0)\Big)}(y,0)\, f(y)\, dy\,,\\
\widetilde{K} f(x) &:= \text{“p.v.”} \int_{\mathbb{R}^n} \Big(\partial_{\nu} E(x,0,\cdot,\cdot)\Big)(y,0)\, f(y)\, dy\,,\\
\mathbf{T} f(x) &:= \text{“p.v.”} \int_{\mathbb{R}^n} \Big(\nabla E\Big)(x,0,y,0) f(y)\, dy\,,
\end{aligned} \tag{1.12}$$

where the “principal value” is purely formal, since we do not actually establish convergence of a principal value. We shall give precise definitions and derive the jump relations for the layer potentials in Section 2. Classically, $\widetilde{K}$ is often denoted K^*, but we avoid this notation here, as $\widetilde{K}$ need not be the adjoint of K unless L is self-adjoint. In fact, using the notation $\mathrm{adj}(T)$ to denote the Hermitian adjoint of an operator T acting in $\mathbb{R}^n$, we have that $\widetilde{K}_L = \mathrm{adj}(K_{L^*})$.

Let us now recall the definitions of the non-tangential maximal operators $N_*, \widetilde{N}_*$, and of the notion of “non-tangential convergence”. Given $x_0 \in \mathbb{R}^n$, define the cone $\Gamma(x_0) := \{(x,t) \in \mathbb{R}^{n+1}_+ : |x_0 - x| < t\}$. Then for measurable functions $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$, define

$$N_* F(x_0) := \sup_{(x,t)\in\Gamma(x_0)} |F(x,t)|,$$

$$\widetilde{N}_* F(x_0) := \sup_{(x,t)\in\Gamma(x_0)} \left(\fint\!\!\fint_{|(x,t)-(y,s)|<t/4} |F(y,s)|^2 dy ds \right)^{1/2}.$$

We shall say that F “converges non-tangentially” to a function $f : \mathbb{R}^n \to \mathbb{C}$, and write $F \xrightarrow{\text{n.t.}} f$, if for a.e. $x \in \mathbb{R}^n$, it holds that

$$\lim_{\Gamma(x)\ni(y,t)\to(x,0)} F(y,t) = f(x)\,.$$

These definitions have obvious analogues in the lower half-space $\mathbb{R}^{n+1}_-$ that we distinguish by writing $\Gamma^\pm, N^\pm_*, \widetilde{N}^\pm_*$, e.g., the cone $\Gamma^-(x_0) := \{(x,t) \in \mathbb{R}^{n+1}_- : |x_0 - x| < -t\}$.

As usual, for $1 < p < \infty$, let $\dot{L}^p_1(\mathbb{R}^n)$ denote the homogenous Sobolev space of order one, which is defined as the completion of $C^\infty_0(\mathbb{R}^n)$, with respect to the norm $\|f\|_{\dot{L}^p_1} := \|\nabla f\|_p$, realized as a subspace of the space $L^1_{\mathrm{loc}}(\mathbb{R}^n)/\mathbb{C}$ of locally integrable functions modulo constant functions.

As usual, for $0 < p \le 1$, let $H^p_{\mathrm{at}}(\mathbb{R}^n)$ denote the classical atomic Hardy space, which is a subspace of the space $\mathbf{S}'(\mathbb{R}^n)$ of tempered distributions (see, e.g., [52, Chapter III] for a precise definition). Also, for $n/(n+1) < p \le 1$, let $\dot{H}^{1,p}_{\mathrm{at}}(\mathbb{R}^n)$ denote the homogeneous "Hardy-Sobolev" space of order one, which is a subspace of $\mathbf{S}'(\mathbb{R}^n)/\mathbb{C}$ (see, e.g., [45, Section 3] for further details). In particular, we call $a \in \dot{L}^2_1(\mathbb{R}^n)$ a *regular atom* if there exists a cube $Q \subset \mathbb{R}^n$ such that

$$\operatorname{supp} a \subset Q, \qquad \|\nabla a\|_{L^2(Q)} \le |Q|^{\frac{1}{2}-\frac{1}{p}},$$

and we define the space

$$\dot{H}^{1,p}_{\mathrm{at}}(\mathbb{R}^n) := \{f \in \mathbf{S}'(\mathbb{R}^n)/\mathbb{C} : \nabla f = \sum_{j=1}^{\infty} \lambda_j \nabla a_j,\ (\lambda_j)_j \in \ell^p,\ a_j \text{ is a regular atom}\},$$

where the series converges in $H^p_{\mathrm{at}}(\mathbb{R}^n)$, and the space is equipped with the quasi-norm $\|f\|_{\dot{H}^{1,p}_{\mathrm{at}}(\mathbb{R}^n)} := \inf\,[\sum_j |\lambda_j|^p]^{1/p}$, where the infimum is taken over all such representations.

We now define the scales

$$H^p(\mathbb{R}^n) := \begin{cases} H^p_{\mathrm{at}}(\mathbb{R}^n)\,, & 0 < p \le 1, \\ L^p(\mathbb{R}^n)\,, & 1 < p < \infty, \end{cases} \qquad \dot{H}^{1,p}(\mathbb{R}^n) := \begin{cases} \dot{H}^{1,p}_{\mathrm{at}}(\mathbb{R}^n)\,, & \frac{n}{n+1} < p \le 1, \\ \dot{L}^p_1(\mathbb{R}^n)\,, & 1 < p < \infty. \end{cases}$$

We recall that, by the classical result of Fefferman and Stein [20], the dual of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$. Moreover, $(H^p_{\mathrm{at}}(\mathbb{R}^n))^* = \dot{C}^\alpha(\mathbb{R}^n)$, if $\alpha := n(1/p - 1) \in (0,1)$, where $\dot{C}^\alpha(\mathbb{R}^n)$ denotes the homogeneous Hölder space of order α. In general, for a measurable set E, and for $0 < \alpha < 1$, the Hölder space $\dot{C}^\alpha(E)$ is defined to be the set of $f \in C(E)/\mathbb{C}$ satisfying

$$\|f\|_{\dot{C}^\alpha} := \sup \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty,$$

where the supremum is taken over all pairs $(x,y) \in E \times E$ such that $x \ne y$. For $0 \le \alpha < 1$, we define the scale

$$\Lambda^\alpha(\mathbb{R}^n) := \begin{cases} \dot{C}^\alpha(\mathbb{R}^n)\,, & 0 < \alpha < 1, \\ BMO(\mathbb{R}^n)\,, & \alpha = 0\,. \end{cases}$$

As usual, we say that a function $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ belongs to the tent space $T^\infty_2(\mathbb{R}^{n+1}_+)$, if it satisfies the Carleson measure condition

$$\|F\|_{T^\infty_2(\mathbb{R}^{n+1}_+)} := \left(\sup_Q \frac{1}{|Q|} \iint_{R_Q} |F(x,t)|^2 \, \frac{dxdt}{t}\right)^{1/2} < \infty\,.$$

Here, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and $R_Q := Q \times (0, \ell(Q))$ is the usual "Carleson box" above Q.

With these definitions and notational conventions in place, we are ready to state the first main result of this paper.

Theorem 1.2. *Suppose that L and L^* satisfy the standard assumptions, let α denote the minimum of the De Giorgi/Nash exponents for L and L^* in (1.3), and set $p_\alpha := n/(n+\alpha)$. Then there exists a constant $p^+ > 2$, depending only on the standard constants, such that the operators $\nabla\mathcal{S}_L$, $\nabla_x S_L$, $\widetilde{K}_L$ and $\mathcal{D}_L$ have unique extensions from $L^2(\mathbb{R}^n)$ satisfying*

$$\sup_{t>0} \|\nabla\mathcal{S}_L f(\cdot,t)\|_{L^p(\mathbb{R}^n,\mathbb{C}^{n+1})} \leq C_p\|f\|_{H^p(\mathbb{R}^n)}, \qquad \forall\, p \in (p_\alpha, p^+)\,, \tag{1.13}$$

$$\|\widetilde{N}_*\left(\nabla\mathcal{S}_L f\right)\|_{L^p(\mathbb{R}^n)} \leq C_p\|f\|_{H^p(\mathbb{R}^n)}, \qquad \forall\, p \in (p_\alpha, p^+)\,, \tag{1.14}$$

$$\|\nabla_x S_L f\|_{H^p(\mathbb{R}^n,\mathbb{C}^n)} \leq C_p\|f\|_{H^p(\mathbb{R}^n)}, \qquad \forall\, p \in (p_\alpha, p^+)\,, \tag{1.15}$$

$$\|\widetilde{K}_L f\|_{H^p(\mathbb{R}^n)} \leq C_p\|f\|_{H^p(\mathbb{R}^n)}, \qquad \forall\, p \in (p_\alpha, p^+)\,, \tag{1.16}$$

$$\|N_*(\mathcal{D}_L f)\|_{L^p(\mathbb{R}^n)} \leq C_p\|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, p \in \left(\frac{p^+}{p^+-1}, \infty\right)\,, \tag{1.17}$$

$$\|t\nabla\mathcal{D}_L f\|_{T_2^\infty(\mathbb{R}^{n+1}_+)} \leq C\|f\|_{BMO(\mathbb{R}^n)}, \tag{1.18}$$

$$\|\mathcal{D}_L f\|_{\dot{C}^\beta(\overline{\mathbb{R}^{n+1}_+})} \leq C_\beta\|f\|_{\dot{C}^\beta(\mathbb{R}^n)}, \qquad \forall\, \beta \in (0,\alpha)\,, \tag{1.19}$$

for all f such that the right-hand side is finite. The analogous results hold for L^* and in the lower half-space.

Remark 1.3. The case $p = 2$ has already been proved in [50] (an earlier result treating complex perturbations of the real symmetric case appeared in [1]). The new results in the present paper concern the case $p \neq 2$, and the spaces BMO and $\dot{C}^\beta$.

To state our second main result, let us recall the definitions of the Neumann and Regularity problems, with (for now) $n/(n+1) < p < \infty$:

$$(\mathrm{N})_p \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n), \\ \partial_\nu u(\cdot,0) = g \in H^p(\mathbb{R}^n), \end{cases} \qquad (\mathrm{R})_p \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n), \\ u(\cdot,0) = f \in \dot{H}^{1,p}(\mathbb{R}^n)\,, \end{cases}$$

where we specify that the solution u, of either $(N)_p$ or $(R)_p$, will assume its boundary data in the the following sense:

- $u(\cdot,0) \in \dot{H}^{1,p}(\mathbb{R}^n)$, and $u \xrightarrow{\text{n.t.}} u(\cdot,0)$;
- $\nabla_x u(\cdot,0)$ and $\partial_\nu u(\cdot,0)$ belong to $H^p(\mathbb{R}^n)$, and are the weak limits, (in L^p, for $p > 1$, and in the sense of tempered distributions, if $p \leq 1$), as $t \to 0$, of $\nabla_x u(\cdot,t)$, and of $-e_{n+1}\cdot A\nabla u(\cdot,t)$, respectively.

We also formulate the Dirichlet problem in L^p, with $1 < p < \infty$:

$$(\mathrm{D})_p \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ N_*(u) \in L^p(\mathbb{R}^n), \\ u(\cdot,0) = f \in L^p(\mathbb{R}^n)\,, \end{cases}$$

and in Λ^α, with $0 \leq \alpha < 1$:

$$(\mathrm{D})_{\Lambda^\alpha} \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ t\nabla u \in T_2^\infty(\mathbb{R}^{n+1}_+) \text{ if } \alpha = 0\,, \text{ or } u \in \dot{C}^\alpha(\overline{\mathbb{R}^{n+1}_+}) \text{ if } 0 < \alpha < 1\,, \\ u(\cdot,0) = f \in \Lambda^\alpha(\mathbb{R}^n)\,. \end{cases}$$

The solution u of $(D)_p$, with data f, satisfies

- $u \xrightarrow{\text{n.t.}} f$, and $u(\cdot,t) \to f$ as $t \to 0$ in $L^p(\mathbb{R}^n)$.

The solution u of $(D)_{\Lambda^\alpha}$, with data f, satisfies

- $u(\cdot,t) \to f$ as $t \to 0$ in the weak* topology on Λ^α, $0 \le \alpha < 1$.
- $u \in \dot{C}^\alpha(\overline{\mathbb{R}^{n+1}_+})$, and $u(\cdot,0) = f$ pointwise, $0 < \alpha < 1$.

Theorem 1.4. *Let $L = -\operatorname{div} A\nabla$ and $L_0 = -\operatorname{div} A_0\nabla$ be as in (1.1) and (1.2) with $A = A(x)$ and $A_0 = A_0(x)$ both t-independent, and suppose that A_0 is real symmetric. There exists $\varepsilon_0 > 0$ and $\epsilon > 0$, both depending only on dimension and the ellipticity of A_0, such that if*

$$\|A - A_0\|_\infty < \varepsilon_0\,,$$

then $(N)_p$, $(R)_p$, $(D)_q$ and $(D)_{\Lambda^\alpha}$ are uniquely solvable for L and L^* when $1-\epsilon < p < 2+\epsilon$, $(2+\epsilon)' < q < \infty$ and $0 \le \alpha < n\epsilon/(1-\epsilon)$, respectively.

Remark 1.5. By Remark 1.1, both L and L^* satisfy the "standard assumptions" under the hypotheses of Theorem 1.4.

Remark 1.6. Theorems 1.2 and 1.4 continue to hold, with the half-space $\mathbb{R}^{n+1}_+$ replaced by a Lipschitz graph domain of the form $\Omega = \{(x,t) \in \mathbb{R}^{n+1} : t > \phi(x)\}$, where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. Indeed, that case may be reduced to that of the half-space by a standard pull-back mechanism. We omit the details.

Let us briefly review some related history. We focus first on the question of boundedness of layer potentials. As we have just noted, our results extended immediately to the setting of a Lipschitz graph domain. The prototypical result in that setting is the result of Coifman, McIntosh and Meyer [15] concerning the L^2 boundedness of the Cauchy integral operator on a Lipschitz curve, which implies L^2 bounds for the layer potentials associated to the Laplacian via the method of rotations. In turn, the corresponding H^p/L^p bounds follow by classical Calderón-Zygmund theory.

For the variable coefficient operators considered here, the L^2 boundedness theory (essentially, the case $p = 2$ of Theorem 1.2, along with L^2 square function estimates) was introduced in [1]. In that paper, it was shown, first, that such L^2 bounds (along with L^2 invertibility for $\pm(1/2)I + K$) are stable under small complex L^∞ perturbations of the coefficient matrix, and second, that these boundedness and invertibility results hold in the case that A is real and symmetric (hence also for complex perturbations of real symmetric A). The case $p = 2$ for A real, but not necessarily symmetric, was treated in [39] in the case $n = 1$ (i.e., in ambient dimension $n + 1 = 2$), and in [25], in all dimensions. Moreover, in hindsight, in the special case that the matrix A is of the "block" form

$$\left[\begin{array}{c|c} & 0 \\ B & \vdots \\ & 0 \\ \hline 0\cdots 0 & 1 \end{array}\right],$$

where $B = B(x)$ is a $n \times n$ matrix, L^2 bounds for layer potentials follow from the solution of the Kato problem [8], since in the block case the single layer potential is given by $\mathcal{S}f(\cdot,t) = (1/2)J^{-1/2}e^{-t\sqrt{J}}$, where $J := -\operatorname{div}_x B(x)\nabla_x$.

Quite recently, the case $p = 2$ of Theorem 1.2 was shown to hold in general, for L and L^* satisfying the "standard assumptions", in work of Rosén [50], in which L^2 bounds for layer potentials are obtained via results of [6] concerning functional calculus of certain first order "Dirac-type" operators[1]. We note further that Rosén's L^2 estimates do not require the DG/N/M hypothesis (rather, just ellipticity and t-independence). On the other hand, specializing to the "block" case mentioned above, we observe that counter-examples in [44] and [22] (along with some observations in [3]), show that the full range of L^p and Hardy space results treated in the present paper cannot be obtained without assuming DG/N/M. It seems very likely that L^p boundedness for some restricted range of p should still hold, even in the absence of DG/N/M, as is true in the special case of the "block matrices" treated in [3], [11], [28], and [42], but we have not considered this question here. We mention also that even in the presence of DG/N/M (in fact, even for A real and symmetric), the constraint on the upper bound on p in (1.13)-(1.14) is optimal. To see this, consider the block case, so that L is of the form $Lu = u_{tt} + \mathrm{div}_x\, B(x)\nabla_x u =: u_{tt} - Ju$, where $B = B(x)$ is an $n\times n$ uniformly elliptic matrix. Thus, $\mathcal{S}f(\cdot,t) = (1/2)J^{-1/2}e^{-t\sqrt{J}}f$, so that, considering only the tangential part of the gradient in (1.13), and letting $t\to 0$, we obtain as a consequence of (1.13) that

$$\|\nabla_x J^{-1/2} f\|_p \lesssim \|f\|_p\,. \tag{1.20}$$

But by Kenig's examples (see [7, pp. 119–120]), for each $p > 2$, there is a J as above for which the Riesz transform bound (1.20) fails. The matrix B may even be taken to be real symmetric. Thus, our results are in the nature of best possible, in the sense that, first, the DG/N/M hypothesis is needed to treat p near (or below) 1, and second, that even with DG/N/M, the exponent p^+ is optimal.

As regards the question of solvability, addressed here in Theorem 1.4, we recall that in the special case of the Laplacian on a Lipschitz domain, solvability of the L^p Dirichlet problem is due to Dahlberg [17], while the Neumann and Regularity problems were treated first, in the case $p = 2$, by Jerison and Kenig [34], and then by Verchota [54], by an alternative proof using the method of layer potentials; and second, in the case $1 < p < 2+\varepsilon$, by Verchota [54] (Regularity problem only), and in the case $1 \le p < 2+\varepsilon$ by Dahlberg and Kenig [18] (Neumann and Regularity), and finally, in the case $1-\varepsilon < p < 1$ by Brown [12] (who then obtained D_{Λ^α} by duality). A conceptually different proof of the latter result has been subsequently given by Kalton and Mitrea in [35] using a general perturbation technique of functional analytic nature[2]. More generally, in the setting of variable coefficients, in the special case that $A = A_0$ (i.e., that A is real symmetric), the L^p results for the Dirichlet problem were obtained by Jerison and Kenig [33], and for the Neumann and Regularity problems by Kenig and Pipher in [38] (the latter authors also treated the analogous Hardy space theory in the case $p = 1$). The case $p = 2$ of Theorem 1.4 (allowing complex coefficients) was obtained first in [1], with an alternative proof given in [4]. The case $n = 1$ (i.e., in ambient dimension $n+1 = 2$) of Theorem 1.4 follows from the work of Barton [9].

In the present work, we consider solvability of boundary value problems only for complex perturbations of real, symmetric operators, but we point out that there has also been some recent progress in the case of non-symmetric t-independent operators. For real, non-symmetric coefficients, the case $n = 1$ has been treated by Kenig, Koch, Pipher and Toro [37] (Dirichlet problem), and by Kenig and Rule [39] (Neumann and Regularity). The

[1] A direct proof of these L^2 bounds for layer potentials, bypassing the functional calculus results of [6], will appear in [24].

[2] Thus answering a question posed by E. Fabes.

work of Barton [9] allows for complex perturbations of the results of [37] and [39]. The higher dimensional case $n > 1$ has very recently been treated in [25] (the Dirichlet problem for real, non-symmetric operators), and in [26] (Dirichlet and Regularity, for complex perturbations of the real, non-symmetric case). In these results for non-symmetric operators, necessarily there are additional restrictions on the range of allowable p, as compared to the symmetric case (see [37]). We remark that in the non-symmetric setting, with $n > 1$, the Neumann problem remains open.

We mention that we have also obtained an analogue of Theorem 1.4 for the Transmission problem, which we plan to present in a forthcoming publication [30].

Finally, let us discuss briefly the role of t-independence in our "standard assumptions". Caffarelli, Fabes and Kenig [13] have shown that some regularity, in a direction transverse to the boundary, is needed to obtain L^p solvability for, say, the Dirichlet problem. Motivated by their work, one may naturally split the theory of boundary value problems for elliptic operators in the half-space[3] into two parts: 1) solvability theory for t-independent operators, and 2) solvability results in which the discrepancy $|A(x,t) - A(x,0)|$, which measures regularity in t at the boundary, is controlled by a Carleson measure estimate of the type considered in [21][4], and in which one has some good solvability result for the operator with t-independent coefficients $A_0(x) := A(x,0)$. The present paper, and its companion article [30], fall into category 1). The paper [29] falls into category 2), and uses our results here to obtain boundedness and solvability results for operators in that category, in which the Carleson measure estimate for the discrepancy is sufficiently small (in this connection, see also the previous work [5], which treats the case $p = 2$).

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2. Jump relations and definition of the boundary integrals

Throughout this section, we impose the "standard assumptions" defined previously. The operators div and ∇ are considered in all $n+1$ variables, and we write div_x and ∇_x when only the first n variables are involved. Also, since we shall consider operators T that may be viewed as acting either in $\mathbb{R}^{n+1}$, or in $\mathbb{R}^n$ with the t variable frozen, we need to distinguish Hermitian adjoints in these two settings. We therefore use T^* to denote the $(n+1)$-dimensional adjoint of T, while $\operatorname{adj}(T)$ denotes the adjoint of T acting in $\mathbb{R}^n$.

As usual, to apply the layer potential method, we shall need to understand the jump relations for the co-normal derivatives of $u^\pm = \mathcal{S}^\pm f$. To this end, let us begin by recording the fact that, by the main result of [50],

$$\sup_{\pm t>0} \|\nabla \mathcal{S}_L^\pm f(\cdot,t)\|_{L^2(\mathbb{R}^n,\mathbb{C}^{n+1})} + \sup_{\pm t>0} \|\nabla \mathcal{S}_{L^*}^\pm f(\cdot,t)\|_{L^2(\mathbb{R}^n,\mathbb{C}^{n+1})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\,. \tag{2.1}$$

Combining the last estimate with [1, Lemma 4.8] (see Lemma 3.2 below), we obtain

$$\|\widetilde{N}_*^\pm(\nabla \mathcal{S}^\pm f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\,. \tag{2.2}$$

Next, we recall the following fact proved in [1]. Recall that $e_{n+1} := (0, \dots, 0, 1)$ denotes the standard unit basis vector in the $t = x_{n+1}$ direction.

[3] There are analogues of the theory in a star-like Lipschitz domain.

[4] The Carleson measure control of [21] is essentially optimal, in view of [13].

Lemma 2.1 ([1, Lemmata 4.1 and 4.3])**.** *Suppose that L and L^* satisfy the standard assumptions. If $Lu = 0$ in $\mathbb{R}^{n+1}_{\pm}$ and $\widetilde{N}^{\pm}_{*}(\nabla u) \in L^2(\mathbb{R}^n)$, then the co-normal derivative $\partial_\nu u(\cdot, 0)$ exists in the variational sense and belongs to $L^2(\mathbb{R}^n)$, i.e., there exists a unique $g \in L^2(\mathbb{R}^n)$, and we set $\partial_\nu u(\cdot, 0) := g$, with $\|g\|_2 \lesssim \|\widetilde{N}^{\pm}_{*}(\nabla u)\|_2$, such that*

- *(i)* $\int_{\mathbb{R}^{n+1}_{\pm}} A\nabla u \cdot \nabla \Phi \, dX = \pm \int_{\mathbb{R}^n} g \, \Phi(\cdot, 0) \, dx$ *for all* $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$.
- *(ii)* $-\langle A\nabla u(\cdot, t), e_{n+1}\rangle \to g$ *weakly in* $L^2(\mathbb{R}^n)$ *as* $t \to 0^{\pm}$.

Moreover, there exists a unique $f \in \dot{L}^2_1(\mathbb{R}^n)$, with $\|f\|_{\dot{L}^2_1(\mathbb{R}^n)} \lesssim \|\widetilde{N}^{\pm}_{*}(\nabla u)\|_2$, such that

- *(iii)* $u \to f$ *non-tangentially.*
- *(iv)* $\nabla_x u(\cdot, t) \to \nabla_x f$ *weakly in* $L^2(\mathbb{R}^n)$ *as* $t \to 0^{\pm}$.

For each $f \in L^2(\mathbb{R}^n)$, it follows from (1.6) and (2.1) that $u := \mathcal{S}^{\pm} f$ is a solution of $Lu = 0$ in $\mathbb{R}^{n+1}_{\pm}$, and this solution has the properties listed in Lemma 2.1 because (2.2) holds. We then have the following result.

Lemma 2.2. *Suppose that L and L^* satisfy the standard assumptions. If $f \in L^2(\mathbb{R}^n)$, then almost everywhere on $\mathbb{R}^n$, we have*

$$\partial_\nu \mathcal{S}^+ f(\cdot, 0) - \partial_\nu \mathcal{S}^- f(\cdot, 0) = f, \tag{2.3}$$

where the co-normal derivatives are defined in the variational sense of Lemma 2.1.

Proof. Let us first suppose that $f \in C_0^\infty(\mathbb{R}^n)$, and introduce

$$u := \begin{cases} \mathcal{S}^+ f & \text{in } \mathbb{R}^{n+1}_{+}, \\ \mathcal{S}^- f & \text{in } \mathbb{R}^{n+1}_{-}, \end{cases}$$

and pick some $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$. By Lemma 2.1 (i) and the property of the fundamental solution in (1.7), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \left\{ \partial_\nu \mathcal{S}^+ f(x,0) - \partial_\nu \mathcal{S}^- f(x,0) \right\} \Phi(x,0) \, dx \\ &\qquad = \int_{\mathbb{R}^n} \partial_\nu \mathcal{S}^+ f(x,0) \, \Phi(x,0) \, dx \; - \int_{\mathbb{R}^n} \partial_\nu \mathcal{S}^- f(x,0) \, \Phi(x,0) \, dx \\ &\qquad = \int_{\mathbb{R}^{n+1}_{+}} \langle A\nabla u, \nabla \Phi \rangle \, dX \; + \int_{\mathbb{R}^{n+1}_{-}} \langle A\nabla u, \nabla \Phi \rangle \, dX \\ &\qquad = \int_{\mathbb{R}^{n+1}} \left\langle A(x) \left(\int_{\mathbb{R}^n} \nabla_{x,t} E(x,t,y,0) f(y) \, dy \right), \nabla \Phi(x,t) \right\rangle dxdt \\ &\qquad = \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^{n+1}} \langle A(x) \nabla_{x,t} E(x,t,y,0), \nabla \Phi(x,t) \rangle \, dxdt \right) dy \\ &\qquad = \int_{\mathbb{R}^n} f(y) \, \Phi(y,0) \, dy. \end{aligned} \tag{2.4}$$

The use of Fubini's theorem in the fifth line is justified by absolute convergence, since $\nabla E(\cdot, Y) \in L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $1 \le p < (n+1)/n$ (see [27, Theorem 3.1]).

Given an arbitrary $f \in L^2(\mathbb{R}^n)$, we may approximate f by $f_k \in C_0^\infty(\mathbb{R}^n)$, and observe that both the first and last lines in (2.4) converge appropriately (for the first line, this follows from (2.2) and Lemma 2.1). Then, since Φ was arbitrary in $C_0^\infty(\mathbb{R}^{n+1})$, (2.3) follows. $\square$

In view of (2.3), we now define the bounded operators K, $\widetilde{K} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\mathbf{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$, as discussed in (1.12), rigorously by

$$\begin{aligned} \widetilde{K}_L f &:= -\tfrac{1}{2} f + \partial_\nu \mathcal{S}_L^+ f(\cdot, 0) = \tfrac{1}{2} f + \partial_\nu \mathcal{S}_L^- f(\cdot, 0) \\ K_L f &:= \operatorname{adj}(\widetilde{K}_{L^*}) f \\ \mathbf{T}_L f &:= \Big(\nabla_x \mathcal{S}_L f\,,\ \tfrac{-1}{A_{n+1,n+1}} \big(\widetilde{K}_L f + \textstyle\sum_{j=1}^n A_{n+1,j}\, \partial_{x_j} \mathcal{S}_L f\big)\Big). \end{aligned} \tag{2.5}$$

We then have the following lemma, which we quote without proof from [1][5], although part (i) below is just a rephrasing of Lemma 2.1(ii) and Lemma 2.2.

Lemma 2.3 ([1, Lemma 4.18]). *Suppose that L and L^* satisfy the standard assumptions. If $f \in L^2(\mathbb{R}^n)$, then*

(i) $\partial_\nu (\mathcal{S}_L^\pm f)(\cdot, 0) = \big(\pm\tfrac{1}{2} I + \widetilde{K}_L\big) f,$
and $-\langle A \nabla \mathcal{S}_L^\pm f(\cdot, t), e_{n+1}\rangle \to \big(\pm\tfrac{1}{2} I + \widetilde{K}_L\big) f$ *weakly in L^2 as $t \to 0^\pm$,*

where the co-normal derivative is defined in the variational sense of Lemma 2.1.

(ii) $\nabla \mathcal{S}_L^\pm f(\cdot, t) \to \Big(\mp \tfrac{1}{2A_{n+1,n+1}} e_{n+1} + \mathbf{T}_L\Big) f$ *weakly in L^2 as $t \to 0^\pm$,*

where the tangential component of $\mathbf{T}_L f$ equals $\nabla_x \mathcal{S}_L f$.

(iii) $\mathcal{D}_L^\pm f(\cdot, t) \to \big(\mp\tfrac{1}{2} I + K_L\big) f$ *weakly in L^2 as $t \to 0^\pm$.*

Improvements of Lemmas 2.1 and 2.3 appear in [5], where strong convergence results are established. The weak convergence results, however, are sufficient for our purposes.

3. A "Calderón-Zygmund" Theory for the boundedness of layer potentials: Proof of Theorem 1.2

We continue to impose the "standard assumptions" throughout this section. We shall work in the upper half-space, the proofs of the analogous bounds for the lower half-space being essentially identical. Our main goal in this section is to prove Theorem 1.2.

We begin with some observations concerning the kernels of the operators $f \mapsto \partial_t \mathcal{S} f(\cdot, t)$ and $f \mapsto \nabla_x \mathcal{S} f(\cdot, t)$, which we denote respectively by

$$K_t(x, y) := \partial_t E(x, t, y, 0) \quad \text{and} \quad \vec{H}_t(x, y) := \nabla_x E(x, t, y, 0).$$

By the DG/N/M estimates (1.3) and (1.4) (see [27, Theorem 3.1] and [1, Lemma 2.5]), for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ such that $|t| + |x - y| > 0$, and for each integer $m \geq 0$, we have

$$|(\partial_t)^m E(x, t, y, 0)| \leq \frac{C_m}{(|t| + |x - y|)^{n+m-1}}\,, \tag{3.1}$$

and when, in addition, $2|h| \leq \max(|x - y|, |t|)$, we have

$$\begin{aligned} &|(\partial_t)^m E(x + h, t, y, 0) - (\partial_t)^m E(x, t, y, 0)| \\ &\quad + |(\partial_t)^m E(x, t, y + h, 0) - (\partial_t)^m E(x, t, y, 0)| \leq C_m \frac{|h|^\alpha}{(|t| + |x - y|)^{n+m-1+\alpha}}, \end{aligned} \tag{3.2}$$

where $\alpha \in (0, 1]$ is the minimum of the De Giorgi/Nash exponents for L and L^* in (1.3). Thus, $K_t(x, y)$ is a standard Calderón-Zygmund kernel, uniformly in t, but $\vec{H}_t(x, y)$ is not,

[5] [1, Lemma 4.18] assumes that (2.1) holds, but as noted above, it is now known that this is always the case, for t-independent divergence form elliptic operators, by the result of [50].

and for this reason the proof of Theorem 1.2 will be somewhat delicate. On the other hand, the lemma below shows that the kernel $\vec{H}_t(x,y)$ does satisfy a sort of weak "1-sided" Calderón-Zygmund condition similar to those considered by Kurtz and Wheeden in [41, Lemma 1]. In particular, the following lemma from [1] is at the core of our proof of Theorem 1.2.

Lemma 3.1 ([1, Lemma 2.13, (4.15) and (2.7)]). *Suppose that L and L^* satisfy the standard assumptions. Consider a cube $Q \subset \mathbb{R}^n$ and fix any points $x, x' \in Q$ and $t, t' \in \mathbb{R}$ such that $|t - t'| < 2\ell(Q)$. For all $(y,s) \in \mathbb{R}^{n+1}$, set*

$$u(y,s) := E(x,t,y,s) - E(x',t',y,s).$$

If $\alpha > 0$ is the Hölder exponent in (3.2), then for all integers $k \geq 4$, we have

$$\sup_{s\in\mathbb{R}} \int_{2^{k+1}Q\setminus 2^kQ} |\nabla u(y,s)|^2\, dy \lesssim 2^{-2\alpha k}\left(2^k\ell(Q)\right)^{-n}.$$

The analogous bound holds with E^* in place of E.

We will also need the following lemma from [1] to deduce (1.14) from (1.13) for $p \geq 2$.

Lemma 3.2 ([1, Lemma 4.8]). *Suppose that L and L^* satisfy the standard assumptions. Let $\mathcal{S}_t$ denote the operator $f \mapsto \mathcal{S}_L f(\cdot,t)$. Then for $1 < p < \infty$,*

$$\|\widetilde{N}_*(\nabla\mathcal{S}_L f)\|_{L^p(\mathbb{R}^n)} \leq C_p\left(1 + \sup_{t>0}\|\nabla\mathcal{S}_t\|_{p\to p}\right)\|f\|_{L^p(\mathbb{R}^n)},$$

where $\|\cdot\|_{p\to p}$ denotes the operator norm in L^p. The analogous bound holds for L^* and in the lower half-space.

To be precise, Lemma 3.2 is a direct combination of (i) and (ii) in [1, Lemma 4.8]. We are now ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. As noted above, we work in $\mathbb{R}^{n+1}_+$ and restrict our attention to the layer potentials for L, as the proofs in $\mathbb{R}^{n+1}_-$ and for L^* are essentially the same. We first consider estimates (1.13)-(1.16), and we separate their proofs into two parts, according to whether $p \leq 2$ or $p > 2$. Afterwards, we prove estimates (1.17)-(1.19).

Part 1: estimates (1.13)-(1.16) in the case $p_\alpha < p \leq 2$. We set $\mathcal{S} := \mathcal{S}_L^+$ to simplify notation. We separate the proof into the following three parts.

Part 1(a): estimate (1.13) in the case $p_\alpha < p \leq 2$. Consider first the case $p \leq 1$. We claim that if $\frac{n}{n+1} < p \leq 1$ and a is an H^p-atom in $\mathbb{R}^n$ with

$$\operatorname{supp} a \subset Q, \qquad \int_{\mathbb{R}^n} a\, dx = 0, \qquad \|a\|_{L^2(\mathbb{R}^n)} \leq \ell(Q)^{n\left(\frac{1}{2}-\frac{1}{p}\right)}, \tag{3.3}$$

then for $\alpha > 0$ as in (3.2), and for each integer $k \geq 4$, we have

$$\sup_{t\geq 0} \int_{2^{k+1}Q\setminus 2^kQ} |\nabla\mathcal{S}a(x,t)|^2\, dx \lesssim 2^{-(2\alpha+n)k}\ell(Q)^{n\left(1-\frac{2}{p}\right)}, \tag{3.4}$$

where $\nabla\mathcal{S}a(\cdot,0)$ is defined on $2^{k+1}Q \setminus 2^kQ$, since $\operatorname{supp} a \subset Q$. Indeed, using the vanishing moment condition of the atom, Minkowski's inequality, and Lemma 3.1 (with the roles of

(x,t) and (y,s), or equivalently, the roles of L and L^*, reversed), we obtain

$$\begin{aligned}
&\int_{2^{k+1}Q\setminus 2^kQ} |\nabla \mathcal{S}a(x,t)|^2\,dx \\
&\qquad = \int_{2^{k+1}Q\setminus 2^kQ} \left| \int_{\mathbb{R}^n} \left[\nabla_{x,t}E(x,t,y,0) - \nabla_{x,t}E(x,t,y_Q,0)\right] a(y)\,dy \right|^2 dx \\
&\qquad \leq \left(\int_{\mathbb{R}^n} |a(y)| \left(\int_{2^{k+1}Q\setminus 2^kQ} \left| \nabla_{x,t}E(x,t,y,0) - \nabla_{x,t}E(x,t,y_Q,0) \right|^2 dx \right)^{1/2} dy \right)^2 \\
&\qquad \lesssim 2^{-2\alpha k}\left(2^k\ell(Q)\right)^{-n} \|a\|^2_{L^1(\mathbb{R}^n)} \lesssim 2^{-(2\alpha+n)k}\ell(Q)^{n\left(1-\frac{2}{p}\right)},
\end{aligned}$$

since $\|a\|_{L^1(\mathbb{R}^n)} \leq |Q|^{1/2}\|a\|_{L^2(\mathbb{R}^n)}$. This proves (3.4) and thus establishes the claim.

With (3.4) in hand, we can now prove (1.13) by a standard argument. We write

$$\int_{\mathbb{R}^n} |\nabla \mathcal{S}a(x,t)|^p\,dx = \int_{16Q} |\nabla \mathcal{S}a(x,t)|^p\,dx + \sum_{k=4}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} |\nabla \mathcal{S}a(x,t)|^p\,dx,$$

where a is an H^p-atom supported in Q as in (3.3). Applying Hölder's inequality with exponent $2/p$, the L^2 estimate for $\nabla\mathcal{S}$ in (2.1), and estimate (3.4) for $t>0$, we obtain

$$\sup_{t>0} \int_{\mathbb{R}^n} |\nabla \mathcal{S}a(x,t)|^p\,dx \lesssim 1,$$

since $\alpha > n(1/p-1)$ in the interval $p_\alpha < p \leq 1$ with $p_\alpha := n/(n+\alpha)$. This proves (1.13) for $p_\alpha < p \leq 1$, and so interpolation with (2.1) proves (1.13) for $p_\alpha < p \leq 2$.

Part 1(b): estimate (1.14) in the case $p_\alpha < p \leq 2$. We first note that as in Part 1(a), by using (2.2) instead of (2.1), we may reduce matters to showing that for $p_\alpha < p \leq 1$ and for each integer $k \geq 10$, we have

$$\int_{2^{k+1}Q\setminus 2^kQ} |\widetilde{N}_*(\nabla \mathcal{S}a)|^p \lesssim 2^{-(\alpha-n(1/p-1))kp},$$

whenever a is an H^p-atom supported in Q as in (3.3), since $\alpha > n(1/p-1)$ in the interval $p_\alpha < p \leq 1$ with $p_\alpha := n/(n+\alpha)$. In turn, using Hölder's inequality with exponent $1/p$ when $p<1$, we need only prove that for each integer $k \geq 10$, we have

$$\int_{2^{k+1}Q\setminus 2^kQ} |\widetilde{N}_*(\nabla \mathcal{S}a)| \lesssim 2^{-\alpha k}|Q|^{1-1/p}. \tag{3.5}$$

To this end, set $u := \mathcal{S}a$, and suppose that $x \in 2^{k+1}Q \setminus 2^kQ$ for some integer $k \geq 10$. We begin with the estimate $\widetilde{N}_* \leq N_1 + N_2$, where

$$N_1(\nabla u)(x) := \sup_{|x-y|<t<2^{k-3}\ell(Q)} \left(\fint_{B((y,t),t/4)} |\nabla u|^2 \right)^{1/2},$$

$$N_2(\nabla u)(x) := \sup_{|x-y|<t,\ t>2^{k-3}\ell(Q)} \left(\fint_{B((y,t),t/4)} |\nabla u|^2 \right)^{1/2}.$$

Following [38], by Caccioppoli's inequality we have

$$\begin{aligned} N_1(\nabla u)(x) &\lesssim \sup_{|x-y|<t<2^{k-3}\ell(Q)} \left(\fint_{B((y,t),t/2)} \frac{|u-c_B|^2}{t^2} \right)^{1/2} \\ &\lesssim \sup_{t<2^{k-3}\ell(Q)} \left\{ \left(\fint_{t/2}^{3t/2} \fint_{|x-y|<3t/2} \frac{|u(y,s)-u(y,0)|^2}{t^2} \right)^{1/2} \right. \\ &\qquad\qquad \left. + \left(\fint_{|x-y|<3t/2} \frac{|u(y,0)-c_B|^2}{t^2} \right)^{1/2} \right\} =: I + II, \end{aligned}$$

where the constant c_B is at our disposal, and $u(y,0) := \mathcal{S}a(y,0) := \mathcal{S}a(y)$.

By the vanishing moment property of a, if z_Q denotes the center of Q, then for all (y,s) and t as in I, we have

$$\begin{aligned} \frac{1}{t}|u(y,s)-u(y,0)| &= \left| \frac{1}{t} \int_0^s \frac{\partial}{\partial \tau} \mathcal{S}a(y,\tau)\, d\tau \right| \\ &\leq \sup_{0<\tau<3t/2} \int_{\mathbb{R}^n} |\partial_\tau E(y,\tau,z,0) - \partial_\tau E(y,\tau,z_Q,0)||a(z)|\, dz \\ &\lesssim \int_{\mathbb{R}^n} \frac{\ell(Q)^\alpha}{|y-z_Q|^{n+\alpha}} |a(z)|\, dz \\ &\lesssim 2^{-\alpha k}(2^k \ell(Q))^{-n} |Q|^{1-1/p}, \end{aligned}$$

where in the next-to-last step we have used (3.2) with $m = 1$, and in the last step we have used that $\|a\|_1 \lesssim |Q|^{1-1/p}$. Thus,

$$\int_{2^{k+1}Q\setminus 2^k Q} I\, dx \leq 2^{-\alpha k}|Q|^{1-1/p},$$

as desired. By Sobolev's inequality, for an appropriate choice of c_B, we have

$$\begin{aligned} II &\lesssim \sup_{0<t<2^{k-3}\ell(Q)} \left(\fint_{|x-y|<3t/2} |\nabla_{\tan} u(y,0)|^{2_*} \right)^{1/2_*} \\ &\lesssim \left(M(|\nabla_{\tan} u(\cdot,0)|^{2_*} \chi_{2^{k+3}Q\setminus 2^{k-2}Q})(x) \right)^{1/2_*}, \end{aligned}$$

where $\nabla_{\tan} u(x,0) := \nabla_x u(x,0)$ is the tangential gradient, $2_* := 2n/(n+2)$, and M, as usual, denotes the Hardy-Littlewood maximal operator. Consequently, we have

$$\begin{aligned} \int_{2^{k+1}Q\setminus 2^k Q} II &\lesssim \left(2^k \ell(Q)\right)^{n/2} \left(\int_{2^{k+3}Q\setminus 2^{k-2}Q} |\nabla_{\tan} u(\cdot,0)|^2 \right)^{1/2} \\ &\lesssim \left(2^k \ell(Q)\right)^{n/2} 2^{-(\alpha+n/2)k} \ell(Q)^{n(1/2-1/p)} \lesssim 2^{-\alpha k}|Q|^{1-1/p}, \end{aligned}$$

where in the second inequality we used estimate (3.4) for $t = 0$, since $u = \mathcal{S}a$ and $\operatorname{supp} a \subset Q$. We have therefore proved that N_1 satisfies (3.5).

It remains to treat N_2. For each $x \in 2^{k+1}Q \setminus 2^k Q$, choose (y_*, t_*) in the cone $\Gamma(x) \subset \mathbb{R}^{n+1}_+$ so that the supremum in the definition of N_2 is essentially attained, i.e., so that

$$N_2(\nabla u)(x) \leq 2 \left(\fint_{B((y_*,t_*),t_*/4)} |\nabla u|^2 \right)^{1/2},$$

with $|x-y_*|<t_*$ and $t_*\geq 2^{k-3}\ell(Q)$. By Caccioppoli's inequality,

$$N_2(\nabla u)(x)\lesssim \frac{1}{t_*}\left(\fint_{B((y_*,t_*),\,t_*/2)}|u|^2\right)^{1/2}.$$

Now for $(y,s)\in B((y_*,t_*),t_*/2)$, by (3.2) with $m=0$, we have

$$\begin{aligned}|u(y,s)| &\leq \int_{\mathbb{R}^n}|E(y,s,z,0)-E(y,s,z_Q,0)||a(z)|\,dz\\ &\lesssim \|a\|_{L^1(\mathbb{R}^n)}\frac{\ell(Q)^\alpha}{s^{n-1+\alpha}}\lesssim \ell(Q)^\alpha t_*^{1-n-\alpha}|Q|^{1-1/p}.\end{aligned}$$

Therefore,

$$N_2(\nabla u)(x)\lesssim \ell(Q)^\alpha t_*^{-n-\alpha}|Q|^{1-1/p}\lesssim 2^{-\alpha k}(2^k\ell(Q))^{-n}|Q|^{1-1/p}.$$

Integrating over $2^{k+1}Q\setminus 2^kQ$, we obtain (3.5) for N_2, hence (1.14) holds for $p_\alpha<p\leq 2$.

Part 1(c): estimates (1.15)-(1.16) in the case $p_\alpha<p\leq 2$. We note that the case $p=2$ holds by (2.1) and Lemma 2.3 (i) and (ii). Thus, by interpolation, it is again enough to treat the case $p_\alpha<p\leq 1$, and in that setting, (1.15)-(1.16) are an immediate consequence of the estimates in (3.6)-(3.7) below, which we note for future reference.

Proposition 3.3. *Suppose that L and L^* satisfy the standard assumptions, let α denote the minimum of the De Giorgi/Nash exponents for L and L^* in (1.3), and set $p_\alpha := n/(n+\alpha)$. Then the operators $\nabla\mathcal{S}_L$, $\nabla_x S_L$ and $\widetilde{K}_L$ have unique extensions from $L^2(\mathbb{R}^n)$ satisfying*

$$\|\nabla_x S_L f\|_{H^p(\mathbb{R}^n,\mathbb{C}^n)}+\sup_{t>0}\|\nabla_x\mathcal{S}_L f(\cdot,t)\|_{H^p(\mathbb{R}^n,\mathbb{C}^n)}\leq C_p\|f\|_{H^p(\mathbb{R}^n)},\qquad \forall\, p\in(p_\alpha,1]\,,\tag{3.6}$$

$$\|\widetilde{K}_L f\|_{H^p(\mathbb{R}^n)}+\sup_{t>0}\|\langle A\nabla\mathcal{S}_L f(\cdot,t),e_{n+1}\rangle\|_{H^p(\mathbb{R}^n)}\leq C_p\|f\|_{H^p(\mathbb{R}^n)},\qquad \forall\, p\in(p_\alpha,1]\,,\tag{3.7}$$

for all $f\in H^p(\mathbb{R}^n)$. The analogous results hold for L^* and in the lower half-space.

Proof. It suffices to show that if a is an $H^p(\mathbb{R}^n)$-atom as in (3.3), and $t>0$, then

$$\begin{aligned}\vec{m}_0 &:= C\nabla_x\mathcal{S}a, & \vec{m}_t &:= C\nabla_x\mathcal{S}a(\cdot,t),\\ m_0 &:= C\widetilde{K}a, & m_t &:= C\langle A\nabla\mathcal{S}a(\cdot,t),e_{n+1}\rangle,\end{aligned}$$

are all molecules adapted to Q, for some harmless constant $C\in(0,\infty)$, depending only on the "standard constants". Recall that, for $n/(n+1)<p\leq 1$, an H^p-molecule adapted to a cube $Q\subset\mathbb{R}^n$ is a function $m\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ satisfying

$$\begin{aligned}&(i)\quad \int_{\mathbb{R}^n}m(x)\,dx=0,\\ &(ii)\quad \left(\int_{16Q}|m(x)|^2\,dx\right)^{1/2}\leq \ell(Q)^{n\left(\frac12-\frac1p\right)},\\ &(iii)\quad \left(\int_{2^{k+1}Q\setminus 2^kQ}|m(x)|^2\,dx\right)^{1/2}\leq 2^{-\varepsilon k}\left(2^k\ell(Q)\right)^{n\left(\frac12-\frac1p\right)},\qquad \forall\, k\geq 4,\end{aligned}\tag{3.8}$$

for some $\varepsilon>0$ (see, e.g., [16], [53]).

Note that for $\vec{m}_t$ and m_t, when $t>0$, property (ii) follows from the L^2 estimate in (2.1), and (iii) follows from (3.4) with $\varepsilon:=\alpha-n(1/p-1)$, which is positive for $p_\alpha<p\leq 1$ with $p_\alpha:=n/(n+\alpha)$. Moreover, these estimates for $\vec{m}_t$ and m_t hold uniformly in t, and since $a\in L^2(\mathbb{R}^n)$, we obtain (ii) and (iii) for $\vec{m}_0$ and m_0 by Lemma 2.3.

Thus, it remains to show that $\vec{m}_t$ and m_t have mean-value zero for all $t \geq 0$. This is nearly trivial for $\vec{m}_t$. For any $R > 1$, choose $\Phi_R \in C_0^\infty(\mathbb{R}^{n+1})$, with $0 \leq \Phi_R \leq 1$, such that

$$\Phi_R \equiv 1 \text{ on } B(0,R), \quad \operatorname{supp}\Phi_R \subset B(0,2R), \quad \|\nabla\Phi_R\|_{L^\infty(\mathbb{R}^{n+1})} \lesssim 1/R\,, \tag{3.9}$$

and let $\phi_R := \Phi_R(\cdot,0)$ denote its restriction to $\mathbb{R}^n\times\{0\}$. For $1 \leq j \leq n$ and $R > C(\ell(Q)+|y_Q|)$ (where y_Q is the center of Q), using that a has mean value zero, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \partial_{x_j}\mathcal{S}a(\cdot,t)\,\phi_R\right| &= \left|\int_{\mathbb{R}^n} \mathcal{S}a(\cdot,t)\,\partial_{x_j}\phi_R\right| \\
&\lesssim \frac{1}{R}\int_{R\leq|x|\leq 2R}\int_Q |E(x,t,y,0 - E(x,t,y_Q,0)|\,|a(y)|\,dy\,dx \\
&\lesssim \frac{1}{R}\int_{R\leq|x|\leq 2R}\int_Q \frac{\ell(Q)^\alpha}{R^{n-1+\alpha}}\,|a(y)|\,dy \\
&\lesssim \left(\frac{\ell(Q)}{R}\right)^\alpha \|a\|_{L^1(\mathbb{R}^n)} \lesssim \left(\frac{\ell(Q)}{R}\right)^\alpha \ell(Q)^{n(1-1/p)}\,,
\end{aligned}$$

where we used the DG/N bound (3.2) with $m = 0$, the Cauchy-Schwarz inequality and the definition of an atom (3.3). Letting $R \to \infty$, we obtain $\int_{\mathbb{R}^n} \nabla_x\mathcal{S}a(\cdot,t) = 0$ for all $t \geq 0$.

Next, let us show that $\widetilde{\mathcal{K}}a$ has mean-value zero. Set $u := \mathcal{S}a$ in $\mathbb{R}^{n+1}_+$, so that matters are reduced to proving that

$$\int_{\mathbb{R}^n} \partial_\nu u(x,0)\,dx = 0,$$

where $\partial_\nu u(\cdot,0)$ is defined in the variational sense of Lemma 2.1. Choose Φ_R, ϕ_R as above, and note that $\partial_\nu u(\cdot,0) \in L^1(\mathbb{R}^n)$, by the bounds (3.8) (ii) and (iii) that we have just established. Then by Lemma 2.1 (i), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \partial_\nu u(\cdot,0)\,dx\right| &= \left|\lim_{R\to\infty}\int_{\mathbb{R}^n} \partial_\nu u(\cdot,0)\,\phi_R\,dx\right| \\
&= \left|\lim_{R\to\infty}\int_{\mathbb{R}^{n+1}_+} \langle A\nabla u, \nabla\Phi_R\rangle\,dX\right| \\
&\lesssim \overline{\lim_{R\to\infty}}\left(\int_{X\in\mathbb{R}^{n+1}_+:R<|X|<2R} |\nabla u|^q\,dX\right)^{1/q}\left(\int_{R<|X|<2R} |\nabla\Phi_R|^{q'}\,dX\right)^{1/q'}\,,
\end{aligned}$$

where $q := p(n+1)/n$ and $q' = q/(q-1)$. Since $0 < \alpha \leq 1$ and $p_\alpha := n/(n+\alpha)$, we have $n/(n+1) < p \leq 1$, hence $1 < q \leq (n+1)/n$ and $n+1 \leq q' < \infty$. Consequently, the second factor above is bounded uniformly in R as $R \to \infty$, whilst the first factor converges to zero by Lemma 6.2 and the dominated convergence theorem, since we have already proven (1.14) in the case $p_\alpha < p \leq 2$. This proves that $\int_{\mathbb{R}^n} \widetilde{\mathcal{K}}a = 0$. The proof that $\int_{\mathbb{R}^n}\langle A\nabla\mathcal{S}a(\cdot,t), e_{n+1}\rangle = 0$ for all $t > 0$ follows in the same way, except we use [1, (4.6)] instead of Lemma 2.1 (i). ◻

This concludes the proof of Part 1 of Theorem 1.2. At this point, we note for future reference the following corollary of (1.16) and Proposition 3.3.

Corollary 3.4. *Suppose that L and L^* satisfy the standard assumptions, and let α denote the minimum of the De Giorgi/Nash exponents for L and L^* in (1.3). Then the operators*

$\mathcal{D}_L$ and K_L have unique extensions from $L^2(\mathbb{R}^n)$ satisfying

$$\sup_{t>0} \|\mathcal{D}_L g(\cdot,t)\|_{\Lambda^\beta(\mathbb{R}^n)} + \|K_L g\|_{\Lambda^\beta(\mathbb{R}^n)} \leq C_\beta \|g\|_{\Lambda^\beta(\mathbb{R}^n)}, \quad \forall \beta \in [0,\alpha), \tag{3.10}$$

for all $g \in \Lambda^\beta(\mathbb{R}^n)$. Moreover, $\mathcal{D}_L 1$ is constant on $\mathbb{R}^{n+1}_+$, and $K_L 1$ is constant on $\mathbb{R}^n$. The analogous results hold for L^* and in the lower half-space.

Proof. For all $t \in \mathbb{R} \setminus \{0\}$, using the notation

$$\partial_{\nu^*} \mathcal{S}_{t,L^*} f := \partial_{\nu^*} \mathcal{S}_{L^*} f(\cdot,t), \qquad \mathcal{D}_{t,L} f := \mathcal{D}_L f(\cdot,t),$$

we have $\mathcal{D}_{t,L} = \operatorname{adj}(\partial_{\nu^*} \mathcal{S}_{-t,L^*})$ and $K_L := \operatorname{adj}(\widetilde{K}_{L^*})$ (initially on $L^2(\mathbb{R}^n)$), by definitions (1.8) and (2.5). Thus, estimates (1.16) and (3.7) imply (3.10) by duality. In particular, this means that $\mathcal{D}_{t,L}$ and K_L are defined on $\Lambda^\beta(\mathbb{R}^n)$ as the operators dual to $\partial_{\nu^*} \mathcal{S}_{-t,L^*}$ and $\widetilde{K}_{L^*}$ via the duality pairing between $\Lambda^\beta(\mathbb{R}^n)$ and $H^{n/(n+\beta)}_{\mathrm{at}}(\mathbb{R}^n)$.

Now consider the upper half-space $\mathbb{R}^{n+1}_+$. The case $\beta = 0$ of (3.10) shows that $\mathcal{D}_L 1(\cdot,t)$ and $K_L 1$ exist in $BMO(\mathbb{R}^n)$, for each $t > 0$. The moment conditions obtained in the proof of Proposition 3.3 show that for any atom a as in (3.3), and for each $t > 0$, we have

$$\langle \mathcal{D}_L 1(\cdot,t), a\rangle = \int_{\mathbb{R}^n} \partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot,-t) = 0, \quad \text{and} \quad \langle K_L 1, a\rangle = \int_{\mathbb{R}^n} \widetilde{K}_{L^*} a = 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $BMO(\mathbb{R}^n)$ and $H^1_{\mathrm{at}}(\mathbb{R}^n)$. This shows, since a was an arbitrary atom, that $\mathcal{D}_L 1(\cdot,t)$ and $K_L 1$ are zero in the sense of $BMO(\mathbb{R}^n)$, hence $\mathcal{D}_L 1(x,t)$ and $K_L 1(x)$ are constant in $x \in \mathbb{R}^n$, for each fixed $t > 0$.

It remains to prove that $\mathcal{D}_L 1(x,t)$ is constant in $t > 0$, for each fixed $x \in \mathbb{R}^n$. To this end, let ϕ_R denote the boundary trace of a smooth cut-off function Φ_R as in (3.9). We write

$$\partial_t \mathcal{D}_L 1(x,t) = \partial_t \mathcal{D}_L \phi_R(x,t) + \partial_t \mathcal{D}_L(1-\phi_R)(x,t),$$

and consider each term separately. Since $\phi_R \in L^2(\mathbb{R}^n)$, and since $\mathcal{D}_L$, acting on Λ^β, is an extension of the operator defined on L^2 in (1.8), by t-independence, we have

$$\begin{aligned}
\partial_t \mathcal{D}_L \phi_R(x,t) &= \int_{\mathbb{R}^n} \overline{(\partial_t \partial_{\nu^*} E^*(\cdot,\cdot,x,t))}(y,0)\, \phi_R(y)\, dy \\
&= \int_{\mathbb{R}^n} \overline{\partial_s e_{n+1} \cdot A^*(y)\,(\nabla_{y,s} E^*(y,s,x,t))}|_{s=0}\, \phi_R(y) dy \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n+1} \int_{\mathbb{R}^n} \overline{A^*_{i,j}(y)\,(\partial_{y_j} E^*(y,s,x,t))}|_{s=0}\, \partial_{y_i} \phi_R(y)\, dy
\end{aligned}$$

(the final equality is justified by [1, Lemma 2.15], since $L^* E^* = 0$ away from the pole at (x,t)). Then for $R > C|x|$ with $C > 1$ sufficiently large,

$$\begin{aligned}
|\partial_t \mathcal{D}_L \phi_R(x,t)| &\lesssim R^{-1} \int_{C^{-1}R<|x-y|<CR} \left|(\nabla E(x,t,\cdot,\cdot))(y,0)\right| dy \\
&\lesssim R^{-1+n/2} \left(\int_{C^{-1}R<|x-y|<CR} \left|(\nabla E(x,t,\cdot,\cdot))(y,0)\right|^2 dy \right)^{1/2} \lesssim R^{-1},
\end{aligned}$$

where in the last step we used the L^2 decay for ∇E from [1, Lemmata 2.5 and 2.8].

Remark 3.5. In order to handle the term $\partial_t \mathcal{D}_L(1-\phi_R)$, and for later use, we note the following general principle. If $\beta \in [0,\alpha)$ and $g \in \Lambda^\beta(\mathbb{R}^n)$, then for all $H^{n/(n+\beta)}(\mathbb{R}^n)$-atoms a, we have

$$\begin{aligned}\langle \mathcal{D}_{t,L} g, a\rangle &= \langle g, \partial_{\nu^*}\mathcal{S}_{-t,L^*}a\rangle \\ &= \int_{\mathbb{R}^n} g(y)\left(\int_{\mathbb{R}^n} -e_{n+1}\cdot \overline{A^*(y)[(\nabla E(x,t,\cdot,\cdot))(y,0)]}\, a(x)\,dx\right) dy,\end{aligned} \tag{3.11}$$

where the passage to an absolutely convergent integral is justified because $\partial_{\nu^*}\mathcal{S}_{L^*}a(\cdot,-t)$ is a constant multiple of an $H^{n/(n+\beta)}(\mathbb{R}^n)$-molecule, as seen from the proof of Proposition 3.3. Therefore, the vanishing moment property of a and Fubini's theorem justify writing

$$\mathcal{D}_L g(x,t) = \int_{\mathbb{R}^n}\left(-e_{n+1}\cdot\overline{A^*(y)(\nabla E(x,t,\cdot,\cdot))(y,0)} - h(y)\right) g(y)\,dy + c \tag{3.12}$$

for some $c \in \mathbb{R}$ and any function h for which the above integral is absolutely convergent.

Applying the above principle to $\lim_{\eta\to 0}\frac{1}{\eta}[\mathcal{D}_{t+\eta,L}(1-\phi_R)-\mathcal{D}_{t,L}(1-\phi_R)]$, then for $R > 2|x|$, we may write

$$\partial_t\mathcal{D}_L(1-\phi_R)(x,t) = \int_{\mathbb{R}^n}\left(-e_{n+1}\cdot\overline{A^*(y)(\partial_t\nabla E(x,t,\cdot,\cdot))(y,0)}\right)(1-\phi_R)(y)\,dy,$$

since the monotone convergence theorem and the L^2 decay estimate for the t-derivative $\partial_t\nabla E$ from [1, Lemmata 2.5 and 2.8] imply that the above integral is bounded by

$$\begin{aligned}&\int_{|x-y|>R/2} |\partial_t(\nabla E(x,t,\cdot,\cdot))(y,0)|\,dy \\ &\qquad\lesssim \sum_{k=0}^{\infty}(2^kR)^{n/2}\left(\int_{2^{k-1}R<|x-y|<2^kR}|\partial_t(\nabla E(x,t,\cdot,\cdot))(y,0)|^2\,dy\right)^{1/2} \\ &\qquad\lesssim \sum_{k=0}^{\infty}(2^kR)^{-1} \lesssim R^{-1}.\end{aligned}$$

Moreover, we have now shown that

$$|\partial_t\mathcal{D}_L 1(x,t)| \le |\partial_t\mathcal{D}_L\phi_R(x,t)| + |\partial_t\mathcal{D}_L(1-\phi_R)(x,t)| \lesssim R^{-1}$$

for all positive R sufficiently large, hence $\mathcal{D}_L 1(x,t)$ is constant in $t>0$, for each fixed $x\in\mathbb{R}^n$, as required. □

Part 2: estimates (1.13)-(1.16) in the case $2<p<p^+$. We begin by stating without proof the following variant of Gehring's lemma as established by Iwaniec [32] (see also [47, Proposition 11.2.3]).

Lemma 3.6. *Suppose that $g, h \in L^p(\mathbb{R}^n)$, with $1<p<\infty$, and that for some $C_0>0$ and for all cubes $Q\subset\mathbb{R}^n$,*

$$\left(\fint_Q g^p\right)^{1/p} \le C_0\fint_{4Q} g + \left(\fint_{4Q} h^p\right)^{1/p}. \tag{3.13}$$

Then there exists $s = s(n,p,C_0) > p$ and $C = C(n,p,C_0)>0$ such that

$$\int_{\mathbb{R}^n} g^s \le C\int_{\mathbb{R}^n} h^s.$$

Remark 3.7. If $1 < r < p$, by replacing g with $\tilde{g} := g^r$, h with $\tilde{h} := h^r$, and p with $\tilde{p} := \frac{p}{r}$, then the conclusion of the Lemma 3.6 holds provided (3.13) is replaced with

$$\left(\fint_Q g^p\right)^{1/p} \leq C_0 \left(\fint_{4Q} g^r\right)^{1/r} + \left(\fint_{4Q} h^p\right)^{1/p}.$$

In this case s also depends on r.

It will be convenient to set $\mathcal{S}_t f := \mathcal{S}_L^{\pm} f(\cdot, t)$ for all $\pm t > 0$, and $\mathcal{S}_0 f := S_L f$ (see (1.10)). We shall apply Remark 3.7 with $g := \nabla_x \mathcal{S}_{t_0} f$, where $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is arbitrary and $t_0 > 0$ is fixed, in the case $p = 2$ and $r = 2_* := 2n/(n+2)$. To be precise, we shall prove that for each fixed $t_0 > 0$, and for every cube $Q \subset \mathbb{R}^n$, we have

$$\left(\fint_Q |\nabla_x \mathcal{S}_{t_0} f|^2 \, dx\right)^{1/2} \lesssim \left(\fint_{4Q} |\nabla_x \mathcal{S}_{t_0} f|^{2_*} \, dx\right)^{1/2_*} + \left(\fint_{4Q} \Big(|f| + N_{**}(\partial_t \mathcal{S}_t f)\Big)^2\right)^{1/2}, \tag{3.14}$$

for all $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where N_{**} is the "two-sided" nontangential maximal operator

$$N_{**}(u)(x) := \sup_{\{(y,t) \in \mathbb{R}^{n+1} : |x-y| < |t|\}} |u(y,t)|.$$

We claim that the conclusion of Part 2 of Theorem 1.2 then follows. Indeed, for each fixed $t \in \mathbb{R}$, $\partial_t \mathcal{S}_t$ is a Calderón-Zygmund operator with a "standard kernel" $K_t(x,y) := \partial_t E(x,t,y,0)$, with Calderón-Zygmund constants that are uniform in t (see (3.1)-(3.2)). Thus, by the L^2 bound (2.1), we have from standard Calderón-Zygmund theory and a variant of the usual Cotlar inequality for maximal singular integrals that

$$\sup_{t>0} \|\partial_t \mathcal{S}_t f\|_p + \|N_{**}(\partial_t \mathcal{S}_t f)\|_p \leq C_p \|f\|_p, \qquad \forall\, p \in (1, \infty). \tag{3.15}$$

Consequently, if (3.14) holds for arbitrary $t_0 > 0$, then there exists $p^+ > 2$ such that

$$\sup_{t>0} \|\nabla \mathcal{S}_t f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, p \in (2, p^+),$$

for all $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, by Lemma 3.6 and Remark 3.7. It would then follow by density that $\nabla \mathcal{S}_t$ has a bounded extension to $L^p(\mathbb{R}^n)$, for all $p \in (2, p^+)$, with the same estimate, which would prove (1.13). We could then use Lemma 3.2 to obtain (1.14), and (1.15)-(1.16) would follow from Lemma 2.3 (i) and (ii), and another density argument. Thus, it is enough to prove (3.14).

To this end, let $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, fix a cube $Q \subset \mathbb{R}^n$ and split

$$f = f_1 + f_2 := f\, 1_{4Q} + f\, 1_{(4Q)^c}, \qquad u = u_1 + u_2 := \mathcal{S}_t f_1 + \mathcal{S}_t f_2.$$

Using (2.1), and the definition of f_1, we obtain

$$\sup_{t>0} \fint_Q |\nabla_x \mathcal{S}_t f_1(x)|^2 \, dx \lesssim \fint_{4Q} |f(x)|^2 \, dx. \tag{3.16}$$

Thus, to prove (3.14) it suffices to establish

$$\left(\fint_Q |\nabla_x u_2(x,t_0)|^2 \, dx\right)^{1/2} \lesssim \left(\fint_{4Q} |\nabla_x u(x,t_0)|^{2_*} \, dx\right)^{1/2_*} + \left(\fint_{4Q} \Big(|f| + N_{**}(\partial_t u)\Big)^2\right)^{1/2}. \tag{3.17}$$

To do this, we shall use the following result from [1].

Proposition 3.8 ([1, Proposition 2.1]). *Let L be as in (1.1)-(1.2). If A is t-independent, then there exists $C_0 > 0$, depending only on dimension and ellipticity, such that for all cubes $Q \subset \mathbb{R}^n$ and $s \in \mathbb{R}$ the following holds: if $Lu = 0$ in $4Q \times (s - \ell(Q), s + \ell(Q))$, then*

$$\frac{1}{|Q|}\int_Q |\nabla u(x,s)|^2 dx \leq C_0 \frac{1}{\ell(Q)^2}\frac{1}{|Q^{**}|}\iint_{Q^{**}} |u(x,t)|^2 dxdt,$$

where $Q^{**} := 3Q \times (s - \ell(Q)/2, s + \ell(Q)/2)$.

Applying Proposition 3.8 to $u_2 = \mathcal{S}_t(f 1_{(4Q)^c})$ (which is a solution of $Lu = 0$ in the infinite strip $4Q \times (-\infty, \infty)$), with $s = t_0$, we obtain

$$\begin{aligned}
\fint_Q |\nabla_x u_2(x,t_0)|^2\, dx &\lesssim \frac{1}{\ell(Q)^2}\fint_{t_0-\ell(Q)/2}^{t_0+\ell(Q)/2}\fint_{3Q} |u_2(x,t) - c_Q|^2 dxdt \\
&\lesssim \frac{1}{\ell(Q)^2}\fint_{t_0-\ell(Q)/2}^{t_0+\ell(Q)/2}\fint_{3Q} |u_2(x,t) - u_2(x,t_0)|^2 dxdt + \frac{1}{\ell(Q)^2}\fint_{3Q} |u_2(x,t_0) - c_Q|^2 dx \\
&\lesssim \frac{1}{\ell(Q)^2}\fint_{t_0-\ell(Q)/2}^{t_0+\ell(Q)/2}\fint_{3Q} \left|\int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \partial_s u_2(x,s)\, ds\right|^2 dxdt + \frac{1}{\ell(Q)^2}\fint_{3Q} |u_2(x,t_0) - c_Q|^2 dx \\
&\lesssim \fint_{3Q}\fint_{t_0-l(Q)/2}^{t_0+\ell(Q)/2} |\partial_s u_2(x,s)|^2 dsdx + \left(\fint_{3Q} |\nabla_x u_2(x,t_0)|^{2_*} dx\right)^{2/2_*} \\
&\lesssim \fint_{3Q} (N_{**}(\partial_t u_2)(x))^2 dx + \left(\fint_{3Q} |\nabla_x u_2(x,t_0)|^{2_*} dx\right)^{2/2_*} \\
&\lesssim \fint_{3Q} (N_{**}(\partial_t u)(x))^2 dx + \left(\fint_{3Q} |\nabla_x u(x,t_0)|^{2_*} dx\right)^{2/2_*} + \fint_{4Q} |f(x)|^2 dx,
\end{aligned}$$

where in the fourth line we have made an appropriate choice of c_Q in order to use Sobolev's inequality, and in the last line we wrote $u_2 = u - u_1$, and then used (3.15) with $p = 2$ to control $N_{**}(\partial_t u_1)$, and (3.16) to control $\nabla_x u_1$. Estimate (3.17) follows, and so the proofs of estimates (1.13)-(1.16) are now complete.

Part 3: proof of estimate (1.17)[6]. We shall actually prove a more general result. It will be convenient to use the following notation. For $f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$, we set

$$(3.18)\quad (\mathcal{S}\nabla)f(x,t) := \int_{\mathbb{R}^n} \Big(\nabla E(x,t,\cdot,\cdot)\Big)(y,0)\cdot f(y)\, dy\,, \qquad (x,t) \in \mathbb{R}^{n+1} \setminus ((\operatorname{supp} f) \times \{0\})\,,$$

where $\Big(\nabla E(x,t,\cdot,\cdot)\Big)(y,0) := \Big(\nabla_{y,s} E(x,t,y,s)\Big)\Big|_{s=0}$. For $t \neq 0$, the integral is absolutely convergent for all $x \in \mathbb{R}^n$ and all $f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$, by the L^2 decay estimates for ∇E in [1, Lemmata 2.5 and 2.8] (see (1.11)). Moreover, for $x \notin \operatorname{supp} f$, the integral is absolutely convergent with $t = 0$. We shall prove that

$$(3.19)\qquad \|\widetilde{N}_*((\mathcal{S}\nabla)f)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n,\mathbb{C}^{n+1})}, \qquad \forall\, p \in \left(\frac{p^+}{p^+ - 1}, \infty\right),$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) \cap L^\infty(\mathbb{R}^n, \mathbb{C}^{n+1})$, which clearly implies (1.17) by density and the definition of $\mathcal{D}$ in (1.8). Moreover, it is enough to work with $\widetilde{N}_*$ rather than N_*, since for solutions, the former controls the latter pointwise, for appropriate choices of aperture, by the Moser estimate (1.4).

[6] We are indebted to S. Mayboroda for suggesting this proof, which simplifies our original argument.

For all $t \in \mathbb{R} \setminus \{0\}$, using the notation $\mathcal{S}_{t,L} f := \mathcal{S}_L f(\cdot, t)$ and $(\mathcal{S}_t \nabla) f := (\mathcal{S}\nabla) f(\cdot, t)$, we have $\mathcal{S}_t \nabla = \operatorname{adj}(\nabla \mathcal{S}_{-t,L^*})$ on $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$. Thus by duality, since (1.13) holds for L^*, as well as for L, and in both half-spaces, we have

$$\text{(3.20)} \qquad \|1_{(\operatorname{supp} f)^c} (\mathcal{S}\nabla) f(\cdot, 0)\|_p + \sup_{t \in \mathbb{R}\setminus\{0\}} \|(\mathcal{S}\nabla) f(\cdot, t)\|_p \le C_p \|f\|_p , \qquad \forall\, p \in \left(\frac{p^+}{p^+ - 1}, \infty \right),$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) \cap L^\infty(\mathbb{R}^n, \mathbb{C}^{n+1})$, where Lemma 2.3 (ii) provides the bound at $t = 0$.

We now fix $p > p^+/(p^+ - 1)$, choose r so that $p^+/(p^+ - 1) < r < p$, and consider $f \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) \cap L^\infty(\mathbb{R}^n, \mathbb{C}^{n+1})$. Let $z \in \mathbb{R}^n$ and $(x, t) \in \Gamma(z)$. For each integer $k \ge 0$, set $\Delta_k := \{y \in \mathbb{R}^n : |y - z| < 2^{k+2} t\}$ and write

$$f = f\, 1_{\Delta_0} + \sum_{k=1}^{\infty} f\, 1_{\Delta_k \setminus \Delta_{k-1}} =: f_0 + \sum_{k=1}^{\infty} f_k =: f_0 + f^\star .$$

Likewise, set $u := (\mathcal{S}\nabla) f$, $u_k := (\mathcal{S}\nabla) f_k$, and $u^\star := (\mathcal{S}\nabla) f^\star$. Also, let $B_{x,t} := B((x,t), t/4)$, $\widetilde{B}_{x,t} := B((x,t), t/2)$ and $B^k_{x,t} := B((x,t), 2^k t)$. Using the Moser estimate (1.5), we have

$$\begin{aligned}
\left(\fint\!\!\fint_{B_{x,t}} |u|^2 \right)^{1/2} &\lesssim \left(\fint\!\!\fint_{\widetilde{B}_{x,t}} |u|^r \right)^{1/r} \\
&\lesssim \left(\fint\!\!\fint_{\widetilde{B}_{x,t}} |u_0|^r \right)^{1/r} + \left(\fint\!\!\fint_{\widetilde{B}_{x,t}} |u^\star(y,s) - u^\star(y,0)|^r \, dyds \right)^{1/r} + \left(\fint_{\Delta_0} |u^\star(\cdot, 0)|^r \right)^{1/r} \\
&=: I + II + III .
\end{aligned}$$

By (3.20), we have

$$I \lesssim \left(\fint_{\Delta_0} |f|^r \right)^{1/r} .$$

To estimate II, note that for all $(y, s) \in B^k_{x,t}$, including when $s = 0$, we have

$$u_k(y, s) := (\mathcal{S}\nabla) f_k(y, s) = \int_{\mathbb{R}^n} \Big(\nabla E(y, s, \cdot, \cdot) \Big)(z, 0) \cdot f_k(z)\, dz ,$$

since $\operatorname{supp} f_k \subset \Delta_k \setminus \Delta_{k-1}$ does not intersect $B^k_{x,t}$. Thus, $L u_k = 0$ in $B^k_{x,t}$, and so by the De Giorgi/Nash estimate (1.3), followed by (1.5) and (3.20), we obtain

$$II \le \sum_{k=1}^{\infty} \sup_{\widetilde{B}_{x,t}} |u_k(y,s) - u_k(y,0)| \lesssim \sum_{k=1}^{\infty} 2^{-\alpha k} \left(\fint\!\!\fint_{B^k_{x,t}} |u_k|^r \right)^{1/r} \lesssim \sum_{k=1}^{\infty} 2^{-\alpha k} \left(\fint_{\Delta_k} |f|^r \right)^{1/r} .$$

We also have $III \le [M(|1_{\Delta_0} u^\star(\cdot, 0)|^r)(z)]^{1/r}$, where M denotes the usual Hardy-Littlewood maximal operator. Altogether, after taking the supremum over $(x, t) \in \Gamma(z)$, we obtain

$$\text{(3.21)} \qquad \widetilde{N}_*((\mathcal{S}\nabla) f)(z) \lesssim [M(|f|^r)(z)]^{1/r} + \Big[M\Big(|1_{\Delta_0} (\mathcal{S}\nabla)(1_{(\Delta_0)^c} f)(\cdot, 0)|^r \Big)(z) \Big]^{1/r}$$

for all $z \in \mathbb{R}^n$. Estimate (3.19), and thus (1.17), then follow readily from (3.21) and (3.20).

Part 4: proof of estimate (1.18). We first recall the following square function estimate, whose proof is given in [29, Section 3]. For all $f \in L^2(\mathbb{R}^n)$, it holds that

$$\iint_{\mathbb{R}^{n+1}_+} |t\, \nabla (\mathcal{S}\nabla) f(x,t)|^2 \, \frac{dxdt}{t} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)} ,$$

where $\mathcal{S}\nabla$ is defined in (3.18). Thus, by the definition of $\mathcal{D}$ in (1.8), this implies that

$$\iint_{\mathbb{R}^{n+1}_+} |t\,\nabla\mathcal{D}f(x,t)|^2\,\frac{dxdt}{t} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}\,, \tag{3.22}$$

for all $f \in L^2(\mathbb{R}^n)$.

We now proceed to prove (1.18) for $f \in BMO(\mathbb{R}^n)$. The definition of $\mathcal{D}f$ in this context is understood via duality, as explained in the proof of Corollary 3.4. The proof follows a classical argument from [20] with some necessary modifications. Fix a cube Q, set $Q_0 := 32Q$, split $f = f_0 + f_1 =: f1_{Q_0} + f1_{(Q_0)^c}$, and assume that $f_{Q_0} := \fint_{Q_0} f = 0$, which we may do without loss of generality, since $\nabla\mathcal{D}1 = 0$ by Corollary 3.4.

By (3.22), since $f_0 \in L^2(\mathbb{R}^n)$, we have

$$\int_0^{\ell(Q)}\int_Q |t\,\nabla\mathcal{D}f_0(x,t)|^2\,\frac{dxdt}{t} \lesssim \int_{Q_0}|f|^2 = \int_{Q_0}|f-f_{Q_0}|^2 \lesssim |Q|\,\|f\|^2_{BMO(\mathbb{R}^n)}\,. \tag{3.23}$$

By Caccioppoli's inequality, since $L(\mathcal{D}f_1) = 0$ in $4Q \times (-4\ell(Q), 4\ell(Q))$, we have

$$\int_0^{\ell(Q)}\int_Q |t\,\nabla\mathcal{D}f_1(x,t)|^2\,\frac{dxdt}{t} \lesssim \fint_{-\ell(Q)}^{2\ell(Q)}\int_{2Q} |\mathcal{D}f_1(x,t) - c_Q|^2\,dxdt\,, \tag{3.24}$$

where the constant c_Q is at our disposal. We now apply the principle in Remark 3.5, and choose $c_Q := \mathcal{D}f_1(x_Q, t_Q)$, where x_Q denotes the center of Q, and t_Q denotes the center of the interval $(-\ell(Q), 2\ell(Q))$, to obtain

$$\begin{aligned}
&|\mathcal{D}f_1(x,t) - c_Q| \\
&\quad = \left|\int_{\mathbb{R}^n} -e_{n+1}\cdot\overline{A^*(y)\nabla_{y,s}(E(x,t,y,s) - E(x_Q,t_Q,y,s))}|_{s=0}\,f_1(y)\,dy\right| \\
&\quad \lesssim \sum_{k=1}^{\infty}\left(\int_{2^{k+1}Q_0\setminus 2^kQ_0} \left|\nabla_{y,s}(E(x,t,y,s) - E(x_Q,t_Q,y,s))|_{s=0}\right|^2\,dy\right)^{1/2} \|1_{2^{k+1}Q_0}f\|_2 \\
&\quad \lesssim \sum_{k=1}^{\infty} 2^{-\alpha k}\left(\fint_{2^{k+1}Q_0}|f|^2\right)^{1/2} = \sum_{k=1}^{\infty} 2^{-\alpha k}\left(\fint_{2^{k+1}Q_0}|f-f_{Q_0}|^2\right)^{1/2} \\
&\quad \lesssim \left(\sum_{k=1}^{\infty} k\,2^{-\alpha k}\right)\|f\|_{BMO(\mathbb{R}^n)} \lesssim \|f\|_{BMO(\mathbb{R}^n)}\,,
\end{aligned} \tag{3.25}$$

for all $(x,t) \in 2Q \times (-\ell(Q), 2\ell(Q))$, where we used the monotone convergence theorem and the Cauchy-Schwarz inequality in the third line, Lemma 3.1 in the fourth line, and the telescoping argument of [20] in the fifth line. We obtain (1.18) by combining (3.23), (3.24) and (3.25).

Part 5: proof of estimate (1.19). Let $f \in \dot{C}^\beta(\mathbb{R}^n)$ for $\beta \in (0,\alpha)$. The definition of $\mathcal{D}f$ and Kf in this context is understood via duality, as explained in the proof of Corollary 3.4. It remains to prove that $\|\mathcal{D}f\|_{\dot{C}^\beta(\mathbb{R}^{n+1}_+)} \le C_\beta\|f\|_{\dot{C}^\beta(\mathbb{R}^n)}$, since then $\mathcal{D}f$ has an extension in $\dot{C}^\beta(\overline{\mathbb{R}^{n+1}_+})$ with the same estimate. In fact, Corollary 3.9 below will show that the extension must satisfy $\mathcal{D}^\pm f(\cdot,0) = (\mp\frac{1}{2}I + K)f$.

For $\tau > 0$, let $I := Q \times [\tau, \tau + \ell(Q)]$ denote any $(n+1)$-dimensional cube contained in $\mathbb{R}^{n+1}_+$, where as usual, Q is a cube in $\mathbb{R}^n$. By the well-known criterion of N. Meyers [43], it is enough to show that for every such I, there is a constant c_I such that

$$\frac{1}{|I|}\iint_I \frac{|\mathcal{D}f(x,t)-c_I|}{\ell(I)^\beta}\,dxdt \,\le\, C_\beta\,\|f\|_{\dot{C}^\beta(\mathbb{R}^n)}\,, \tag{3.26}$$

for some constant C_β depending only on β and the standard constants. To do this, we make the same splitting $f = f_0 + f_1 =: f1_{Q_0} + f1_{(Q_0)^c}$ as in Part 4 above and assume that $f_{Q_0} := \fint_{Q_0} f = 0$, which we may do without loss of generality, since $\nabla\mathcal{D}1 = 0$ by Corollary 3.4.

By (1.17), since $f_0 \in L^2(\mathbb{R}^n)$, we have

$$\sup_{t>0}\|\mathcal{D}f_0(\cdot,t)\|_{L^2(\mathbb{R}^n)} \lesssim \|f_0\|_{L^2(\mathbb{R}^n)}\,.$$

Consequently, since $\ell(I) = \ell(Q)$, we have

$$\begin{aligned}\frac{1}{|I|}\iint_I \frac{|\mathcal{D}f_0(x,t)|}{\ell(I)^\beta}\,dxdt \,&\le\, \frac{1}{\ell(Q)^\beta}\sup_{t>0}\fint_Q |\mathcal{D}f_0(x,t)|\,dx\\ &\lesssim\, \ell(Q)^{-\beta}\left(\fint_{Q_0}|f|^2\right)^{1/2} \,=\, \ell(Q)^{-\beta}\left(\fint_{Q_0}|f-f_{Q_0}|^2\right)^{1/2} \,\le\, \|f\|_{\dot{C}^\beta(\mathbb{R}^n)}\,.\end{aligned} \tag{3.27}$$

We now apply the principle in Remark 3.5, and choose $c_I := \mathcal{D}f_1(x_Q,t_I)$, where x_Q denotes the center of Q, and t_I denotes the center of the interval $(\tau,\tau+\ell(Q))$, to obtain

$$\begin{aligned}\frac{|\mathcal{D}f_1(x,t)-c_I|}{\ell(I)^\beta} \,&\lesssim\, \frac{1}{\ell(Q)^\beta}\sum_{k=1}^{\infty}2^{-\alpha k}\left(\fint_{2^{k+1}Q_0}|f|^2\right)^{1/2}\\ &\lesssim\, \left(\sum_{k=1}^{\infty}k\,2^{-(\alpha-\beta)k}\right)\|f\|_{\dot{C}^\beta(\mathbb{R}^n)} \,\le\, C_\beta\,\|f\|_{\dot{C}^\beta(\mathbb{R}^n)}\,,\end{aligned} \tag{3.28}$$

for all $(x,t)\in I$, as in (3.25) but with t_I replacing t_Q. We obtain (3.26) by combining (3.27) and (3.28), which in turn proves (1.19) and completes the proof of Theorem 1.2. $\square$

We conclude this section with the following immediate corollary of Theorem 1.2. We will obtain more refined versions of some of these convergence results in Section 5.

Corollary 3.9. *Suppose that L and L^* satisfy the standard assumptions, let α denote the minimum of the De Giorgi/Nash exponents for L and L^* in (1.3), set $p_\alpha := n/(n+\alpha)$ and let $p^+ > 2$ be as in Theorem 1.2.*

If $1 < p < p^+$ and $p^+/(p^+-1) < q < \infty$, then for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, one has

- *(i) $-\langle A\nabla\mathcal{S}_L^\pm f(\cdot,t), e_{n+1}\rangle \to \left(\pm\frac{1}{2}I + \widetilde{K}_L\right)f$ weakly in L^p as $t\to 0^\pm$.*
- *(ii) $\nabla\mathcal{S}_L^\pm f(\cdot,t) \to \left(\mp\frac{1}{2A_{n+1,n+1}}e_{n+1} + \mathbf{T}_L\right)f$ weakly in L^p as $t\to 0^\pm$.*
- *(iii) $\mathcal{D}_L^\pm g(\cdot,t) \to \left(\mp\frac{1}{2}I + K_L\right)g$ weakly in L^q as $t\to 0^\pm$.*

If $p_\alpha < p \le 1$ and $0 \le \beta < \alpha$, then for all $f \in H^p(\mathbb{R}^n)$ and $g \in \Lambda^\beta(\mathbb{R}^n)$, one has

- *(iv) $-\langle A\nabla\mathcal{S}_L^\pm f(\cdot,t), e_{n+1}\rangle \to \left(\pm\frac{1}{2}I + \widetilde{K}_L\right)f$ in the sense of tempered distributions as $t\to 0^\pm$.*
- *(v) $\nabla_x\mathcal{S}_L^\pm f(\cdot,t) \to \nabla_x S_L f$ in the sense of tempered distributions as $t\to 0^\pm$.*
- *(vi) $\mathcal{D}_L^\pm g(\cdot,t) \to \left(\mp\frac{1}{2}I + K_L\right)g$ in the weak* topology on $\Lambda^\beta(\mathbb{R}^n)$, $0\le\beta<\alpha$, as $t\to 0^\pm$. Moreover, if $0<\beta<\alpha$, then $\mathcal{D}_L^\pm g(\cdot,0) = \left(\mp\frac{1}{2}I + K_L\right)g$ in the sense of $\dot{C}^\beta(\mathbb{R}^n)$.*

If $p_\alpha < p < p^+$, then for all $f \in H^p(\mathbb{R}^n)$, one has *(vii)* $\mathcal{S}_L^{\pm} f(\cdot, t) \to S_L f$ *in the sense of tempered distributions (modulo constants in the case* $n \le p < p^+$*) as* $t \to 0^{\pm}$.

The analogous results hold for L^*.

Proof. Items (i)-(v) of the corollary follow immediately from Theorem 1.2, Proposition 3.3, Lemma 2.3, and the fact that $L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$ for $0 < p < \infty$. We omit the details. Item (vii) is "elementary" in the case $p > 1$, by (3.1)-(3.2). The case $p_\alpha < p \le 1$ follows readily from the case $p = 2$, the density of $L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ in $H^p(\mathbb{R}^n)$, Proposition 3.3, and the bound

$$\|S_L f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}\,, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}\,.$$

The latter holds for all $f \in H^p(\mathbb{R}^n)$, since the kernel estimates (3.1)-(3.2), and the fact that $p > n/(n+\alpha)$, allow one to follow the proof, *mutatis mutandi*, of the analogous estimate for I_1 (see, e.g., [52, III.5.21]). Again we omit the routine details.

We prove item (vi) as follows, treating only layer potentials for L in $\mathbb{R}^{n+1}_+$, as the proofs for L^* and in $\mathbb{R}^{n+1}_-$ are the same. We recall that $\Lambda^\beta(\mathbb{R}^n) = (H^p_{\mathrm{at}}(\mathbb{R}^n))^*$, with $p = n/(n+\beta)$ (so that, in particular, $n/(n+1) < p \le 1$). It is therefore enough to prove that

$$\langle \mathcal{D}_L g(\cdot, t), a\rangle \xrightarrow{t\to 0^+} \Big\langle \big((-1/2)I + K_L\big) g, a\Big\rangle\,, \tag{3.29}$$

where a is an $H^p(\mathbb{R}^n)$-atom supported in a cube $Q \subset \mathbb{R}^n$ as in (3.3), and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\Lambda^\beta(\mathbb{R}^n)$ and $H^p_{\mathrm{at}}(\mathbb{R}^n)$. Using Corollary 3.4 and dualizing, we have

$$\begin{aligned}
\langle \mathcal{D}_L g(\cdot, t), a\rangle &= \langle \mathcal{D}_L (g - g_Q)(\cdot, t), a\rangle \\
&= \langle g - g_Q, \partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot, -t)\rangle \\
&= \int_{\mathbb{R}^n} (g - g_Q)\, \partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot, -t) \\
&= \int_{\lambda Q} (g - g_Q)\, \partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot, -t) \;+\; \int_{(\lambda Q)^c} (g - g_Q)\, \partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot, -t) \\
&=: I_t(\lambda) + II_t(\lambda)\,,
\end{aligned}$$

where $g_Q := \fint_Q g$, and $\lambda > 0$ is at our disposal. We note that the passage to an absolutely convergent integral in the third equality above is justified because $\partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot, -t)$ is a constant multiple of an $H^p(\mathbb{R}^n)$-molecule, as can be seen from the proof of Proposition 3.3. By Lemma 2.3, since $a \in L^2(\mathbb{R}^n)$ and $1_{\lambda Q}(g - g_Q) \in L^2(\mathbb{R}^n)$, we have

$$I_t(\lambda) \xrightarrow{t\to 0^+} \int_{\lambda Q} (g - g_Q)\, ((-1/2)I + \widetilde{K}_{L^*}) a.$$

Setting $R_j := 2^{j+1}\lambda Q \setminus 2^j \lambda Q$, we have

$$\begin{aligned}
|II_t(\lambda)| &\le \sum_{j=0}^{\infty} \|\partial_{\nu^*} \mathcal{S}_{L^*} a(\cdot, -t)\|_{L^2(R_j)}\, \|g - g_Q\|_{L^2(R_j)} \\
&\lesssim \sum_j (2^j\lambda)^{-\alpha - \frac{n}{2}} |Q|^{\frac{1}{2} - \frac{1}{p}}\, \|g - g_Q\|_{L^2(R_j)} \;\lesssim\; \sum_j (2^j\lambda)^{(\beta-\alpha)/2} \|g\|_{\Lambda^\beta(\mathbb{R}^n)} \approx \lambda^{(\beta-\alpha)/2} \|g\|_{\Lambda^\beta(\mathbb{R}^n)},
\end{aligned}$$

where in the second and third inequalities, we used (3.4) and then a telescoping argument. We observe that the bound (3.4) is uniform in t, so that

$$\lim_{\lambda\to\infty} II_t(\lambda) = 0\,, \quad \text{uniformly in } t\,.$$

Similarly, using Corollary 3.4 and the fact that $((-1/2)I + \widetilde{K}_{L^*})a$ is a constant multiple of an $H^p(\mathbb{R}^n)$-molecule, as can be seen from the proof of Proposition 3.3, we have

$$\Big\langle \big((-1/2)I + K_L\big)g, a\Big\rangle = \int_{\mathbb{R}^n} (g - g_Q)\,((-1/2)I + \widetilde{K}_{L^*})a.$$

Moreover, by (3.8) (iii) with $m := ((-1/2)I + \widetilde{K}_{L^*})a$, we have

$$\left|\int_{(\lambda Q)^c} (g - g_Q)\,((-1/2)I + \widetilde{K}_{L^*})a\right| \lesssim \lambda^{(\beta-\alpha)/2}\|g\|_{\Lambda^\beta(\mathbb{R}^n)} \xrightarrow{\lambda\to\infty} 0\,,$$

from which (3.29) follows. $\square$

4. Solvability via the method of layer potentials: Proof of Theorem 1.4

The case $p = 2$ of Theorem 1.4 was proved in [1] via the method of layer potentials. We shall now use Theorem 1.2 and perturbation techniques to extend that result to the full range of indices stated in Theorem 1.4.

Proof of Theorem 1.4. Let $L := -\operatorname{div} A\nabla$ and $L_0 := -\operatorname{div} A_0\nabla$ be as in (1.1) and (1.2), with A and A_0 both t-independent, and suppose that A_0 is real symmetric. Let $\varepsilon_0 > 0$ and suppose that $\|A - A_0\|_\infty < \varepsilon_0$. We suppose henceforth that $\varepsilon_0 > 0$ is small enough (but not yet fixed) so that, by Remark 1.1, every operator $L_\sigma := (1-\sigma)L_0 + \sigma L$, $0 \le \sigma \le 1$, along with its Hermitian adjoint, satisfies the standard assumptions, with uniform control of the "standard constants". For the remainder of this proof, we will let ϵ denote an arbitrary small positive number, not necessarily the same at each occurrence, but ultimately depending only on the standard constants and the perturbation radius ε_0.

Let us begin with the Neumann and Regularity problems. As mentioned in the introduction, for real symmetric coefficients (the case $A = A_0$), solvability of $(N)_p$ and $(R)_p$ was obtained in [38] in the range $1 \le p < 2 + \epsilon$. Moreover, although not stated explicitly in [38], the methods of that paper provide the analogous Hardy space results in the range $1 - \epsilon < p < 1$, but we shall not use this fact here. We begin with two key observations.

Our first observation is that, by Theorem 1.2 and analytic perturbation theory,

$$\begin{aligned}\|(\widetilde{K}_L - \widetilde{K}_{L_0})f\|_{H^p(\mathbb{R}^n)} &+ \|\nabla_x(\mathcal{S}_L - \mathcal{S}_{L_0})\,f\|_{H^p(\mathbb{R}^n)} \\ &\le C_p\,\|A - A_0\|_{L^\infty(\mathbb{R}^n)}\,\|f\|_{H^p(\mathbb{R}^n)}\,, \quad 1-\epsilon < p < 2+\epsilon\,.\end{aligned} \tag{4.1}$$

We may verify (4.1) by following the arguments in [26, Section 9].

Our second observation is that we have the pair of estimates

$$\|f\|_{H^p(\mathbb{R}^n)} \le C_p\,\|((\pm 1/2)I + \widetilde{K}_{L_0})f\|_{H^p(\mathbb{R}^n)}\,, \quad 1 \le p < 2+\epsilon\,, \tag{4.2}$$

and

$$\|f\|_{H^p(\mathbb{R}^n)} \le C_p\,\|\nabla_x \mathcal{S}_{L_0} f\|_{H^p(\mathbb{R}^n)}\,, \quad 1 \le p < 2+\epsilon\,. \tag{4.3}$$

We verify (4.2) and (4.3) by using Verchota's argument in [54] as follows. First, it is enough to establish these estimates for $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$, which is dense in $H^p(\mathbb{R}^n)$. Then, by the triangle inequality, we have

$$
\begin{aligned}
C_p \|f\|_{H^p(\mathbb{R}^n)} &\leq \|((1/2)I + \widetilde{K}_{L_0})f\|_{H^p(\mathbb{R}^n)} + \|((1/2)I - \widetilde{K}_{L_0})f\|_{H^p(\mathbb{R}^n)} \\
&= \|\partial_\nu u_0^+\|_{H^p(\mathbb{R}^n)} + \|\partial_\nu u_0^-\|_{H^p(\mathbb{R}^n)}\,,
\end{aligned} \tag{4.4}
$$

where $u_0^\pm := \mathcal{S}_{L_0}^\pm f$, and we have used the jump relation formula in Lemma 2.3 (i). Moreover, by the solvability of $(N)_p$ and $(R)_p$ in [38], which we apply in both the upper and lower half-spaces, and the fact that the tangential gradient of the single layer potential does not jump across the boundary, we have that

$$
\begin{aligned}
\|\partial_\nu u_0^+\|_{H^p(\mathbb{R}^n)} &\approx \|\nabla_x u_0^+(\cdot,0)\|_{H^p(\mathbb{R}^n)} \\
&= \|\nabla_x u_0^-(\cdot,0)\|_{H^p(\mathbb{R}^n)} \approx \|\partial_\nu u_0^-\|_{H^p(\mathbb{R}^n)}\,, \quad 1 \leq p < 2+\epsilon\,,
\end{aligned}
$$

where the implicit constants depend only on p, n and ellipticity. Combining the latter estimate with (4.4), we obtain (4.2) and (4.3).

With (4.2) and (4.3) in hand, we obtain invertibility of the mappings

$$
(\pm 1/2)I + \widetilde{K}_{L_0} : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)\,, \text{ and } S_{L_0} : H^p(\mathbb{R}^n) \to \dot{H}_1^p(\mathbb{R}^n)\,,
$$

by a method of continuity argument which connects L_0 to the Laplacian $-\Delta$ via the path $\tau \to L_\tau := (1-\tau)(-\Delta) + \tau L_0$, $0 \leq \tau \leq 1$. Indeed, the "standard constants" are uniform for every L_τ in the family, so we have the analogue of (4.1), with L and L_0 replaced by L_{τ_1} and L_{τ_2}, for any $\tau_1, \tau_2 \in [0,1]$. We omit the details.

We now fix $\varepsilon_0 > 0$ small enough, depending on the constants in (4.1)-(4.3), so that (4.2) and (4.3) hold with L in place of L_0. Consequently, by another method of continuity argument, in which we now connect L to L_0, via the path $\sigma \mapsto L_\sigma := (1-\sigma)L_0 + \sigma L$, $0 \leq \sigma \leq 1$, and use that (4.1) holds not only for L and L_0, but also uniformly for any intermediate pair L_{σ_i} and L_{σ_2}, we obtain invertibility of the mappings

$$
(\pm 1/2)I + \widetilde{K}_L : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)\,, \text{ and } S_L : H^p(\mathbb{R}^n) \to \dot{H}_1^p(\mathbb{R}^n)\,, \tag{4.5}
$$

initially in the range $1 \leq p < 2+\epsilon$. Again we omit the routine details. Moreover, since (1.15)-(1.16) hold in the range $n/(n+\alpha) < p < p^+$, we apply the extension of Sneiberg's Theorem obtained in [35] to deduce that the operators in (4.5) are invertible in the range $1-\epsilon < p < 2+\epsilon$. We then apply the extension of the open mapping theorem obtained in [46, Chapter 6], which holds on quasi-Banach spaces, to deduce that the inverse operators are bounded.

At this point, we may construct solutions of $(N)_p$ and $(R)_p$ as follows. Given Neumann data $g \in H^p(\mathbb{R}^n)$, or Dirichlet data $f \in \dot{H}^{1,p}(\mathbb{R}^n)$, with $1-\epsilon < p < 2+\epsilon$, we set

$$
u_N := \mathcal{S}_L \left((1/2)I + \widetilde{K}_L\right)^{-1} g\,, \qquad u_R := \mathcal{S}_L(S_L^{-1} f)
$$

and observe that u_N then solves $(N)_p$, and u_R solves $(R)_p$, by (1.14), the invertibility of $(1/2)I + \widetilde{K}_L$ and of S_L (respectively), and Corollary 3.9 (which guarantees weak convergence to the data; we defer momentarily the matter of non-tangential convergence).

Next, we consider the Dirichlet problem. Since the previous analysis also applies to L^*, we dualize our estimates for $(\pm 1/2)I + \widetilde{K}_{L^*}$ to obtain that $(\pm 1/2)I + K_L$ is bounded and invertible from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $2-\epsilon < q < \infty$, and from $\Lambda^\beta(\mathbb{R}^n)$ to $\Lambda^\beta(\mathbb{R}^n)$, $0 \leq \beta < \epsilon$.

Given Dirichlet data f in $L^q(\mathbb{R}^n)$ or $\Lambda^\beta(\mathbb{R}^n)$, in the stated ranges, we then construct the solution to the Dirichlet problem by setting

$$u := \mathcal{D}_L\Big((-1/2)I + K_L\Big)^{-1} f\,,$$

which solves $(D)_q$ (at least in the sense of weak convergence to the data) or $(D)_{\Lambda^\beta}$, by virtue of (1.17)-(1.19) and Corollary 3.9.

We note that, at present, our solutions to $(N)_p$, $(R)_p$, and $(D)_q$ assume their boundary data in the weak sense of Corollary 3.9. In the next section, however, we establish some results of Fatou type (see Lemmata 5.1, 5.2 and 5.3), which allow us to immediately deduce the stronger non-tangential and norm convergence results required here.

It remains to prove that our solutions to $(N)_p$ and $(R)_p$ are unique among the class of solutions satisfying $\widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n)$, and that our solutions to $(D)_q$ (resp. $(D)_{\Lambda^\beta}$) are unique among the class of solutions satisfying $N_*(u) \in L^q(\mathbb{R}^n)$ (resp. $t\nabla u \in T^\infty_2(\mathbb{R}^{n+1}_+)$, if $\beta = 0$, or $u \in \dot{C}^\beta(\mathbb{R}^n)$, if $\beta > 0$). We refer the reader to [29, Proposition 8.19] for a proof of the uniqueness for $(N)_p$, $(R)_p$ and $(D)_q$ for a more general class of operators. Also, we refer the reader to [10, Corollary 3.21] for a proof of the uniqueness for $(D)_{\Lambda^\beta}$ in the case $\beta > 0$. Finally, the uniqueness for $(D)_{\Lambda^0} = (D)_{BMO}$ follows by combining Theorem 1.2 and Corollaries 3.4 and 3.9 with the uniqueness result in Proposition 4.1 below (see also Remark 4.2). □

We conclude this section by proving a uniqueness result for $(D)_{\Lambda^0} = (D)_{BMO}$.

Proposition 4.1. *Suppose that the hypotheses of Theorem 1.4 hold. If $Lu = 0$ in $\mathbb{R}^{n+1}_+$ with*

$$\sup_{t>0} \|u(\cdot,t)\|_{BMO(\mathbb{R}^n)} < \infty, \tag{4.6}$$

$$\|u\|_{BMO(\mathbb{R}^{n+1}_+)} < \infty, \tag{4.7}$$

and $u(\cdot,t) \to 0$ in the weak topology on $BMO(\mathbb{R}^n)$ as $t \to 0^+$, then $u = 0$ in $\mathbb{R}^{n+1}_+$ in the sense of $BMO(\mathbb{R}^n)$. The analogous results hold for L^* and in the lower half-space.*

Remark 4.2. We note that (4.7) follows from the Carleson measure estimate

$$\sup_Q \frac{1}{|Q|} \iint_{R_Q} |\nabla u(x,t)|^2\, t\, dxdt < \infty\,, \tag{4.8}$$

by the Poincaré inequality of [31], but we will not make explicit use of (4.8) in the proof of Proposition 4.1.

Proof. For each $\varepsilon > 0$, we set $u_\varepsilon(x,t) := u(x,t+\varepsilon)$ and $f_\varepsilon := u_\varepsilon(\cdot,0) = u(\cdot,\varepsilon)$. First, note that $u_\varepsilon \in \dot{C}^\beta(\overline{\mathbb{R}^{n+1}_+})$ for $0 < \beta \leq \alpha$, with a bound that depends on ε, where α is the De Giorgi/Nash exponent for L in (1.3). To see this, we use the "mean oscillation" characterization of $\dot{C}^\beta$ due to N. Meyers (see (3.26)). In particular, for $(n+1)$-dimensional boxes I with side length $\ell(I) \leq \varepsilon/2$, we use the DG/N estimate (1.3), and for boxes with $\ell(I) \geq \varepsilon/2$, we use (4.7). We omit the routine details. By the uniqueness of $(D)_{\Lambda^\beta}$ for $\beta > 0$ (see [10]), we must then have

$$u_\varepsilon(\cdot,t) = \mathcal{P}_t f_\varepsilon := \mathcal{D}_L((-1/2)I + K_L)^{-1} f_\varepsilon, \qquad \forall\, \varepsilon > 0\,. \tag{4.9}$$

Next, by (4.6), we have $\sup_{\varepsilon>0} \|f_\varepsilon\|_{BMO(\mathbb{R}^n)} < \infty$, and so there exists a subsequence f_{ε_k} converging in the weak* topology on $BMO(\mathbb{R}^n)$ to some f in $BMO(\mathbb{R}^n)$. Let g denote a finite linear combination of H^1-atoms, and for each $t > 0$, set $g_t := \mathrm{adj}(\mathcal{P}_t)g$, where adj denotes the n-dimensional Hermitian adjoint. Let $\langle\cdot,\cdot\rangle$ denote the (complex-valued) dual pairing between $BMO(\mathbb{R}^n)$ and $H^1_{\rm at}(\mathbb{R}^n)$. Then, since $\mathrm{adj}(\mathcal{P}_t)$ is bounded on $H^1(\mathbb{R}^n)$, uniformly in $t>0$, by Theorem 1.2, we have

$$\int_{\mathbb{R}^n}(\mathcal{P}_t f)\,\overline{g} = \langle f\,,\, g_t\rangle = \lim_{k\to\infty}\langle f_{\varepsilon_k}\,,\, g_t\rangle$$
$$= \lim_{k\to\infty}\int_{\mathbb{R}^n}(\mathcal{P}_t f_{\varepsilon_k})\,\overline{g} = \lim_{k\to\infty}\int_{\mathbb{R}^n} u(\cdot, t+\varepsilon_k)\,\overline{g} = \int_{\mathbb{R}^n} u(\cdot,t)\,\overline{g}\,,$$

where in the next-to-last step we used (4.9), and in the last step we used the DG/N estimate (1.3), and the fact that g is a finite linear combination of atoms. Since g was an arbitrary element of a dense subset of $H^1(\mathbb{R}^n)$, this shows that $u(\cdot,t)=\mathcal{P}_t f$.

Now, since $u(\cdot,t)=\mathcal{P}_t f$ for some f in $BMO(\mathbb{R}^n)$, by Corollary 3.9, we have $u(\cdot,t)\to f$ in the weak* topology as $t\to 0^+$. On the other hand, we assumed that $u(\cdot,t)\to 0$ in the weak* topology, thus $f=0$, and so $u(\cdot,t)=\mathcal{P}_t f\; = 0$ in the sense of $BMO(\mathbb{R}^n)$. □

5. Boundary behavior of solutions

In this section, we present some results of "Fatou-type", which show that Theorem 1.4 is optimal, in the sense that, necessarily, the data must belong to the stated space, in order to obtain the desired quantitative estimate for the solution or its gradient. The results also show that in some cases, our solutions enjoy convergence to the data in a stronger sense than that provided by Corollary 3.9. The results are contained in three lemmata. The first two results below are of an a priori nature and pertain to the Neumann and Regularity problems.

Lemma 5.1. *Let $n/(n+1)<p<\infty$. Suppose that L and L^* satisfy the standard assumptions. If $Lu=0$ in $\mathbb{R}^{n+1}_\pm$ and $\widetilde{N}^\pm_*(\nabla u)\in L^p(\mathbb{R}^n)$, then the co-normal derivative $\partial_\nu u(\cdot,0)$ exists in the variational sense and belongs to $H^p(\mathbb{R}^n)$, i.e., there exists a unique $g\in H^p(\mathbb{R}^n)$, and we set $\partial_\nu u(\cdot,0):=g$, with*

$$\|g\|_{H^p(\mathbb{R}^n)}\leq C_p\,\|\widetilde{N}^\pm_*(\nabla u)\|_{L^p(\mathbb{R}^n)}\,, \tag{5.1}$$

such that

$$\int_{\mathbb{R}^{n+1}_\pm} A\nabla u\cdot\nabla\Phi\, dX = \pm\langle g\,,\Phi(\cdot,0)\rangle\,,\qquad \forall\,\Phi\in C_0^\infty(\mathbb{R}^{n+1})\,, \tag{5.2}$$

where $\langle g\,,\Phi(\cdot,0)\rangle := \int_{\mathbb{R}^n} g(x)\,\Phi(x,0)\,dx$, if $p\geq 1$, and $\langle g\,,\Phi(\cdot,0)\rangle$ denotes the usual pairing of the distribution g with the test function $\Phi(\cdot,0)$, if $p<1$. Moreover, there exists a unique $f\in \dot{H}^{1,p}(\mathbb{R}^n)$, and we set $u(\cdot,0):=f$, with

$$\|f\|_{\dot{H}^{1,p}(\mathbb{R}^n)}\leq C_p\|\widetilde{N}^\pm_*(\nabla u)\|_{L^p(\mathbb{R}^n)}\,, \tag{5.3}$$

such that $u\to f$ non-tangentially.

Lemma 5.2. *Suppose that L and L^* satisfy the standard assumptions. Suppose that $Lu=0$ in $\mathbb{R}^{n+1}_\pm$ and $\widetilde{N}^\pm_*(\nabla u)\in L^p(\mathbb{R}^n)$ for some $n/(n+1)<p<\infty$. Then there exists $\epsilon>0$, depending only on the standard constants, such that in the case $1<p<2+\epsilon$, one has*

$$\sup_{\pm t>0}\|\nabla u(\cdot,t)\|_{L^p(\mathbb{R}^n)}\leq C_p\|\widetilde{N}^\pm_*(\nabla u)\|_{L^p(\mathbb{R}^n)}\,, \tag{5.4}$$

$$-\,e_{n+1}\cdot A\nabla u(\cdot,t)\to\partial_\nu u(\cdot,0) \text{ weakly in } L^p \text{ as } t\to 0^\pm\,, \tag{5.5}$$

$$\nabla_x u(\cdot,t)\to\nabla_x u(\cdot,0) \text{ weakly in } L^p \text{ as } t\to 0^\pm\,, \tag{5.6}$$

where $\partial_\nu u(\cdot,0) \in L^p(\mathbb{R}^n)$ and $u(\cdot,0) \in \dot{L}_1^p(\mathbb{R}^n)$ denote the variational co-normal and non-tangential boundary trace, respectively, defined in Lemma 5.1.

Also, in the case $n/(n+1) < p \le 1$, if $\sup_{\pm t>0} \|\nabla_x u(\cdot,t)\|_{H^p(\mathbb{R}^n)} < \infty$ and there exists some $h \in \dot{H}^{1,p}(\mathbb{R}^n)$ such that $\nabla_x u(\cdot,t) \to \nabla_x h$ in the sense of tempered distributions, then $u(\cdot,0) = h$ in the sense of $\dot{H}^{1,p}(\mathbb{R}^n)$, where $u(\cdot,0) \in \dot{H}^{1,p}(\mathbb{R}^n)$ denotes the non-tangential boundary trace defined in Lemma 5.1.

The third and final result below concerns solutions to the Dirichlet problem.

Lemma 5.3. *Suppose that the hypotheses of Theorem 1.4 hold and let $2-\epsilon < p < \infty$ denote the range of well-posedness of $(D)_p$.*[7] *If $Lu = 0$ in $\mathbb{R}^{n+1}_\pm$ and*

$$\|N_*^\pm(u)\|_{L^p(\mathbb{R}^n)} < \infty\,, \tag{5.7}$$

then there exists a unique $f \in L^p(\mathbb{R}^n)$, and we set $u(\cdot,0) := f$, such that

$$u \to f \text{ non-tangentially, and } u(\cdot,t) \to f \text{ in } L^p(\mathbb{R}^n) \text{ as } t \to 0^\pm\,. \tag{5.8}$$

Proof of Lemma 5.1. We suppose that $\widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n)$, and we seek a variational co-normal $g \in H^p(\mathbb{R}^n)$, and a non-tangential limit $f \in \dot{H}^{1,p}(\mathbb{R}^n)$, satisfying the bounds (5.1) and (5.3). The case $p > 1$ may be obtained by following, *mutatis mutandi*, the proof of [38, Theorem 3.1] (see also [1, Lemma 4.3], stated in this paper as Lemma 2.1, which treats the case $p = 2$ by following [38, Theorem 3.1]). We omit the details. The case $p \le 1$, which is a bit more problematic, is treated below.

First, we consider the existence of the non-tangential limit $f \in \dot{H}^{1,p}(\mathbb{R}^n)$, assuming now that $\widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n)$ with $n/(n+1) < p \le 1$. In fact, following the proof of [38, Theorem 3.1, p. 462]), we see that the non-tangential limit $f(x)$ exists at every point $x \in \mathbb{R}^n$ for which $\widetilde{N}_*(\nabla u)(x)$ is finite (thus, a.e. in $\mathbb{R}^n$, no matter the value of p), and moreover, for any pair of points $x, y \in \mathbb{R}^n$ at which $\widetilde{N}_*(\nabla u)(x)$ and $\widetilde{N}_*(\nabla u)(y)$ are finite, we have the pointwise estimate

$$|f(x) - f(y)| \le C\,|x-y| \left(\widetilde{N}_*(\nabla u)(x) + \widetilde{N}_*(\nabla u)(y)\right),$$

where C depends only on the standard constants. Thus, by the criterion of [40], we obtain immediately that $f \in \dot{H}^{1,p}(\mathbb{R}^n)$, with $\|f\|_{\dot{H}^{1,p}(\mathbb{R}^n)} \lesssim \|\widetilde{N}_*(\nabla u)\|_{L^p(\mathbb{R}^n)}$.

Next, we consider the existence of the co-normal derivative $g \in H^p(\mathbb{R}^n)$. We use $\langle\cdot,\cdot\rangle$ to denote the usual pairing of tempered distributions $\mathbf{S}'(\mathbb{R}^d)$ and Schwartz functions $\mathbf{S}(\mathbb{R}^d)$, where d may be either n or $n+1$ (the usage will be clear from the context). By Lemma 6.2, for all $0 < q \le 2n/(n+1)$, we have

$$\|\nabla u\|_{L^{q(n+1)/n}(\mathbb{R}^{n+1}_+)} \le C(q,n)\, \|\widetilde{N}_*(\nabla u)\|_{L^q(\mathbb{R}^n)}\,, \tag{5.9}$$

and since $\widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n)$, this implies that $\nabla u \in L^r(\mathbb{R}^{n+1}_+)$, with $r := p(n+1)/n > 1$. We may then define a linear functional $\Lambda = \Lambda_u \in \mathbf{S}'(\mathbb{R}^{n+1})$ by

$$\langle \Lambda, \Phi\rangle := \iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \nabla\Phi\,, \qquad \forall\, \Phi \in \mathbf{S}(\mathbb{R}^{n+1})\,.$$

For $\varphi \in \mathbf{S}(\mathbb{R}^n)$, we say that $\Phi \in \mathbf{S}(\mathbb{R}^{n+1})$ is an extension of φ if $\Phi(\cdot,0) = \varphi$. We now define a linear functional $g \in \mathbf{S}'(\mathbb{R}^n)$ by setting

$$\langle g, \varphi\rangle := \langle \Lambda, \Phi\rangle\,, \qquad \forall\, \varphi \in \mathbf{S}(\mathbb{R}^n)\,,$$

[7] In particular, we then have solvability with estimates for the dual problem $(R)_{p'}$ by Theorem 1.4.

where Φ is any extension of φ. Since such an extension of φ need not be unique, however, we must verify that g is well-defined. To this end, fix $\varphi \in \mathbf{S}(\mathbb{R}^n)$, and let $\Phi_1, \Phi_2 \in \mathbf{S}(\mathbb{R}^{n+1})$ denote any two extensions of φ. Then $\Psi := \Phi_1 - \Phi_2 \in \mathbf{S}(\mathbb{R}^{n+1})$, with $\Psi(\cdot, 0) \equiv 0$, and so $\langle \Lambda, \Psi \rangle = 0$, by the definition of a (weak) solution. Thus, the linear functional g is well-defined, and so u has a variational co-normal $\partial_\nu u(\cdot, 0) := g$ in $\mathbf{S}'(\mathbb{R}^n)$ satisfying (5.2).

It remains to prove (5.1). For $\varphi \in \mathbf{S}(\mathbb{R}^n)$, we set $M_\varphi f := \sup_{t>0} |\varphi_t * f|$, where as usual $\varphi_t(x) := t^{-n}\varphi(x/t)$. We recall that a tempered distribution f belongs to $H^p(\mathbb{R}^n)$ if and only if $M_\varphi f \in L^p(\mathbb{R}^n)$, for some $\varphi \in \mathbf{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi = 1$ (see, e.g., [52, Theorem 1, p. 91]), and we have the equivalence $\|f\|_{H^p(\mathbb{R}^n)} \approx \|M_\varphi f\|_{L^p(\mathbb{R}^n)}$. We now fix $\varphi \in C_0^\infty(\mathbb{R}^n)$, with $\varphi \geq 0$, $\int \varphi = 1$, and $\operatorname{supp}\varphi \subset \Delta(0,1) := \{x \in \mathbb{R}^n : |x| < 1\}$, so we have

$$\|\partial_\nu u\|_{H^p(\mathbb{R}^n)} \lesssim \|M_\varphi(\partial_\nu u)\|_{L^p(\mathbb{R}^n)},$$

and it suffices to show that

$$\|M_\varphi(\partial_\nu u)\|_{L^p(\mathbb{R}^n)} \lesssim \|\widetilde{N}_*(\nabla u)\|_{L^p(\mathbb{R}^n)}.$$

We claim that

$$M_\varphi(\partial_\nu u) \lesssim \left(M\left(\widetilde{N}_*(\nabla u)\right)^{n/(n+1)}\right)^{(n+1)/n}, \tag{5.10}$$

pointwise, where M denotes the usual Hardy-Littlewood maximal operator. Taking the claim for granted momentarily, we see that

$$\int_{\mathbb{R}^n} M_\varphi(\partial_\nu u)^p \lesssim \int_{\mathbb{R}^n} \left(M\left(\widetilde{N}_*(\nabla u)\right)^{n/(n+1)}\right)^{p(n+1)/n} \lesssim \int_{\mathbb{R}^n} \left(\widetilde{N}_*(\nabla u)\right)^p,$$

as desired, since $p(n+1)/n > 1$.

It therefore remains to establish (5.10). To this end, we fix $x \in \mathbb{R}^n$ and $t > 0$, set $B := B(x,t) := \{Y \in \mathbb{R}^{n+1} : |Y - x| < t\}$, and fix a smooth cut-off function $\eta_B \in C_0^\infty(2B)$, with $\eta_B \equiv 1$ on B, $0 \leq \eta_B \leq 1$, and $|\nabla \eta_B| \lesssim 1/t$. Then

$$\Phi_{x,t}(y,s) := \eta_B(y,s)\,\varphi_t(x-y)$$

is an extension of $\varphi_t(x - \cdot)$, with $\Phi_{x,t} \in C_0^\infty(2B)$, which satisfies

$$0 \leq \Phi_{x,t} \lesssim t^{-n}, \qquad |\nabla_Y \Phi_{x,t}(Y)| \lesssim t^{-n-1}.$$

We then have

$$\begin{aligned} |(\varphi_t * \partial_\nu u)(x)| &= |\langle \partial_\nu u, \varphi_t(x-\cdot)\rangle| = \left|\iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \nabla \Phi_{x,t}\right| \\ &\lesssim t^{-n-1} \iint_{\mathbb{R}^{n+1}_+ \cap 2B} |\nabla u| \lesssim t^{-n-1}\left(\int_{\mathbb{R}^n} \left(\widetilde{N}_*(|\nabla u| 1_{2B})(y)\right)^{n/(n+1)} dy\right)^{(n+1)/n}, \end{aligned}$$

where in the last step we have used (5.9) with $q = n/(n+1)$. For $C > 0$ chosen sufficiently large, simple geometric considerations then imply that

$$\widetilde{N}_*(|\nabla u| 1_{2B})(y) \leq \widetilde{N}_*(\nabla u)(y)\, 1_{\Delta(x,Ct)}(y),$$

where $\Delta(x, Ct) := \{y \in \mathbb{R}^n : |x - y| < Ct\}$. Combining the last two estimates, we obtain

$$|(\varphi_t * \partial_\nu u)(x)| \lesssim \left(t^{-n} \int_{|x-y|<Ct} \left(\widetilde{N}_*(\nabla u)(y)\right)^{n/(n+1)} dy\right)^{(n+1)/n}.$$

Taking the supremum over $t > 0$, we obtain (5.10), as required. □

Proof of Lemma 5.2. We begin with (5.4) and follow the proof in the case $p = 2$ from [1]. The desired bound for $\partial_t u$ follows readily from t-independence and the Moser local boundedness estimate (1.4). Thus, we only need to consider $\nabla_x u$. Let $\vec{\psi} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$, with $\|\vec{\psi}\|_{p'} = 1$. For $t > 0$, let $\mathbb{D}_t$ denote the grid of dyadic cubes Q in $\mathbb{R}^n$ with side length satisfying $\ell(Q) \leq t < 2\ell(Q)$, and for $Q \in \mathbb{D}_t$, set $Q^* := 2Q \times (t/2, 3t/2)$. Then, using the Caccioppoli-type estimate on horizontal slices in [1, (2.2)], we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \nabla_y u(y,t) \cdot \vec{\psi}(y)\, dy \right| &\leq \left(\int_{\mathbb{R}^n} |\nabla_y u(y,t)|^p \, dy \right)^{1/p} \|\vec{\psi}\|_{p'} \\
&= \left(\sum_{Q \in \mathbb{D}_t} \frac{1}{|Q|} \int_Q \int_Q |\nabla_y u(y,t)|^p dy\, dx \right)^{1/p} \\
&\lesssim \left(\sum_{Q \in \mathbb{D}_t} \int_Q \left(\frac{1}{|Q^*|} \iint_{Q^*} |\nabla_y u(y,s)|^2 \, dyds \right)^{p/2} dx \right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}^n} \left(\widetilde{N}_*(\nabla u) \right)^p \right)^{1/p} .
\end{aligned}$$

This concludes the proof of (5.4).

Next, we prove (5.5). By (5.1) and (5.4), and the density of $C_0^\infty(\mathbb{R}^n)$ in $L^{p'}(\mathbb{R}^n)$, it is enough to prove that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} \vec{N} \cdot A(x) \nabla u(x,t)\, \phi(x)\, dx = \int_{\mathbb{R}^n} \partial_\nu u(x,0)\, \phi(x)\, dx\,, \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^n)\,,$$

where $\vec{N} := -e_{n+1}$. For $\phi \in C_0^\infty(\mathbb{R}^n)$, let Φ denote a $C_0^\infty(\mathbb{R}^{n+1})$ extension of ϕ to $\mathbb{R}^{n+1}$, so by (5.2), it suffices to show that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} \vec{N} \cdot A \nabla u(\cdot, t)\, \phi = \iint_{\mathbb{R}^{n+1}_+} A \nabla u \cdot \nabla \Phi\,. \tag{5.11}$$

Let P_ε be an approximate identity in $\mathbb{R}^n$ with a smooth, compactly supported convolution kernel. Integrating by parts, we see that for each $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} \vec{N} \cdot P_\varepsilon(A \nabla u(\cdot, t))\, \phi = \iint_{\mathbb{R}^{n+1}_+} P_\varepsilon \left(A \nabla u(\cdot, t+s) \right)(x) \cdot \nabla \Phi(x,s)\, dxds, \tag{5.12}$$

since the assumption that $Lu = 0$ and the t-independence of the coefficients imply that

$$\begin{aligned}
&\iint_{\mathbb{R}^{n+1}_+} \operatorname{div} \Big(P_\varepsilon(A \nabla u(\cdot, t + \cdot)) \Big)(x,s)\, \Phi(x,s)\, dxds \\
&\qquad = \lim_{\delta \to 0} \iint_{\mathbb{R}^{n+1}_+} \operatorname{div} \Big(\widetilde{P}_\delta \circ P_\varepsilon(A \nabla u(\cdot, t + \cdot)) \Big)(x,s)\, \Phi(x,s)\, dxds = 0,
\end{aligned}$$

where $\widetilde{P}_\delta$ denotes an approximate identity in $\mathbb{R}_+$, acting in the transversal coordinate s, with a smooth, compactly supported convolution kernel. By the dominated convergence theorem, we may pass to the limit as $\varepsilon \to 0$ in (5.12) to obtain

$$\int_{\mathbb{R}^n} \vec{N} \cdot A\nabla u(\cdot, t)\,\phi = \iint_{\mathbb{R}^{n+1}_+} A(x)\nabla u(x, t+s) \cdot \nabla\Phi(x, s)\,dxds$$
$$= \int_t^\infty \int_{\mathbb{R}^n} A(x)\nabla u(x, s) \cdot \nabla\Big(\Phi(x, s-t) - \Phi(x, s)\Big)\,dxds$$
$$+ \int_t^\infty \int_{\mathbb{R}^n} A(x)\nabla u(x, s) \cdot \nabla\Phi(x, s)\,dxds =: I(t) + II(t)\,.$$

By Lemma 6.2 and the dominated convergence theorem, we have $I(t) \to 0$, as $t \to 0$, and $II(t) \to \iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \nabla\Phi$, as $t \to 0$, hence (5.11) holds.

Next, we prove (5.6). By (5.3) and (5.4), and the density of $C_0^\infty(\mathbb{R}^n)$ in $L^{p'}(\mathbb{R}^n)$, it is enough to prove that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} u(x,t)\ \mathrm{div}_x\,\vec{\psi}(x)\,dx \ = \int_{\mathbb{R}^n} u(x,0)\ \mathrm{div}_x\,\vec{\psi}(x)\,dx\,, \quad \forall\,\vec{\psi} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)\,. \tag{5.13}$$

Following the proof of [38, Theorem 3.1, p. 462]), we obtain

$$|u(x,t) - u(x,0)| \lesssim t\widetilde{N}_*(\nabla u)(x)\,, \qquad \text{for a.e. } x \in \mathbb{R}^n\,, \tag{5.14}$$

whence (5.13) follows.

Finally, we consider the case $n/(n+1) < p \leq 1$, and we assume that

$$\sup_{\pm t>0} \|\nabla_x u(\cdot,t)\|_{H^p(\mathbb{R}^n)} < \infty\,, \tag{5.15}$$

and that there exists $h \in \dot{H}^{1,p}(\mathbb{R}^n)$ such that $\nabla_x u(\cdot,t) \to \nabla_x h$ in the sense of tempered distributions. By Sobolev embedding, $u(\cdot,0)$ and $u(\cdot,t)$ each have a realization belonging (uniformly in t by (5.15)) to $L^q(\mathbb{R}^n)$, with $1/q = 1/p - 1/n$. Note that $q > 1$, since $p > n/(n+1)$. For all $\varepsilon \in (0,1)$, by combining the pointwise estimate (5.14), which still holds in this case, with the trivial bound $|u(\cdot,t) - u(\cdot,0)| \leq |u(\cdot,t)| + |u(\cdot,0)|$, we obtain

$$|u(x,t) - u(x,0)| \lesssim \Big(t\widetilde{N}_*(\nabla u)(x)\Big)^\varepsilon \left(|u(x,t)| + |u(\cdot,0)(x)|\right)^{1-\varepsilon}, \qquad \text{for a.e. } x \in \mathbb{R}^n\,.$$

For p, q as above, set $r = q/(1-\varepsilon)$, $s = p/\varepsilon$, and choose $\varepsilon \in (0,1)$, depending on p and n, so that $1/r + 1/s = 1$. Then by Hölder's inequality, for all $\psi \in \mathbf{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} (u(x,t) - u(x,0))\psi(x)\,dx\right| \\ &\lesssim \|\psi\|_\infty\, t^\varepsilon \left(\int_{\mathbb{R}^n} \left(\widetilde{N}_*(\nabla u)\right)^p\right)^{1/s} \left(\|u(\cdot,0)\|_q + \sup_{t>0}\|u(\cdot,t)\|_q\right)^{1-\varepsilon} \to 0\,,\end{aligned} \tag{5.16}$$

as $t \to 0$. On the other hand, for all $\vec{\phi} \in \mathbf{S}(\mathbb{R}^n, \mathbb{C}^n)$, we have

$$\int_{\mathbb{R}^n} (u(x,t) - h(x))\,\mathrm{div}_x\,\vec{\phi}(x)\,dx \to 0\,.$$

Combining the latter fact with (5.16), applied with $\psi = \mathrm{div}_x\,\vec{\phi}$, we obtain

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{div}_x\,\vec{\phi}(x)\,dx = \int_{\mathbb{R}^n} u(x,0)\,\mathrm{div}_x\,\vec{\phi}(x)\,dx\,, \qquad \forall \vec{\phi} \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^n)\,,$$

thus $\nabla_x h = \nabla_x u(\cdot,0)$ as tempered distributions, and since each belongs to $H^p(\mathbb{R}^n)$, we also have $\nabla_x h = \nabla_x u(\cdot,0)$ in $H^p(\mathbb{R}^n)$, hence $u(\cdot,0) = h$ in the sense of $\dot{H}^{1,p}(\mathbb{R}^n)$. □

Proof of Lemma 5.3. We first prove that (5.8) holds in the case that $u = \mathcal{D}h$ for some $h \in L^p(\mathbb{R}^n)$. Indeed, in that scenario, the case $p = 2$ has been treated in [1, Lemma 4.23]. To handle the remaining range of p, we observe that by Theorem 1.2, we have

$$\|N_*(\mathcal{D}h)\|_{L^p(\mathbb{R}^n)} \leq C_p \|h\|_{L^p(\mathbb{R}^n)} .$$

We may therefore exploit the usual technique, whereby a.e. convergence on a dense class (in our case $L^2 \cap L^p$), along with L^p bounds on the controlling maximal operator, imply a.e. convergence for all $h \in L^p(\mathbb{R}^n)$. We omit the standard argument. Convergence in $L^p(\mathbb{R}^n)$ then follows by the dominated convergence theorem.

Thus, it is enough to show that $u = \mathcal{D}h$ for some $h \in L^p(\mathbb{R}^n)$. We follow the corresponding argument for the case $p = 2$ given in [1], which in turn follows [51, pp. 199–200], substituting $\mathcal{D}$ for the classical Poisson kernel. For each $\varepsilon > 0$, set $f_\varepsilon := u(\cdot, \varepsilon)$, and let $u_\varepsilon := \mathcal{D}((-1/2)I + K)^{-1} f_\varepsilon$ denote the layer potential solution with data f_ε. We claim that $u_\varepsilon(x, t) = u(x, t+\varepsilon)$. To prove this, we set $U_\varepsilon(x, t) := u(x, t+\varepsilon) - u_\varepsilon(x, t)$, and observe that

(i) $LU_\varepsilon = 0$ in $\mathbb{R}^{n+1}_+$ (by t-independence of coefficients).
(ii) Estimate (5.7) holds for U_ε, uniformly in $\varepsilon > 0$.
(iii) $U_\varepsilon(\cdot, 0) = 0$ and $U_\varepsilon(\cdot, t) \to 0$ non-tangentially and in L^p, as $t \to 0$.

Item (iii) relies on interior continuity (1.3) and smoothness in t, in combination with the non-tangential and strong L^p convergence results for layer potentials proved in the preceding paragraph, applied to the term $u_\varepsilon := \mathcal{D}((-1/2)I + K)^{-1} f_\varepsilon$. The claim then follows by the uniqueness for $(D)_p$, which is proved in [29, Proposition 8.19(i)] for a more general class of operators.

We now complete the proof of the lemma. For convenience of notation, for each $t > 0$, we set $\mathcal{D}_t h := \mathcal{D}h(\cdot, t)$. By (5.7), $\sup_\varepsilon \|f_\varepsilon\|_{L^p(\mathbb{R}^n)} < \infty$, and so there exists a subsequence f_{ε_k} converging in the weak* topology on $L^p(\mathbb{R}^n)$ to some $f \in L^p(\mathbb{R}^n)$. For each $g \in L^{p'}(\mathbb{R}^n)$, we set $g_1 := \mathrm{adj}\left((-1/2)I + K\right)^{-1} \mathrm{adj}(\mathcal{D}_t) g$, and observe that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left[\mathcal{D}_t((-1/2)I + K)^{-1} f\right] \overline{g} = \int_{\mathbb{R}^n} f\, \overline{g_1} &= \lim_{k\to\infty} \int_{\mathbb{R}^n} f_{\varepsilon_k} \overline{g_1} \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} \left[\mathcal{D}_t((-1/2)I + K)^{-1} f_{\varepsilon_k}\right] \overline{g} \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} u(\cdot, t+\varepsilon_k)\, \overline{g} = \int_{\mathbb{R}^n} u(\cdot, t)\, \overline{g}.
\end{aligned}$$

It follows that $u = \mathcal{D}h$, with $h = ((-1/2)I + K)^{-1} f$ in $L^p(\mathbb{R}^n)$, as required. □

6. Appendix: Auxiliary lemmata

We now return to prove some technical results that were used to prove Proposition 3.3 and Lemmata 5.1-5.2. The results are stated in the more general setting of a Lipschitz graph domain of the form $\Omega := \{(x, t) \in \mathbb{R}^{n+1} : t > \phi(x)\}$, where $\phi : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. We set $M := \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} < \infty$, and consider constants

$$0 < \eta < \frac{1}{M}, \qquad 0 < \beta < \min\left\{1, \frac{1}{M}\right\}. \tag{6.1}$$

We define the cone

$$\Gamma := \{X = (x, t) \in \mathbb{R}^{n+1} : |x| < \eta t\}.$$

For $X \in \Omega \subset \mathbb{R}^{n+1}$, we use the notation $\delta(X) := \operatorname{dist}(X, \partial\Omega)$. For $u \in L^2_{\rm loc}(\Omega)$, we set

$$\widetilde{N}_*(u)(Q) := \sup_{X \in Q+\Gamma} \left(\fint_{B(X,\beta\delta(X))} |u(Y)|^2\, dY \right)^{1/2}, \qquad Q \in \partial\Omega, \tag{6.2}$$

and

$$N_*(u)(Q) := \sup_{X \in Q+\Gamma} |u(X)|, \qquad Q \in \partial\Omega. \tag{6.3}$$

If we want to emphasize the dependence on η and β, then we shall write Γ_η, $\widetilde{N}_{*,\eta,\beta}$, $N_{*,\eta}$. The lemma below shows that the choice of η and β, within the permissible range in (6.1), is immaterial for $L^p(\partial\Omega)$ estimates of $\widetilde{N}_{*,\eta,\beta}$.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ denote a Lipschitz graph domain. For each $p \in (0,\infty)$ and*

$$0 < \eta_1,\, \eta_2 < \frac{1}{M}, \qquad 0 < \beta_1,\, \beta_2 < \min\Big\{1\,,\, \frac{1}{M}\Big\},$$

there exist constants $c, C \in (0,\infty)$, depending on $M, p, \eta_1, \eta_2, \beta_1, \beta_2$, such that

$$c\|\widetilde{N}_{*,\eta_2,\beta_2} u\|_{L^p(\partial\Omega)} \le \|\widetilde{N}_{*,\eta_1,\beta_1} u\|_{L^p(\partial\Omega)} \le C\|\widetilde{N}_{*,\eta_2,\beta_2} u\|_{L^p(\partial\Omega)} \tag{6.4}$$

for all $u \in L^2_{\rm loc}(\Omega)$.

Proof. First, a straightforward adaptation of the argument in [52, p. 62] gives

$$\|\widetilde{N}_{*,\eta_2,\beta} u\|_{L^p(\partial\Omega)} \le C\|\widetilde{N}_{*,\eta_1,\beta} u\|_{L^p(\partial\Omega)}, \tag{6.5}$$

whenever $0 < \eta_1 < \eta_2 < 1/M$, $p \in (0,\infty)$ and $\beta \in (0,1)$. The opposite inequality is trivially true (with $C = 1$). Thus, since $\frac{1-M\eta}{M+\eta} \nearrow \frac{1}{M}$ as $\eta \searrow 0$, estimate (6.4) will follow as soon as we prove that for any

$$0 < \eta < \frac{1}{M}, \qquad 0 < \beta_1 < \beta_2 < \min\Big\{1\,,\, \frac{1-M\eta}{M+\eta}\Big\}, \tag{6.6}$$

there exists a finite constant $C > 0$, depending on $M, p, \eta, \beta_1, \beta_2$, such that

$$\|\widetilde{N}_{*,\eta,\beta_2} u\|_{L^p(\partial\Omega)} \le C\|\widetilde{N}_{*,\eta,\beta_1} u\|_{L^p(\partial\Omega)}$$

for all $u \in L^2_{\rm loc}(\Omega)$. To this end, let η, β_1, β_2 be as in (6.6) and consider two arbitrary points, $Q \in \partial\Omega$ and $X \in Q + \Gamma_{\eta_2}$, as well as two parameters, $\beta' \in (0,\beta_2)$ and $\varepsilon > 0$, to be chosen later. The parameter $\varepsilon > 0$ and Euclidean geometry ensure that

$$\begin{aligned} &|X-Y| < \beta_2\delta(X) \Longrightarrow \\ &\qquad |B(X,\beta'\delta(X))| \le C_{n,\beta_2,\beta',\varepsilon}\, |B(Y,(\beta_2-\beta'+\varepsilon)\delta(X)) \cap B(X,\beta'\delta(X))|. \end{aligned}$$

We also have

$$|X-Z| < \beta'\,\delta(X) \Longrightarrow \frac{1}{1+\beta'}\delta(Z) \le \delta(X) \le \frac{1}{1-\beta'}\delta(Z), \tag{6.7}$$

and

$$B(X,\beta'\delta(X)) \subset Q + \Gamma_\kappa, \quad \text{where } \kappa := \frac{\eta+\beta'}{1-\beta'\eta}.$$

Note that, due to our assumptions, $0 < \kappa < 1/M$. Using Fubini's Theorem and the preceding considerations, we may then write

$$\begin{aligned}
\fint_{B(X,\beta_2\delta(X))} |u(Y)|^2\,dY &= \fint_{B(X,\beta_2\delta(X))} \left(\fint_{B(Y,(\beta_2-\beta'+\varepsilon)\delta(X))\cap B(X,\beta'\delta(X))} 1\,dZ\right)|u(Y)|^2\,dY \\
&= C_{n,\beta_2,\beta',\varepsilon} \fint_{B(X,\beta'\delta(X))} \left(\fint_{B(X,\beta_2\delta(X))} 1_{B(Z,(\beta_2-\beta'+\varepsilon)\delta(X))}(Y)\,|u(Y)|^2\,dY\right) dZ \\
&= C_{n,\beta_2,\beta',\varepsilon} \fint_{B(X,\beta'\delta(X))} \left(\fint_{B(Z,(\beta_2-\beta'+\varepsilon)\delta(X))} |u(Y)|^2\,dY\right) dZ \\
&\leq C_{n,\beta_2,\beta',\varepsilon} \fint_{B(X,\beta'\delta(X))} \left(\fint_{B(Z,\frac{\beta_2-\beta'+\varepsilon}{1-\beta'}\delta(Z))} |u(Y)|^2\,dY\right) dZ \\
&\leq C_{n,\beta_2,\beta',\varepsilon} \left(\widetilde{N}_{*,\kappa,\frac{\beta_2-\beta'+\varepsilon}{1-\beta'}}(u)(Q)\right)^2.
\end{aligned}$$

We now choose $\varepsilon \in (0, \beta_1(1-\beta_2))$ and set $\beta' := \frac{\beta_2-\beta_1+\varepsilon}{1-\beta_1}$ to ensure that $\beta' \in (0, \beta_2)$ and $\frac{\beta_2-\beta'}{1-\beta'} = \beta_1$, so the inequality above further yields

$$\widetilde{N}_{*,\eta,\beta_2}(u)(Q) \leq C\widetilde{N}_{*,\kappa,\beta_1}(u)(Q) \quad \text{for some } \kappa = \kappa(\beta_1,\beta_2,\eta) \in (0, 1/M). \tag{6.8}$$

Consequently, by (6.8) and (6.5), we have

$$\|\widetilde{N}_{*,\eta,\beta_2}u\|_{L^p(\partial\Omega)} \leq C\|\widetilde{N}_{*,\kappa,\beta_1}u\|_{L^p(\partial\Omega)} \leq C\|\widetilde{N}_{*,\eta,\beta_1}u\|_{L^p(\partial\Omega)}.$$

This finishes the proof of the lemma. □

We now prove a self-improvement property for $L^p(\Omega)$ estimates of solutions.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ denote a Lipschitz graph domain. Suppose that $w \in L^2_{\mathrm{loc}}(\Omega)$, and that $\widetilde{N}_*(w) \in L^p(\partial\Omega)$ for some $p \in (0,\infty)$. First, if $0 < p \leq 2n/(n+1)$, then*

$$w \in L^{p(n+1)/n}(\Omega) \quad \textit{and} \quad \|w\|_{L^{p(n+1)/n}(\Omega)} \leq C_{\partial\Omega,p}\,\|\widetilde{N}_*(w)\|_{L^p(\partial\Omega)}. \tag{6.9}$$

Second, if $0 < p < \infty$, and if $Lw = 0$ in Ω, then (6.9) holds. Finally, there exists $q = q(n,\Lambda) > 2$ such that if $0 < p < qn/(n+1)$, and if $w = \nabla u$ for some solution $u \in L^2_{1,\mathrm{loc}}(\Omega)$ of $Lu = 0$ in Ω, then (6.9) holds.

We note that estimate (6.9) is a special case of [14, Theorem 2.6], but we provide an alternative proof below for the reader's convenience.

Proof. Fix η, β as in (6.1). We observe that by Lemma 6.1, the choice of β within the permissible range is immaterial. We now choose β' so that $0 < \beta' < \beta/2 < 1/2$. Then (6.7) holds, and we have $\beta'/(1-\beta') < \beta$.

Case 1. Suppose that $w \in L^2_{\mathrm{loc}}(\Omega)$, and that $\widetilde{N}_*(w) \in L^p(\partial\Omega)$ for some $0 < p \leq 2n/(n+1)$. To prove (6.9), we set

$$F(Z) := \left(\fint_{B(Z,\beta\delta(Z))} |w(X)|^2\,dX\right)^{1/2}, \qquad Z \in \Omega,$$

and observe that

$$\begin{aligned}\iint_\Omega |w(X)|^{p(n+1)/n}\,dX &= \iint_\Omega \left(\fint_{|X-Z|<\beta'\delta(X)} dZ\right) |w(X)|^{p+p/n}\,dX\\ &\lesssim \iint_\Omega \left(\fint_{|X-Z|<\beta\delta(Z)} |w(X)|^{p+p/n}\,dX\right) dZ\\ &\lesssim \iint_\Omega \left(\fint_{|X-Z|<\beta\delta(Z)} |w(X)|^2 dX\right)^{(p+p/n)/2} dZ\\ &=: \iint_\Omega F(Z)^{p+p/n}\,dZ \lesssim \|\mu\|_C \int_{\partial\Omega} N_*(F)^p,\end{aligned}$$

where we have used Fubini's Theorem, (6.7) and the fact that $\beta'/(1-\beta') < \beta$ in the first inequality, the fact that $p(n+1)/n \leq 2$ in the second, and Carleson's lemma (which still holds in the present setting) in the third. In particular, we are using $\|\mu\|_C$ to denote the Carleson norm of the measure

$$d\mu(Z) := F(Z)^{p/n}\, 1_\Omega(Z)\, dZ.$$

Also, by definition, $N_*(F) = \widetilde{N}_*(w)$ (see (6.2) and (6.3)), and so

$$\|N_*(F)\|_{L^p(\partial\Omega)} = \|\widetilde{N}_*(w)\|_{L^p(\partial\Omega)} < \infty.$$

Thus, to finish the proof of Case 1, it is enough to observe that for every "surface ball" $\Delta(P,r) := B(P,r)\cap\partial\Omega$, where $P := (x,\varphi(x)) \in \partial\Omega$ and $r>0$, we have

$$\begin{aligned}\frac{1}{|\Delta(P,r)|}\iint_{B(P,r)\cap\Omega} F(Z)^{p/n}\,dZ &\lesssim r^{-n}\int_{|x-z|<r}\int_{\varphi(z)}^{\varphi(z)+2r} F(z,s)^{p/n}\,ds dz\\ &\lesssim r \fint_{|x-z|<r} \Big(N_*(F)(z,\varphi(z))\Big)^{p/n} dz\\ &\lesssim \left(\int_{|x-z|<r} \Big(N_*(F)(z,\varphi(z))\Big)^{p} dz\right)^{1/n} \lesssim \|N_*(F)\|_{L^p(\partial\Omega)}^{p/n},\end{aligned}$$

since the bound (6.9) follows, as required.

Case 2. Now suppose that $Lw = 0$ in Ω, and that $\widetilde{N}_*(w) \in L^p(\partial\Omega)$ for some $p\in(0,\infty)$. By Moser's sub-mean inequality (1.4), we have $\widetilde{N}_*(w)(Q) \approx \|w\|_{L^\infty(Q+\Gamma)} =: N_*(w)(Q)$, uniformly for $Q\in\partial\Omega$, at least if $\beta>0$ is sufficiently small. Under this assumption, estimate (6.9) can then be proved as in Case 1, except that invoking Hölder's inequality, which was the source of the restriction $p \leq 2n/(n+1)$, is unnecessary. This completes the proof of Case 2, since the restriction on the size of β is immaterial by Lemma 6.1.

Case 3. Finally, suppose that $w = \nabla u$ for some solution $u \in L^2_{1,\mathrm{loc}}(\Omega)$ of $Lu = 0$ in Ω, and that $\widetilde{N}_*(w) \in L^p(\partial\Omega)$ for some $p\in(0,\infty)$. It is well-known (see, e.g., [36]) that there exists $q = q(n,\Lambda) > 2$ such that

$$\left(\fint_{B(X,\beta\delta(X))} |w(Y)|^q\,dY\right)^{1/q} \lesssim \left(\fint_{B(X,2\beta\delta(X))} |w(Y)|^2\,dY\right)^{1/2}.$$

The proof of (6.9) when $0<p<qn/(n+1)$ then proceeds as in Case 1, where Lemma 6.1 is used once more to readjust the size of the balls. □

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