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Hamilton cycles in quasirandom hypergraphs[*]

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Abstract

We show that, for a natural notion of quasirandomness in k-uniform hypergraphs, any quasirandom k-uniform hypergraph on n vertices with constant edge density and minimum vertex degree $\Omega(n^{k-1})$ contains a loose Hamilton cycle. We also give a construction to show that a k-uniform hypergraph satisfying these conditions need not contain a Hamilton ℓ-cycle if $k - \ell$ divides k. The remaining values of ℓ form an interesting open question.

1 Introduction

A *k-uniform hypergraph*, or *k-graph H* consists of a set of vertices $V(H)$ and a set of edges $E(H)$, where each edge consists of k vertices. We say that a k-graph C is an *ℓ-cycle* if its vertices can be cyclically ordered in such a way that each edge of C consists of k consecutive vertices, and so that each edge intersects the subsequent edge in ℓ vertices. This generalises the notion of a cycle in a graph in a natural manner, though various other definitions of cycles in hypergraphs have also been considered, such as a Berge cycle [2]. Note in particular that each edge of an ℓ-cycle k-graph C has $k - \ell$ vertices which were not contained in the previous edge, so the number of vertices of C must be divisible by $k - \ell$. We say that a k-graph H on n vertices contains a *Hamilton ℓ-cycle* if it contains an n-vertex ℓ-cycle as a subgraph; as before, this is only possible if $k - \ell$ divides n. We refer to 1-cycles and $(k-1)$-cycles as *loose*

cycles and *tight cycles* respectively, and in the same way refer to *loose Hamilton cycles* and *tight Hamilton cycles*.

Finding sufficient conditions which ensure that a k-graph contains a Hamilton ℓ-cycle (or other spanning structure) has been a highly active area of research in recent years, with particular interest in minimum degree conditions and quasirandomness conditions.

1.1 Minimum degree conditions

Sufficient minimum degree conditions which ensure that a k-graph contains a Hamilton ℓ-cycle can be seen as hypergraph analogues of the well-known Dirac's theorem [5], which states that any graph G on $n \geq 3$ vertices with $\delta(G) \geq n/2$ contains a Hamilton cycle. For a k-graph H and a set $S \subseteq V(H)$, the *degree* of S, denoted $d_H(S)$ or $d(S)$ (when H is clear from the context), is the the number of edges of H which contain S as a subset. If $S = \{v\}$ is a singleton then we write simply $d_H(v)$ or $d(v)$ rather than $d(\{v\})$. The *minimum s-degree* of H, written $\delta_s(H)$, is the minimum of $d(S)$ taken over all s-sets of vertices of H. In particular we refer to the minimum 1-degree and the minimum $k - 1$ degree of H as the *minimum vertex degree* and *minimum codegree* of H respectively. The following theorem gives an Dirac-type result for k-graphs with high minimum codegree; simple constructions show that for any k and ℓ this minimum codegree condition is best possible up to the ηn error term.

Theorem 1. *For any $k \geq 3$, $1 \leq \ell \leq k - 1$ and $\eta > 0$, there exists n_0 such that if $n \geq n_0$ is divisible by $k - \ell$ and H is a k-graph on n vertices with*

$$
\delta_{k-1}(H) \geq \begin{cases} \left(\frac{1}{2} + \eta\right) n & \text{if } k - \ell \text{ divides } k, \\ \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \eta\right) n & \text{otherwise,} \end{cases}
$$

then H contains a Hamilton ℓ-cycle.

The case $\ell = k - 1$ of Theorem 1 was proved by Rödl, Ruciński and Szemerédi [24], confirming a conjecture of Katona and Kierstead [13], by their innovative 'absorbing method' (the approach we follow in this paper). The same authors then showed that for the case $k = 3$ and $\ell = 2$ the ηn error term can be removed [23]. The remaining cases of Theorem 1 with $k - \ell$ divides k follow immediately from the case $\ell = k - 1$, since if $k - \ell$ divides n then any $(k - 1)$-cycle of order n contains an ℓ-cycle on the same vertex set as a subgraph. The remaining cases of Theorem 1 were subsequently proven in a sequence of papers by Kühn and Osthus [16], Keevash, Kühn, Mycroft, and Osthus [14], Hàn and Schacht [8] and Kühn, Mycroft, and Osthus [15], while more recently Han and Zhao [9] showed that the ηn error term can also be removed for $\ell < k/2$.

Much less is known for other minimum degree conditions. In particular, an analogous result to Theorem 1 for minimum vertex degree conditions is only known for $k = 3$ and $\ell = 1$. This is the following theorem due to Han and Zhao [10] (an asymptotic version of this result was given previously by Buß, Hàn and Schacht [3]).

Theorem 2. *There exists n_0 such that if $n \geq n_0$ is even and H is a 3-graph on n vertices with*

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{\lfloor \frac{3n}{4} \rfloor}{2} + 2 - n \bmod 2$$

then H contains a Hamilton 1-cycle.

1.2   Quasirandomness conditions

Another type of sufficient condition for the existence of a Hamilton ℓ-cycle in a k-graph H is a quasirandomness condition, where we assume that H satisfies some property (or properties) typical of a random k-graph on the same vertex set; there are many candidate properties which could be considered. In this vein, Frieze and Krivelevich [6] defined a form of quasirandomness for k-graphs termed (p, ε)-regularity, and proved for $1 \leq \ell \leq k/2$ that not only must any (p, ε)-regular k-graph H contain a Hamilton ℓ-cycle, but in fact that any such H contains a collection of edge-disjoint Hamilton ℓ-cycles covering almost all of the edges of H. Frieze, Krivelevich and Loh [7] then proved an analogous result for tight Hamilton cycles in 3-graphs, following which Bal and Frieze [1] proved an analogous result for Hamilton ℓ-cycles in k-graphs with $k/2 < \ell \leq k - 1$ (each of the latter two papers used a somewhat different definition of (p, ε)-regularity tailored to the problem in question).

However, the notions of quasirandomness used for these results are very strong; each involves a 'generalised codegree condition' which gives the approximate size of the intersection of the neighbourhoods of a small number of sets of vertices. Weaker notions of quasirandomness in hypergraphs have also been studied; Lenz and Mubayi [20] determined the poset of implications between many such notions, and also demonstrated that each such notion is equivalent to the existence of a large spectral gap for an appropriate definition of first and second eigenvalues [21]. In this paper we consider the weakest of these forms of quasirandomness, which Lenz and Mubayi referred to as $\texttt{Expand}_p[1+\ldots+1]$, and which is a natural generalisation to k-graphs of a notion of quasirandomness for graphs that originated in early work of Thomason [25, 26] and Chung, Graham and Wilson [4].

Definition. *Let $k \geq 2$, let $0 < \mu, p < 1$, and let H be a k-graph on n vertices. We say that H is (p, μ)-dense if for any $X_1, \ldots, X_k \subseteq V(H)$ we have*

$$e(X_1, \ldots, X_k) \geq p|X_1| \cdots |X_k| - \mu n^k,$$

where $e(X_1, \ldots, X_k)$ is the number of k-tuples $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ such that $\{x_1, \ldots, x_k\} \in H$ (note that if the sets X_i overlap an edge might be counted more than once). We say that H is an (n, p, μ) k-graph if H has n vertices and is (p, μ)-dense. Finally, for $0 < \alpha < 1$, an (n, p, μ, α) k-graph is an (n, p, μ) k-graph H which also satisfies $\delta_1(H) \geq \alpha\binom{n}{k-1}$.

Note that an (n, p, μ) k-graph may contain isolated vertices, so this notion of quasirandomness by itself does not imply the existence of even a single Hamilton ℓ-cycle. Lenz and Mubayi [18] previously studied perfect packings in (n, p, μ, α) k-graphs, showing that

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any such k-graph must contain a perfect F-packing for any fixed linear k-graph F (for n sufficiently large, μ sufficiently small, and subject to certain natural divisibility conditions); they then continued this line of research through similar packing results for various related notions of quasirandomness in [19].

1.3 New results

Our main result in this paper is the following theorem, which states that any k-graph satisfying our quasirandomness condition, whose minimum vertex degree is not too small, must contain a loose Hamilton cycle. This is the first example of a connected spanning structure whose existence is guaranteed by this notion of quasirandomness.

Theorem 3. *Let $k \geq 2$. For any $0 < p, \alpha < 1$ there exist n_0 and $\mu > 0$ such that if H is an (n, p, μ, α) k-graph, where $n \geq n_0$ is divisible by $k - 1$, then H contains a Hamilton 1-cycle.*

Remarks.

- The minimum vertex degree condition cannot be dropped from the statement of Theorem 3. Indeed, fix $p \in (0, 1)$, let $f(n) = o(n)$ and consider the following k-graph sequence: take the disjoint union of the random k-graph $G^{(k)}(n - f(n), p)$ and a clique of size $f(n)$. The minimum vertex degree is $\binom{f(n)-1}{k-1}$, there is no Hamilton 1-cycle, and for all $\mu > 0$, the k-graph is still (p, μ)-dense with high probability.

- It is not true that a k-graph H which satisfies the conditions of Theorem 3 must contain edge-disjoint Hamilton 1-cycles covering almost all of the edges of H (as was the case for the (p, ε)-regular k-graphs discussed in Section 1.2). For example, let H be formed from the complete k-graph on n vertices by fixing a vertex v and arbitrarily deleting $\frac{2}{3}\binom{n-1}{k-1}$ edges which contain v. Then, providing n is sufficiently large compared to μ, H is an $(n, \frac{1}{2}, \mu, \frac{1}{4})$ k-graph. Furthermore, the size of any collection of edge-disjoint 1-cycles in H is at most the degree of v, that is, $\frac{1}{3}\binom{n-1}{k-1}$. Since a Hamilton 1-cycle contains $\frac{n}{k-1}$ edges, we conclude that any collection of edge-disjoint Hamilton 1-cycles in H covers at most $\frac{n}{3(k-1)}\binom{n-1}{k-1} = \frac{k}{3(k-1)}\binom{n}{k}$ edges, that is, at most around two-thirds of the edges of H.

- The definition of (p, μ)-dense cannot be changed to $e(X) \geq p\binom{|X|}{k} - \mu n^k$ for all vertex sets X, where $e(X)$ is the number of edges in X. Indeed, consider the disjoint union G of two cliques of size $n/2$. For all $\mu > 0$ and $X \subseteq V(G)$, we have $e(X) \geq 2^{-k}\binom{|X|}{k} - \mu n^k$ for large n. Clearly G has minimum vertex degree $\Omega(n^{k-1})$ and no Hamilton 1-cycle.

- Our proof of Theorem 3 is valid for graphs (i.e. when $k = 2$), but Theorem 3 was already known in this case. Indeed, Kühn, Osthus and Treglown [17] showed that any large 'robust expander' graph with high minimum degree must contain a Hamilton cycle, and it is straightforward to show that any (n, p, μ, α) 2-graph has this form (actually their result was stated for directed graphs, but the analogue for undirected graphs

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follows immediately). Very recently a proof of this result which does not use graph regularity (and so applies for much smaller graphs) was given by Lo and Patel [22]. Other 'expansion' properties which ensure the existence of Hamilton cycles in graphs were established by Hefetz, Krivelevich and Szabó [11]. Our results can be seen as providing a hypergraph analogue of these results; our quasirandomness condition is an 'expansion' property which guarantees the existance of a Hamilton cycle.

- We do not treat the case $p = o(1)$ as $n \to \infty$, but the techniques likely extend to the sparse setting.

It is natural to ask whether analogous results to Theorem 3 hold for Hamilton ℓ-cycles when $\ell \geq 2$. Since consecutive edges of a Hamilton ℓ-cycle intersect in ℓ vertices, a necessary prerequisite for this is that we strengthen the minimum vertex degree condition $\delta_1(H) \geq \alpha\binom{n}{k-1}$ of Theorem 3 to a minimum ℓ-degree condition $\delta_\ell(H) \geq \alpha\binom{n}{k-\ell}$. Indeed, in Section 4 we give a construction of an (n, p, μ) k-graph H with $\delta_j(H) \geq \alpha\binom{n}{k-j}$ for any $1 \leq j \leq \ell - 1$ which contains a vertex x such that the intersection of any edge containing x and any edge not containing x has size at most $\ell - 1$; it follows that H contains no Hamilton ℓ-cycle.

Interestingly, in the case where $k \geq 3$ and $k - \ell$ divides k, the analogous statement to Theorem 3 for Hamilton ℓ-cycles does not hold even after this strengthening (i.e. assuming also that H satisfies the minimum ℓ-degree condition $\delta_\ell(H) \geq \alpha\binom{n}{k-\ell}$). Indeed, by adapting a construction of Lenz and Mubayi [18, 19] we prove the following proposition.

Proposition 4. *Let k and ℓ be integers such that $k \geq 3$, $2 \leq \ell \leq k - 1$ and $k - \ell$ divides k. Then for any $\mu > 0$ there exists n_0 such that for any $n \geq n_0$ which is divisible by $2k$, there is a k-graph H on n vertices such that*

(a) H is $(2^{-\binom{k}{\ell}}, \mu)$-dense,
(b) $\delta_j(H) \geq (2^{-\binom{k}{\ell}} - \mu)\binom{n}{k-j}$ for any $1 \leq j \leq \ell$,
(c) H does not contain a Hamilton ℓ-cycle.

In particular, we cannot guarantee the existence of a tight Hamilton cycle in an (n, p, μ, α) k-graph H even if we also assume that H has minimum codegree $\delta_{k-1}(H) \geq \alpha n$.

In summary, we have shown that any (n, p, μ) k-graph H with $\delta_\ell(H) \geq \alpha\binom{n}{k-1}$ must contain a Hamilton ℓ-cycle if $\ell = 1$, whilst for $k \geq 3$ this statement is false if $k - \ell$ divides k. This leaves the remaining cases as an interesting open problem.

Problem. Fix $\ell > 1$ and $k \geq 4$ such that $k - \ell$ does not divide k. Is the following statement true? For all $0 < p, \alpha < 1$ there exist n_0 and $\mu > 0$ such that if H is an (n, p, μ) k-graph with $\delta_\ell(H) \geq \alpha\binom{n}{k-\ell}$, where $n \geq n_0$ is divisible by $k - \ell$, then H contains a Hamilton ℓ-cycle.

1.4 Structure and notation of this paper

The remainder of this paper is organized as follows. In Section 2, we outline the 'absorbing method' introduced by Rödl, Ruciński and Szemerédi, and state three key lemmas needed

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to apply this method in the context of Theorem 3. We then combine these key lemmas to prove Theorem 3. In Section 3 we prove each of the three key lemmas, for which our main tool is an extension lemma for quasirandom k-graphs developed by Lenz and Mubayi [18]. Finally, in Section 4 we prove Proposition 4.

Throughout this paper we identify a k-graph H with its edge set, for example writing $|H|$ for the number of edges of H and $e \in H$ to mean $e \in E(H)$. For a set A we write $\binom{A}{k}$ to denote the collection of subsets of A of size k. We omit floors and ceilings wherever these do not affect the argument.

2 The absorbing method

Loosely speaking, the absorbing method proceeds as follows to find a Hamilton cycle in a k-graph H (formal definitions will follow this outline). First, we find an 'absorbing path' in H. This is a path P_0 which can 'absorb' any small set S of vertices of H not in P_0, meaning that for any such S there is a path Q with vertex set $V(P_0) \cup S$ which has the same endvertices as P_0. Next, we cover almost all of the remaining vertices of H with vertex-disjoint paths $P_1, \ldots, P_r$ which do not intersect P_0. Having done this, we find 'connecting' paths $Q_0, \ldots, Q_r$ which are vertex-disjoint from each other and have only endvertices in common with the paths $P_0, \ldots, P_r$; these endvertices are chosen so that $P_0, Q_0, P_1, Q_1, \ldots, P_r, Q_r$ forms a cycle C. Since the paths $P_0, P_1, \ldots, P_r$ covered almost all of the vertices of H, only a small number of vertices of H are not in C, so we can apply the absorbing property of P_0 to 'absorb' these vertices and so give a Hamilton cycle in H. The fact that we can achieve each of these steps is guaranteed by our three key lemmas.

A *loose path* P is a k-graph whose vertices can be linearly ordered in such a way that each edge of P consists of k consecutive vertices, and so that each edge intersects the subsequent edge in precisely one vertex. Since this is the only type of path we will discuss in this paper, we will refer to such paths simply as paths. The *length* of a path is the number of edges it contains. If P is a path, an *endvertex pair* of P is a pair of vertices (x, y) such that x is a degree one vertex in the first edge of P and y is a degree one vertex in the last edge of P. (Note there are multiple endvertex pairs if $k > 2$.)

Our first key lemma states that, under the conditions of Theorem 3, we can find an 'absorbing path'.

Lemma 5 (Absorbing path lemma). *Fix $k \geq 2$, $0 < p, \alpha < 1$, and $0 < \varepsilon < \frac{\alpha^4 p^2}{400k^2}$. There exists $c_5 > 0$ depending only on p, α, and k and there exist $\mu > 0$ and n_0 depending on p, α, k, and ε such that the following holds. Let H be an (n, p, μ, α) k-graph with $n \geq n_0$. Then there exists a path P in H with endvertex pair (u, v) with at most εn vertices, with the property that for any set $X \subseteq V(H) \setminus V(P)$ such that $k - 1$ divides $|X|$ and $|X| \leq c_5 \varepsilon n$, there is a path P^* in H such that $V(P^*) = V(P) \cup X$ and P^* has endvertex pair (u, v).*

The next key lemma is a connecting lemma, stating that we can find a vertex set B which is disjoint from a given small set A (i.e. the vertices of the absorbing path) such that for

any small collection of pairs of vertices, we can use B to find vertex-disjoint constant-length paths with the given endvertex pairs.

Lemma 6 (Connecting lemma). *Fix $k \geq 2$, $0 < p, \alpha < 1$, and $0 < \varepsilon < \frac{\alpha^2 p}{20k}$. There exists $0 < c_6 < 1$ depending only on p, α, and k and there exist $\mu > 0$ and n_0 depending on p, α, k, and ε such that the following holds. Let H be an (n, p, μ, α) k-graph with $n \geq n_0$ and let $A \subseteq V(H)$ with $|A| \leq \varepsilon n$. Then there exists a set $B \subseteq V(H) \setminus A$ with $|B| \leq \varepsilon^2 n$ such that, for any $t \leq c_6 \varepsilon^2 n$ and any $2t$ distinct vertices $u_1, \ldots, u_t, v_1, \ldots, v_t$ of H, there exist vertex-disjoint paths $Q_1, \ldots, Q_t$ of length three such that, for each $1 \leq i \leq t$, (u_i, v_i) is an endvertex pair of Q_i and $V(Q_i) \subseteq B \cup \{u_i, v_i\}$.*

Finally, the path cover lemma states that we can cover almost all of the vertices of H by a constant number of vertex-disjoint paths.

Lemma 7 (Path cover lemma). *Fix $k \geq 2$, $0 < p < 1$, and $0 < \varepsilon < \frac{p}{2k \cdot k!}$. There exist $\mu > 0$ and n_0 such that the following holds. If H is an (n, p, μ) k-graph with $n \geq n_0$ then there exists a collection $\mathcal{P}$ of at most $1/\varepsilon^3$ vertex-disjoint paths in H such that at most $\varepsilon^2 n$ vertices of H are not covered by any path $P \in \mathcal{P}$.*

We can now prove Theorem 3 by combining the three key lemmas as outlined earlier.

Proof of Theorem 3. Let c_5 and c_6 be the constants from Lemmas 5 and 6 respectively. Define

$$\varepsilon := \frac{1}{2} \min \left\{ c_5, \frac{\alpha^4 p^2}{400 k^2}, \frac{p}{2k \cdot k!} \right\}.$$

(Note this is a valid definition because c_5 only depends on α, p, and k.) Also, assume $\mu > 0$ is small enough and $n_0 \geq \frac{2}{c_6 \varepsilon^5}$ is large enough to apply Lemmas 5, 6, and 7 (the latter with $(1 - 2\varepsilon)n_0$ here in place of n_0 there and $\frac{\mu}{(1-2\varepsilon)^k}$ here in place of μ there).

First, apply Lemma 5 to obtain a path P with endvertex pair (u, v), at most εn vertices, and such that for any set $X \subseteq V(H) \setminus V(P)$ where $k - 1$ divides $|X|$ and $|X| \leq c_5 \varepsilon n$, there is a path P^* in H such that $V(P^*) = V(P) \cup X$ and P^* has endvertex pair (u, v). Next, let $A = V(P)$ and apply Lemma 6 to obtain a set $B \subseteq V(H) \setminus A$ with $|B| \leq \varepsilon^2 n$ such that any collection of at most $c_6 \varepsilon^2 n$ pairs of vertices can be connected using B.

Let $H' = H \setminus (A \cup B)$ and let $n' = |V(H')|$. Note that $n' \geq (1 - 2\varepsilon)n$ and that H' is an $(n', p, \frac{\mu}{(1-2\varepsilon)^k})$ k-graph. Indeed, for any $X_1, \ldots, X_k \subseteq V(H')$, since H is (n, p, μ)-dense we have that

$$e(X_1, \ldots, X_k) \geq p|X_1| \cdots |X_k| - \mu n^k \geq p|X_1| \cdots |X_k| - \frac{\mu}{(1 - 2\varepsilon)^k}(n')^k.$$

Apply Lemma 7 to H' to produce a collection of vertex-disjoint paths $\mathcal{P} = \{P_1, \ldots, P_t\}$ such that $t \leq \frac{1}{\varepsilon^3}$ and at most $\varepsilon^2 n' \leq \varepsilon^2 n$ vertices of H' are not covered by paths in $\mathcal{P}$. Let (u_i, v_i) be an endvertex pair of P_i for each i. Also, recall that the absorbing path P has endvertex pair (u, v). By choice of B we can choose vertex-disjoint paths $Q_0, \ldots, Q_t$ such

that path Q_0 has endvertex pair (v, u_1), path Q_t has endvertex pair (v_t, u), and for each $1 \le i \le t-1$ the path Q_i has endvertex pair (v_i, u_{i+1}), and such that these endvertices are the only vertices of the paths Q_i which do not lie in B. Indeed, we can do this since the number of pairs is $t + 1 \le \frac{1}{\varepsilon^3} + 1 \le \frac{2}{\varepsilon^3}$, which is at most $c_6 \varepsilon^2 n$ by choice of n_0. Observe that $C = P, Q_0, P_1, Q_1, \ldots, P_t, Q_t$ is then a cycle in H.

Let X be the set of all vertices of H not covered by C. So X consists of the at most $\varepsilon^2 n$ vertices of H' not covered by $\mathcal{P}$, as well as the at most $\varepsilon^2 n$ vertices of B not covered by the paths Q_i. Since $\varepsilon \le c_5/2$, we have $|X| \le 2\varepsilon^2 n \le c_5 \varepsilon n$. Furthermore, since n and $|V(C)|$ are both divisible by $k-1$, $|X|$ is divisible by $k-1$ also. So by choice of P there is a path P^* in H such that $V(P^*) = V(P) \cup X$ and P^* has endvertex pair (u, v) (i.e. the same endvertex pair as P). Replacing P by P^* in C gives a loose Hamilton cycle in H. $\qquad \square$

3 Proofs of the key lemmas

This section contains proofs of the three key lemmas: the connecting lemma, the absorbing path lemma, and the path cover lemma. For both the connecting lemma and absorbing path lemma, the key element is an extension lemma for quasirandom hypergraphs proved by Lenz and Mubayi [18]. We actually only need the following special case of the lemma, which is obtained from the full version [18, Lemma 11] by taking $Z_{m+1} = \cdots = Z_f = V(H)$; it is easily checked that equations (1) and (2) of the full version of the lemma are then both satisfied.

Lemma 8 (Extension Lemma). *Fix $k \ge 2$, $0 < p, \alpha, \gamma < 1$ and integers $0 \le m \le f$. Suppose that F is an f-vertex k-graph with vertex set $V(F) = \{s_1, \ldots, s_m, t_{m+1}, \ldots, t_f\}$ such that*

(a) any edge $e \in F$ satisfies $|e \cap \{s_1, \ldots, s_m\}| \le 1$,
(b) any two distinct edges $e, e' \in F$ have $|e \cap e'| \le 1$, and
(c) any two distinct edges $e, e' \in F$ with $e \cap \{s_1, \ldots, s_m\} \ne \emptyset$ and $e' \cap \{s_1, \ldots, s_m\} \ne \emptyset$ satisfy $e \cap e' \cap \{t_{m+1}, \ldots, t_f\} = \emptyset$.

Then there exist n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ, α) k-graph with $n \ge n_0$ and let $y_1, \ldots, y_m \in V(H)$. Then the number of edge-preserving injections from $V(F)$ to $V(H)$ which map s_i to y_i for each $1 \le i \le m$ is at least

$$\alpha^{d_F(s_1)} \cdots \alpha^{d_F(s_m)} p^{|F| - \sum_{i=1}^{m} d_F(s_i)} n^{f-m} - \gamma n^{f-m}.$$

We will also require the following well-known concentration bound on sums of indicator random variables; the form we use is Corollary 2.3 of [12].

Lemma 9 (Chernoff bound). *Let $0 < p < 1$, let $X_1, \ldots, X_m$ be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for any $1 \le i \le m$, and let $X = \sum_{i=1}^{m} X_i$. Then for any $0 < a \le 3/2$ we have*

$$\mathbb{P}[|X - \mathbb{E}[X]| > a\mathbb{E}[X]] \le 2e^{-a^2 \mathbb{E}[X]/3}.$$

3.1 The connecting lemma

Let u and v be distinct vertices of a k-graph H, and let C be a set of $3k - 4$ vertices of H. We say that C is a *connecting set* for the pair (u, v) if $H[C \cup \{u, v\}]$ contains a path of length three with endvertex pair (u, v).

Lemma 10. *For any $k \geq 2$ and $0 < p, \alpha < 1$, there exist $\mu > 0$ and n_0 such that the following holds. If H is an (n, p, μ, α) k-graph with $n \geq n_0$, then for any $u, v \in V(H)$ with $u \neq v$ there are at least $\frac{1}{2}\alpha^2 p\binom{n}{3k-4}$ connecting sets for (u, v).*

Proof. Let $\gamma = \frac{1}{2}\alpha^2 p$, let F be a path of length three, let $m = 2$, let $f = 3k - 2$ and let (s_1, s_2) be an endvertex pair of F. Choose $\mu > 0$ and n_0 such that we can apply Lemma 8 with these inputs. Then for any $u, v \in V(H)$ with $u \neq v$, Lemma 8 states that there exist at least $\frac{1}{2}\alpha^2 pn^{3k-4}$ edge-preserving injections from $V(F)$ to $V(H)$ such that s_1 maps to u and s_2 maps to v. The image of $V(F) \setminus \{s_1, s_2\}$ under any such injection is a connecting set for (u, v). Since each connecting set is given by at most $(3k - 4)!$ injections, we conclude that the number of connecting sets for (u, v) is at least $\frac{1}{2(3k-4)!}\alpha^2 pn^{3k-4} \geq \frac{1}{2}\alpha^2 p\binom{n}{3k-4}$. $\square$

Proof of Lemma 6. Let $0 < p, \alpha < 1$, let $\nu = \frac{\alpha^2 p}{4}$ and let $c_6 = \frac{\nu}{50k^2}$. Also fix $0 < \varepsilon < \frac{\nu}{5k} = \frac{\alpha^2 p}{20k}$, and choose $\mu > 0$ small enough and $n_0 \geq 10k$ large enough to apply Lemma 10 and for the union bound later in the proof. Let H be an (n, p, μ, α) k-graph with $n \geq n_0$ and fix $A \subseteq V(H)$ with $|A| \leq \varepsilon n$. Define $a = |A|$ and $q = \frac{\varepsilon^2 n}{20k}\binom{n}{3k-4}^{-1}$.

Now form a collection $\mathcal{B} \subseteq \binom{V(H) \setminus A}{3k-4}$ by including each element of $\binom{V(H) \setminus A}{3k-4}$ with probability q and independently of all other choices. The expected size of $\mathcal{B}$ is $q\binom{n-a}{3k-4} \leq \frac{\varepsilon^2 n}{20k}$ so, by Markov's inequality, with probability at least $3/4$ we have $|\mathcal{B}| \leq \frac{\varepsilon^2 n}{5k}$. Similarly, the expected number of ordered pairs of elements from $\mathcal{B}$ which intersect is at most

$$q^2 \binom{n-a}{3k-4}(3k - 4)\binom{n-a}{3k-5} = q^2(3k - 4)\frac{3k-4}{n-a-3k+5}\binom{n-a}{3k-4}^2 \leq \frac{\varepsilon^4 n}{20}$$

so, by Markov's inequality, with probability at least $3/4$ at most $\frac{\varepsilon^4 n}{5} < \frac{\nu^2 \varepsilon^2 n}{125k^2}$ members of $\mathcal{B}$ intersect another member of $\mathcal{B}$.

For each pair of distinct vertices $u, v \in V(H)$ choose a collection $\Gamma_{u,v}$ of $\nu\binom{n}{3k-4}$ connecting sets C for (u, v) with $C \cap A = \emptyset$. This is possible since by Lemma 10 there are at least $2\nu\binom{n}{3k-4}$ connecting sets for (u, v), and at most $|A|\binom{n}{3k-5} \leq \varepsilon n\frac{3k-4}{n-3k+5}\binom{n}{3k-4} < \nu\binom{n}{3k-4}$ of these sets contain a vertex of A. Then for any u and v the expected size of $\Gamma_{u,v} \cap \mathcal{B}$ is $q\nu\binom{n}{3k-4} = \frac{\nu\varepsilon^2 n}{20k}$, so by Lemma 9 we have

$$\mathbb{P}\left[\left||\Gamma_{u,v} \cap \mathcal{B}| - \frac{\nu\varepsilon^2 n}{20k}\right| > \frac{\nu\varepsilon^2 n}{40k}\right] \leq 2e^{-\nu\varepsilon^2 n/480k}.$$

Taking a union bound over all of the $\binom{n}{2}$ pairs (u, v), we conclude that with probability at least $3/4$ the collection $\mathcal{B}$ satisfies $|\Gamma_{u,v} \cap \mathcal{B}| \geq \frac{\nu\varepsilon^2 n}{40k}$ for every $u \neq v$.

We may therefore fix a collection $\mathcal{B}$ which satisfies each of the three described events of probability at least $3/4$. We then form $\mathcal{B}' \subseteq \mathcal{B}$ by deleting from $\mathcal{B}$ the at most $\frac{\nu^2 \varepsilon^2 n}{125 k^2}$ members of $\mathcal{B}$ which intersect another member of $\mathcal{B}$. The family $\mathcal{B}'$ then satisfies $|\mathcal{B}'| \leq |\mathcal{B}| \leq \frac{\varepsilon^2 n}{5k}$ and has the property that for any distinct vertices $u, v \in V(H)$ at least $\frac{\nu \varepsilon^2 n}{40k} - \frac{\nu^2 \varepsilon^2 n}{125 k^2} \geq \frac{\nu \varepsilon^2 n}{50k} \geq c_6 \varepsilon^2 n$ members of $\mathcal{B}'$ are connecting sets for (u, v). Let $B = \bigcup \mathcal{B}'$, so $|B| = (3k - 4)|\mathcal{B}'| \leq \varepsilon^2 n$. Finally, for any $t \leq c_6 \varepsilon^2 n$ and any $2t$ distinct vertices $u_1, \ldots, u_t, v_1, \ldots, v_t$ of H, we can greedily choose for each pair (u_i, v_i) a unique member C_i of $\mathcal{B}'$ which is a connecting set for (u_i, v_i); the chosen connecting sets give the required paths. $\qquad\square$

3.2 The absorbing path lemma

To prove the absorbing path lemma, we will apply Lemma 8 with the following k-graph F.

Lemma 11. *For any $k \geq 2$ there exists a k-graph F with $3k^2 - 7k + 5$ vertices and $5k - 7$ edges, a set $S \subseteq V(F)$ of $k - 1$ vertices, and distinct vertices $u, v \in V(F)$, such that*

(a) *F contains a path with endvertex pair (u, v) as a spanning subgraph,*
(b) *$F \setminus S$ contains a path with endvertex pair (u, v) as a spanning subgraph,*
(c) *no edge $E \in F$ has $|E \cap S| \geq 2$,*
(d) *for any distinct edges $E_1, E_2 \in F$ we have $|E_1 \cap E_2| \leq 1$, and*
(e) *for any distinct edges $E_1, E_2 \in F$ with $E_1 \cap S \neq \emptyset$ and $E_2 \cap S \neq \emptyset$ we have $(E_1 \cap E_2) \setminus S = \emptyset$.*

Proof. To define the k-graph F, we first define a 2-graph F' and then form F by adding $k - 2$ vertices to each edge of F'. Let $V(F') = \{a_i, s_i, t_i : 1 \leq i \leq k - 1\} \cup \{b_i : 1 \leq i \leq k - 2\}$ and $E(F') = \{a_i s_i, s_i b_i, a_i t_i, t_i b_i, b_i a_{i+1} : 1 \leq i \leq k - 2\} \cup \{a_{k-1} s_{k-1}, a_{k-1} t_{k-1}, s_{k-1} t_{k-1}\}$. Also, let $u = a_1$, $v = t_{k-1}$, and $S = \{s_1, \ldots, s_{k-1}\}$. Note that $F' \setminus S$ has a spanning (u, v)-path and F' has a (u, v)-path which covers all vertices except $t_1, \ldots, t_{k-2}$.
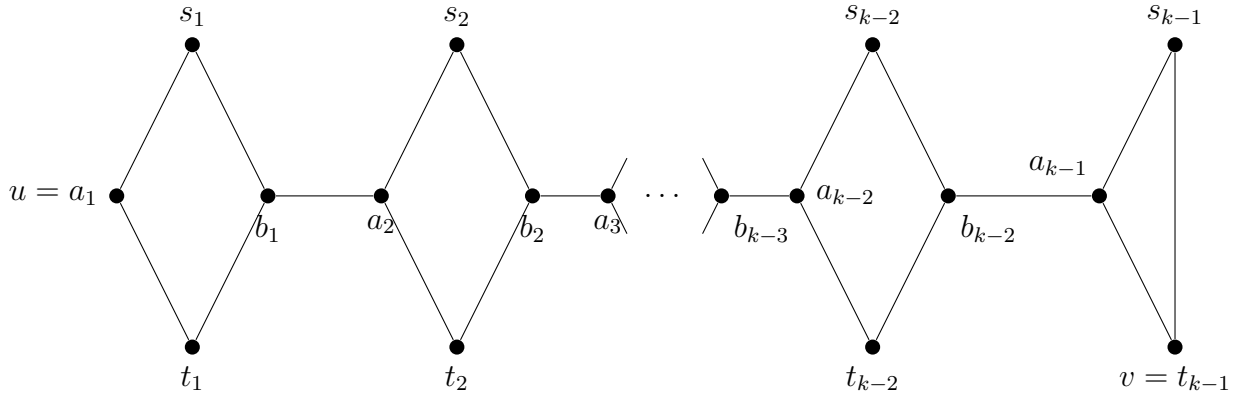

Figure 1: The graph F'.

Let $V(F) = V(F') \cup \{c_{i,j} : 0 \leq i \leq 2k - 4, 1 \leq j \leq k - 2\} \cup \{d_{i,j} : 1 \leq i, j \leq k - 2\}$. We now insert exactly $k - 2$ vertices into each edge of F' to form the hyperedges of F. For

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example, into the edge $a_1 s_1 \in E(F')$, we insert the vertex set $X_0 = \{c_{0,1}, c_{0,2}, \ldots, c_{0,k-2}\}$. In general, define

- $C := \{c_{i,j} : 0 \le i \le 2k - 4, 1 \le j \le k - 2\}$,
- $D := \{d_{i,j} : 1 \le i, j \le k - 2\}$,
- $W_i := \{d_{i,1}, \ldots, d_{i,k-2}\}$ for all $1 \le i \le k - 2$,
- $X_i := \{c_{i,1}, \ldots, c_{i,k-2}\}$ for all $0 \le i \le 2k - 4$,
- $Y_i := \{c_{i+1,1}, c_{i+2,2}, \ldots, c_{i+j,j}, \ldots, c_{i+k-2,k-2}\}$ for all $0 \le i \le 2k - 4$, where the first index of c is taken modulo $2k - 3$, and
- $Z := \{t_1, \ldots, t_{k-2}\}$.

Let the edges of F be the following:

- $\{a_i, s_i\} \cup X_{2i-2}$ for all $1 \le i \le k - 1$,
- $\{a_i, t_i\} \cup Y_{2i-2}$ for all $1 \le i \le k - 1$,
- $\{s_i, b_i\} \cup X_{2i-1}$ for all $1 \le i \le k - 2$,
- $\{t_i, b_i\} \cup Y_{2i-1}$ for all $1 \le i \le k - 2$,
- $\{b_i, a_{i+1}\} \cup W_i$ for all $1 \le i \le k - 2$,
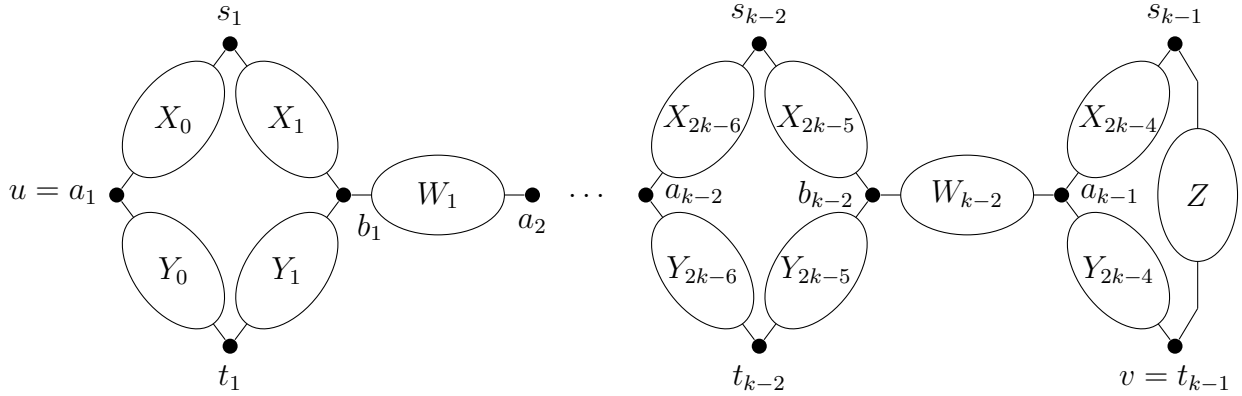- $\{s_{k-1}, t_{k-1}\} \cup Z$.

Figure 2: The k-graph F (note that the sets of added vertices are not disjoint).

We now verify that the properties stated in the lemma hold for F. Recall that $V(F) = V(F') \cup C \cup D$, the sets $X_0, \ldots, X_{2k-4}$ partition C, the sets $Y_0, \ldots, Y_{2k-4}$ also partition C, and $W_1, \ldots, W_{k-2}$ partition D. First note that, by construction, no two vertices of S lie in the same hyperedge of F, so we have (c).

For (a), recall that F' contains a (u, v)-path covering all vertices except the vertices in Z. The vertices in Z are inserted into the edge $s_{k-1} t_{k-1}$, the sets $X_0, \ldots, X_{2k-4}$ partition C, and the sets $W_1, \ldots, W_{k-2}$ partition D so the corresponding hyperedges in F form a spanning path with endvertex pair (u, v). In Figure 2, the path consists of the upper hyperedges.

For (b), recall that $F' \setminus S$ contains a spanning (u, v)-path and similarly the corresponding hyperedges in F form a spanning path with endvertex pair (u, v). Indeed, $Y_0, \ldots, Y_{2k-4}$

partition C and $W_1, \ldots, W_{k-2}$ partition D, so all vertices of $V(F') \setminus S$ are used exactly once in this path. In Figure 2, the path consists of the lower hyperedges.

For (d), consider two distinct hyperedges E_1 and E_2 of F and let E_1' and E_2' be the corresponding edges in F' (that is, E_1 was formed by adding vertices to E_1' and similarly for E_2' and E_2). If E_1 or E_2 was formed by inserting a W_i, then $|E_1 \cap E_2| = |E_1' \cap E_2'| \leq 1$. Indeed, each vertex in W_i is inserted into at most one hyperedge so will never contribute to the intersection, and F' is a graph so $|E_1' \cap E_2'| \leq 1$. Now suppose that E_1 or E_2 was formed by inserting Z, say $E_1 = \{s_{k-1}, t_{k-1}\} \cup Z$. If $Z \cap E_2 = \emptyset$ then similarly we have $|E_1 \cap E_2| = |E_1' \cap E_2'| \leq 1$. On the other hand, if $Z \cap E_2 \neq \emptyset$ then we have $t_i \in E_2$ for some $1 \leq i \leq k - 2$, so either $E_2 = \{a_i, t_i\} \cup Y_{2i-2}$ or $E_2 = \{t_i, b_i\} \cup Y_{2i-1}$, and in either case we have $|E_1 \cap E_2| = 1$.

Now consider when E_1 and E_2 are both formed by inserting one of the Xs or Ys. The sets $X_0, \ldots, X_{2k-4}$ form a partition of C, so if E_1 and E_2 were both formed by inserting one of the Xs, then $|E_1 \cap E_2| = |E_1' \cap E_2'| \leq 1$. Similarly, the sets $Y_0, \ldots, Y_{2k-4}$ form a partition of C, so if both E_1 and E_2 were formed by inserting one of the Ys, then $|E_1 \cap E_2| = |E_1' \cap E_2'| \leq 1$. Thus without loss of generality assume E_1 was formed by inserting X_i and E_2 was formed by inserting Y_ℓ. By construction we know that $|X_i \cap Y_\ell| \leq 1$ since $X_i \cap Y_\ell = \{c_{i, i-\ell}\}$ if $1 \leq i - \ell \leq k - 2$ and is empty otherwise. Thus if $E_1' \cap E_2' = \emptyset$, then $|E_1 \cap E_2| \leq 1$. If $|E_1' \cap E_2'| = 1$, then by construction we must have $i = \ell$ since that is the only situation in which graph edges which insert one of the Xs and one of the Ys share a vertex. Since $X_i \cap Y_i = \emptyset$, we have that $|E_1 \cap E_2| \leq 1$.

Finally, for (e) consider $E_1, E_2 \in E(F)$ with $E_1 \neq E_2$, $E_1 \cap S \neq \emptyset$ and $E_2 \cap S \neq \emptyset$. If both E_1 and E_2 were formed by inserting Xs, then since $X_0, \ldots, X_{2k-4}$ is a partition of C we have that E_1 and E_2 do not intersect outside S. Now assume without loss of generality that $E_1 = \{s_{k-1}, t_{k-1}\} \cup Z$. In this case, since $Z \cap C = \emptyset$, we also have that $(E_1 \cap E_2) \setminus S = \emptyset$. $\square$

For the remainder of this subsection, fix such a k-graph F, a set $S \subseteq V(F)$ and distinct vertices $u, v \in V(F)$. For any k-graph H and any set $Y \subseteq V(H)$ with $|Y| = k - 1$, we say that a set $Z \subseteq V(H)$ with $|Z| = 3k^2 - 8k + 6$ is an *absorbing set for* Y if $H[Y \cup Z]$ contains a copy of F in which Y corresponds to S.

Lemma 12. *For any $k \geq 2$ and $0 < p, \alpha < 1$, there exist $\mu > 0$ and n_0 such that the following holds. If H is an (n, p, μ, α) k-graph with $n \geq n_0$, then for any set $Y \subseteq V(H)$ with $|Y| = k - 1$, there are at least $\frac{1}{2} \alpha^{2k-2} p^{3k-5} \binom{n}{3k^2 - 8k + 6}$ absorbing sets for Y.*

Proof. Let $\gamma = \frac{1}{2} \alpha^{2k-2} p^{3k-5}$, and choose $\mu > 0$ and n_0 for which we can apply Lemma 8 with these inputs and our chosen k-graph F. Then for any set $Y \subseteq V(H)$ of size $|Y| = k - 1$, Lemma 8 states that there are at least $\frac{1}{2} \alpha^{2k-2} p^{3k-5} n^{3k^2 - 8k + 6}$ edge-preserving injections from $V(F)$ to $V(H)$ such that the vertices of S are mapped to the vertices of Y. The image of $V(F) \setminus S$ under such an injection is an absorbing set for Y, and each absorbing set is given by at most $(3k^2 - 8k + 6)!$ injections, so we conclude that the number of absorbing sets for Y is at least $\frac{1}{2(3k^2 - 8k + 6)!} \alpha^{2k-2} p^{3k-5} n^{3k^2 - 8k + 6} \geq \frac{1}{2} \alpha^{2k-2} p^{3k-5} \binom{n}{3k^2 - 8k + 6}$. $\square$

Proof of Lemma 5. Let $0 < p, \alpha < 1$, let $\nu = \frac{1}{2}\alpha^{2k-2}p^{3k-5}$, and let c_6 be the constant from Lemma 6 for these values of p and α. Define $c_5 = \frac{c_6\nu^2}{10k^2}$, and fix ε with $0 < \varepsilon < \frac{\alpha^4 p^2}{400k^2}$. We will apply Lemma 6 with $\sqrt{\varepsilon}$ in place of ε; note for this that $0 < \sqrt{\varepsilon} < \frac{\alpha^2 p}{20k}$. Assume that μ is small enough and $n_0 \geq 10k^2$ is large enough for this application of Lemma 6, and also to apply Lemma 12 and for the union bound later in the proof. Let H be an (n, p, μ, α) k-graph with $n \geq n_0$ and let

$$q := \frac{c_6\varepsilon\nu n}{4k^2}\binom{n}{3k^2 - 8k + 6}^{-1}.$$

Now form a collection $\mathcal{Z}$ of subsets of $V(H)$ by including each set $Z \subseteq V(H)$ of size $3k^2 - 8k + 6$ at random with probability q and independently of all other choices. The expected size of $\mathcal{Z}$ is $q\binom{n}{3k^2-8k+6} = \frac{c_6\varepsilon\nu n}{4k^2}$ so by Markov's inequality, with probability at least $3/4$ we have $|\mathcal{Z}| \leq \frac{c_6\varepsilon\nu n}{k^2}$. Similarly, the expected number of ordered pairs of members of $\mathcal{Z}$ which intersect is

$$q^2\binom{n}{3k^2 - 8k + 6}(3k^2 - 8k + 6)\binom{n}{3k^2 - 8k + 5} \leq \left(\frac{c_6\varepsilon\nu n}{4k^2}\right)^2 \frac{(3k^2 - 8k + 6)^2}{n - (3k^2 - 8k + 5)} \leq c_6^2\varepsilon^2\nu^2 n,$$

so by Markov's inequality, with probability at least $3/4$ we have that at most $4c_6^2\varepsilon^2\nu^2 n \leq \frac{c_6\varepsilon\nu^2 n}{100k^2}$ members of $\mathcal{Z}$ intersect another member of $\mathcal{Z}$.

For each $Y \subseteq V(H)$ of size $|Y| = k - 1$, choose a collection Γ_Y of $\nu\binom{n}{3k^2-8k+6}$ absorbing sets for Y. Such a subset Γ_Y exists by Lemma 12. Then for any fixed Y the expected size of $\Gamma_Y \cap \mathcal{Z}$ is $q\nu\binom{n}{3k^2-8k+6} = \frac{c_6\varepsilon\nu^2 n}{4k^2}$, so by Lemma 9 we have

$$\mathbb{P}\left[\left||\Gamma_Y \cap \mathcal{Z}| - \frac{c_6\varepsilon\nu^2 n}{4k^2}\right| > \frac{c_6\varepsilon\nu^2 n}{8k^2}\right] \leq 2e^{-c_6\varepsilon\nu^2 n/96k^2}.$$

Taking a union bound over all of the $\binom{n}{k-1}$ sets Y of $k - 1$ vertices of H, we find that with probability at least $3/4$ the collection $\mathcal{Z}$ contains at least $\frac{c_6\varepsilon\nu^2 n}{8k^2}$ absorbing sets for each Y.

We may therefore fix a collection $\mathcal{Z}$ such that each of the three events described as having probability at least $3/4$ hold. Let $\mathcal{Z}'$ be the subfamily of $\mathcal{Z}$ obtained by deleting the at most $\frac{c_6\varepsilon\nu^2 n}{100k^2}$ members of $\mathcal{Z}$ which intersect another member of $\mathcal{Z}$, as well as any member of $\mathcal{Z}$ which is not an absorbing set for any $Y \in \binom{V(H)}{k-1}$, and define $t = |\mathcal{Z}'|$. Then $t \leq |\mathcal{Z}| \leq \frac{c_6\varepsilon\nu n}{k^2} \leq c_6\varepsilon n$ and, for any $Y \subseteq V(H)$ with $|Y| = k - 1$, at least $\frac{c_6\varepsilon\nu^2 n}{8k^2} - \frac{c_6\varepsilon\nu^2 n}{100k^2} \geq \frac{c_6\nu^2\varepsilon n}{10k^2} = c_5\varepsilon n$ members of $\mathcal{Z}'$ are absorbing sets for Y.

Write $\mathcal{Z}' = \{Z_1, \ldots, Z_t\}$ and $A = \bigcup \mathcal{Z}'$. For each i, since Z_i is an absorbing set for some $Y \in \binom{V(H)}{k-1}$, we know that Z_i induces a copy of $F \setminus S$ in H; let u_i and v_i be the vertices corresponding to u and v in this copy. In particular $H[Z_i]$ contains a spanning path P_i with endvertex pair (u_i, v_i). We now apply Lemma 6 with $\sqrt{\varepsilon}$ in place of ε to obtain vertex-disjoint paths Q_i of length 3 for $1 \leq i \leq t - 1$ such that the endvertex pair of Q_i is (v_i, u_{i+1}), and such that the paths Q_i contain no vertices of A except for the vertices of these endvertex pairs. This is possible because $|A| \leq t(3k^2 - 8k + 6) \leq \sqrt{\varepsilon}n$ and $t \leq c_6\varepsilon n$.

Having chosen the paths Q_i, we define the path $P = P_1, Q_1, P_2, Q_2, \ldots, P_{t-1}, Q_{t-1}, P_t$, so P has $t(3k^2 - 8k + 6) + (t-1)(3k - 4) \leq \varepsilon n$ vertices and endvertex pair (u, v), where $u = u_1$ and $v = v_t$.

Now consider any set $X \subseteq V(H) \setminus V(P)$ such that $|X| \leq c_5 \varepsilon n$ and $|X|$ is divisible by $k-1$. Partition X arbitrarily into sets $Y_1, \ldots, Y_{|X|/(k-1)}$ of size $k-1$ and greedily assign each Y_i to a unique member $Z_{f(i)}$ of $\mathcal{Z}'$ which is an absorbing set for Y_i. Such an assignment is possible since for each Y_i at least $c_5 \varepsilon n$ members of $\mathcal{Z}'$ are absorbing sets for Y. Since $Z_{f(i)}$ is an absorbing set for Y_i there is then a path $P^*_{f(i)}$ in H with $V(P^*_{f(i)}) = V(P_{f(i)}) \cup Y_i$ which has the same endvertex pair $(u_{f(i)}, v_{f(i)})$ as $P_{f(i)}$. By replacing the path $P_{f(i)}$ with $P^*_{f(i)}$ in P for each $1 \leq i \leq t$ we obtain a path P^* in H with the same endvertex pair (u, v) as P such that $V(P^*) = V(P) \cup X$, as required. $\qquad\square$

3.3 The path cover lemma

Since a quasirandom k-graph remains quasirandom even after the deletion of almost all of its vertices, the main difficulty in proving the path cover lemma is to show that H contains a single path of linear length. We can then greedily choose and delete paths to obtain the desired collection of paths.

Lemma 13. *For any $k \geq 2$, $0 < p < 1$ and any $0 < \varepsilon < \frac{p}{2k \cdot k!}$, there exists $\mu > 0$ and n_0 such that the following holds. Let H be an (n, p, μ) k-graph with $n \geq n_0$, and let $X \subseteq V(H)$ be such that $|X| \geq \varepsilon^2 n$. Then $H[X]$ contains a path of length at least $\varepsilon^3 n$.*

Proof. Let $\mu = \frac{p}{2} \varepsilon^{2k}$, let $H' = H[X]$, and let $m = |X|$ (so $m \geq \varepsilon^2 n$). We first claim that the average vertex degree of vertices in H' is at least $\frac{p}{2} \frac{m^k}{(k-1)!}$. Indeed, since H is (p, μ)-dense, $e(X, \ldots, X) \geq pm^k - \mu n^k \geq \frac{p}{2} m^k$. Since $|H'| = \frac{1}{k!} e(X, \ldots, X)$, the average vertex degree of H' is at least $\frac{p}{2} \frac{m^{k-1}}{(k-1)!}$. It follows that there exists a subgraph H'' of H' with minimum vertex degree $\delta_1(H'') \geq \frac{p}{2k} \frac{m^{k-1}}{(k-1)!}$, since the deletion of any vertex whose vertex degree is smaller than this increases the average vertex degree.

Now let P be a longest path in H'' and let (x, y) be an endvertex pair of P. Then all edges of H'' containing y must also contain another vertex of P or we could extend P. If the length of P is less than $\varepsilon^3 n$, then P has fewer than $k\varepsilon^3 n$ vertices, which implies that $d_{H''}(y) \leq k\varepsilon^3 n m^{k-2} \leq k\varepsilon m^{k-1}$. Since $\varepsilon < \frac{p}{2k^2 (k-1)!}$, this contradicts the minimum degree of H''. So P is a path in $H[X]$ of length at least $\varepsilon^3 n$. $\qquad\square$

Proof of Lemma 7. We repeatedly apply Lemma 13 to choose a path of length at least $\varepsilon^3 n$ in H to add to $\mathcal{P}$. In each application we take X to be the set of all vertices of H not covered by previously-chosen members of $\mathcal{P}$, and we continue until $|X| < \varepsilon^2 n$, at which point we can no longer apply Lemma 13. At this point at most $\varepsilon^2 n$ vertices of H are not covered by paths in $\mathcal{P}$. Moreover, the paths in $\mathcal{P}$ are vertex-disjoint by our choice of X, and $\mathcal{P}$ therefore has size at most $1/\varepsilon^3$ since each member of $\mathcal{P}$ has length at least $\varepsilon^3 n$. $\qquad\square$

This completes the proof of the last of our three key lemmas, and so concludes the proof of Theorem 3.

4 Avoiding Hamilton ℓ-cycles

In this section we prove Proposition 4 using the following construction, which was presented for 3-graphs in [18].

Construction. For integers ℓ, k and n with $2 \le \ell \le k - 1$, we form a random k-graph $H = H(n, k, \ell)$ on n vertices as follows. Let X and Y be disjoint sets such that $|X \cup Y| = n$, $n/2 - 1 \le |X| < n/2 + 1$, and $|X|$ is odd. Let $V := X \cup Y$, and let $G = G^{(\ell)}(n, \frac{1}{2})$ be the random ℓ-graph on vertex set V in which each edge is included with probability $1/2$, independently of all other choices. We take V to be the vertex set of H, and say that a k-tuple e of vertices of V forms an edge of H if either

(a) $|e \cap X|$ is even and e induces a clique in G, or
(b) $|e \cap X|$ is odd and e induces an independent set in G.

The following proposition shows that, whenever $k - \ell$ divides k and $2k$ divides n, the graph H constructed above does not contain a Hamilton ℓ-cycle.

Proposition 14. *Let $k \ge 3$ and $2 \le \ell \le k - 1$ be integers such that $k - \ell$ divides k. Then for any integer n which is divisible by $2k$ the k-graph $H = H(n, k, \ell)$ does not contain a Hamilton ℓ-cycle.*

Proof. Let $L := n/(k - \ell)$. Suppose that C is a Hamilton ℓ-cycle in H with edges $e_1, \ldots, e_L$ indexed in the order they appear in C. For each $1 \le i \le L$ define the *block* $B_i := e_i \setminus e_{i+1}$ (with indices taken modulo L), so $|B_i| = k - \ell$. Then for any $1 \le i \le L$ we have $e_i = B_i \cup \cdots \cup B_{i+k/(k-\ell)-1}$ and $e_{i+1} = B_{i+1} \cup \cdots \cup B_{i+k/(k-\ell)}$. Since $|e_i \cap e_{i+1}| = \ell$, by definition of H either both e_i and e_{i+1} induce cliques in G, or both e_i and e_{i+1} induce independent sets in G. In either case we find that both $|e_i \cap X|$ and $|e_{i+1} \cap X|$ have the same parity, and therefore that $|B_i \cap X|$ and $|B_{i+k/(k-\ell)} \cap X|$ have the same parity. Since $L(k - \ell)/k = n/k$ is even, it follows that for any $1 \le i \le k/(k - \ell)$, the set

$$D_i = \bigcup_{0 \le j < L(k-\ell)/k} B_{i+jk/(k-\ell)}$$

has the property that $|D_i \cap X|$ is even. However, since the sets $D_1, \ldots, D_{k/(k-\ell)}$ partition the vertex set V, it follows that $|V \cap X|$ is even also, contradicting the fact that $|X|$ is odd. $\square$

Proof of Proposition 4. By Proposition 14 it suffices to show that for any $\mu > 0$ there exists n_0 such that if $n \ge n_0$ then the random graph $H(n, k, \ell)$ satisfies properties (a) and (b) of Proposition 4 with positive probability. For the case $k = 3$ short proofs of these statements were given in [18, Lemmas 20 and 21], and similar arguments hold for any $k \ge 3$ (we omit the details). $\square$

We conclude by giving the construction mentioned in Section 1.3, of an (n, p, μ) k-graph H' with $\delta_j(H') \ge \alpha \binom{n}{k-j}$ for any $1 \le j \le \ell - 1$ which contains a vertex x such that the intersection of any edge containing x and any edge not containing x has size at

most $\ell - 1$ (where $2 \leq \ell \leq k - 1$); this is similar to the construction given above. Fix a set V of n vertices, and choose some $x \in V$. Let $G = G^{(\ell)}(n, \frac{1}{2})$ be the random ℓ-graph on vertex set V in which each edge is included with probability $1/2$, independently of all other choices. We define H' to be the k-graph on V whose edges are all k-tuples e of vertices of V such that either $x \in e$ and e induces a clique in G, or $x \notin e$ and e induces an independent set in G. Since a clique and an independent set in an ℓ-graph can have at most $\ell - 1$ vertices in common, H' has the desired property that the intersection of any edge containing x and any edge not containing x has size at most $\ell - 1$. Moreover, standard probabilistic arguments similar to those of [18, Lemmas 20 and 21] show that, for any fixed $\mu > 0$, with high probability H' is indeed an $(n, 2^{-\binom{k}{\ell}}, \mu)$ k-graph with $\delta_j(H') \geq 2^{-\binom{k}{\ell}} \binom{n}{k-j}$ for any $1 \leq j \leq \ell - 1$.

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