# The BEM with graded meshes for the electric field integral equation on polyhedral surfaces

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# The BEM with graded meshes for the electric field integral equation on polyhedral surfaces

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**Abstract**

We consider the variational formulation of the electric field integral equation on a Lipschitz polyhedral surface $\Gamma$. We study the Galerkin boundary element discretisations based on the lowest-order Raviart-Thomas surface elements on a sequence of anisotropic meshes algebraically graded towards the edges of $\Gamma$. We establish quasi-optimal convergence of Galerkin solutions under a mild restriction on the strength of grading. The key ingredient of our convergence analysis are new componentwise stability properties of the Raviart-Thomas interpolant on anisotropic elements.

*Key words*: electromagnetic scattering, electric field integral equation, Galerkin discretisation, boundary element method, Raviart-Thomas interpolation, anisotropic elements, graded mesh
*AMS Subject Classification*: 65N38, 65N12, 78M15

## 1 Introduction

In this paper, we study the Galerkin boundary element method (BEM) on graded meshes for numerical solution of the electric field integral equation (EFIE) on a Lipschitz polyhedral surface $\Gamma$ in $\mathbb{R}^3$ (i.e., $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron). The EFIE models the scattering of time-harmonic electromagnetic waves at a perfect conductor, and the Galerkin BEM is widely used in engineering practice for simulation of this physical phenomenon.

The Galerkin BEM considered in this paper employs $\mathrm{div}_\Gamma$-conforming lowest-order Raviart-Thomas surface elements to discretise the variational formulation of the EFIE (known as Rumsey's principle). This approach is referred to as the natural BEM for the

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EFIE (there exist other approaches, e.g., based on a stable mixed reformulation of Rumsey's principle, see [16]). Non-coercivity of the bilinear form in Rumsey's principle (due to the infinite-dimensional kernel of $\mathrm{div}_\Gamma$, cf. (2.1)) significantly complicates the convergence analysis of Galerkin schemes. This problem can be overcome by using appropriate decompositions of vector fields in order to isolate the kernel of $\mathrm{div}_\Gamma$ (we refer to discussion in [18, Section 3], to an abstract theory in [13, 12], and we outline available techniques for constructing such decompositions in Section 4). These ideas have led to major advances in the convergence analysis and a priori error analysis of the BEM for the EFIE on (open and closed) Lipschitz surfaces, see [24, 16, 13, 19, 18] for the $h$-version of the BEM and [5, 8, 4, 7] for high-order methods ($p$- and $hp$-BEM). All these results, however, assume shape-regularity of the underlying meshes on $\Gamma$.

It is well-known that convergence rates of the $h$-BEM with quasi-uniform and shape-regular meshes are bounded by the poor regularity of solutions to the EFIE on non-smooth surfaces. For example, on a closed polyhedral surface $\Gamma = \partial\Omega$, the solution may be only $\mathbf{H}^\varepsilon(\Gamma)$-regular (with a small $\varepsilon > 0$ in the case of non-convex polyhedron $\Omega$, cf. [20, Section 4.4.2]), and convergence rate of the $h$-BEM is only $\frac{1}{2} + \varepsilon$ in this case, whereas in the case of smooth solutions the lowest-order $h$-BEM converges with the optimal rate of $\frac{3}{2}$ (see [24, Theorem 8.2] and [4, Theorem 2.2]). Taking the cue from the $h$-BEM results for the Laplacian (see [28, 27]), we expect that an optimal convergence rate of the $h$-BEM for the EFIE can be recovered on the non-smooth surface $\Gamma$, if one employs the meshes that are appropriately graded towards the edges of $\Gamma$. These meshes contain highly anisotropic elements along the edges of $\Gamma$, and none of the results mentioned above is applicable in this case. Moreover, to the best of our knowledge, the quasi-optimality of the Galerkin $h$-BEM with graded meshes for the EFIE has not been studied in the literature, and with this paper we fill this theoretical gap.

In the next section, we introduce necessary notation and formulate the EFIE in its variational form. In Section 3, we construct graded meshes on $\Gamma$, introduce the boundary element space, and formulate the main result of the paper— Theorem 3.1—that establishes quasi-optimal convergence of Galerkin solutions on graded meshes. The proof of Theorem 3.1 follows the approach suggested in [13, 19], summarised in [18, Section 9.1], and extended to a general class of operators in [12, Section 3]. At the heart of this approach is the decomposition technique described in Section 4. Section 5 is instrumental in the construction of the corresponding discrete decomposition: here we establish new stability properties of the Raviart-Thomas interpolant of low-regular vector fields on anisotropic elements. In Section 6, we introduce the discrete decomposition and complete the proof of Theorem 3.1. An essential ingredient here is the projection operator $\mathcal{Q}_h$ with enhanced approximation properties (see Proposition 6.1). The proof of Proposition 6.1 is given in Section 7.

# 2 The electric field integral equation

The variational formulation of the EFIE is posed on the Hilbert space

$$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}.$$

Here, $\mathrm{div}_\Gamma$ denotes the surface divergence operator, $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ is the dual space of $\mathbf{H}_\parallel^{1/2}(\Gamma)$ (the tangential trace space of $\mathbf{H}^1(\Omega)$ on $\Gamma$, see [14, 17]), and $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$. The space $\mathbf{X}$ is equipped with its graph norm $\|\cdot\|_\mathbf{X}$. We refer to [14, 15, 17, 18] for definitions and properties of $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and other involved trace spaces. We also recall from [14, 17] that $\mathbf{X}$ is the natural tangential trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$.

In the present article, we use the same notation as in [4], where we recalled definitions of the full range of Sobolev spaces and differential operators needed for convergence analysis of the BEM for the EFIE (see Section 3.1 therein). In particular, we use a traditional notation for the Sobolev spaces (of scalar functions) $H^s$, $\tilde{H}^s$ $(s \in [-1, 1])$, $H_0^s$ $(s \in (0, 1])$ and their norms on Lipschitz domains and surfaces (see [25, 26]). The norm and inner product in $L^2(D) = H^0(D)$ on a domain or surface $D$ will be denoted by $\|\cdot\|_{0,D}$ and $(\cdot,\cdot)_{0,D}$, respectively. The notation $(\cdot,\cdot)_{0,D}$ will be used also for appropriate duality pairings extending the $L^2(D)$-pairing for functions on $D$.

For vector fields we will use boldface symbols (e.g., $\mathbf{u} = (u_1, u_2)$), and the spaces (or sets) of vector fields are also denoted in boldface (e.g., $\mathbf{H}^s(D) = (H^s(D))^2$ with $D \subset \mathbb{R}^2$). The norms and inner products in these spaces are defined componentwise. The notation for the Sobolev spaces of tangential vector fields on $\Gamma$ follows [14, 15, 17]. In particular, $\mathbf{L}_\mathrm{t}^2(\Gamma)$ denotes the space of two-dimensional, tangential, square integrable vector fields on $\Gamma$. The norm and inner product in this space will be denoted by $\|\cdot\|_{0,\Gamma}$ and $(\cdot,\cdot)_{0,\Gamma}$, respectively, and we will also use $(\cdot,\cdot)_{0,\Gamma}$ for appropriate duality pairings extending the $\mathbf{L}_\mathrm{t}^2(\Gamma)$-pairing for tangential vector fields on $\Gamma$. The similarity of this notation with the one for scalar functions should not lead to any confusion, as the meaning will always be clear from the context .

For a fixed wave number $k > 0$ and for a given source functional $\mathbf{f} \in \mathbf{X}'$, the variational formulation for the EFIE reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \Psi_k \mathrm{div}_\Gamma\, \mathbf{u}, \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \boldsymbol{\Psi}_k \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{2.1}$$

Here, $\Psi_k$ (resp., $\boldsymbol{\Psi}_k$) denote the scalar (resp., vectorial) single layer boundary integral operator on $\Gamma$ for the Helmholtz operator $-\Delta - k^2$, see [16, Section 4.1] (resp., [18, Section 5]).

To ensure the uniqueness of the solution to (2.1) we always assume that $k^2$ is not an electrical eigenvalue of the interior problem in $\Omega$.

# 3 Galerkin BEM on graded meshes. The main result.

For approximate solution of (2.1) we apply the natural BEM based on Galerkin discretisations with lowest-order Raviart-Thomas spaces on graded meshes.

First, let us describe the construction of graded meshes on individual faces of $\Gamma$. Here, we follow [28, Section 3]. For simplicity, we can assume that all faces of $\Gamma$ are triangles. On general polygonal faces the construction is similar, or one can first subdivide the polygon into triangles. On a triangular face $F \subset \Gamma$, we first draw three lines through the centroid and parallel to the sides of $F$. This makes $F$ divided into three parallelograms and three triangles (see Figure 1). Each of the three parallelograms can be mapped onto the unit square $\widehat{Q} = (0,1)^2$ by a linear transformation such that the vertex $(0,0)$ of $\widehat{Q}$ is the image of a vertex of $F$. Analogously, each of the three sub-triangles can be mapped onto the unit triangle $\widehat{T} = \{\mathbf{x} = (x_1, x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\} \subset \widehat{Q}$ such that the vertex $(1,1)$ of $\widehat{T}$ is the image of the centroid of $F$. Next, the graded mesh on $\widehat{Q}$ (and hence on $\widehat{T}$) is generated by the lines

$$x_1 = \left(\frac{i}{N}\right)^\beta, \quad x_2 = \left(\frac{j}{N}\right)^\beta, \quad i, j = 0, 1, \ldots, N.$$

Here, $\beta \geq 1$ is the grading parameter, and $N \geq 1$ corresponds to the level of refinement. Mapping each cell of these meshes back onto the face $F$, we obtain a graded mesh of triangles and parallelograms on $F$ (see Figure 1). Note that the diameter of the largest element of this mesh is proportional to $\beta N^{-1}$. Hence, $h = 1/N$ defines the mesh parameter, and we will denote by $\mathcal{T} = \{\Delta_h^\beta\}$ a family of graded meshes $\Delta_h^\beta = \{K;\ \cup \bar{K} = \bar{\Gamma}\}$ generated on $\Gamma$ by following the procedure described above.

Let us now introduce the boundary element space $\mathbf{X}_h$. It is known that Raviart-Thomas surface elements provide an affine equivalent family of $\mathrm{div}_\Gamma$-conforming finite elements under the Piola transformation, see [11, Section III.3]. We will write $\boldsymbol{\mathcal{RT}}_0(K)$ for the local lowest-order Raviart-Thomas space on a generic (triangular or quadrilateral) element $K$, and we denote by $\mathbf{X}_h = \boldsymbol{\mathcal{RT}}_0(\Delta_h^\beta)$ the corresponding space of $\mathrm{div}_\Gamma$-conforming boundary elements over the graded mesh $\Delta_h^\beta$.

The following theorem states the unique solvability and quasi-optimal convergence of the Galerkin BEM on graded meshes for the EFIE.

**Theorem 3.1** *There exists $h_0 < 1$ such that for any $\mathbf{f} \in \mathbf{X}'$ and for any graded mesh $\Delta_h^\beta$ with $h \leq h_0$ and $\beta \in [1,3)$, the Galerkin boundary element discretisation of* (2.1) *admits a unique solution $\mathbf{u}_h \in \mathbf{X}_h$ and the h-version of the Galerkin BEM on graded meshes $\Delta_h^\beta$ converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}} \leq C \inf_{\mathbf{v} \in \mathbf{X}_h} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}}, \tag{3.1}$$

*where the constant $C$ may depend only on the geometry of $\Gamma$ and the grading parameter $\beta$.*

The proof of Theorem 3.1 relies on an abstract theory for analysing convergence of Galerkin discretisations for non-coercive variational problems like (2.1). This theory was developed in [13], [19], [18, Section 9.1], and in [12, Section 3]. In particular, it follows from the latter article that in order to prove Theorem 3.1 we need to establish the following properties:

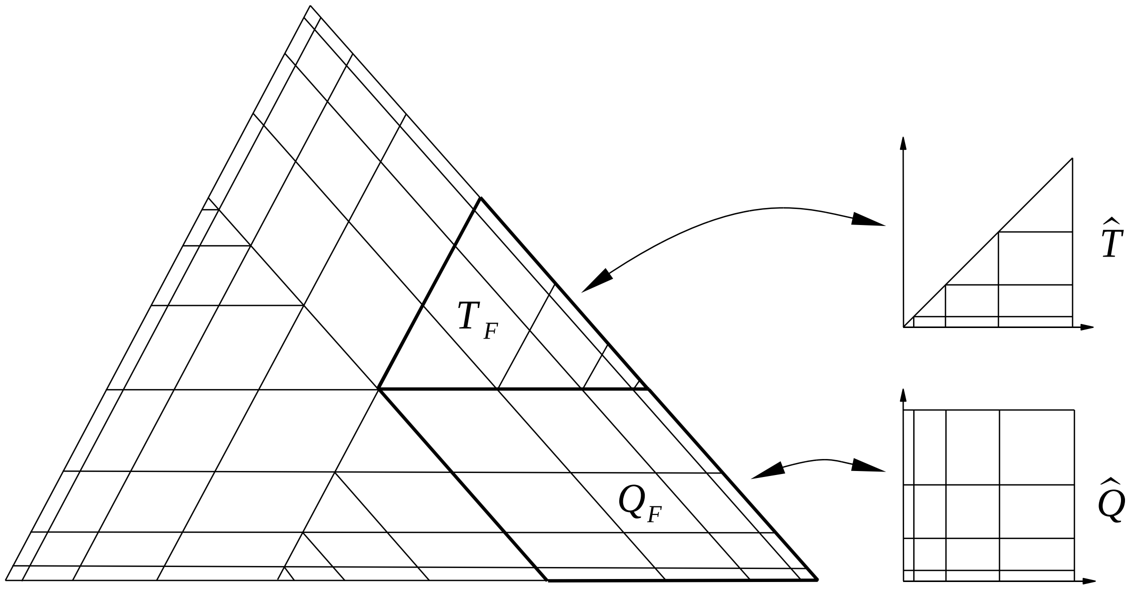


Figure 1: Graded mesh on the triangular face $F \subset \Gamma$. The triangular (resp., parallelogram) block of elements $T_F$ (resp., $Q_F$) is the image of the graded mesh on the unit triangle $\widehat{T}$ (resp., the unit square $\widehat{Q}$).

**(A)** the existence of a stable direct decomposition $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$ such that $a|_{\mathbf{V}\times\mathbf{V}}$ and $-a|_{\mathbf{W}\times\mathbf{W}}$ are both $\mathbf{X}$-coercive, and $a|_{\mathbf{V}\times\mathbf{W}}$ and $a|_{\mathbf{W}\times\mathbf{V}}$ are both compact;

**(B)** the existence of the corresponding discrete decomposition $\mathbf{X}_h = \mathbf{V}_h + \mathbf{W}_h$, $\mathbf{W}_h \subset \mathbf{W}$, that is uniformly stable with respect to the mesh parameter $h$;

**(C)** the gap property

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \inf_{\mathbf{v} \in \mathbf{V}} \frac{\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{X}}}{\|\mathbf{v}_h\|_X} \le \varepsilon(h) \quad \text{with} \quad \varepsilon(h) \to 0 \quad \text{as} \quad h \to 0. \tag{3.2}$$

We will prove Theorem 3.1 by verifying these properties in Sections 4 and 6 below.

**Remark 3.1** *Theorem* 3.1 *remains valid if* $\Gamma$ *is a piecewise plane orientable open surface (see* [13] *for the problem formulation and the underlying tangential trace spaces in this case). The proof repeats the arguments in Sections* 4 *and* 6 *below by using a specific construction of the decomposition* $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$ *as described in* [8, Section 3].

**Remark 3.2** *If some information about the regularity of the solution* $\mathbf{u}$ *to* (2.1) *is available, then convergence result of Theorem* 3.1 *translates into an a priori error estimate in the* *natural* $\mathbf{X}$*-norm. For scattering problems with sufficiently smooth source functional* $\mathbf{f}$ *(e.g., with* $\mathbf{f}$ *representing the excitation by an incident plane wave), the regularity of the solution depends only on the geometry of* $\Gamma$*. In particular, nonsmoothness of* $\Gamma$ *leads to singularities in the solution of the EFIE, severely affecting convergence rates of the h-BEM on shape-regular meshes. However, similar to the case of the Laplacian in* [28, 27]*, by employing the graded meshes with sufficiently large grading parameter* $\beta$ *(depending on the strength of singularities in* $\mathbf{u}$*) one may hope to recover the optimal convergence rate (i.e., the rate of the h-BEM on quasi-uniform meshes in the case of a smooth solution). The main question here is whether the restriction on the grading parameter* $\beta$ *that guarantees quasi-optimality of the Galerkin BEM in Theorem* 3.1*, is sufficient for recovering this optimal convergence. We will address this issue in the forthcoming article* [10].

Throughout the paper, $C$, $C_1$, *etc.* denote generic positive constants that are independent of the mesh parameter $h$ and involved functions but may depend on the geometry of $\Gamma$ and the grading parameter $\beta$. We will also write $a \lesssim b$ and $A \simeq B$, which means the existence of generic positive constants $C$, $C_1$, $C_2$ such that $a \le Cb$ and $C_1 B \le A \le C_2 B$, respectively.

# 4 Decomposition technique

Let us address property **(A)** from Section 3. One way to obtain a suitable splitting is to employ the Hodge decomposition of $\mathbf{X}$, cf. [15]. This idea was successfully exploited in [16, 24, 19, 13] in the context of the $h$-BEM on shape-regular meshes and, in a modified form, in [5] for analysing the $p$-BEM on plane open screens. When attempting to use the Hodge decomposition for convergence analysis of the $h$-BEM on graded meshes, poor regularity of vector fields in the $\mathbf{V}$-component leads to severe restrictions on the grading parameter $\beta$.

An alternative technique employs a regularising projection $\mathsf{R} : \mathbf{X} \to \mathbf{X}$ to construct a decomposition of $\mathbf{X}$ with enhanced regularity of the $\mathbf{V}$-component (see [23], [18, Section 3], and [12, Section 4.3.1]). More specifically, there exists a projection $\mathsf{R} : \mathbf{X} \to \mathbf{X}$ (see [18, Lemma 2] for details of construction of $\mathsf{R}$) such that

$$\mathrm{div}_\Gamma\, \mathsf{R}\,\mathbf{u} = \mathrm{div}_\Gamma \mathbf{u} \quad \text{for any } \mathbf{u} \in \mathbf{X} \tag{4.1}$$

and

$$\exists C = C(\Gamma) > 0 \;\; \text{such that} \;\; \|\,\mathsf{R}\,\mathbf{u}\|_{\mathbf{H}^{1/2}_\perp(\Gamma)} \le C\, \|\mathrm{div}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma)} \;\; \forall\, \mathbf{u} \in \mathbf{X}, \tag{4.2}$$

where $\mathbf{H}^{1/2}_\perp(\Gamma) \subset \mathbf{X}$ is the rotated tangential trace space of $\mathbf{H}^1(\Omega)$ on $\Gamma = \partial\Omega$, see [14, 17].

Then, one can define the decomposition

$$\mathbf{X} = \mathbf{V} \oplus \mathbf{W} \quad \text{with} \quad \mathbf{V} := \mathsf{R}(\mathbf{X}) \subset \mathbf{H}^{1/2}_\perp(\Gamma) \quad \text{and} \quad \mathbf{W} := (\mathrm{Id} - \mathsf{R})(\mathbf{X}). \tag{4.3}$$

Decomposition (4.3) was used in [8] to prove unique solvability and quasi-optimal convergence of the $hp$-BEM with locally variable polynomial degrees on shape-regular meshes.

As we will see in this paper, the same decomposition technique can be used effectively in the analysis of the $h$-BEM on graded meshes.

By (4.1) we conclude that $\mathbf{W}$ comprises $\mathrm{div}_\Gamma$-free vector fields. Stability of the decomposition in (4.3) follows from inequality (4.2) and the continuous embedding $\mathbf{H}_\perp^{1/2}(\Gamma) \hookrightarrow \mathbf{X}$. Furthermore, the embedding $\mathbf{V} \hookrightarrow \mathbf{L}_t^2(\Gamma)$ is compact by (4.2) and Rellich's theorem. Thus, thanks to the $H^{-1/2}(\Gamma)$-coercivity (resp., $\mathbf{H}_\perp^{-1/2}(\Gamma)$-coercivity) of $\Psi_k$ (resp. $\mathbf{\Psi}_k$) (see [18, Lemmas 8, 7]), the $\mathbf{X}$-coercivity of $a|_{\mathbf{V}\times\mathbf{V}}$ and $-a|_{\mathbf{W}\times\mathbf{W}}$ is proved by the same arguments as in [13, proof of Theorem 3.4]. The compactness of $a|_{\mathbf{V}\times\mathbf{W}}$ and $a|_{\mathbf{W}\times\mathbf{V}}$ is due to the continuity of $\mathbf{\Psi}_k : \mathbf{H}_\perp^{-1/2}(\Gamma) \to \mathbf{H}_\perp^{1/2}(\Gamma)$ and the compactness of the embedding $\mathbf{V} \hookrightarrow \mathbf{H}_\perp^{-1/2}(\Gamma)$ (see [18, Lemma 9]). This proves **(A)**.

Before we can define the discrete counterpart of decomposition (4.3), we need to find a suitable projector onto the space of Raviart-Thomas surface elements. Stability of the discrete decomposition will follow from stability properties of the Raviart-Thomas interpolation on anisotropic elements, which is the subject of the next section.

# 5 Raviart-Thomas interpolation on anisotropic elements

In this section, we establish new stability properties of the Raviart-Thomas interpolant on anisotropic elements. We will also prove the corresponding interpolation error estimates.

In the context of the finite element method, the analysis of interpolation operators on anisotropic elements can be found in [3, 1]. For the Raviart-Thomas interpolation (see, e.g., [1, Section 3]), the main idea is to study componentwise stability of the interpolant on a reference element $\widehat{K}$. For sufficiently regular vector fields, this study relies on the fact that the standard (scalar) trace operator is well defined for functions in $W^{1,p}(\widehat{K})$ for any $p > 1$. In our BEM application, however, the stability result is needed for low-regular vector fields living in fractional Sobolev spaces $\mathbf{H}^s(\widehat{K}) \cap \mathbf{H}(\mathrm{div}, \widehat{K})$ with $0 < s \le 1/2$. For such vector fields, the trace of the normal component only exists in a weak sense. Therefore, instead of the standard trace argument used in [1], we use Green's formula (5.3) and consistently employ the anisotropic seminorms defined below. More precisely, for $s \in (0, 1/2]$ we introduce the $H^s$-seminorms of anisotropic type. On the reference square $\widehat{Q} = (0,1)^2$ these are defined as follows:

$$|u|^2_{AH_1^s(\widehat{Q})} = \int_0^1 |u(\cdot, x_2)|^2_{H^s(0,1)}\, dx_2,$$

$$|u|^2_{AH_2^s(\widehat{Q})} = \int_0^1 |u(x_1, \cdot)|^2_{H^s(0,1)}\, dx_1.$$

These definitions are meaningful for all $u \in H^s(\widehat{Q})$ due to [25, Theorem 10.2] which yields that

$$\|u\|_{H^s(\widehat{Q})} \simeq \|u\|_{0,\widehat{Q}} + |u|_{AH_1^s(\widehat{Q})} + |u|_{AH_2^s(\widehat{Q})}. \tag{5.1}$$

On the reference triangle $\widehat{T} = \{(x_1, x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\}$, the following seminorms

$$|u|^2_{AH^s_1(\widehat{T})} = \int_0^1 |u(\cdot, x_2)|^2_{H^s(x_2,1)}\, dx_2,$$
$$|u|^2_{AH^s_2(\widehat{T})} = \int_0^1 |u(x_1, \cdot)|^2_{H^s(0,x_1)}\, dx_1$$

are also well defined for all $u \in H^s(\widehat{T})$. Indeed, by Theorem 1.4.3.1 of [21] there exists a continuous linear operator (called the extension operator) $\mathsf{E} : H^s(\widehat{T}) \to H^s(\widehat{Q})$ such that

$$\mathsf{E}\, u|_{\widehat{T}} = u, \quad \forall u \in H^s(\widehat{T}).$$

Hence, applying (5.1) to $\mathsf{E}\, u$, we have

$$|u|_{AH^s_1(\widehat{T})} + |u|_{AH^s_2(\widehat{T})} \lesssim \|u\|_{H^s(\widehat{T})}. \tag{5.2}$$

We recall that the Raviart-Thomas interpolant $\Pi_{\mathrm{RT}}\mathbf{u}$ is well-defined for any $\mathbf{u} \in \mathbf{H}^s(K)$ ($s > 0$) such that $\operatorname{div} \mathbf{u} \in L^2(K)$, where $K$ is any triangle or rectangle. Indeed, for such vector fields the following Green's formula is meaningful (see, e.g., [6, Lemma 2.1])

$$(\mathbf{u}, \nabla\varphi)_{0,K} + \int_K \operatorname{div} \mathbf{u}\, \varphi = (\mathbf{u} \cdot \mathbf{n}, \varphi)_{0,\partial K}, \quad \forall \varphi \in H^{1-\varepsilon}(K), \tag{5.3}$$

with $\varepsilon \in (0, s)$ and $\mathbf{n}$ denoting the outward unit normal to $\partial K$. Hence, taking $\varphi \in H^{1-\varepsilon}(K)$ such that $\varphi = 1$ on the edge $e \subset \partial K$ and $\varphi = 0$ on $\partial K \backslash e$, we can define $(\mathbf{u} \cdot \mathbf{n}, 1)_{0,e} := (\mathbf{u} \cdot \mathbf{n}, \varphi)_{0,\partial K}$.

In what follows, we will denote by $\widehat{\Pi}_{\mathrm{RT}}$ the Raviart-Thomas interpolation operator on the reference element $\widehat{K}$ ($\widehat{K} = \widehat{T}$ or $\widehat{Q}$). Throughout this section, we will refer to two components of a vector field using the indices $l,\ l' \in \{1, 2\}$ such that $l \neq l'$ (e.g., $l' = 3 - l$ for $l = 1,\ 2$).

## 5.1 The reference element

Let $\widehat{K}$ be the reference element ($\widehat{K} = \widehat{T}$ or $\widehat{Q}$) and $\widehat{e}_i$ ($i = 1, 2, 3$ or $i = 1, 2, 3, 4$) denote the edges of $\widehat{K}$. Specifically, on the references triangle $\widehat{T}$, we denote by $\widehat{e}_1$ (resp. $\widehat{e}_2$) the edge on the $x_1$-axis (resp., parallel to the $x_2$-axis), and by $\widehat{e}_3$ the oblique edge (see Figure 2). We recall from [11] that the lowest-order Raviart-Thomas elements on $\widehat{K}$ are defined as

$$\boldsymbol{\mathcal{RT}}_0(\widehat{K}) = \begin{cases} \{(a, b)^\top + c(x_1, x_2)^\top;\ a, b, c \in \mathbb{R}\} & \text{if } \widehat{K} = \widehat{T}, \\ \{(a + cx_1, b + dx_2)^\top;\ a, b, c, d \in \mathbb{R}\} & \text{if } \widehat{K} = \widehat{Q}, \end{cases}$$

and that the associated degrees of freedom are given by

$$\int_{\widehat{e}_i} \mathbf{u} \cdot \mathbf{n}\, ds.$$

In the case of the reference triangle $\widehat{T}$ we have the following result.

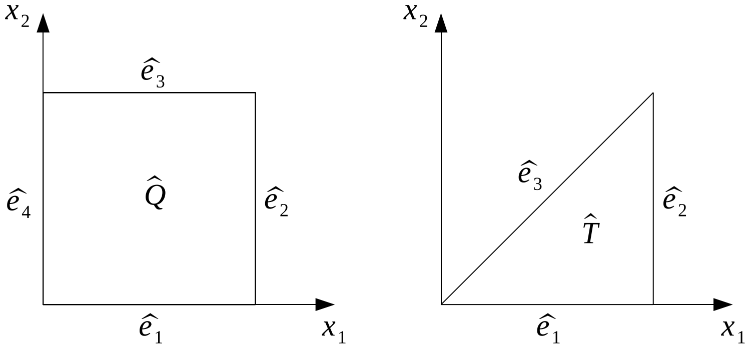


Figure 2: The reference square $\widehat{Q}$ and the reference triangle $\widehat{T}$.

**Theorem 5.1** (i) *For all* $\mathbf{u} \in \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ *with* $s > 1/2$*, we have for* $l = 1, 2$

$$\|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_l\|_{0,\widehat{T}} \lesssim \|u_l\|_{H^s(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}. \tag{5.4}$$

(ii) *For all* $\mathbf{u} \in \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ *with* $0 < s \le 1/2$*, we have for* $l = 1, 2$

$$\|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_l\|_{0,\widehat{T}} \lesssim \|u_l\|_{H^s(\widehat{T})} + |u_{l'}|_{AH^s_{l'}(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}. \tag{5.5}$$

**Proof.** We will prove both statements for $l = 1$. The proof is analogous in the case $l = 2$. One has $\widehat{\Pi}_{\mathrm{RT}}\mathbf{u} = (a, b)^\top + c(x_1, x_2)^\top$ with $a, b, c \in \mathbb{R}$. Hence

$$\|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_1\|_{0,\widehat{T}} = \|a + cx_1\|_{0,\widehat{T}} \lesssim |a| + |c| \le |a + c| + 2|c|.$$

Observe that

$$2c = \mathrm{div}\,\widehat{\Pi}_{\mathrm{RT}}\mathbf{u} = 2\int_{\widehat{T}} \mathrm{div}\,\mathbf{u}.$$

By the Cauchy-Schwarz inequality this yields

$$|c| \lesssim \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}.$$

Therefore in order to estimate $\|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_1\|_{0,\widehat{T}}$ one needs to bound $|a + c| = |(\mathbf{u} \cdot \mathbf{n}, 1)_{0,\widehat{e}_2}|$.

If $\mathbf{u} \in \mathbf{H}^s(\widehat{T})$ with $s > 1/2$, then we can use the trace theorem to estimate

$$|a + c| = |(\mathbf{u} \cdot \mathbf{n}, 1)_{0,\widehat{e}_2}| = |(u_1, 1)_{0,\widehat{e}_2}| \lesssim \|u_1\|_{H^s(\widehat{T})}.$$

This proves statement (i) (for $l = 1$).

Now, let us consider $\mathbf{u} \in \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ for $0 < s \le 1/2$. In order to estimate $(\mathbf{u} \cdot \mathbf{n}, 1)_{0,\widehat{e}_2}$ in this case, we fix a function $\varphi \in H^{1-\varepsilon}(\widehat{T})$, $\varepsilon \in (0, s)$, such that $\varphi = 1$ on $\widehat{e}_2$ and $\varphi = 0$ on $\partial\widehat{T}\backslash\widehat{e}_2$. Then by Green's formula (5.3) we have

$$(\mathbf{u} \cdot \mathbf{n}, 1)_{0,\widehat{e}_2} = (u, \nabla\varphi)_{0,\widehat{T}} + \int_{\widehat{T}} \mathrm{div}\mathbf{u}\,\varphi = (u_1, \partial_1\varphi)_{0,\widehat{T}} + (u_2, \partial_2\varphi)_{0,\widehat{T}} + \int_{\widehat{T}} \mathrm{div}\mathbf{u}\,\varphi. \tag{5.6}$$

For the term $(u_1, \partial_1\varphi)_{0,\widehat{T}}$, we first use a standard duality argument,

$$|(u_1, \partial_1\varphi)_{0,\widehat{T}}| \leq \|u_1\|_{H^\varepsilon(\widehat{T})} \|\partial_1\varphi\|_{H^{-\varepsilon}(\widehat{T})},$$

and by the continuity property of $\partial_1 : H^{1-\varepsilon}(\widehat{T}) \to H^{-\varepsilon}(\widehat{T})$ (see Theorem 1.4.4.6 of [21]) we find

$$|(u_1, \partial_1\varphi)_{0,\widehat{T}}| \lesssim \|u_1\|_{H^s(\widehat{T})} \|\varphi\|_{H^{1-\varepsilon}(\widehat{T})}. \tag{5.7}$$

The term $(u_2, \partial_2\varphi)_{0,\widehat{T}}$ requires more subtle analysis. Let $I(x_1)$ denote the interval $(0, x_1)$ for any $x_1 \in \widehat{I} := (0, 1)$. First, by (5.2) and Fubini's theorem, we may write

$$(u_2, \partial_2\varphi)_{0,\widehat{T}} = \int_0^1 (u_2(x_1, \cdot), \partial_2\varphi(x_1, \cdot))_{0,I(x_1)}\, dx_1. \tag{5.8}$$

Now we use a density argument to show that

$$\int_0^1 g(x_1)(1, \partial_2\varphi(x_1, \cdot))_{0,I(x_1)}\, dx_1 = 0, \tag{5.9}$$

for all $g \in L^2(\widehat{I})$. Indeed, fix a sequence of smooth function $\varphi_n$ such that

$$\varphi_n \to \varphi \text{ in } H^{1-\varepsilon}(\widehat{T}) \text{ as } n \to \infty.$$

Then for all $x_1 \in \widehat{I} = (0, 1)$, we have

$$(1, \partial_2\varphi_n(x_1, \cdot))_{0,I(x_1)} = \int_0^{x_1} \partial_2\varphi_n(x_1, x_2)\, dx_2 = \varphi_n(x_1, x_1) - \varphi_n(x_1, 0).$$

For any $g \in L^2(\widehat{I})$, multiplying this identity by $g(x_1)$ and integrating the result in $x_1 \in (0, 1)$, we obtain

$$\int_0^1 g(x_1)(1, \partial_2\varphi_n(x_1, \cdot))_{0,I(x_1)}\, dx_1 = \int_{\widehat{e}_3} g\varphi_n - \int_{\widehat{e}_1} g\varphi_n.$$

Hence, as $n \to \infty$ we find that

$$\int_0^1 g(x_1)(1, \partial_2\varphi(x_1, \cdot))_{0,I(x_1)}\, dx_1 = \int_{\widehat{e}_3} g\varphi - \int_{\widehat{e}_1} g\varphi,$$

which proves (5.9) by recalling that $\varphi = 0$ on $\widehat{e}_1$ and $\widehat{e}_3$.

Coming back to (5.8) and using (5.9), we have

$$(u_2, \partial_2\varphi)_{0,\widehat{T}} = \int_0^1 \big(u_2(x_1, \cdot) - \mathcal{M}_{I(x_1)}(u_2(x_1, \cdot)), \partial_2\varphi(x_1, \cdot)\big)_{0,I(x_1)}\, dx_1,$$

where $\mathcal{M}_{I(x_1)}(u_2(x_1,\cdot)) = \frac{1}{x_1}\int_0^{x_1} u_2(x_1,x_2)\,dx_2$ is the mean of $u_2(x_1,\cdot)$ on $I(x_1) = (0,x_1)$. Then a duality argument yields

$$|(u_2,\partial_2\varphi)_{0,\widehat{T}}| \le \int_0^1 \|u_2(x_1,\cdot) - \mathcal{M}_{I(x_1)}(u_2(x_1,\cdot))\|_{H^\varepsilon(I(x_1))}\|\partial_2\varphi(x_1,\cdot)\|_{H^{-\varepsilon}(I(x_1))}\,dx_1. \quad (5.10)$$

Using a scaling argument and Friedrichs' inequality we estimate

$$\begin{aligned}\|u_2(x_1,\cdot) - \mathcal{M}_{I(x_1)}(u_2(x_1,\cdot))\|_{H^\varepsilon(I(x_1))} &\le \|u_2(x_1,\cdot) - \mathcal{M}_{I(x_1)}(u_2(x_1,\cdot))\|_{H^s(I(x_1))}\\ &\le C|u_2(x_1,\cdot)|_{H^s(I(x_1))}, \end{aligned}\quad (5.11)$$

with $C > 0$ independent of $x_1 \in (0,1)$.

For the second factor in the integrand in (5.10), in order to apply Theorem 1.4.4.6 of [21] on a fixed domain, we first notice that $\varphi(x_1,\cdot) \in H_0^{1-\varepsilon}(I(x_1))$ a. e. in $(0,1) \ni x_1$. Therefore, for almost all $x_1 \in (0,1)$ there exists a sequence of functions $\varphi_n \in C_0^\infty(I(x_1))$, $n = 1,2,\ldots$, such that $\varphi_n \to \varphi(x_1,\cdot)$ in $H_0^{1-\varepsilon}(I(x_1))$ as $n \to \infty$ and

$$\int_0^{x_1} \partial\varphi_n(x_2)\,dx_2 = 0 \quad \forall n = 1,2,\ldots.$$

Using a scaling argument, we deduce that

$$\|\partial\varphi_n\|_{H^{-\varepsilon}(I(x_1))} = \sup_{\substack{v\in H^\varepsilon(I(x_1))\\ v\neq 0}} \frac{\int_0^{x_1}\partial\varphi_n\,v\,dx_2}{\|v\|_{H^\varepsilon(I(x_1))}} \le x_1^{-\frac{1}{2}+\varepsilon} \sup_{\substack{\widehat{v}\in H^\varepsilon(0,1)\\ \widehat{v}\neq 0}} \frac{\int_0^1 \partial\widehat{\varphi}_n\,\widehat{v}\,d\widehat{x}_2}{|\widehat{v}|_{H^\varepsilon(0,1)}}.$$

As $\partial\widehat{\varphi}_n$ has zero average, we can estimate

$$\begin{aligned}\|\partial\varphi_n\|_{H^{-\varepsilon}(I(x_1))} &\le x_1^{-\frac{1}{2}+\varepsilon} \sup_{\substack{\widehat{v}\in H^\varepsilon(0,1)\\ \widehat{v}\neq 0,\, \int_0^1\widehat{v}=0}} \frac{\int_0^1 \partial\widehat{\varphi}_n\widehat{v}\,d\widehat{x}_2}{|\widehat{v}|_{H^\varepsilon(0,1)}}\\ &\lesssim x_1^{-\frac{1}{2}+\varepsilon} \sup_{\substack{\widehat{v}\in H^\varepsilon(0,1)\\ \widehat{v}\neq 0,\, \int_0^1\widehat{v}=0}} \frac{\int_0^1 \partial\widehat{\varphi}_n\widehat{v}\,d\widehat{x}_2}{\|\widehat{v}\|_{H^\varepsilon(0,1)}} \le x_1^{-\frac{1}{2}+\varepsilon}\|\partial\widehat{\varphi}_n\|_{H^{-\varepsilon}(0,1)}.\end{aligned}$$

Hence, by Theorem 1.4.4.6 of [21] we prove that

$$\|\partial\varphi_n\|_{H^{-\varepsilon}(I(x_1))} \lesssim x_1^{-\frac{1}{2}+\varepsilon}\|\widehat{\varphi}_n\|_{H^{1-\varepsilon}(0,1)} \lesssim x_1^{-\frac{1}{2}+\varepsilon}|\widehat{\varphi}_n|_{H^{1-\varepsilon}(0,1)}.$$

Mapping back to the interval $I(x_1) = (0,x_1)$ we have

$$\|\partial\varphi_n\|_{H^{-\varepsilon}(I(x_1))} \lesssim |\varphi_n|_{H^{1-\varepsilon}(I(x_1))}.$$

As $n \to \infty$ we find

$$\|\partial_2\varphi(x_1,\cdot)\|_{H^{-\varepsilon}(I(x_1))} \le C_1|\varphi(x_1,\cdot)|_{H^{1-\varepsilon}(I(x_1))} \quad \text{a. e. on } (0,1) \ni x_1, \quad (5.12)$$

with $C_1 > 0$ independent of $x_1$.

Using estimates (5.11) and (5.12) in (5.10) we arrive at

$$\left|(u_2, \partial_2\varphi)_{0,\widehat{T}}\right| \lesssim \int_0^1 |u_2(x_1,\cdot)|_{H^s(0,x_1)} \|\varphi(x_1,\cdot)\|_{H^{1-\varepsilon}(0,x_1)}\, dx_1.$$

Then the Cauchy-Schwarz inequality yields

$$\left|(u_2, \partial_2\varphi)_{0,\widehat{T}}\right| \lesssim |u_2|_{AH_2^s(\widehat{T})} \Big( \int_0^1 \|\varphi(x_1,\cdot)\|^2_{H^{1-\varepsilon}(0,x_1)}\, dx_1 \Big)^{\frac{1}{2}}. \tag{5.13}$$

The last term on the right-hand side of (5.6) is estimated by using the Cauchy-Schwarz inequality:

$$\int_{\widehat{T}} \mathrm{div}\mathbf{u}\,\varphi \le \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}\, \|\varphi\|_{0,\widehat{T}}.$$

Using this estimate and inequalities (5.7), (5.13) in (5.6), we obtain (5.5) (for $l = 1$). □

**Corollary 5.1** (i) *For all* $\mathbf{u} \in \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ *with* $s > 1/2$*, we have for* $l = 1, 2$

$$\|u_l - (\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_l\|_{0,\widehat{T}} \lesssim |u_l|_{H^s(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}.$$

(ii) *For all* $\mathbf{u} \in \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ *with* $0 < s \le 1/2$*, we have for* $l = 1, 2$

$$\|u_l - (\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_l\|_{0,\widehat{T}} \lesssim |u_l|_{H^s(\widehat{T})} + |u_{l'}|_{AH^s_{l'}(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}.$$

**Proof.** It is sufficient to prove only statement (ii). For $l = 1$, we take $\widetilde{\mathbf{u}} = \mathbf{u} - (\mathcal{M}_{\widehat{T}} u_1, 0)^\top$, where $\mathcal{M}_{\widehat{T}} u_1 = \fint_{\widehat{T}} u_1$. One has

$$\widetilde{\mathbf{u}} - \widehat{\Pi}_{\mathrm{RT}}\widetilde{\mathbf{u}} = \mathbf{u} - \widehat{\Pi}_{\mathrm{RT}}\mathbf{u}, \qquad \mathrm{div}\,\widetilde{\mathbf{u}} = \mathrm{div}\,\mathbf{u},$$

and

$$|u_1|_{H^s(\widehat{T})} = |\tilde{u}_1|_{H^s(\widehat{T})} \simeq \|\tilde{u}_1\|_{H^s(\widehat{T})}.$$

The assertion then follows by applying estimate (5.5) to $\widetilde{\mathbf{u}}$. The proof is analogous for $l = 2$. □

**Counterexample 5.1** *Here we provide a counterexample which demonstrates that for low-regular vector fields the term* $|u_{l'}|_{AH^s_{l'}(\widehat{T})}$ *in* (5.5) *cannot be omitted. This is in contrast to the case of sufficiently-regular fields in* (5.4) *(see also Lemma* 3.3 *in* [1]*). In particular, if we assume that*

$$\|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_2\|_{0,\widehat{T}} \lesssim \|u_2\|_{H^{1/2}(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}} \tag{5.14}$$

*for all* $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$*, then we will arrive at a contradiction. Indeed, inspired by Example* 2.6 *of* [3]*, we define on* $\widehat{T} \times (0,1)$

$$v^{\varepsilon}(x_1, x_2, x_3) = (x_1 - 1) w^{\varepsilon}(x_2, x_3) \quad \text{with} \quad w^{\varepsilon}(x_2, x_3) = \min\left\{1, \varepsilon \log\log \frac{e}{r}\right\} \quad \text{for any } \varepsilon > 0.$$

*Here,* $r = (x_2^2 + x_3^2)^{\frac{1}{2}}$*, and* $e$ *is the Euler number. Taking* $\nabla v^{\varepsilon} \times \boldsymbol{\nu}$ *on* $\widehat{T}$ *(here* $\boldsymbol{\nu} = (0,0,-1)$*), we find a divergence-free vector field*

$$\mathbf{u}^{\varepsilon}(x_1, x_2) = \big((1 - x_1)\partial_{x_2} w^{\varepsilon}(x_2, 0), w^{\varepsilon}(x_2, 0)\big)^{\top}.$$

*Simple calculations show that* $\widehat{\Pi}_{\mathrm{RT}} \mathbf{u}^{\varepsilon} = (0,1)^{\top}$*, and by the trace theorem we have*

$$\|(\mathbf{u}^{\varepsilon})_2\|_{H^{1/2}(\widehat{T})} = \|w^{\varepsilon}(x_2, 0)\|_{H^{1/2}(\widehat{T})} \lesssim \|w^{\varepsilon}\|_{H^1(\widehat{T} \times (0,1))}.$$

*Since* $\|w^{\varepsilon}\|_{H^1(\widehat{T} \times (0,1))} \to 0$ *as* $\varepsilon \to 0$ *(see* [3, Example 2.6]*), we conclude that* $(\mathbf{u}^{\varepsilon})_2 \to 0$ *in* $H^{\frac{1}{2}}(\widehat{T})$ *as* $\varepsilon \to 0$*. This seems to contradict* (5.14) *but not directly because the first component of* $\mathbf{u}^{\varepsilon}$ *is not in* $H^{\frac{1}{2}}(\widehat{T})$*. Hence, in order to arrive at a contradiction, we need to show that if* (5.14) *holds for all* $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$*, then* $\mathbf{u}^{\varepsilon}$ *satisfies* (5.14) *(with a constant independent of* $\varepsilon$*). Indeed, for a fixed* $\varepsilon > 0$*, as* $w^{\varepsilon}(\cdot, 0) \in H^1(0,1)$*, we can consider a sequence of smooth functions* $w_n \in C^{\infty}([0,1])$ *such that*

$$w_n \to w^{\varepsilon}(\cdot, 0) \text{ in } H^1(0,1) \text{ as } n \to \infty.$$

*Then we define*

$$\mathbf{u}_n(x_1, x_2) = ((1 - x_1)\partial w_n(x_2), w_n(x_2))^{\top}.$$

*One has* $\mathbf{u}_n \in \mathbf{H}^{\frac{1}{2}}(\widehat{T})$ *and* $\mathrm{div}\, \mathbf{u}_n = 0$*. Moreover,* $\widehat{\Pi}_{\mathrm{RT}} \mathbf{u}_n \to \widehat{\Pi}_{\mathrm{RT}} \mathbf{u}^{\varepsilon}$ *as* $n \to \infty$*. Therefore, applying estimate* (5.14) *to* $\mathbf{u}_n$ *and letting* $n \to \infty$*, we conclude that* $\mathbf{u}^{\varepsilon}$ *satisfies* (5.14)*.*

**Remark 5.1** *By Counterexample* 5.1 *we can easily show that a result similar to Lemma* 3.3 *in* [1] *for the Nédélec interpolant on the tetrahedron* $\widehat{T}_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_i > 0,\ i = 1,2,3 \text{ and } 0 < x_1 + x_2 + x_3 < 1\}$ *is not valid. In other words, the anisotropic estimate*

$$\|(\widehat{\Pi}_{\mathrm{Ned}} \mathbf{v})_l\|_{0,\widehat{T}_3} \lesssim \|v_l\|_{\mathbf{H}^1(\widehat{T}_3)} + \|\mathbf{curl}\, \mathbf{v}\|_{0,\widehat{T}_3}, \quad l = 1,2,3$$

*does not hold for all* $\mathbf{v} \in \mathbf{H}^1(\widehat{T}_3) = \big(H^1(\widehat{T}_3)\big)^3$*.*

In the case of the reference square $\widehat{Q}$, the following results analogous to those in Theorem 5.1 and Corollary 5.1 hold (we refer to [9, Section 5.1] for details of the proofs).

**Theorem 5.2** (i) *For all* $\mathbf{u} \in \mathbf{H}^s(\widehat{Q})$ *with* $s > 1/2$*, we have for* $l = 1$, 2

$$\|(\widehat{\Pi}_{\mathrm{RT}} \mathbf{u})_l\|_{0,\widehat{Q}} \lesssim \|u_l\|_{H^s(\widehat{Q})} \quad \text{and} \quad \|u_l - (\widehat{\Pi}_{\mathrm{RT}} \mathbf{u})_l\|_{0,\widehat{Q}} \lesssim |u_l|_{H^s(\widehat{Q})}.$$

(ii) *For all* $\mathbf{u} \in \mathbf{H}^s(\widehat{Q}) \cap \mathbf{H}(\mathrm{div}, \widehat{Q})$ *with* $0 < s \leq 1/2$*, we have for* $l = 1$, 2

$$\|(\widehat{\Pi}_{\mathrm{RT}} \mathbf{u})_l\|_{0,\widehat{Q}} \lesssim \|u_l\|_{H^s(\widehat{Q})} + |u_{l'}|_{AH^s_{l'}(\widehat{Q})} + \|\mathrm{div}\, \mathbf{u}\|_{0,\widehat{Q}},$$

$$\|u_l - (\widehat{\Pi}_{\mathrm{RT}} \mathbf{u})_l\|_{0,\widehat{Q}} \lesssim |u_l|_{H^s(\widehat{Q})} + |u_{l'}|_{AH^s_{l'}(\widehat{Q})} + \|\mathrm{div}\, \mathbf{u}\|_{0,\widehat{Q}}.$$

## 5.2 Anisotropic elements

In this subsection we will denote the functions on the elements $K$ and $\widehat{K}$ by $\mathbf{u}$ and $\widehat{\mathbf{u}}$, respectively. Analogous notation will be used for coordinates (e.g., $\mathbf{x} \in K$ and $\widehat{\mathbf{x}} \in \widehat{K}$) and for differential operators (e.g., div and $\widehat{\mathrm{div}}$).

First, let us prove the following auxiliary result.

**Lemma 5.1** *Let $K$ be the image of the reference element $\widehat{K}$ ($\widehat{K} = \widehat{T}$ or $\widehat{K} = \widehat{Q}$) under diagonal scaling with matrix $B = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$, where $h_l > 0$. Then for any $u \in H^{-1/2}(K)$ there holds*

$$\|\widehat{u}\|_{H^{-1/2}(\widehat{K})} \lesssim \frac{\max\{h_1^{1/2},\, h_2^{1/2}\}}{h_1 h_2} \|u\|_{H^{-1/2}(K)}, \tag{5.15}$$

*where $\widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = \mathbf{u}(B\widehat{\mathbf{x}})$, $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2) \in \widehat{K}$.*

**Proof.** By the definition of the dual norm

$$\|\widehat{u}\|_{H^{-1/2}(\widehat{K})} = \sup_{\widehat{v} \in \tilde{H}^{1/2}(\widehat{K})} \frac{(\widehat{u}, \widehat{v})_{0,\widehat{K}}}{\|\widehat{v}\|_{\tilde{H}^{1/2}(\widehat{K})}}. \tag{5.16}$$

We now estimate the norm $\|\widehat{v}\|_{\tilde{H}^{1/2}(\widehat{K})}$. If $\widehat{v} \in H^1(\widehat{K})$, then diagonal scaling yields

$$\|\widehat{v}\|^2_{0,\widehat{K}} = (h_1 h_2)^{-1} \|v\|^2_{0,K},$$

$$\|\widehat{\partial}_1 \widehat{v}\|^2_{0,\widehat{K}} \simeq h_1 h_2^{-1} \|\partial_1 v\|^2_{0,K}, \quad \|\widehat{\partial}_2 \widehat{v}\|^2_{0,\widehat{K}} \simeq h_1^{-1} h_2 \|\partial_2 v\|^2_{0,K}.$$

Therefore,

$$\|\widehat{v}\|^2_{H^1_0(\widehat{K})} = \|\widehat{\partial}_1 \widehat{v}\|^2_{0,\widehat{K}} + \|\widehat{\partial}_2 \widehat{v}\|^2_{0,\widehat{K}} \gtrsim \frac{\min\{h_1^2, h_2^2\}}{h_1 h_2} \|v\|^2_{H^1_0(K)},$$

and by interpolation between $L^2$ and $H^1_0$ we find that

$$\|\widehat{v}\|^2_{\tilde{H}^{1/2}(\widehat{K})} \gtrsim \frac{\min\{h_1, h_2\}}{h_1 h_2} \|v\|^2_{\tilde{H}^{1/2}(K)} = \frac{1}{\max\{h_1, h_2\}} \|v\|^2_{\tilde{H}^{1/2}(K)} \quad \forall \widehat{v} \in \tilde{H}^{1/2}(\widehat{K}). \tag{5.17}$$

Since $(\widehat{u}, \widehat{v})_{0,\widehat{K}} = (h_1 h_2)^{-1} (u, v)_{0,K}$, we use (5.17) in (5.16) to obtain inequality (5.15). $\Box$

Now, we are in a position to prove the stability result and the corresponding error estimate for the Raviart-Thomas interpolation on anisotropic elements.

**Theorem 5.3** *Let $K$ be either the triangle $T$ with vertices $(0,0)$, $(h_1, 0)$, $(h_1, h_2)$, or the rectangle $Q$ with vertices $(0,0)$, $(h_1, 0)$, $(0, h_2)$, $(h_1, h_2)$, where $h_l > 0$. Denote $h_{\max} := \max\{h_1, h_2\}$. Then for any $\mathbf{u} \in \mathbf{H}^{1/2}(K)$ with $\mathrm{div}\,\mathbf{u} \in \mathbb{R}$ there holds for $l = 1, 2$*

$$\|(\Pi_{\mathrm{RT}} \mathbf{u})_l\|^2_{0,K} \lesssim \|u_l\|^2_{0,K} + \frac{h^3_{\max}}{h_1 h_2} \left( |u_l|^2_{H^{1/2}(K)} + |u_{l'}|^2_{AH^{1/2}_{l'}(K)} + \|\mathrm{div}\,\mathbf{u}\|^2_{H^{-1/2}(K)} \right) \tag{5.18}$$

*and*

$$\|u_l - (\Pi_{\mathrm{RT}} \mathbf{u})_l\|^2_{0,K} \lesssim \frac{h^3_{\max}}{h_1 h_2} \left( |u_l|^2_{H^{1/2}(K)} + |u_{l'}|^2_{AH^{1/2}_{l'}(K)} + \|\mathrm{div}\,\mathbf{u}\|^2_{H^{-1/2}(K)} \right). \tag{5.19}$$

**Proof.** We consider only the case of the triangle, $K = T$. The proof is similar for $K = Q$. We use the Piola transformation to define $\widehat{\mathbf{u}} \in \mathbf{H}^{1/2}(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ on the reference triangle $\widehat{T}$ as follows:

$$\widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = h_1 h_2 B^{-1} \mathbf{u}(B\widehat{\mathbf{x}}) \quad \text{with} \quad B = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\|\widehat{u}_1\|_{0,\widehat{T}}^2 &= h_2^2 \, (h_1 h_2)^{-1} \, \|u_1\|_{0,T}^2 = h_2 \, h_1^{-1} \, \|u_1\|_{0,T}^2, \\
|\widehat{u}_1|_{H^{1/2}(\widehat{T})}^2 &= \int_{\widehat{T}} \int_{\widehat{T}} \frac{|\widehat{u}_1(\widehat{\mathbf{x}}) - \widehat{u}_1(\widehat{\mathbf{y}})|^2}{(|\widehat{x}_1 - \widehat{y}_1|^2 + |\widehat{x}_2 - \widehat{y}_2|^2)^{3/2}} d\widehat{\mathbf{x}} \, d\widehat{\mathbf{y}} \\
&= h_2^2 \, (h_1 h_2)^{-2} \int_T \int_T \frac{|u_1(\mathbf{x}) - u_1(\mathbf{y})|^2}{(h_1^{-2}|x_1 - y_1|^2 + h_2^{-2}|x_2 - y_2|^2)^{3/2}} d\mathbf{x} \, d\mathbf{y} \\
&\le h_1^{-2} \, h_{\max}^3 \, |u_1|_{H^{1/2}(T)}^2,
\end{aligned}$$

and

$$\begin{aligned}
|\widehat{u}_2|_{AH_2^{1/2}(\widehat{T})}^2 &= \int_0^1 |\widehat{u}_2(\widehat{x}_1, \cdot)|_{H^{1/2}(0,\widehat{x}_1)}^2 \, d\widehat{x}_1 \\
&= h_1^{-1} \int_0^{h_1} \left[ \int_0^{h_2 x_1/h_1} \int_0^{h_2 x_1/h_1} \frac{h_1^2 |u_2(x_1, x_2) - u_2(x_1, y_2)|^2}{h_2^{-2} |x_2 - y_2|^2} \, \frac{dx_2 \, dy_2}{h_2^2} \right] dx_1 \\
&= h_1 \, |u_2|_{AH_2^{1/2}(T)}^2.
\end{aligned}$$

Furthermore, the standard properties of the Piola transformation yield

$$\Pi_{\mathrm{RT}} \mathbf{u}(\mathbf{x}) = \frac{1}{h_1 h_2} B \widehat{\Pi}_{\mathrm{RT}} \widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = \begin{pmatrix} 1/h_2 & 0 \\ 0 & 1/h_1 \end{pmatrix} \widehat{\Pi}_{\mathrm{RT}} \widehat{\mathbf{u}}(\widehat{\mathbf{x}}) \quad \text{and} \quad \widehat{\mathrm{div}} \, \widehat{\mathbf{u}} = h_1 h_2 \, \mathrm{div} \, \mathbf{u} \in \mathbb{R}.$$

Therefore, applying Theorem 5.1 (ii) and Lemma 5.1 and using the fact that $\widehat{\mathrm{div}} \, \widehat{\mathbf{u}} \in \mathbb{R}$ (hence, $\|\widehat{\mathrm{div}} \, \widehat{\mathbf{u}}\|_{0,\widehat{T}} \simeq \|\widehat{\mathrm{div}} \, \widehat{\mathbf{u}}\|_{H^{-1/2}(\widehat{T})}$), we obtain

$$\begin{aligned}
\|(\Pi_{\mathrm{RT}} \mathbf{u})_1\|_{0,T}^2 &= h_2^{-2} h_1 h_2 \, \|(\widehat{\Pi}_{\mathrm{RT}} \widehat{\mathbf{u}})_1\|_{0,\widehat{T}}^2 \\
&\lesssim h_1 h_2^{-1} \left( \|\widehat{u}_1\|_{0,\widehat{T}}^2 + |\widehat{u}_1|_{H^{1/2}(\widehat{T})}^2 + |\widehat{u}_2|_{AH_2^{1/2}(\widehat{T})}^2 + \|\widehat{\mathrm{div}} \, \widehat{\mathbf{u}}\|_{0,\widehat{T}}^2 \right) \\
&\lesssim \|u_1\|_{0,T}^2 + \frac{h_{\max}^3}{h_1 h_2} |u_1|_{H^{1/2}(T)}^2 + \frac{h_1^2}{h_2} |u_2|_{AH_2^{1/2}(T)}^2 + \frac{h_1}{h_2} \, h_{\max} \|\mathrm{div} \, \mathbf{u}\|_{H^{-1/2}(T)}^2.
\end{aligned}$$

Recalling that $h_{\max} = \max\{h_1, h_2\}$, it is easy to see that

$$\frac{h_1^2}{h_2} \le \frac{h_1}{h_2} \, h_{\max} \le \frac{h_{\max}^3}{h_1 h_2},$$

and then inequality (5.18) follows (for $l = 1$ and $K = T$).

Arguing as above and using Corollary 5.1 (ii) instead of Theorem 5.1 (ii) we establish the error estimate in (5.19) for $l = 1$. The proof is analogous in the case $l = 2$. $\square$

We can now estimate the $\mathbf{L}^2$-error of the Raviart-Thomas interpolation on the graded mesh $\Delta_h^\beta$ on $\Gamma$. The specific estimate that we need is for $\mathbf{H}^{1/2}$-regular vector fields with discrete divergence.

**Lemma 5.2** *For any* $\mathbf{u} \in \mathbf{H}_\perp^{1/2}(\Gamma)$ *such that* $\mathrm{div}_\Gamma \mathbf{u} \in \mathrm{div}_\Gamma \mathbf{X}_h$ *there holds*

$$\|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|_{0,\Gamma} \lesssim h^{1-\beta/2} \, \|\mathbf{u}\|_{\mathbf{H}_\perp^{1/2}(\Gamma)}. \tag{5.20}$$

**Proof.** Let $F$ be a face of $\Gamma$, and let $T_F \subset F$ be a triangular block of elements, see Figure 1 (the arguments are analogous for the parallelogram block of elements $Q_F$). The triangle $T_F$ is mapped onto the unit triangle $\widehat{T}$ by the affine transformation which is independent of the mesh parameter $h$. Let us first establish the error estimate for the Raviart-Thomas interpolation on the unit triangle $\widehat{T}$ partitioned into elements as shown in Figure 1.

The graded mesh on $\widehat{T}$ comprises the quadrilaterals $K_{ij} = I_i \times I_j$ $(i, j = 1, \ldots, N,\ i > j)$ isomorphic to $(0, h_i) \times (0, h_j)$ with $h_i \geq h_j$ and the triangles $K_{ii}$ isomorphic to the triangle with vertices $(0, 0)$, $(0, h_i)$, $(h_i, h_i)$. Applying error estimates from Theorem 5.3 on each element $K_{ij}$ $(i \geq j)$, we have for $l = 1, 2$:

$$\|u_l - (\Pi_{\mathrm{RT}}\mathbf{u})_l\|^2_{0,K_{ij}} \lesssim h_i^2 h_j^{-1} \left( |u_l|^2_{H^{1/2}(K_{ij})} + |u_{l'}|^2_{AH^{1/2}_{l'}(K_{ij})} + \|\mathrm{div}\,\mathbf{u}\|^2_{H^{-1/2}(K_{ij})} \right).$$

Summing these estimates over all elements in $\widehat{T}$ and recalling that $h_i^2 h_j^{-1} \lesssim h^{2-\beta}$ for $1 \leq i, j \leq N$, we obtain

$$\|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|^2_{0,\widehat{T}} \lesssim h^{2-\beta} \sum_{\substack{i,j=1 \\ i \geq j}}^{N} \left( |\mathbf{u}|^2_{\mathbf{H}^{1/2}(K_{ij})} + \sum_{l=1}^{2} |u_l|^2_{AH_l^{1/2}(K_{ij})} + \|\mathrm{div}\,\mathbf{u}\|^2_{H^{-1/2}(K_{ij})} \right). \tag{5.21}$$

Note that by a standard superposition argument

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j=1}^{i} |u_2|^2_{AH_2^{1/2}(K_{ij})} &\lesssim \sum_{i=1}^{N} \int_{I_i} |u_2(x_1, \cdot)|^2_{H^{1/2}(0,x_1)} \, dx_1 \\ &= \int_0^1 |u_2(x_1, \cdot)|^2_{H^{1/2}(0,x_1)} \, dx_1 = |u_2|^2_{AH_2^{1/2}(\widehat{T})} \overset{(5.2)}{\lesssim} \|u_2\|^2_{H^{1/2}(\widehat{T})}, \end{aligned} \tag{5.22}$$

and similarly for $u_1$. Hence, using standard superadditivity properties of the squared $H^{1/2}$-seminorm and the squared $H^{-1/2}$-norm (see, e.g., [2, Theorem 4.1]), we deduce from (5.21) the following error estimate on $\widehat{T}$:

$$\|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|_{0,\widehat{T}} \lesssim h^{1-\beta/2} \left( \|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{H^{-1/2}(\widehat{T})} \right).$$

Applying now the Piola transformation associated with the mapping $T_F \to \widehat{T}$, patching together all individual blocks of elements on all faces of $\Gamma$, and using the superadditivity of $H^{1/2}$- and $H^{-1/2}$-norms (as the functions of subdomains), we obtain

$$\|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|_{0,\Gamma} \lesssim h^{1-\beta/2} \left( \|\mathbf{u}\|_{\mathbf{H}_-^{1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma)} \right) \tag{5.23}$$

(here and below we use the space $\mathbf{H}_-^s(\Gamma)$, $s > 0$, which is defined in a piecewise fashion by localisation to each face of $\Gamma$, with the norm $\|\mathbf{u}\|^2_{\mathbf{H}_-^s(\Gamma)} := \sum_{F\subset\Gamma} \|\mathbf{u}|_F\|^2_{\mathbf{H}^s(F)}$).

Inequality (5.20) follows from (5.23) due to the continuity property of $\mathrm{div}_\Gamma$: $\mathbf{H}_\perp^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ (see [15, Section 4.2]). □

# 6 Discrete decomposition and the gap property

Following the ideas from [13] and [18, Section 9.1], we can use the Raviart-Thomas interpolation operator $\Pi_{\mathrm{RT}}$ to define the discrete counterparts of $\mathbf{V}$ and $\mathbf{W}$ in (4.3) (e.g., we can set $\mathbf{V}_h := \Pi_{\mathrm{RT}}(\mathsf{R}(\mathbf{X}_h))$, where $\mathsf{R}$ is the regularised projector introduced in Section 4). However, as it follows from the results in Section 5, the Raviart-Thomas interpolation of low-regular vector fields on graded meshes $\Delta_h^\beta$ is only stable (with respect to the $\mathbf{L}^2$-norm) when $\beta < 2$. Since the definition of the energy space $\mathbf{X}$ for the EFIE involves the dual space $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ with a weaker norm than $\|\cdot\|_{0,\Gamma}$, we can relax the restriction on the grading parameter $\beta$ by employing a different projection onto the boundary element space and using a duality argument on individual faces of $\Gamma$. This approach was successfully used by Hiptmair and Schwab in [24, Section 8] and by Buffa and Christiansen in [13, Section 4.2.2] in the context of the $h$-BEM with shape-regular meshes for the EFIE (see [5, 4] for applications of these ideas to the analysis of the $p$-BEM and the $hp$-BEM with quasi-uniform meshes). We will demonstrate below that using these techniques together with stability properties and error estimates for the Raviart-Thomas interpolation on anisotropic elements, one can design a stable discrete decomposition of the boundary element space on $\Delta_h^\beta$ and prove the corresponding gap property (3.2) for any $\beta < 3$.

The construction of the desired projection operator is technically involved. Therefore, we formulate here the final result relevant to our discussion and give a detailed proof in the next section. In the Proposition below, $\Pi_0$ denotes the $L^2(\Gamma)$-projection onto the space of piecewise constant functions over the mesh $\Delta_h^\beta$, and $\mathbf{H}_-^{-1/2}(\Gamma)$ denotes the dual space of $\mathbf{H}_-^{1/2}(\Gamma)$ (with $\mathbf{L}_t^2(\Gamma)$ as pivot space).

**Proposition 6.1** *There exists an operator $\mathcal{Q}_h : \mathbf{H}_-^s(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_h$ ($s > 0$) such that*

$$\mathrm{div}_\Gamma \circ \mathcal{Q}_h = \Pi_0 \circ \mathrm{div}_\Gamma, \tag{6.1}$$

*and for any $\varepsilon > 0$*

$$\|\mathbf{u} - \mathcal{Q}_h\mathbf{u}\|_{\mathbf{H}_-^{-1/2}(\Gamma)} \lesssim h^{1/2-\varepsilon} \|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)} \quad \forall\, \mathbf{u} \in \mathbf{H}_-^s(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma). \tag{6.2}$$

Thus, the operator $\mathcal{Q}_h$ inherits the crucial commuting diagram property (6.1) of the classical RT-interpolation operator and, at the same time, allows to gain an extra power of $h$ when estimating the error $(\mathbf{u} - \mathcal{Q}_h\mathbf{u})$ in the dual norm.

**Corollary 6.1** *For any* $\mathbf{u} \in \mathbf{H}_\perp^{1/2}(\Gamma)$ *such that* $\mathrm{div}_\Gamma \mathbf{u} \in \mathrm{div}_\Gamma \mathbf{X}_h$, *one has* $\mathcal{Q}_h\mathbf{u} \in \mathbf{X}_h$, $\mathrm{div}_\Gamma \mathcal{Q}_h\mathbf{u} = \mathrm{div}_\Gamma \mathbf{u}$, *and for any* $\varepsilon > 0$ *there holds*

$$\|\mathbf{u} - \mathcal{Q}_h\mathbf{u}\|_{\mathbf{X}} = \|\mathbf{u} - \mathcal{Q}_h\mathbf{u}\|_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} \lesssim h^{3/2-\beta/2-\varepsilon}\|\mathbf{u}\|_{\mathbf{H}_\perp^{1/2}(\Gamma)}. \tag{6.3}$$

Since $\|\cdot\|_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} \lesssim \|\cdot\|_{\mathbf{H}_-^{-1/2}(\Gamma)}$, this result immediately follows from Proposition 6.1 and Lemma 5.2 due to the commuting diagram property for $\Pi_{\mathrm{RT}}$.

Since $\mathsf{R}\,\mathbf{X}_h \subset \mathbf{H}_\perp^{1/2}(\Gamma)$ (see (4.2)) and $\mathrm{div}_\Gamma \mathsf{R}\,\mathbf{X}_h = \mathrm{div}_\Gamma \mathbf{X}_h$ (see (4.1)), the following definitions are valid thanks to Proposition 6.1:

$$\mathbf{V}_h := (\mathcal{Q}_h \circ \mathsf{R})\mathbf{X}_h, \quad \mathbf{W}_h := (\mathrm{Id} - \mathcal{Q}_h \circ \mathsf{R})\mathbf{X}_h.$$

Then the commuting diagram property (6.1) and the error estimate (6.3) with $\beta < 3$ pave the way for verifying properties **(B)** and **(C)** from Section 3 in exactly the same way as demonstrated in [8, Section 6] (see also [9, Section 6] for details). This completes the proof of Theorem 3.1.

# 7 Proof of Proposition 6.1

In this section, we give a constructive proof of Proposition 6.1. For any $\mathbf{u} \in \mathbf{H}_-^s(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ we construct $\mathcal{Q}_h\mathbf{u}$ in the Raviart-Thomas spaces on individual faces of $\Gamma$. Let $F$ be a single face of $\Gamma$, and let $\Delta_h^\beta(F)$ denote the restriction of the graded mesh $\Delta_h^\beta$ onto $F$. For the sake of simplicity of notation we will omit the subscript $F$ for differential operators over this face, e.g., we will write div for $\mathrm{div}_F$. We will also write $(\cdot,\cdot)$ for the $L^2(F)$- and $\mathbf{L}^2(F)$-inner products, and similarly $\|\cdot\|$ for the corresponding norms of scalar functions and vector fields. First, let us prove the following auxiliary result.

**Lemma 7.1** *For any* $s > 1/2$, *the Raviart-Thomas interpolation operator* $\Pi_{\mathrm{RT}} : \mathbf{H}^s(F) \cap \mathbf{H}(\mathrm{div}, F) \to \boldsymbol{\mathcal{RT}}_0(\Delta_h^\beta(F))$, *is* $\mathbf{L}^2(F)$*-stable, i.e., there exists a constant* $C > 0$ *independent of* $h$ *such that*

$$\|\Pi_{\mathrm{RT}}\mathbf{u}\|_{0,F} \le C\big(\|\mathbf{u}\|_{\mathbf{H}^s(F)} + \|\mathrm{div}\,\mathbf{u}\|_{0,F}\big) \quad \forall \mathbf{u} \in \mathbf{H}^s(F) \cap \mathbf{H}(\mathrm{div}, F). \tag{7.1}$$

**Proof.** Similarly to the proof of Lemma 5.2, it is sufficient to establish (7.1) for the unit triangle $\widehat{T}$ partitioned into elements as shown in Figure 1 (this is because the affine transformations that map triangular blocks of elements $T_F \subset F$ onto $\widehat{T}$ are independent of $h$).

The graded mesh on $\widehat{T}$ (see Figure 1) comprises anisotropic quadrilaterals $K_{ij} = I_i \times I_j$ $(i, j = 1, \dots, N,\ i > j)$ isomorphic to $(0, h_i) \times (0, h_j)$ with $h_i \ge h_j$ and shape-regular triangles $K_{ii}$ isomorphic to the triangle with vertices $(0, 0)$, $(0, h_i)$, $(h_i, h_i)$. Using the

Piola transform associated with the mapping $\widehat{K} \to K_{ij}$ ($i \geq j$; $\widehat{K} = \widehat{T}$ or $\widehat{Q}$), we define $\widehat{\mathbf{u}} \in \mathbf{H}^s(\widehat{K}) \cap \mathbf{H}(\mathrm{div}, \widehat{K})$. Then, by the standard properties of the Piola transform we have

$$\|\widehat{u}_1\|^2_{0,\widehat{K}} \simeq h_i^{-1}\, h_j\, \|u_1\|^2_{0,K_{ij}}, \quad \|\widehat{\mathrm{div}\, \mathbf{u}}\|^2_{0,\widehat{K}} \simeq h_i h_j\, \|\mathrm{div}\, \mathbf{u}\|^2_{0,K_{ij}},$$

$$\|(\Pi_{\mathrm{RT}}\mathbf{u})_1\|^2_{0,K_{ij}} \simeq h_i\, h_j^{-1}\, \|(\widehat{\Pi}_{\mathrm{RT}}\widehat{\mathbf{u}})_1\|^2_{0,\widehat{K}}.$$

The application of the scaling argument yields:

$$\begin{aligned}
|\widehat{u}_1|^2_{H^s(\widehat{T})} &= \int_{\widehat{T}}\int_{\widehat{T}} \frac{|\widehat{u}_1(\widehat{\mathbf{x}}) - \widehat{u}_1(\widehat{\mathbf{y}})|^2}{(|\widehat{x}_1 - \widehat{y}_1|^2 + |\widehat{x}_2 - \widehat{y}_2|^2)^{1+s}} d\widehat{\mathbf{x}}\, d\widehat{\mathbf{y}} \\
&\simeq h_i^2\, (h_i h_i)^{-2} \int_{K_{ii}}\int_{K_{ii}} \frac{|u_1(\mathbf{x}) - u_1(\mathbf{y})|^2}{(h_i^{-2}|x_1 - y_1|^2 + h_i^{-2}|x_2 - y_2|^2)^{1+s}} d\mathbf{x}\, d\mathbf{y} \\
&= h_i^{2s}\, |u_1|^2_{H^s(K_{ii})},
\end{aligned}$$

$$\begin{aligned}
|\widehat{u}_1|^2_{AH_1^s(\widehat{Q})} &= \int_0^1 |\widehat{u}_1(\cdot, \widehat{x}_2)|^2_{H^s(0,1)}\, d\widehat{x}_2 \\
&= h_j^{-1} \int_0^{h_j} \left[ \int_0^{h_i}\int_0^{h_i} \frac{h_j^2 |u_1(x_1, x_2) - u_1(y_1, x_2)|^2}{h_i^{-1-2s}|x_1 - y_1|^{1+2s}} \frac{dx_1\, dy_1}{h_i^2} \right] dx_2 \\
&\simeq h_i^{2s-1} h_j\, |u_1|^2_{AH_1^s(K_{ij})} \quad (i > j),
\end{aligned}$$

and analogously,

$$|\widehat{u}_1|^2_{AH_2^s(\widehat{Q})} \simeq h_i^{-1} h_j^{1+2s}\, |u_1|^2_{AH_2^s(K_{ij})} \quad (i > j).$$

Therefore, applying Theorem 5.1 (i), we obtain

$$\begin{aligned}
\|(\Pi_{\mathrm{RT}}\mathbf{u})_1\|^2_{0,K_{ii}} &\simeq \|(\widehat{\Pi}_{\mathrm{RT}}\widehat{\mathbf{u}})_1\|^2_{0,\widehat{T}} \lesssim \|\widehat{u}_1\|^2_{H^s(\widehat{T})} + \|\widehat{\mathrm{div}\, \mathbf{u}}\|_{0,\widehat{T}} \\
&\simeq \|u_1\|^2_{0,K_{ii}} + h_i^{2s}\, |u_1|^2_{H^s(K_{ii})} + h_i^2\, |\mathrm{div}\, \mathbf{u}|^2_{0,K_{ii}}.
\end{aligned} \tag{7.2}$$

Similarly, applying Theorem 5.2 (i) and recalling (5.1), we have for $i > j$

$$\begin{aligned}
\|(\Pi_{\mathrm{RT}}\mathbf{u})_1\|^2_{0,K_{ij}} &\simeq h_i\, h_j^{-1}\, \|(\widehat{\Pi}_{\mathrm{RT}}\widehat{\mathbf{u}})_1\|^2_{0,\widehat{Q}} \lesssim h_i h_j^{-1}\, \|\widehat{u}_1\|^2_{H^s(\widehat{Q})} \\
&\simeq h_i h_j^{-1} \left( \|\widehat{u}_1\|^2_{0,\widehat{Q}} + |\widehat{u}_1|^2_{AH_1^s(\widehat{Q})} + |\widehat{u}_1|^2_{AH_2^s(\widehat{Q})} \right) \\
&\simeq \|u_1\|^2_{0,K_{ij}} + h_i^{2s}\, |u_1|^2_{AH_1^s(K_{ij})} + h_j^{2s}\, |u_1|^2_{AH_2^s(K_{ij})}.
\end{aligned} \tag{7.3}$$

The estimates analogous to (7.2) and (7.3) are also valid for $\|(\Pi_{\mathrm{RT}}\mathbf{u})_2\|_{0,K_{ij}}$ with $i \geq j$.

Combining the estimates for both components of $\Pi_{\mathrm{RT}}\mathbf{u}$ over all elements in $\widehat{T}$ and then using the superposition argument as in (5.22) for anisotropic seminorms and the superadditivity property of the $H^{1/2}$-norm, we arrive at the desired result. □

Our construction of the operator $\mathcal{Q}_h$ follows the technique used by Hiptmair and Schwab in the proof of Lemma 8.1 in [24] but relies on stability properties of the Raviart-Thomas interpolation on graded meshes over individual faces of $\Gamma$. Given $\mathbf{u} \in \mathbf{H}^s(F) \cap \mathbf{H}(\mathrm{div}, F)$, $s > 0$, we consider the following mixed problem: *Find* $(\mathbf{z}, f) \in \mathbf{H}(\mathrm{div}, F) \times L^2_*(F)$ *such that*

$$\begin{aligned} (\mathbf{z}, \mathbf{v}) + (\mathrm{div}\, \mathbf{v}, f) &= (\mathbf{u}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{div}, F), \\ (\mathrm{div}\, \mathbf{z}, g) &= (\mathrm{div}\, \mathbf{u}, g) && \forall g \in L^2_*(F), \\ \mathbf{z} \cdot \tilde{\mathbf{n}} &= \mathbf{u} \cdot \tilde{\mathbf{n}} && \text{on } \partial F. \end{aligned} \tag{7.4}$$

Here, $L^2_*(F) := \{v \in L^2(F);\ (v, 1) = 0\}$, $\tilde{\mathbf{n}}$ is the unit outward normal vector to $\partial F$, and $\mathbf{H}_0(\mathrm{div}, F) := \{\mathbf{v} \in \mathbf{H}(\mathrm{div}, F);\ \mathbf{v} \cdot \tilde{\mathbf{n}}|_{\partial F} = 0\}$.

The unique solvability of (7.4) is proved by standard techniques (see [11, Chapter II]). In fact, it is clear that the pair $(\mathbf{u}, 0)$ solves (7.4).

A conforming Galerkin approximation of problem (7.4) with Raviart-Thomas elements on the graded mesh $\Delta^\beta_h(F)$ reads as: *Find* $(\mathbf{z}_h, f_h) \in \mathbf{X}_h(F) \times R_h(F)$ *such that*

$$\begin{aligned} (\mathbf{z}_h, \mathbf{v}) + (\mathrm{div}\, \mathbf{v}, f_h) &= (\mathbf{u}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F), \\ (\mathrm{div}\, \mathbf{z}_h, g) &= (\mathrm{div}\, \mathbf{u}, g) && \forall g \in R_h(F), \\ \mathbf{z}_h \cdot \tilde{\mathbf{n}} &= \Pi_{\mathrm{RT}} \mathbf{u} \cdot \tilde{\mathbf{n}} && \text{on } \partial F. \end{aligned} \tag{7.5}$$

Here, $\mathbf{X}_h(F)$ denotes the restriction of $\mathbf{X}_h$ onto the face $F$, and $R_h(F) := \{g \in L^2(F);\ g|_K = \mathrm{const},\ \forall K \in \Delta^\beta_h(F) \text{ and } (g, 1) = 0\}$.

Note that the third equation in (7.5) implies $(\mathrm{div}(\mathbf{u} - \mathbf{z}_h), 1) = 0$. Hence, the second identity in (7.5) holds for any piecewise constant function $g \in \mathrm{div}\, \mathbf{X}_h(F)$. Thus, $\mathrm{div}\, \mathbf{z}_h$ is the $L^2(F)$-projection of $\mathrm{div}\, \mathbf{u}$ onto $\mathrm{div}\, \mathbf{X}_h(F)$. In particular, if $\mathrm{div}\, \mathbf{u} \in \mathrm{div}\, \mathbf{X}_h(F)$ then $\mathrm{div}\, \mathbf{z}_h = \mathrm{div}\, \mathbf{u}$.

We now prove the unique solvability of (7.5). First, for any $g_h \in R_h(F)$ we find a function $\phi \in H^1_*(F) := \{\phi \in H^1(F);\ (\phi, 1) = 0\}$ solving the variational problem

$$(\nabla \phi, \nabla \tilde{\phi}) = (g_h, \tilde{\phi}) \quad \forall \tilde{\phi} \in H^1_*(F). \tag{7.6}$$

Applying the standard regularity result for problem (7.6) (see, e.g., [22, p. 82]), we conclude that $\phi \in H^{1+r}(F)$ with some $r \in (\frac{1}{2}, \frac{\pi}{\omega})$ (here, $\omega < 2\pi$ denotes the maximal internal angle at the vertices of $F$), and

$$\|\nabla \phi\|_{\mathbf{H}^r(F)} \lesssim \|\phi\|_{H^{1+r}(F)} \lesssim \|g_h\|. \tag{7.7}$$

Therefore, $\nabla \phi \in \mathbf{H}^r(F) \cap \mathbf{H}_0(\mathrm{div}, F)$, $r > \frac{1}{2}$, and the interpolant $\Pi_{\mathrm{RT}} \nabla \phi \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F)$ is well defined and stable, due to Lemma 7.1. Moreover, $\mathrm{div}\, (\Pi_{\mathrm{RT}} \nabla \phi) = \Pi_0(\mathrm{div}\, \nabla \phi) \overset{(7.6)}{=} g_h$. Hence, using (7.1) and (7.7) we prove the discrete inf-sup condition:

$$\begin{aligned} \sup_{\substack{\mathbf{v}_h \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F) \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{(\mathrm{div}\, \mathbf{v}_h, g_h)}{\|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div}, F)}} &\geq \frac{(\mathrm{div}\, (\Pi_{\mathrm{RT}} \nabla \phi), g_h)}{\|\Pi_{\mathrm{RT}} \nabla \phi\|_{\mathbf{H}(\mathrm{div}, F)}} \\ &\geq \frac{\|g_h\|^2}{C \left( \|\nabla \phi\|_{\mathbf{H}^r(F)} + \|\mathrm{div}\, \nabla \phi\| \right) + \|\mathrm{div}\, (\Pi_{\mathrm{RT}} \nabla \phi)\|} \\ &\geq \tilde{C}\, \|g_h\| \quad \forall g_h \in R_h(F). \end{aligned}$$

This condition along with the property $\operatorname{div}\big(\mathbf{X}_h(F) \cap \mathbf{H}_0(\operatorname{div}, F)\big) = R_h(F)$ ensures existence, uniqueness, and quasi-optimality of the solution $(\mathbf{z}_h, f_h)$ to (7.5) (see [11]). In particular, using the quasi-optimality and recalling that $\mathbf{z} = \mathbf{u}$, $f = 0$, we estimate

$$\begin{aligned}
\|\mathbf{u} - \mathbf{z}_h\|_{\mathbf{H}(\operatorname{div},F)} &\lesssim \inf_{\substack{\mathbf{v}_h \in \mathbf{X}_h(F) \\ (\mathbf{v}_h - \Pi_{\mathrm{RT}}\mathbf{u})\cdot\tilde{\mathbf{n}}|_F = 0}} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}(\operatorname{div},F)} + \inf_{g_h \in R_h(F)} \|f - g_h\| \\
&\lesssim \|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|_{\mathbf{H}(\operatorname{div},F)}. \qquad (7.8)
\end{aligned}$$

We now estimate $\|\mathbf{u} - \mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2}(F)}$. One has for any $\varepsilon \in (0, \frac{1}{2})$

$$\|\mathbf{u} - \mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2}(F)} \le \|\mathbf{u} - \mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2+\varepsilon}(F)} = \sup_{\mathbf{w} \in \mathbf{H}^{1/2-\varepsilon}(F)\setminus\{\mathbf{0}\}} \frac{|(\mathbf{u} - \mathbf{z}_h, \mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)}}. \qquad (7.9)$$

For a given $\mathbf{w} \in \mathbf{H}^{1/2-\varepsilon}(F)$, we solve the following problem: *Find $\varphi \in H^1_*(F)$ such that*

$$(\nabla \varphi, \nabla \phi) = -(\mathbf{w}, \nabla \phi) \quad \forall \phi \in H^1_*(F). \qquad (7.10)$$

Similarly to (7.7), the regularity result for $\varphi$ reads as

$$\varphi \in H^{3/2-\varepsilon}(F), \quad \|\varphi\|_{H^{3/2-\varepsilon}(F)} \lesssim \|\tilde{f}\|_{(H^{1/2+\varepsilon}(F))'} \lesssim \|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)}, \qquad (7.11)$$

where $\tilde{f} \in (H^{1/2+\varepsilon}(F))'$ is defined by $\tilde{f}(\phi) = -(\mathbf{w}, \nabla \phi)$, $\forall \phi \in H^{1/2+\varepsilon}(F)$.

Then we set

$$\mathbf{q} := \mathbf{w} + \nabla \varphi \in \mathbf{H}^{1/2-\varepsilon}(F) \cap \mathbf{H}_0(\operatorname{div}, F). \qquad (7.12)$$

It also follows from (7.10) that $\operatorname{div}\mathbf{q} = \operatorname{div}\mathbf{w} + \operatorname{div}\nabla\varphi = 0$. Furthermore, we have by (7.11)–(7.12) that

$$\|\mathbf{q}\|_{\mathbf{H}^{1/2-\varepsilon}(F)} \lesssim \|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)} + \|\varphi\|_{H^{3/2-\varepsilon}(F)} \lesssim \|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)}. \qquad (7.13)$$

We now use (7.12) and integration by parts to represent the numerator in (7.9) as

$$\begin{aligned}
(\mathbf{u} - \mathbf{z}_h, \mathbf{w}) &= (\mathbf{u} - \mathbf{z}_h, \mathbf{q}) - (\mathbf{u} - \mathbf{z}_h, \nabla \varphi) \\
&= (\mathbf{u} - \mathbf{z}_h, \mathbf{q}) + (\operatorname{div}(\mathbf{u} - \mathbf{z}_h), \varphi) - ((\mathbf{u} - \mathbf{z}_h)\cdot\tilde{\mathbf{n}}, \varphi)_{0,\partial F}.
\end{aligned}$$

Hence, using (7.4), (7.5) and recalling that $\mathbf{z} = \mathbf{u}$, $f = 0$, we find for any $\mathbf{q}_h \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\operatorname{div}, F)$ and arbitrary $\varphi_h \in R_h(F)$

$$\begin{aligned}
|(\mathbf{u} - \mathbf{z}_h, \mathbf{w})| &= |(\mathbf{u} - \mathbf{z}_h, \mathbf{q} - \mathbf{q}_h) + (\mathbf{u} - \mathbf{z}_h, \mathbf{q}_h) \\
&\quad + (\operatorname{div}(\mathbf{u} - \mathbf{z}_h), \varphi - \varphi_h) - ((\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u})\cdot\tilde{\mathbf{n}}, \varphi)_{0,\partial F}| \\
&= |(\mathbf{u} - \mathbf{z}_h, \mathbf{q} - \mathbf{q}_h) + (\operatorname{div}\mathbf{q}_h, f_h) \\
&\quad + (\operatorname{div}(\mathbf{u} - \mathbf{z}_h), \varphi - \varphi_h) - ((\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u})\cdot\tilde{\mathbf{n}}, \varphi)_{0,\partial F}| \\
&\le \|\mathbf{u} - \mathbf{z}_h\|\,\|\mathbf{q} - \mathbf{q}_h\| + |(\operatorname{div}\mathbf{q}_h, f_h)| + \|\operatorname{div}(\mathbf{u} - \mathbf{z}_h)\|\,\|\varphi - \varphi_h\| \\
&\quad + \|(\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)}\,\|\varphi\|_{H^{1-\varepsilon}(\partial F)}. \qquad (7.14)
\end{aligned}$$

Let $\Pi_{\mathrm{RT}}^{\mathrm{q/u}}$ denote the Raviart-Thomas interpolation operator on the 'coarse' quasi-uniform and shape-regular mesh $\Delta_h^{\mathrm{q/u}}(F)$ obtained from the graded mesh $\Delta_h^{\beta}(F)$ by patching together long and thin elements (see Figure 3). We also denote by $\Pi_0^{\mathrm{q/u}}$ the $L^2(F)$-projector onto the space of piecewise constant functions on $\Delta_h^{\mathrm{q/u}}(F)$. Then we set

$$\mathbf{q}_h := \Pi_{\mathrm{RT}}^{\mathrm{q/u}}\mathbf{q} \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F) \quad \text{and} \quad \varphi_h := \Pi_0^{\mathrm{q/u}}\varphi \in R_h(F).$$

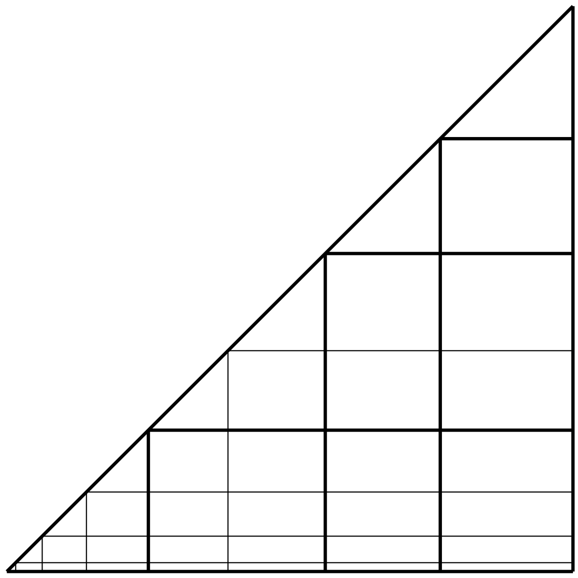

Figure 3: An example of shape-regular quasi-uniform mesh (thicker lines) obtained by patching together elements of the graded mesh (thinner lines).

By the standard properties of the Raviart-Thomas interpolation and the $L^2$-projection on quasi-uniform and shape-regular meshes, we have

$$\mathrm{div}\,\mathbf{q}_h = \Pi_0^{\mathrm{q/u}}\mathrm{div}\,\mathbf{q} = 0, \tag{7.15}$$

$$\|\mathbf{q} - \mathbf{q}_h\| \lesssim h^{1/2-\varepsilon}\,\|\mathbf{q}\|_{H^{1/2-\varepsilon}(F)} \overset{(7.13)}{\lesssim} h^{1/2-\varepsilon}\,\|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}, \tag{7.16}$$

$$\|\varphi - \varphi_h\| \lesssim h\,\|\varphi\|_{H^{3/2-\varepsilon}(F)} \overset{(7.11)}{\lesssim} h\,\|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}. \tag{7.17}$$

To estimate $\|(\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)}$ we recall that $\int_{e_h}(\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}} = 0$ for any element edge $e_h \subset \partial F$. Therefore, we can use a standard duality argument to prove (cf. [13, p. 259])

$$\|(\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)} \lesssim \Big(\max_{e_h \subset \partial F} |e_h|\Big)^{1-\varepsilon} \|(\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}}\|_{L^2(\partial F)}.$$

Then by interpolation we obtain

$$\|(\mathbf{u}-\Pi_{\mathrm{RT}}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)} \lesssim h^{1/2-\varepsilon}\,\|(\mathbf{u}-\Pi_{\mathrm{RT}}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1/2}(\partial F)} \lesssim h^{1/2-\varepsilon}\,\|\mathbf{u}-\Pi_{\mathrm{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},F)} \tag{7.18}$$

(here, we also used the continuity of the normal trace operator $\mathbf{v}\mapsto\mathbf{v}\cdot\tilde{\mathbf{n}}|_{\partial F}$ as a mapping $\mathbf{H}(\mathrm{div},F)\to H^{-1/2}(\partial F)$).

Furthermore, one has

$$\|\varphi\|_{H^{1-\varepsilon}(\partial F)} \lesssim \|\varphi\|_{H^{3/2-\varepsilon}(F)} \overset{(7.11)}{\lesssim} \|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}. \tag{7.19}$$

Now, using (7.15)–(7.19) in (7.14) and recalling (7.8) we find

$$|(\mathbf{u}-\mathbf{z}_h,\mathbf{w})| \lesssim h^{1/2-\varepsilon}\,\|\mathbf{u}-\Pi_{\mathrm{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},F)}\,\|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}.$$

Using this estimate in (7.9) we obtain

$$\|\mathbf{u}-\mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2}(F)} \lesssim h^{1/2-\varepsilon}\,\|\mathbf{u}-\Pi_{\mathrm{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},F)}. \tag{7.20}$$

Now we can prove the desired result.

**Proof of Proposition 6.1.** For any $\mathbf{u}\in\mathbf{H}^s_-(\Gamma)\cap\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)$, we define $\mathcal{Q}_h\mathbf{u}\in\mathbf{X}_h$ face by face as $\mathcal{Q}_h\mathbf{u}|_F := \mathbf{z}_h$ for any face $F\subset\Gamma$, where $\mathbf{z}_h$ is a unique (vectorial) solution to (7.5). Then the commuting diagram property (6.1) follows from the second identity in (7.5), and inequality (7.20) yields estimate (6.2). □

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# References

[1] G. Acosta, T. Apel, R. G. Durán, and A. L. Lombardi, *Error estimates for Raviart-Thomas interpolation of any order on anisotropic tetrahedra*, Math. Comp., 80 (2011), pp. 141–163.

[2] M. Ainsworth, W. McLean, and T. Tran, *The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling*, SIAM J. Numer. Anal., 36 (1999), pp. 1901–1932.

[3] T. Apel, *Anisotropic Finite Elements: Local Estimates and Applications*, Advances in Numerical Mathematics, Teubner, 1999.

[4] A. Bespalov and N. Heuer, *The hp-BEM with quasi-uniform meshes for the electric field integral equation on polyhedral surfaces: a priori error analysis*, Appl. Numer. Math., 60 (2010), pp. 705–718.

[5] ——, *Natural p-BEM for the electric field integral equation on screens*, IMA J. Numer. Anal., 30 (2010), pp. 595–628.

[6] ——, *A new* $\mathbf{H}(\mathrm{div})$*-conforming p-interpolation operator in two dimensions*, ESAIM: Mathematical Modelling and Numerical Analysis, 45 (2011), pp. 255–275.

[7] ——, *Natural hp-BEM for the electric field integral equation with singular solutions*, Numer. Methods Partial Differential Equations, 28 (2012), pp. 1466–1480.

[8] A. Bespalov, N. Heuer, and R. Hiptmair, *Convergence of the natural hp-BEM for the electric field integral equation on polyhedral surfaces*, SIAM J. Numer. Anal., 48 (2010), pp. 1518–1529.

[9] A. Bespalov and S. Nicaise, *The BEM with graded meshes for the electric field integral equation on polyhedral surfaces*, Preprint, arXiv:1504.02647v1 [math.NA], 2015. `http://arxiv.org/abs/1504.02647`.

[10] ——, *A priori error analysis of the BEM with graded meshes for the electric field integral equation on polyhedra*, in preparation.

[11] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, no. 15 in Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[12] A. Buffa, *Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations*, SIAM J. Numer. Anal., 43 (2005), pp. 1–18.

[13] A. Buffa and S. H. Christiansen, *The electric field integral equation on Lipschitz screens: definitions and numerical approximation*, Numer. Math., 94 (2003), pp. 229–267.

[14] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations, Part I: An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[15] ——, *On traces for functional spaces related to Maxwell's equations, Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., 24 (2001), pp. 31–48.

[16] A. Buffa, M. Costabel, and C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Mat., 92 (2002), pp. 679–710.

[17] A. Buffa, M. Costabel, and D. Sheen, *On traces for H(curl, Ω) in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[18] A. Buffa and R. Hiptmair, *Galerkin boundary element methods for electromagnetic scattering*, in Topics in computational wave propagation, vol. 31 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2003, pp. 83–124.

[19] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab, *Boundary element methods for Maxwell transmission problems in Lipschitz domains*, Numer. Math., 95 (2003), pp. 459–485.

[20] M. Costabel and M. Dauge, *Singularities of electromagnetic fields in polyhedral domains*, Arch. Rational Mech. Anal., 151 (2000), pp. 221–276.

[21] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing Inc., Boston, 1985.

[22] ——, *Singularities in Boundary Value Problems*, no. 22 in Research Notes in Applied Mathematics, Masson, Paris, 1992.

[23] R. Hiptmair, *Coupling of finite elements and boundary elements in electromagnetic scattering*, SIAM J. Numer. Anal., 41 (2003), pp. 919–944.

[24] R. Hiptmair and C. Schwab, *Natural boundary element methods for the electric field integral equation on polyhedra*, SIAM J. Numer. Anal., 40 (2002), pp. 66–86.

[25] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, New York, 1972.

[26] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[27] T. von Petersdorff, *Randwertprobleme der Elastizitätstheorie für Polyeder – Singularitäten und Approximation mit Randelementmethoden*, PhD thesis, Technische Hochschule Darmstadt, Germany, 1989.

[28] T. von Petersdorff and E. P. Stephan, *Regularity of mixed boundary value problems in* $\mathbb{R}^3$ *and boundary element methods on graded meshes*, Math. Methods Appl. Sci., 12 (1990), pp. 229–249.