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RICHARD KAYE

Abstract. This paper addresses the structures (M, ω) and $(\omega, \mathrm{SSy}(M))$, where M is a nonstandard model of PA and ω is the standard cut. It is known that $(\omega, \mathrm{SSy}(M))$ is interpretable in (M, ω). Our main technical result is that there is an reverse interpretation of (M, ω) in $(\omega, \mathrm{SSy}(M))$ which is 'local' in the sense of Visser [11]. We also relate the model theory of (M, ω) to the study of transplendent models of PA [2].

This yields a number of model theoretic results concerning the ω-models (M, ω) and their standard systems $\mathrm{SSy}(M, \omega)$, including the following.

- $(M, \omega) \prec (K, \omega)$ if and only if $M \prec K$ and $(\omega, \mathrm{SSy}(M)) \prec (\omega, \mathrm{SSy}(K))$.
- $(\omega, \mathrm{SSy}(M)) \prec (\omega, \mathscr{P}(\omega))$ if and only if $(M, \omega) \prec (M^*, \omega)$ for some ω-saturated M^*.
- $M \prec_{\mathrm{e}} K$ implies $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(K, \omega)$, but cofinal extensions do not necessarily preserve standard system in this sense.
- $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$ if and only if $(\omega, \mathrm{SSy}(M))$ satisfies the full comprehension scheme.
- If $\mathrm{SSy}(M, \omega)$ is uniformly defined by a single formula (analogous to a β function), then $(\omega, \mathrm{SSy}(M, \omega))$ satisfies the full comprehension scheme; and there are models M for which $\mathrm{SSy}(M, \omega)$ is not uniformly defined in this sense.

§1. Introduction. It is very natural, when considering nonstandard models $M \models$ PA of Peano Arithmetic expressed in the first order language with $+, \cdot, <, 0, \text{and} 1$, to expand the model to (M, ω) by adding a unary predicate N to be interpreted by the standard cut ω. For example, the expansion of M to (M, ω) is useful when one might want to consider theories of truth over M for standard formulas. A predicate for standardness is clearly required in such situations.

The language $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ is the usual language of arithmetic $\mathscr{L}_{\mathrm{A}}$ with the predicate N added. We shall denote formulas of $\mathscr{L}_{\mathrm{A}}$ in the usual way as $\theta(\bar{x})$, etc., and formulas of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ will be denoted as if ω (the standard interpretation for N) is included as a parameter, i.e., as $\theta(\bar{x}, \omega)$ etc. The inclusion of the 'second-order parameter' ω is not strictly necessary, but serves as a useful reminder as to whether a formula does or does not involve this extra predicate. An introductory paper by Roman Kossak, Tin Lok Wong, and the present author [7] sets out the background to the study of models (M, ω). This paper continues with the theme, in particular by comparing (M, ω) with the ω-model of second order arithmetic $(\omega, \mathrm{SSy}(M))$. Our notation

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and terminology is standard, as in books by Kaye [4] and Kossak and Schmerl [5], and apart from this required background the current paper is as far as possible self-contained.

In particular, we shall assume the arithmetization of syntax in PA, and, when talking about formulas and truth, it is convenient to identify a formula with its Gödel number, and take a relaxed view on variables. (Provably in PA or indeed in much weaker systems, such details can always be handled by definable primitive recursive functions.) The Σ_n/Π_n hierarchy is the usual hierarchy of first-order formulas of PA. We note that by the formulas Sat_{Σ_n} defining truth for Σ_n formulas any nonstandard model of PA is Σ_n-recursively saturated, i.e., a recursive set of Σ_n formulas that is finitely satisfied in M is realized in M.

We will also look at second-order arithmetic here. An ω-model of second order arithmetic is a structure $(\omega, \mathscr{X}, 0, 1, +, \cdot, <, \in)$, where $\omega = \{0, 1, \ldots\}$, the operations on ω are the usual ones and $\mathscr{X} \subseteq \mathscr{P}(\omega)$. By a common abuse of notation, this is abbreviated $(\omega, \mathscr{X})$ or even as just $\mathscr{X}$. The language of second-order arithmetic is denoted by $\mathscr{L}_{\mathrm{II}}$ and much of this paper compares the language $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ of (M, ω) with $\mathscr{L}_{\mathrm{II}}$. Where we need second order variables and parameters we indicate them with a superscript. For example, Σ_n^0 is essentially the same as Σ_n except that second order set parameters may occur.

One interesting feature of the theory of (M, ω) is the close connections with transplendency [2]. To summarize the connections, we repeat some basic definitions and observations here.

Definition 1.1. The *standard system*, $\mathrm{SSy}(M)$, of a model M is the set of reals $A \subseteq \omega$, such that there is $a \in M$ and a formula $\theta(x, y)$ of $\mathscr{L}_{\mathrm{A}}$ in two free variables, such that
$$A = \{n \in \omega : M \vDash \theta(n, a)\}.$$
The set $\mathrm{Rep}(M)$ is the same as $\mathrm{SSy}(M)$ except that the formula $\theta(x)$ of $\mathscr{L}_{\mathrm{A}}$ is not allowed to contain a parameter a.

Definition 1.2. The *standard system*, $\mathrm{SSy}(M, \omega)$, of a model (M, ω) is the set of reals $A \subseteq \omega$ such that there is $a \in M$ and a formula $\theta(x, y, \omega)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ in two free variables possibly involving ω (or N) as a predicate, such that
$$A = \{n \in \omega : (M, \omega) \vDash \theta(n, a, \omega)\}.$$
The set $\mathrm{Rep}(M, \omega)$ is the same except that $\theta(x)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ may not contain a parameter.

When $M \vDash \mathrm{PA}$ is nonstandard, the systems $\mathrm{SSy}(M)$ and $\mathrm{SSy}(M, \omega)$ are *Scott sets*, i.e., $(\omega, \mathrm{SSy}(M))$ and $(\omega, \mathrm{SSy}(M, \omega))$ both satisfy the second-order system WKL_0 of Friedman. (See e.g., Kaye [4] and Simpson [9] for details.)

Definition 1.3. Let $M \vDash \mathrm{PA}$ be nonstandard. We say M is *full* if $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$ and *fully saturated* if it is full and recursively saturated.

Definition 1.4. Let $M \vDash \mathrm{PA}$ be nonstandard. We say M is *semi-full* if there is some formula $\theta(x, y, \omega)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$, such that each $A \in \mathrm{SSy}(M, \omega)$ is defined by $\{x \in \omega : (M, \omega) \vDash \theta(x, a, \omega)\}$ for some $\bar{a} \in M$.

We will see below that not every nonstandard model of PA is semi-full.

Engström and Kaye [2] introduced a notion, called *transplendency*, which is a variation on the idea of *resplendency* of Barwise, Schlipf, and Ressayre [1, 6] concerning expansions of 'rich' models. In broad terms, a model M is resplendent if it has expansions $(M, \dots) \vDash T$ to any theory T consistent with it. (An *expansion* of M, in contrast to an *extension*, adds structure to the model M without adding additional elements to the domain, whereas an extension adds elements to the domain but preserves the signature.) A *transplendent* model is similar except that the theory satisfied in the expansion $(M, \dots)$ is of the form $T + p\uparrow$ stating some first order axioms T hold and a type p is omitted. The notion of when such as theory is consistent has to be modified too. The following definition is intended to capture the idea that asking for the type p to be omitted does not have any consequences on which types in the original language of M are or are not omitted in elementary extensions of M.

Definition 1.5 (Engström and Kaye). A countable model M is *transplendent* if whenever $\bar{a} \in M$ is a tuple of finitely many parameters from M, $\mathscr{L}'$ is a recursive first-order language recursively extending the language $\mathscr{L}_{\bar{a}}$ of $(M, \bar{a})$, T is a recursive set of first order sentences of language $\mathscr{L}'$ and p is a set of formulas $\phi(\bar{x})$ of $\mathscr{L}'$ in finitely many variables $\bar{x}$, then provided $\mathrm{Th}(M, \bar{a})$ has an ω-saturated model with an expansion satisfying $\mathrm{Th}(M, \bar{a}) + T + p\uparrow$ there is an expansion $(M, \dots) \vDash T + p\uparrow$ of the original model.

Obviously, transplendent models are resplendent and hence recursively saturated. Countable transplendent models exist but no 'nice' closure condition on $\mathrm{SSy}(M)$ is known to be equivalent to transplendency for countable recursively saturated models, though a somewhat technical closure condition was given by Engström and Kaye. Without giving the exact details, we will say that a Scott set $\mathscr{X}$ is *transplendent closed* if it is closed under this condition. Then a countable recursively saturated model of PA with standard system $\mathscr{X}$ is transplendent if and only if $\mathscr{X}$ is transplendent closed.

For transplendent models of arithmetic, $M \vDash \mathrm{PA}$, the key examples of $T + p\uparrow$ are in the case when the new language contains a predicate N for a cut and p is the set of formulas $\{\mathsf{N}(x)\} \cup \{x \neq k : k \in \omega\}$ so that $p\uparrow$ expresses the statement that N is the standard cut ω. Additional properties of ω can then be expressed in a first order way in T. Thus, the study of transplendent models of PA and expansions of models adding a predicate for the standard cut are intimately linked.

Two easy observations [7] on these lines are given next.

Proposition 1.6. *Suppose M is a countable transplendent model of* PA. *Then, M is full.*

Corollary 1.7. *Let $\mathscr{X}$ be a countable Scott set which is transplendent closed. Then there is a countable $M \vDash$* PA *with* $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M) = \mathscr{X}$.

The immediate questions these results suggest is whether converses exist: is a countable fully saturated model M necessarily transplendent? and what closure conditions can be proved for $\mathrm{SSy}(M, \omega)$ when $M \vDash \mathrm{PA}$ is nonstandard, especially without any additional conditions such as fullness? These are the questions that will be addressed in this paper.

Our main technique is to relate these questions to ω-models of second order arithmetic $(\omega, \mathscr{X})$, where $\mathscr{X}$ is a Scott set, in particular $\mathscr{X} = \mathrm{SSy}(M)$ or $\mathscr{X} = \mathrm{SSy}(M, \omega)$. The comprehension axiom scheme in $\mathscr{L}_{\mathrm{II}}$ is particularly relevant.

Definition 1.8. The theory CA_0 is the second order theory consisting of base axioms RCA_0 together with the full second order comprehension axiom scheme,

$$\forall \bar{A}, \bar{a}\ \exists X\ (\forall n\ (n \in X \leftrightarrow \theta(n, \bar{A}, \bar{a}))),$$

for all formulas θ of second order arithmetic.

It has been observed (in joint work with Wong and Kossak) that fullness implies that $(\omega, \mathrm{SSy}(M))$ is a model of CA_0. This result will be discussed and extended further below.

§2. Transplendency and the theory of (M, ω).

We start with the observation that, although for a given complete extension T of PA there is no unique theory of a pair (M, ω), where M is a nonstandard model of T, there is at least a canonical choice for such a theory, where M is 'as rich as possible'.

From the point of view of countable models of PA (and a great deal more besides) these are very 'large' models. Indeed, any model of PA elementarily embeds in an ω-saturated model, and if we measure 'the amount of saturation' in terms of the size of the standard system $\mathrm{SSy}(M)$, the Scott set $\mathscr{P}(\omega)$ is of course the largest set of reals one can have.

Moreover, any two ω-saturated models of the same complete theory T extending PA are back-and-forth equivalent (i.e., $\mathscr{L}_{\omega_1\omega}$ equivalent) and such an equivalence necessarily preserves the standard cut ω. Thus, if M_1^*, M_2^* are ω-saturated models of the same complete theory T, then $(M_1^*, \omega) \equiv (M_2^*, \omega)$. So T induces a canonical $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ theory $T^\omega = \mathrm{Th}(M^*, \omega)$, where $M^* \vDash T$ is ω-saturated. This is our canonical induced $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-theory extending T.

The argument applies equally well to theories of PA with parameters, and if $\bar{a} \in M \vDash T$ we use $\mathrm{tp}_M(\bar{a})$ or $\mathrm{tp}(\bar{a})$ to denote the complete type of $\bar{a}$ in M. This is such a complete extension of PA with parameters, so induces a canonical set of formulas $\mathrm{tp}_M^\omega(\bar{a})$ which is the complete type of $\bar{a}' \in M^*$ in the pair (M^*, ω), where $M^* \vDash T$ is ω-saturated and $\bar{a}' \in M^*$ realizes $\mathrm{tp}_M(\bar{a})$. Once again we observe that this choice of $\mathrm{tp}_M^\omega(\bar{a})$ is independent of the choice of M^* and $\bar{a}'$. In general we use a superscript ω to denote the induced theory or type in an ω-saturated elementary extension.

Of course there is no particular reason to expect a tuple $\bar{a} \in M \vDash \mathrm{PA}$ to realize $\mathrm{tp}_M^\omega(\bar{a})$ in (M, ω) itself.

Definition 2.1. Given $M \vDash \mathrm{PA}$, we say that M is *ω-elementary* if whenever $\bar{a} \in M$ then $\bar{a}$ realizes $\mathrm{tp}_M^\omega(\bar{a})$ in (M, ω).

It follows that ω-elementary models are necessarily nonstandard.

Proposition 2.2. *Suppose that $M \vDash \mathrm{PA}$. Then M is ω-elementary if and only if for some ω-saturated M^*, $(M, \omega) \prec (M^*, \omega)$.*

Proof. If $M^* \succ M$ is ω-saturated then each $\bar{a} \in M$ realizes $\mathrm{tp}_M^\omega(\bar{a})$ in M. Hence, $(M, \omega) \prec (M^*, \omega)$ is precisely the condition that each $\bar{a} \in M$ realizes $\mathrm{tp}_M^\omega(\bar{a})$. $\dashv$

It is now obvious from the downward Löwenheim–Skolem Theorem that countable ω-elementary models M exist. The link with transplendency is the following.

PROPOSITION 2.3. *If $M \models$ PA is transplendent, then M is ω-elementary.*

PROOF. For each $\bar{a}$ and each $\sigma(\bar{a}, \omega) \in \mathrm{tp}^{\omega}_{M}(\bar{a})$, consider the statements of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ stating that: N is a proper cut; $\sigma(\bar{a}, \mathsf{N})$; and $\{x \in \mathsf{N}\} \cup \{x \neq k : k \in \omega\}$ is omitted. In an ω-saturated $M^* \succ M$ this, together with $\sigma(\bar{a}, \omega)$ is true in $(M^*, \bar{a}, \omega)$ so by transplendency of M, $(M, \omega) \models \sigma(\bar{a}, \omega)$. ⊣

It is easy to see that being ω-elementary already very strong consequences for the standard system.

PROPOSITION 2.4. *Suppose $M \models$ PA is ω-elementary. Then M is full.*

PROOF. Let $A \in \mathrm{SSy}(M, \omega)$ be defined by a formula $\theta(x, \bar{a}, \omega)$. Then the same formula defines A in (M^*, ω), where M^* is an ω-saturated elementary extension of M. Since $A \in \mathscr{P}(\omega) = \mathrm{SSy}(M^*)$, we have

$$(M^*, \omega) \models \exists b\, \forall n \in \omega\, (\theta(n, \bar{a}, \omega) \leftrightarrow (b)_n \neq 0)$$

and this is also true in M, by elementarity. ⊣

Proposition 2.6 below will show that another known consequence of transplendency that in fact follows from ω-elementarity.

DEFINITION 2.5. A Scott set $\mathscr{X}$ is a *β-model* if $(\omega, \mathscr{X}) \prec_{\Sigma^1_1} (\omega, \mathscr{P}(\omega))$. It is a *$\beta_\omega$-model* if $(\omega, \mathscr{X}) \prec (\omega, \mathscr{P}(\omega))$.

PROPOSITION 2.6. *Suppose $M \models$ PA is nonstandard and $(M, \omega) \prec (M^*, \omega)$ for some ω-saturated model $M^* \models$ PA. Then $\mathrm{SSy}(M)$ is a β_ω-model.*

PROOF. If $A \in \mathrm{SSy}(M)$ is coded by some $a \in M$ then each statement $\theta(A)$ of second order arithmetic can be translated as a statement $\hat{\theta}(a)$ in $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$. (See Section 3 for details and variations on translations like this.) But then since $\mathrm{SSy}(M^*) = \mathscr{P}(\omega)$ we have that $(\omega, \mathscr{P}(\omega)) \models \theta(A)$ holds if and only if $(M^*, \omega) \models \hat{\theta}(a)$ which by assumption holds just in case $(M, \omega) \models \hat{\theta}(a)$ i.e., just in case $(M, \mathrm{SSy}(M)) \models \theta(A)$. ⊣

§3. Between (M, ω) and second order arithmetic.

This section introduces the main technical devices we will use in this paper.

It is well known that (M, ω) interprets the ω-model $(\omega, \mathrm{SSy}(M))$ of second order arithmetic. Indeed the 'strength' of the Scott set $\mathrm{SSy}(M)$ is often characterized in terms of second order axioms true in $(\omega, \mathrm{SSy}(M))$. Thus (for example) $(\omega, \mathrm{SSy}(M)) \models \mathrm{WKL}_0$ says that $\mathrm{SSy}(M)$ is a Scott set and $(\omega, \mathrm{SSy}(M)) \models \mathrm{ACA}_0$ says that $\mathrm{SSy}(M)$ is closed under jump.

To be more precise, we can set up a translation of formulas of second order arithmetic to $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$. In the following, we shall use $n, m, k, \ldots$ as number variables (ranging over ω) $A, B, C, \ldots$ as set variables (over subsets of ω) and $a, b, c, \ldots$ as variables over elements of a models M of PA. We will have some fixed correspondence between variables $A, B, C, \ldots$ and $a, b, c, \ldots$, indicated here by using the upper/lower case of the same letter, with or without subscripts.

DEFINITION 3.1. The translation $\boldsymbol{\sigma} : \mathscr{L}_{\mathrm{II}} \to \mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ is defined by

(a) $\psi(\bar{n})^{\boldsymbol{\sigma}}$ is $\psi(\bar{n})$ for $\psi(\bar{n})$ a Δ_0 formulas with no set parameters.
(b) $(n \in A)^{\boldsymbol{\sigma}}$ is $(a)_n \neq 0$, using the usual notion of coding in PA, where the variable a of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ corresponds to A.

(c) $(\forall m \in \omega\ \phi(\bar{n}, m, \bar{A}))^{\sigma}$ is $\forall m \in \omega\ (\phi(\bar{n}, m, \bar{a}))^{\sigma}$.
(d) $(\forall B \subseteq \omega\ \phi(\bar{n}, \bar{A}, B))^{\sigma}$ is $\forall b\ \phi^{\sigma}(\bar{n}, \bar{a}, b)$, where b corresponds to B.
(e) For all other formulas, $\boldsymbol{\sigma}$ commutes with $\wedge, \vee, \neg$: $(\phi \wedge \psi)^{\sigma}$ is $(\phi^{\sigma} \wedge \psi^{\sigma})$; $(\phi \vee \psi)^{\sigma}$ is $(\phi^{\sigma} \vee \psi^{\sigma})$; and $(\neg\phi)^{\sigma}$ is $\neg\phi^{\sigma}$.

For a given set variable $A \subseteq \omega$ and $M \models \mathrm{PA}$ a suitable variable $a \in M$ corresponding to A is any element of M coding A (if such exists).

The following is by an easy induction on formulas.

Proposition 3.2. *For a nonstandard $M \models \mathrm{PA}$ and θ from $\mathscr{L}_{\mathrm{II}}$, if $\bar{n} \in \omega$, and $\bar{A} \in \mathrm{SSy}(M)$, and $a_i \in M$ codes the set $A_i \subseteq \omega$ for each i, then*

$$(M, \omega) \models \theta^{\sigma}(\bar{n}, \bar{a}) \ \Leftrightarrow\ (\omega, \mathrm{SSy}(M)) \models \theta(\bar{n}, \bar{A}).$$

In general, it seems that there can be no similar translation of $(\omega, \mathrm{SSy}(M, \omega))$ into (M, ω). The problem is that a quantification over $\mathrm{SSy}(M, \omega)$ must describe all formulas defining sets $A \in \mathrm{SSy}(M, \omega)$ and it is not clear how this might be done. In one specific case, however, such a result is possible.

Proposition 3.3. *For each nonstandard semi-full $M \models \mathrm{PA}$, $(\omega, \mathrm{SSy}(M, \omega)) \models \mathrm{CA}_0$.*

Proof. Using the assumption that M is semi-full, let $\alpha(x, \bar{y}, \omega)$ be an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ formula such that each $A \in \mathrm{SSy}(M, \omega)$ is $\{x \in \omega : (M, \omega) \models \alpha(x, \bar{a}, \omega)\}$ for some $\bar{a} \in M$. Given $\bar{A} \in \mathrm{SSy}(M, \omega)$ choose $\bar{a}_i \in M$ so that formulas $\alpha(x, \bar{a}_i, \omega)$ defines A_i. We use this and a variation of the translation $\boldsymbol{\sigma}$ above to translate a formula $\theta(x, \bar{A})$ of the language of second order arithmetic to $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$.

As for $\boldsymbol{\sigma}$, we may replace each statement '$u \in A_i$' with $\alpha(x, \bar{a}_i, \omega)$. We translate number quantifiers $\forall n \ldots$ to quantifiers $\forall n \in \omega \ldots$ relativized to ω, and translate set quantifiers such as

$$\forall V\ (\ldots u \in V \ldots w \in V \ldots)$$

to

$$\forall \bar{v}\ (\ldots \alpha(u, \bar{v}, \omega) \ldots \alpha(w, \bar{v}, \omega) \ldots).$$

The point is that since M is semi-full, this allows us to quantify over all possible sets in $\mathrm{SSy}(M, \omega)$ in a uniform way.

This gives a translated formula $\theta^{\omega}(x, \bar{a}, \omega)$ such that for all $x \in \omega$

$$(M, \omega) \models \theta^{\omega}(x, \bar{a}, \omega) \ \Leftrightarrow\ (\omega, \mathrm{SSy}(M, \omega)) \models \theta(x, \bar{A}).$$

It follows that there is a set $B = \{x \in \omega : (M, \omega) \models \theta^{\omega}(x, \bar{a}, \omega)\}$ in $\mathrm{SSy}(M, \omega)$, and hence CA_0 holds. $\dashv$

We now look at the reverse direction, of describing (M, ω) in $(\omega, \mathrm{SSy}(M))$. At first sight this might appear impossible: (M, ω) may say rather more about the standard cut and its induced structure than is available in $(\omega, \mathrm{SSy}(M))$, i.e., it may be that there is other structure on ω that can be interpreted in (M, ω) other than that given by $\mathrm{SSy}(M)$. We shall prove this is not the case. The main difficulty is that the translation of statements about (M, ω) into statements about $(\omega, \mathrm{SSy}(M))$ is not uniform, but is 'local' in the sense of Visser [11]—that is a family of translations must be given for each possible quantifier complexity.

(From a historical point-of-view, it is perhaps worth noting that Kanovei [3] proves a related result on these lines, that $(\omega, \mathrm{SSy}(M))$ is equivalent to the structure on ω induced from (M, ω) *for certain models* $M \models \mathrm{Th}(\mathbb{N})$ *only*. His proof is rather long and uses a tricky forcing construction, but he requires a conclusion that avoids the use of the parameters we have here, so his result is somewhat different.)

Although the translation we present below is essentially a syntactic affair, it is helpful to have some $M \models \mathrm{PA}$ (not necessarily countable) in mind. We will take for our translation a sequence of normal first order variables $x_1, x_2, \ldots$ of the first order language $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$. The variables that we take as parameters are written $\bar{a}$ and these correspond in a slightly different way to second order parameters: any tuple $\bar{a}$ of variables denoting arbitrary elements of M (called, below, *nonstandard variables*, though they may represent standard or nonstandard values) will correspond to a sequence of set variables $A_k^{\bar{a}}$ with $k \in \omega$ varying. The value for a particular $A_k^{\bar{a}}$ that we take will be the set $\Sigma_k{-}\mathrm{tp}(\bar{a})$ of the Gödel numbers of all Σ_k formulas true of $\bar{a}$ in M, for the usual language $\mathscr{L}_{\mathrm{A}}$. Here the type is, as usual a set of formulas in the free variables $x_1, \ldots, x_l$, where l is the length of the tuple $\bar{a}$. Since $M \models \mathrm{PA}$ is nonstandard this set $A_k^{\bar{a}}$ is always coded. The theory of the model M can be regarded as the type of an empty tuple of variables, and this is coded as the sequence $A_k^{\emptyset} = \Sigma_k{-}\mathrm{Th}(M)$. At least one of these sets is always needed as a parameter so our translation does not take sentences to sentences.

The language $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ will be assumed to be built from $\mathscr{L}_{\mathrm{A}}$ with the usual propositional connectives $\wedge, \vee, \neg$ and quantifiers $\forall n \in \omega \ldots$ and $\forall a \ldots$. The predicate $x \in \omega$ is not needed as it can be written $\neg\forall n \in \omega\ \neg(x = n)$.

DEFINITION 3.4. For $k \in \omega$, we define a family of partial translations $\boldsymbol{\tau}k$ from $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ to $\mathscr{L}_{\mathrm{II}}$. For a given formula $\theta(\bar{n}, \bar{a}, \omega)$, $\theta^{\tau k}(\bar{n}, \bar{A})$ will be defined for all sufficiently large $k \in \omega$. The translation is given as follows.

(a) $\psi(\bar{n})^{\tau k}$ is $\psi(\mathrm{clterm}(\bar{n}))$ for $\psi(\bar{n})$ a Δ_0 formula with no nonstandard variables, where

$$\mathrm{clterm}(n) = (\cdots(0 + \overbrace{1) + \cdots 1}^{n})$$

is a canonical $\mathscr{L}_{\mathrm{A}}$ term for n and is calculated by a standard primitive recursive function represented in the language.

(b) If $\psi(\bar{n}, \bar{a})$ has nonstandard variables $\bar{a} = a_1, \ldots, a_n$ but does not involve ω, we use the variables $A_k^{\bar{a}}$. We let $(\psi(\bar{n}, \bar{a}))^{\tau k}$ be the formula

$$\psi(\mathrm{clterm}(\bar{n}), x_1, \ldots, x_n) \in A_k^{\bar{a}}$$

and this is only defined when k is sufficiently large that ψ is Σ_k.

(c) If the formula starts with an ω-bounded quantifier, i.e., $\forall m \in \omega\ \psi(\bar{n}, m, \bar{a})$ then its translation $(\forall m \in \omega\ \psi(\bar{n}, m, \bar{a}))^{\tau k}$ is $\forall m \in \omega\ \psi^{\tau k}$ for k sufficiently large that this is defined. Note that this formula will contain free set variables $A_k^{\bar{b}}$ for certain tuples $\bar{b}$ taken from $\bar{a}$.

(d) If the formula $\phi(\bar{n}, \bar{a})$ starts with an unbounded quantifier, i.e., $\forall b\ \psi(\bar{n}, \bar{a}, b)$ then its translation $\phi^{\tau k}$ is

$$\forall A_k^{\bar{a},b}\ (A_k^{\bar{a},b} \text{ extends } A_k^{\bar{a}} \rightarrow (\psi(\bar{n}, \bar{a}, b))^{\tau k})$$

where this is defined. Here, '$A_k^{\bar{a},b}$ extends $A_k^{\bar{a}}$' means that $A_k^{\bar{a},b}$ is a set of Gödel numbers of Σ_k formulas in free variables $x_1, x_2, \ldots, x_l, x_{l+1}$ (where l is the length of $\bar{a}$), $A_k^{\bar{a},b}$ is a superset of $A_k^{\bar{a}}$ (which is a set of Σ_k formulas in free variables $x_1, x_2, \dots, x_l$) and

$$\{\sigma(x_1, x_2, \dots, x_l, x_{l+1}) : \sigma \in A_k^{\bar{a},b} \text{ is } \Sigma_k\}$$
$$\cup \{\neg\tau(x_1, x_2, \dots, x_l, x_{l+1}) : \tau \notin A_k^{\bar{a},b} \text{ is } \Sigma_k\}$$
$$\cup\, \Pi_{k+1}\text{–Th}(M)$$

is consistent. Consistency means in the sense of the usual Π_1^0 Gödel formula of $\mathscr{L}_{\mathrm{II}}$ expressing this. Given $A_k^{\bar{a},b}$ as above, for any other tuple $\bar{c}$ from $\bar{a}, b$, a similar set that is complete for Σ_k and Π_k formulas can be obtained from $A_k^{\bar{a},b}$ by renaming or substitution of variables using primitive recursive functions definable in $\mathscr{L}_{\mathrm{II}}$ in the usual way and where necessary we will assume this is done and described in the language $\mathscr{L}_{\mathrm{II}}$ as part of the translated formula.

(e) For all other formulas, $\boldsymbol{\tau}k$ commutes with $\wedge, \vee, \neg$ whenever possible, i.e., if k is large enough that the following are defined, we have: $(\phi \wedge \psi)^{\tau k}$ is $(\phi^{\tau k} \wedge \psi^{\tau k})$; $(\phi \vee \psi)^{\tau k}$ is $(\phi^{\tau k} \vee \psi^{\tau k})$; and $(\neg\phi)^{\tau k}$ is $\neg\phi^{\tau k}$.

Remark 3.5. The definition above is made complicated by the requirement to use Σ_k types for suitable fixed k. If one were to use complete types (unbounded in complexity) instead, it might seem that the definition can be simplified, and this seems possible in the case when the model M is recursively saturated so that $\mathrm{Th}(M)$ and each $\mathrm{tp}(\bar{a})$ is coded in M. However, one still needs to identify (by a formula of $\mathscr{L}_{\mathrm{II}}$) when it is the case that a number a codes a set that is a complete consistent theory: this will be problematic in the case when $M \vDash \neg\,\mathrm{Con(PA)}$.

Proposition 3.6. *Let $M \vDash \mathrm{PA}$ be nonstandard, $\bar{n} \in \omega$ and $\bar{a} \in M$. Suppose $\theta(\bar{n}, \bar{a}, \omega)$ is a $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ formula with the free variables shown. Then for some $k_0 \in \omega$ $\theta(\bar{n}, \bar{a}, \omega)^{\tau k}$ is defined for all $k \geqslant k_0$, and whenever $\theta(\bar{n}, \bar{a}, \omega)^{\tau k}$ is defined and*

$$A_k^{\bar{a}} = \{\sigma(x_1, \dots, x_l) : \sigma \in \Sigma_k \text{ and } M \vDash \sigma(a_1, \dots, a_l)\}$$

then

$$(M, \omega) \vDash \theta(\bar{n}, \bar{a}, \omega) \iff (\omega, \mathrm{SSy}(M)) \vDash \theta^{\tau k}(A_k^{\bar{a}}).$$

Proof. Observe that as $M \vDash \mathrm{PA}$ is nonstandard, the set $A_k^{\bar{a}}$ is indeed coded in M for all k.

The proof is by induction on formulas. The induction step is easy in all cases. Note that for the induction step for (d) (the unbounded quantifier) since second order logic number quantifiers quantify over true ω, a set of formulas

$$\{\sigma(x_1, x_2, \dots, x_l, x_{l+1}) : \sigma \in A_k^{\bar{a},b} \text{ is } \Sigma_k\}$$
$$\cup \{\neg\tau(x_1, x_2, \dots, x_l, x_{l+1}) : \tau \notin A_k^{\bar{a},b} \text{ is } \Sigma_k\}$$
$$\cup\, \Pi_{k+1}\text{–Th}(M)$$

is consistent in the sense of the Gödel Π_1^0 formula of $\mathscr{L}_{\mathrm{II}}$ if and only if it is finitely satisfied in M, i.e., it really is consistent. Also, by the induction axioms of PA and the $\mathrm{Sat}_{\Sigma_k}()$ predicates any such coded finitely satisfied set of formulas of bounded complexity (regarded as a type over M) is realized in M. Conversely, by the induction axioms and the $\mathrm{Sat}_{\Sigma_k}()$ of PA again, any Σ_k-type of a tuple $\bar{a} \in M$ is coded in M. ⊣

Proposition 3.7. *A nonstandard* $M \models \mathrm{PA}$ *is full if and only if* $(\omega, \mathrm{SSy}(M)) \models \mathrm{CA}_0$.

Proof. For one direction, if M is full then we have $\mathrm{SSy}(M) = \mathrm{SSy}(M, \omega)$. Also $(\omega, \mathrm{SSy}(M, \omega)) \models \mathrm{CA}_0$ by Proposition 3.3, so $(\omega, \mathrm{SSy}(M)) \models \mathrm{CA}_0$.

Now suppose $(\omega, \mathrm{SSy}(M)) \models \mathrm{CA}_0$. Suppose the formula $\theta(x, \bar{a}, \omega)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ defines some $B \in \mathrm{SSy}(M, \omega)$ where the parameters $\bar{a}$ are from M. Then for some $k \in \omega$ the translation $\theta^{\tau k}(x, A)$ is valid, where $A = \Sigma_k\text{–}\mathrm{tp}(\bar{a})$. By the assumption $(\omega, \mathrm{SSy}(M)) \models \mathrm{CA}_0$ there is $B \in \mathrm{SSy}(M)$ such that

$$B = \{x \in \omega : (\omega, \mathrm{SSy}(M)) \models \theta^{\tau k}(x, A)\} = \{x \in \omega : (M, \omega) \models \theta(x, \bar{a}, \omega)\},$$

as required. $\dashv$

Corollary 3.8. *Suppose a countable Scott set* $\mathscr{X}$ *satisfies* $(\omega, \mathscr{X}) \models \mathrm{CA}_0$. *Then* $\mathscr{X}$ *is* $\mathrm{SSy}(M, \omega)$ *for some nonstandard full* $M \models \mathrm{PA}$.

Proof. Let $(\omega, \mathscr{X}) \models \mathrm{CA}_0$ be nonstandard and $M \models \mathrm{PA}$ with $\mathrm{SSy}(M) = \mathscr{X}$. By Proposition 3.7, M is full, hence $\mathrm{SSy}(M, \omega) = \mathscr{X}$. $\dashv$

There are many possible choices for M in the last corollary. For example, we may take M to be recursively saturated (in which case M is fully saturated) or we could take M to be prime (in which case $\mathscr{X} = \mathrm{Rep}(M) = \mathrm{Rep}(M, \omega)$).

Corollary 3.9. *There are fully saturated models* $M \models \mathrm{PA}$ *for which* $\mathrm{SSy}(M)$ *is not a* β*-model.*

Proof. By a standard result (see Simpson [9]) there are ω-models of CA_0 that are not β-models. $\dashv$

The interpretations just given give a further characterization of ω-elementary models, and shed light on the property of elementarity $(N, \omega) \prec (M, \omega)$ between ω-models. Note that no saturation assumptions are required on the models in the following proofs.

Theorem 3.10. *For nonstandard models* $M \subseteq K$ *of* PA *we have* $(M, \omega) \prec (K, \omega)$ *if and only if* $M \prec K$ *and* $(\omega, \mathrm{SSy}(M)) \prec (\omega, \mathrm{SSy}(K))$.

Proof. Left-to-right is by the interpretation $\boldsymbol{\sigma}$. Given $(M, \omega) \prec (K, \omega)$, $M \prec K$ is obvious and for $\bar{A} \in \mathrm{SSy}(M)$ let $\bar{a}$ code the sets in $\bar{A}$ so $(\omega, \mathrm{SSy}(M)) \models \Theta(\bar{A})$ iff $(M, \omega) \models \Theta^{\sigma}(\bar{a})$ iff $(K, \omega) \models \Theta^{\sigma}(\bar{a})$ iff $(\omega, \mathrm{SSy}(K)) \models \Theta(\bar{A})$.

For right-to-left suppose $M \prec K$ and $(\omega, \mathrm{SSy}(M)) \prec (\omega, \mathrm{SSy}(K))$ and $\bar{a} \in M$ and $\theta(\bar{a}, \omega)$ is a formula of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$. We prove by induction on the complexity of θ that $(M, \omega) \models \theta(\bar{a}, \omega)$ if and only if $(K, \omega) \models \theta(\bar{a}, \omega)$. The base case is when θ does not involve ω and is covered by the assumption $M \prec K$. We look at the only tricky case, that of a formula with a universal quantifier $\forall x\ \psi(x, \bar{a}, \omega)$ being preserved upwards.

Let k be sufficiently large that the interpretation $\boldsymbol{\tau} k$ is correct, and let $A_k^{\bar{a}} \in \mathrm{SSy}(M)$ code the Σ_k-type of $\bar{a}$. Note that $A_k^{\bar{a}} \in \mathrm{SSy}(K)$ as $M \prec K$. Then if $(M, \omega) \models \forall x\ \psi(x, \bar{a}, \omega)$

$$(\omega, \mathrm{SSy}(M)) \models \forall A_k^{\bar{a}, b}\ (A_k^{\bar{a}, b} \text{ extends } A_k^{\bar{a}} \to (\psi(\bar{n}, \bar{a}, b))^{\tau k})$$

so the sake is true in $(\omega, \mathrm{SSy}(K))$, so $(K, \omega) \models \forall x\ \psi(x, \bar{a}, \omega)$ which suffices. $\dashv$

Corollary 3.11. *A nonstandard model* $M \models \mathrm{PA}$ *is* ω*-elementary if and only if* $\mathrm{SSy}(M)$ *is a* β_ω*-model.*

Proof. Let $M' \succ M$ be ω-saturated. Then $(M, \omega) \prec (M', \omega)$ holds if and only if $(\omega, \mathrm{SSy}(M)) \prec (\omega, \mathrm{SSy}(M')) = (\omega, \mathscr{P}(\omega))$ i.e., if and only if $\mathrm{SSy}(M)$ is a β_ω-model. ⊣

End-extensions of nonstandard models are well known to preserve the standard system. This together with Theorem 3.10 tells us that elementary end-extensions preserve formulas in $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$, a result previously shown by Smith [10].

We conclude this section by comparing the results above with an argument due to Schmerl. Schmerl's reflection principle [8, Theorem 2.1] applies to structures of the form[1] $\mathfrak{A} = (M, \dots, N)$, where N is a proper initial segment satisfying the full induction scheme $\mathrm{IA}(\mathscr{L})$,

$$\forall \bar{a}\ (\theta(0, \bar{a}) \wedge \forall x \in N\ (\theta(x, \bar{a}) \rightarrow \theta(x+1, \bar{a})) \rightarrow \forall x \in N\ \theta(x, \bar{a})),$$

for θ in the expanded language (which should be countable and relational) and M satisfies some weak axioms FS of arithmetic. (PA certainly suffices; we refer the reader to Schmerl's paper for the details which are rather more general than stated here.) Thus $(M, \dots, \omega)$ is one possibility; so would be an elementary extension of $(M, \dots, \omega)$. Under these circumstances Schmerl constructs $\mathfrak{B} = (B, \dots, N)$ which is locally interpreted in $\mathfrak{A}$, with the arithmetical part N the same as the arithmetical part N of $\mathfrak{A}$ and for which the rest of the domain B is contained in a separate disjoint copy of N. This is done in such a way that the theory of the original $\mathfrak{A}$ is locally interpreted in $\mathfrak{B}$. $\mathfrak{B}$ is constructed by a careful amalgamation argument, with two kinds of induction: an external induction corresponding to the description of the language of $\mathfrak{A}$ in stages; and an internal induction over N and N-finite substructures. Thus the construction yields $\mathfrak{B}$ which has exactly the same information as $(N, \mathscr{X})$ where $X = \mathrm{SSy}_N(M, \dots)$ and the notation $\mathrm{SSy}_N(M, \dots)$ refers to the system of coded subsets of N, i.e., sets that are the intersection of N with sets that are definable in $(M, \dots)$.

Thus (with the obvious reading of the ellipsis and notations $\mathrm{SSy}(M, \dots)$ for 'standard system in expanded languages') we obtain,

Theorem 3.12. *Let $M \models \mathrm{PA}$ be nonstandard and $\mathfrak{A} = (M, \dots, \omega)$ an expansion of M to a countable language. Then $\mathrm{Th}(\mathfrak{A}, n)_{n \in \omega}$ is locally interpreted in the model of the second order arithmetic $(\omega, \mathrm{SSy}(M, \dots))$.*

In the case when $\mathfrak{A} = (M, N)$ and N is nonstandard, the induction axiom for N corresponds to 'strength' of N and we obtain,

Theorem 3.13. *Let $M \models \mathrm{PA}$ be nonstandard and N a proper strong initial segment of M, then $\mathrm{Th}(M, N, n)_{n \in N}$ is locally interpreted in $(N, \mathrm{SSy}_N(M), n)_{n \in N}$.*

Clearly these are powerful techniques: Theorem 3.12 showing that these local interpretations apply to other expansions of (M, ω) (something that might have been observed from the proof of Proposition 3.6 above); and Theorem 3.13 showing that local interpretations apply to other cuts other than ω. Both these results are important and their consequences should be investigated, but as they take us away from our original questions concerning (M, ω), we shall not take this any further here.

[1] Fractur symbols used here correspond to those in Schmerl's paper.

§4. The standard systems of M and (M,ω). This section is devoted to applications of the interpretations given earlier for $\mathrm{SSy}(M,\omega)$. A sample question is

QUESTION 4.1. *Characterize the countable Scott sets that arise as* $\mathrm{SSy}(M,\omega)$ *for some nonstandard model* $M \vDash \mathrm{PA}$.

DEFINITION 4.2. Given a Scott set $\mathscr{X}$, $\mathrm{Def}(\omega,\mathscr{X})$ is the set of definable subsets of ω, definable with parameters in $(\omega,\mathscr{X})$. Similarly, $\mathrm{Rep}(\omega,\mathscr{X})$ is the set of 0-definable subsets of ω, i.e., definable without parameters.

THEOREM 4.3. *For a nonstandard model* $M \vDash \mathrm{PA}$, *we have* $\mathrm{SSy}(M,\omega) = \mathrm{Def}(\omega,\mathrm{SSy}(M))$.

PROOF. Given $\bar{A} \in \mathrm{SSy}(M)$ and $\theta(x,\bar{A})$ defining some $B \in \mathrm{Def}(\omega,\mathrm{SSy}(M))$ we have $B = \{n \in \omega : (M,\omega) \vDash \theta^\sigma(n,\bar{a})\}$, where $\bar{a}$ codes $\bar{A}$. Conversely, for $C \in \mathrm{SSy}(M,\omega)$ defined by $\psi(x,\bar{a},\omega)$, the set C is defined in $(\omega,\mathrm{SSy}(M))$ by $\psi^{\tau k}(x,A_k^{\bar{a}},\omega)$ for some sufficiently large standard k. ⊣

The argument above shows $\mathrm{Rep}(M,\omega) = \mathrm{Rep}(\omega,\mathrm{SSy}(M))$ when $\mathrm{Th}(M)$ is in $\mathrm{Rep}(\omega,\mathrm{SSy}(M))$. Note that, even if no parameters $\bar{a}$ are required in the definition of $C \in \mathrm{SSy}(M,\omega)$, to show $C \in \mathrm{Def}(\omega,\mathrm{SSy}(M))$ in general seems to require a parameter $A_k^\emptyset$ for the Σ_k-theory of M.

The following corollary is of interest since it is not *a priori* obvious that end-extensions preserve the more complicated notion of standard system $\mathrm{SSy}(-,\omega)$ for the language $\mathscr{L}_\mathrm{A}^{\mathrm{cut}}$.

COROLLARY 4.4. *If* $M \prec_e K \vDash \mathrm{PA}$ *and* M *is nonstandard then* $\mathrm{SSy}(M,\omega) = \mathrm{SSy}(K,\omega)$.

PROOF. Use the facts that $\mathrm{SSy}(M) = \mathrm{SSy}(K)$, $\mathrm{SSy}(M,\omega) = \mathrm{Def}(\omega,\mathrm{SSy}(M))$ and $\mathrm{SSy}(K,\omega) = \mathrm{Def}(\omega,\mathrm{SSy}(K))$. ⊣

It is worth contrasting this with the following result that shows cofinal ω-elementary extensions do not preserve ω-standard systems, answering a question by Kossak.

PROPOSITION 4.5. *If* $M \vDash \mathrm{PA}$ *is nonstandard and countable and for some uncountable* $\mathscr{X} \subseteq \mathscr{P}(\omega)$ *we have* $(\omega,\mathrm{SSy}(M)) \prec (\omega,\mathscr{X})$, *then there is some countable* $M \prec_{cf} K$ *such that* $(M,\omega) \prec (K,\omega)$ *and* $\mathrm{SSy}(M,\omega) \subsetneq \mathrm{SSy}(K,\omega)$.

PROOF. Let M be as given and $(\omega,\mathrm{SSy}(M)) \prec (\omega,\mathscr{X})$ with $\mathscr{X}$ uncountable. Let $A \in \mathscr{X} \setminus \mathrm{Def}(\omega,\mathrm{SSy}(M))$, which exists by countability, and let $(\omega,\mathscr{Y}) \prec (\omega,\mathscr{X})$ where $\mathscr{Y}$ is countable and contains A, all $B \in \mathscr{X}$, and all complete types $\mathrm{tp}_M(\bar{a})$ realized in M. Now let K' be a countable recursively saturated model of $\mathrm{Th}(M)$ with standard system $\mathrm{SSy}(K') = \mathscr{Y}$. Then by standard methods [4] there is an elementary embedding of M into K'. We take K to be the initial segment of K' determined by M, so $M \prec K$ by Gaifman's splitting theorem and $\mathrm{SSy}(K) = \mathscr{Y}$.

As $A \in \mathrm{SSy}(K)$ we have $A \in \mathrm{SSy}(K,\omega) \setminus \mathrm{Def}(\omega,\mathrm{SSy}(M))$, and as $(\omega,\mathscr{X}) \prec (\omega,\mathscr{Y})$ we have $\mathrm{SSy}(M,\omega) \subsetneq \mathrm{SSy}(K,\omega)$ by Theorem 3.10. ⊣

Of course the hypotheses of the Proposition 4.5 apply in a large number of cases: for any uncountable Scott set $\mathscr{X}$ there is a countable $M \vDash \mathrm{PA}$ with $(\omega,\mathrm{SSy}(M)) \prec (\omega,\mathscr{X})$, by the downward Löwenheim–Skolem Theorem and Scott's characterization of Scott sets. Also, if a countable M is given where $\mathrm{SSy}(M)$ is a β_ω-model then $(\omega,\mathrm{SSy}(M)) \prec (\omega,\mathscr{P}(\omega))$.

Theorem 4.3 reduces the question of understanding the standard systems $\mathrm{SSy}(M, \omega)$ to understanding second order definability in $(\omega, \mathrm{SSy}(M))$, especially in the case where the full comprehension axiom scheme is not true. Thus Question 4.1 can be viewed entirely as a question about Scott sets. This in turn involves understanding quantification over sets in $\mathrm{SSy}(M)$. It is not obvious at this stage whether this question might be solved best in the context of models of arithmetic or as a question of recursion theory or second order arithmetic, and in general it appears to be quite difficult.

For the remainder of this section then, we will quote the necessary conditions known for the question, and aim to give more straightforward examples and applications of Theorem 4.3 in a case where quantification over the Scott set can be grounded in an arithmetic way.

Quantification over ω is obviously available in (M, ω) so the following result clearly holds.

Proposition 4.6. *Let $M \vDash \mathrm{PA}$ be nonstandard. Then $\mathrm{SSy}(M, \omega)$ is a Scott set closed under jump.*

In the case when $M \vDash \mathrm{Th}(\mathbb{N})$ slightly more can be said.

Proposition 4.7 (Kanovei [3]). *When $M \vDash \mathrm{Th}(\mathbb{N})$ is nonstandard $\mathrm{Th}(\mathbb{N}) \in \mathrm{SSy}(M, \omega)$, i.e., $\mathrm{SSy}(M, \omega)$ contains a set of Turing degree $\mathbf{0}^{(\omega)}$.*

Proposition 4.7 was re-worked by Kaye, Kossak, and Wong [7], who showed the following.

Proposition 4.8. *Let $M \vDash \mathrm{PA}$ be nonstandard in which ω is a strong cut. Then $\mathrm{SSy}(M, \omega)$ is closed under the ωth jump operation $\mathbf{a} \mapsto \mathbf{a}^{(\omega)}$.*

Propositions 4.6 and 4.7 show that $\mathrm{SSy}(M, \omega)$ contains the arithmetic sets and, if additionally $M \vDash \mathrm{Th}(\mathbb{N})$ then $\mathrm{SSy}(M, \omega)$ contains all sets arithmetic in $\mathrm{Th}(\mathbb{N})$. The following shows that the collection of arithmetic sets is $\mathrm{SSy}(M, \omega)$ for some M.

Theorem 4.9. *Let T be an arithmetic completion of PA and let $\mathscr{X} \subseteq \mathscr{P}(\omega)$ be the set of arithmetic sets. Then the prime model $M = P_T \vDash T$ of T has $\mathrm{SSy}(M, \omega) = \mathscr{X}$.*

Proof. We have easily that $\mathrm{SSy}(M) = \mathrm{Rep}(T) \subseteq \mathscr{X} \subseteq \mathrm{SSy}(M, \omega)$. It suffices to show that each $A \in \mathrm{SSy}(M, \omega)$ is arithmetic. Let $\theta(x, \bar{a}, \omega)$ define A in (M, ω). Translate this as $\theta^{\tau k}(x, \bar{A}_k^{\bar{a}})$, an $\mathscr{L}_{\mathrm{II}}$ formula with parameters from $\mathrm{SSy}(M)$. We need to show that $\theta^{\tau k}(x, \bar{A}_k^{\bar{a}})$ defines an arithmetic set in $(\omega, \mathrm{SSy}(M))$. To do this it suffices to show how to handle set quantification $\exists X\ \theta(\bar{x}, X, \bar{B})$ in $\mathscr{L}_{\mathrm{II}}$.

Recall that a set $B \in \mathrm{SSy}(M)$ is coded by some $b \in M$, and as M is prime, this b is definable by some formula $\eta(v)$, i.e., b satisfies this formula and $T \vdash \exists! v\ \eta(v)$. Thus, to search over all such B it suffices to search for a formula η satisfying $T \vdash \exists! v\ \eta(v)$ and consider

$$B = \{n \in \omega : T \vdash \forall v\ (\eta(v) \to `n \in v')\},$$

where '$n \in v$' is the formula used (e.g., based on the Gödel β function) to define $\mathrm{SSy}(M)$. This search can be done using an oracle for T. Thus, the quantifier is arithmetically computable. ⊣

An easy relativization yields the following.

Theorem 4.10. *Let $A \subseteq \omega$ and let T be a complete theory in the language $\mathscr{L}_{\mathrm{A}}$ with a constant symbol a added, so that T extends PA plus the additional axioms*

'$n \in a$' ($for\ n \in A$) and '$n \notin a$' ($for\ n \notin A$). Suppose also T is arithmetic in A. Then the model $M = \mathrm{cl}_M(a) \models T$ consisting of all T-definable points in $\mathscr{L}_\mathrm{A} \cup \{a\}$ has $\mathrm{SSy}(M, \omega)$ *equal to the collection of sets arithmetic in A.*

Thus, all Scott sets $\mathscr{X}$ that are finitely generated from a single $A \in \mathscr{P}(\omega)$ by jump, the boolean operations and Turing reducibility are of the form $\mathrm{SSy}(M, \omega)$ for some M. In the special case when $\mathscr{X}$ is so-generated from $A \oplus \mathrm{Th}(\mathbb{N})$ (where $\oplus$ is the recursive join of two sets), the theory T in Theorem 4.10 may obviously extend $\mathrm{Th}(\mathbb{N})$, so $\mathscr{X} = \mathrm{SSy}(M, \omega)$ for some $M \models \mathrm{Th}(\mathbb{N})$.

§**5. Some questions.** The current paper answers a large number of questions that were raised while working on the previous paper by Kaye, Kossak, and Wong [7]. We conclude by listing some questions and conjectures that remain.

The main outstanding question is Question 4.1. Theorem 4.3 rephrases this as a question about Scott sets in a way that is independent of nonstandard models. In view of the evidence provided by the examples in Section 4, we would make the following conjecture.

CONJECTURE 5.1. *A countable Scott set $\mathscr{X}$ is* $\mathrm{SSy}(M, \omega)$ *for some $M \models$* PA *if and only if $\mathscr{X}$ is closed under jump. It is* $\mathrm{SSy}(M, \omega)$ *for some $M \models$* $\mathrm{Th}(\mathbb{N})$ *if and only if $\mathscr{X}$ is closed under jump and additionally contains* $\mathbf{0}^{(\omega)}$.

The conditions are necessary, and examples of Scott sets $\mathrm{SSy}(M, \omega)$ with hardly any extra closure properties are given in Theorem 4.10. We do not have an analog of Theorem 4.10 for Scott sets closed under the ωth jump operation that yields models in which ω is strong. Thus, the evidence is somewhat weaker for the following.

CONJECTURE 5.2. *A countable Scott set $\mathscr{X}$ is* $\mathrm{SSy}(M, \omega)$ *for some $M \models$* PA *in which ω is strong if and only if $\mathscr{X}$ is closed under the ωth jump,* $\mathbf{a} \mapsto \mathbf{a}^{(\omega)}$.

Proposition 4.8 shows the necessity of the condition here.

QUESTION 5.3. *Characterize the countable Scott sets that arise as* $\mathrm{Rep}(M, \omega)$ *for some nonstandard model $M \models$* PA.

Kanovei [3] answered this question for $\mathrm{Th}(\mathbb{N})$. One might conjecture that the answer for PA is the same as that conjectured for $\mathrm{SSy}(M, \omega)$, i.e., those Scott sets closed under jump. Our results (in particular the proof of Theorem 4.3) give some partial information. In the case when M is recursively saturated, this shows that each set in $\mathrm{Rep}(M, \omega)$ is definable in $(\omega, \mathrm{SSy}(M))$ where $T = \mathrm{Th}(M) \in \mathrm{SSy}(M)$ is the only parameter required. (In the case when M is not recursively saturated, various parameters $\Sigma_n\text{--}\mathrm{Th}(T)$ for the Σ_n theory of T may be required.) The problem of characterizing $\mathrm{Rep}(M, \omega)$ appears to have a different flavor to the work here, requiring some diagonalization or other techniques to define sets without the use of parameters and is left for further work.

Finally, the work above goes some way to explain the ideas of full and semi-full models and connect these to the comprehension axioms. In particular, we have seen that not every model is semi-full, and for these models no single formula θ of $\mathscr{L}_\mathrm{A}^{\mathrm{cut}}$ defines all sets in $\mathrm{SSy}(M, \omega)$. On the other hand, in a full model the formula obtained from the Gödel β function does indeed define all sets in $\mathrm{SSy}(M) = \mathrm{SSy}(M, \omega)$. The situation is somewhat unsatisfactory as we have no other examples of semi-full models.

Question 5.4. *Do there exist countable models $M \models \mathrm{PA}$ which are semi-full but not full*?

This question would seem to require a criterion that one could use to show a model is semi-full. Other that using the definition directly, we do not know any such criterion. A positive answer to the following question, if such an answer could be found (something that seems unlikely) might help significantly.

Question 5.5. *Does the converse to Proposition* 3.3 *hold*? *I.e., is every $M \models \mathrm{PA}$ with $(\omega, \mathrm{SSy}(M, \omega)) \models \mathrm{CA}_0$ semi-full*?

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