ForMaRE: Formal Mathematical Reasoning in Economics

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1 Motivation of the ForMaRE Project

Theoretical economics may be regarded as a branch of applied mathematics, drawing on a wide range of mathematics to explore and prove properties of stylized economic environments. Proofs are error prone since typically for any new axiom set humans have initially no or only limited intuition. This way it is easy to assume false theorems and to overlook cases in proofs. Proofs found in mathematics in general and in theoretical economics in particular, can be viewed from a logical point of view more like proof plans. That is, not all details are given, hidden assumptions may be overlooked, proof steps may be incorrect, generalizations may not hold. Thus, any mathematical discipline, including theoretical economics, can benefit from formalizing proofs since this will make proofs much more reliable. However, there are other potential benefits. For instance, in experimenting with axiomatizations it is much easier to reuse proof efforts. Furthermore the dependencies of assertions can be accessed more easily and experiments with the computational content of theorems becomes possible which without computer support would be time consuming and error-prone.

Mathematical formalization and mechanized reasoning have been applied to economics before, most prominently to social choice theory (cf., e.g., [9], [14], [1]) and game theory (cf., e.g., [11]). Immediately preceding the ForMaRE project, we have ourselves formalized pillage games, a particular form of cooperative games, and motivated this as follows [4]:

1. Economics as a whole, but cooperative game theory in particular, is a relatively new area for mechanized reasoning (still in 2013) and therefore presents a new set of canonical examples and challenge problems.

2. Economics typically involves new mathematics in that axioms particular to economics are postulated. One of the intriguing aspects of cooperative game theory is that, while the mathematical concepts involved are often intelligible to even undergraduate mathematicians, general theories are elusive. This has made pillage games more amenable to formalization than research level mathematics.

Despite these potential benefits, formalization of economics has so far been carried out almost exclusively by computer scientists, not by economists.

2 Auction Theory

Our initial focus in the ForMaRE project were pillage games. But then we became aware of exciting work that has been done in areas with broader audiences than cooperative games and refocused and are currently looking at auction theory (other areas of interest are mentioned in the next section). In particular, we have formalized Vickrey's theorem on second price auctions which establishes that nobody can do better in such an auction than just bidding the own valuation of the good independent of what the other bidders do.

Our starting point is Maskin's work who collected high level versions of Vickrey's theorem and 12 others in a review [6] of an influential auction theory textbook. This sets the roadmap for building an Auction Theory Toolbox – a collaborative effort, to which we invite volunteers (see project home page). Four different formalizations have been done (or are currently done). The one in Isabelle [3] (which has been much improved with input from Makarius Wenzel) is finished as is the one in Mizar (by Marco B. Caminati). One in Theorema 2.0 (with Wolfgang Windsteiger) and one in CASL (with Till Mossakowski) are currently developed.

One of the insights is that the formalization as it is found in the published paper (i.e., Maskin's) is not detailed enough to be input directly into a system. Another is that the proof structure is not necessarily ideal for a formal proof. Concretely, we would have to deal with many more cases than necessary if we took the proof outline as guideline for the formal proof. Furthermore, even when a concrete structure is given, provers typically cannot prove the related lemmas directly but additional auxiliary lemmas need to be introduced.

In building the Auction Theory Toolbox, we are com-

paring [5] these four different systems, whose philosophies cover a large subset of the spectrum: Isabelle (interactive theorem prover, HOL, accessible via a document-oriented IDE [13]), CASL/Hets (uniform GUI front end to a wide range of automated FOL provers [8]), Theorema (automated but configurable theorem prover, HOL appearing as FOL plus set theory in the textbook-like notation of Mathematica notebooks [15]), and Mizar (automated proof checker, FOL plus set theory [7]).

3 Possible Application Areas

In addition to auction theory we also want to study matching markets and finance markets regulation. Our aim is to establish new results and think that some important fields such as those we cite in the following are amenable to formal reasoning: **auctions** are widely used for allocating goods and services. Novel auctions have recently been designed for allocating new top-level Internet domains [2], but it is not known for sure whether they are efficient, i.e. giving a domain to the registrar who values it highest and is therefore expected to utilize it best. **Matching** problems occur, e.g., in health care (matching kidney donors to patients) and in education (children to schools) [10]. Impossibility results are of particular interest here; they rely on finding rich counterexamples. Finally, modern **finance** relies on models to price assets or to compute risk, but banks and regulation authorities still validate and check such models manually. One research challenge is to develop minimal test portfolios that ensure that capital models incorporate relevant risk factors [12].

3.1 Enabling Economists to use Formalized Reasoning

Ultimately we aim at enabling economists to formalize their own designs and validate them themselves. For users without a strong computer science background, this is aggravated by the complexity and abundance of formalized languages and proof assistants. Ideally there would be toolboxes of ready-to-use formalizations of basic concepts, including definitions and essential properties, and guides to extending and applying these toolboxes for different areas of theoretical economics. In order to build them, this means:

1. identifying languages
   (a) that are sufficiently expressive while still exhibiting efficient reasoning tasks,
   (b) that are learnable for people used to informal textbook notation,
   (c) and that have rich libraries of mathematical foundations.
2. identifying proof assistants
   (a) that facilitate reuse from the toolbox,
   (b) whose output is sufficiently comprehensible to help non-experts understand, e.g., why a proof attempt failed, and

(c) whose community is supportive towards non-experts.

4 Conclusion

Theoretical economics is an important field whose results have significant impact on all of us. For instance, auctions allocate trillions of dollars in goods and services every year, but their design is still "far less a science than an art" [6]. Our Auction Theory Toolbox of basic auction theory formalizations aims at making it more of a science, by enabling auction designers to verify their own designs. We hope that tools from the automated reasoning community can support theoretical economics to make its results more dependable.

References

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