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# Approximate maximum likelihood direction of arrival estimation for two closely spaced sources

François Vincent , Olivier Besson  
Dept of Electronics Optronics and Signal  
University of Toulouse-ISAE  
Toulouse, France  
francois.vincent@isae.fr, olivier.besson@isae.fr

Eric Chaumette  
DEMR/TSI ONERA, the French Aerospace Lab, Palaiseau  
Ecole Normale Supérieure de Cachan/SATIE, France  
eric.chaumette@onera.fr

**Abstract**—Most high resolution direction of arrival (DoA) estimation algorithms exploit an eigen decomposition of the sample covariance matrix (SCM). However, their performance dramatically degrades in case of correlated sources or low number of snapshots. In contrast, the maximum likelihood (ML) DoA estimator is more robust to these drawbacks but suffers from a too expensive computational cost which can prevent its use in practice. In this paper, we propose an asymptotic simplification of the ML criterion in the case of two closely spaced sources. This approximated ML estimator can be implemented using only 1-D Fourier transforms. We show that this solution is as accurate as the exact ML one and outperforms all high-resolution techniques in case of correlated sources. This solution can also be used in the single snapshot case where very few algorithms are known to be effective.

## I. INTRODUCTION

The problem of identifying superimposed exponential signals in noise has been one of the most addressed signal processing problem during the last forty years. Two cases of interest have mainly been tackled. The first one consists in frequency estimation from one or multiple time observations (snapshots) while the second one consists in DoA estimation of plane waves impinging on an array of sensors. Many high resolution techniques have been developed to improve the standard Rayleigh resolution. Among the most popular algorithms, one can cite Capon [1], MUSIC [2] [3] or root-MUSIC [4], [5], ESPRIT [6] and Min-Norm [7] in the case of a uniform linear array (ULA). Exploiting the centrohermitian property of the asymptotic covariance matrix, improved versions of these techniques have been proposed, such as unitary root-MUSIC [8] or unitary-ESPRIT [9]. These techniques can reach nearly optimal performance, i.e., the corresponding mean square error (MSE) of the frequency estimates come very close to the Cramér Rao Bound (CRB) [10] in ideal cases. However, when sources become correlated (whose limiting case is multipath propagation) or when the number of snapshots becomes less than the number of sources (non stationary environments), the performance of these methods significantly degrades. To overcome the problem of source correlation, one can use spatial smoothing techniques [11] but the price to be paid is

a loss of resolution. Moreover this method is difficult to be used in the case of arrays with few antennas.

Maximum Likelihood (ML) techniques can handle these cases. Yet, a main drawback of ML algorithms lies in their complexity due to the need to solve a multi-dimensional optimization problem. To overcome this disadvantage, iterative procedures have been proposed. For instance the Alternating Projection algorithm [12], based on a relaxation procedure has been introduced. Weiss [13] proposed to use the Expectation Maximization principle [14] to convert the multi dimensional search procedure into successive one dimensional simpler optimizations. Moreover, in the case of a ULA, Bresler has developed the so-called IQML algorithm [15] and Stoica the so-called IMODE algorithm [16] that are both recursive simplified algorithms. Nevertheless, all these recursive procedures are not guaranteed to converge to the global maximum.

In this article, we propose a direct and low computational cost procedure to maximize the ML criterion in the case of two closely spaced sources. We focus on the two sources case as it constitutes the basic scenario to address the resolution issue. Moreover, in many real-life applications such as radar, communications or navigation problems, DoA estimation occurs after range or Doppler filtering and the effective number of sources to be identified is usually less than two.

This paper is organized as follows. We first introduce in section II the framework at hand and recall the formulation of both the ML estimator and the CRB. Then, we perform a Taylor series expansion for closely spaced sources in section III resulting in a closed-form expression of the frequency difference estimate. Using this expression, a one dimensional search only is needed to estimate the two frequencies. In section IV, numerical simulations illustrate that this solution is as precise as the exact ML procedure and outperforms all classical subspace-based techniques in the case of correlated sources or single snapshot. Finally, section V concludes this paper.

## II. DATA MODEL

We consider a narrow-band uniform linear array of  $M$  sensors with inter-element spacing  $d$ . We assume that two closely spaced plane waves impinge on the array with respective DoA  $\theta_1$  and  $\theta_2$ . Let  $f_i = \frac{d}{\lambda} \sin \theta_i$ ,  $i = 1, 2$  denote the corresponding spatial frequencies and let us re-parameterize the problem

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in terms of  $f_1$  and  $\Delta_f = f_2 - f_1$ , where, by assumption,  $\Delta_f \ll 1/M$ . The model at hand can then be written as follows:

$$\mathbf{x}_t = \mathbf{A}(f_1, \Delta_f) \mathbf{s}_t + \mathbf{n}_t \quad t = 0, \dots, (N-1) \quad (1)$$

where

- $\mathbf{A}(f_1, \Delta_f) = [\mathbf{a}(f_1) \mathbf{a}(f_1 + \Delta_f)] \in \mathbb{C}^{M \times 2}$  with  $\mathbf{a}(f) = \frac{1}{\sqrt{M}} [1 \ e^{2i\pi f} \dots e^{2i\pi f(M-1)}]^T$  denoting the normalized steering vector.
- $\mathbf{s}_t \in \mathbb{C}^2$  stands for the vector of unknown deterministic amplitudes of the sources.
- $\mathbf{n}_t \in \mathbb{C}^M$  denotes the noise vector and is assumed to be zero-mean circularly Gaussian with covariance matrix  $\sigma^2 \mathbf{I}$  where  $\sigma^2$  is an unknown scalar. Moreover,  $\mathbf{n}_t$  is supposed to be temporally white, so that  $\mathcal{E}\{\mathbf{n}_t \mathbf{n}_s^H\} = \sigma^2 \mathbf{I} \delta_{t-s}$  and  $\mathcal{E}\{\mathbf{n}_t \mathbf{n}_s^T\} = \mathbf{0}$ .

It can be noticed that in the single snapshot case ( $N = 1$ ), this model is also valid for spectral analysis in time series analysis. The problem at hand consists in estimating  $f_1$  and the frequency difference  $\Delta_f$ . The ML solution is obtained by maximizing the log-likelihood function with respect to the unknown parameters. Concentrating the likelihood function with respect to  $\sigma^2$  and all  $\mathbf{s}_t$ , it is well known that the ML estimator of  $f_1$  and  $\Delta_f$  is given by [17]

$$\hat{f}_1, \hat{\Delta}_f = \arg \max_{f_1, \Delta_f} \sum_{t=0}^{N-1} \|\mathbf{P}(f_1, \Delta_f) \mathbf{x}_t\|^2 \quad (2)$$

where  $\mathbf{P}(f_1, \Delta_f)$  is the projection onto the subspace spanned by the columns of  $\mathbf{A}(f_1, \Delta_f)$  (signal subspace) and  $\mathbf{P}^\perp(f_1, \Delta_f) = \mathbf{I} - \mathbf{P}(f_1, \Delta_f)$  is the projection onto the noise subspace. Associated with this deterministic model, one can also derive the associated CRB as [17]

$$\mathbf{B}_c = \frac{\sigma^2}{2N} \text{Re} \left[ \hat{\mathbf{R}}_s \odot \mathbf{H}^T \right]^{-1} \quad (3)$$

where  $\hat{\mathbf{R}}_s = \frac{1}{N} \sum_{t=0}^{N-1} \mathbf{s}_t \mathbf{s}_t^H$  is the sources amplitude covariance matrix estimate of  $\mathbf{R}_s = \mathcal{E}\{\mathbf{s}_t \mathbf{s}_t^H\}$ , and  $\mathbf{H} = \Delta^H \mathbf{P}^\perp \Delta$  with  $\Delta = [\frac{\partial \mathbf{a}(f)}{\partial f}|_{f_1} \quad \frac{\partial \mathbf{a}(f)}{\partial f}|_{f_1 + \Delta_f}]$ .

This lower bound on the frequency parameters depends on  $\mathbf{s}_t$  and can thus vary from one set of observed data to another. Hence, one may wish to compare the performance of any estimator to the asymptotic ( $N \rightarrow \infty$ ) limit of (3), viz.,

$$\mathbf{B}_c^{as} = \frac{\sigma^2}{2N} \text{Re} \left[ \mathbf{R}_s \odot \mathbf{H}^T \right]^{-1}. \quad (4)$$

### III. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION FOR TWO CLOSELY SPACED SOURCES

As stated before, we focus on the case of two close-frequency signals. We can then conduct a Taylor series expansion of the ML criterion with respect to the frequency difference  $\Delta_f$ . Let us start with the signal subspace projection matrix. From the definition of  $\mathbf{A}(f_1, \Delta_f)$  we have

$$\left( \mathbf{A}^H(f_1, \Delta_f) \mathbf{A}(f_1, \Delta_f) \right)^{-1} = \frac{1}{1 - |c(\Delta_f)|^2} \begin{pmatrix} 1 & -c(\Delta_f) \\ -c(\Delta_f)^* & 1 \end{pmatrix} \quad (5)$$

with  $c(\Delta_f) = \mathbf{a}(f_1)^H \mathbf{a}(f_1 + \Delta_f)$ . In the case of a ULA one simply has  $c(\Delta_f) = \frac{1}{M} e^{i\pi(M-1)\Delta_f} \frac{\sin \pi M \Delta_f}{\sin \pi \Delta_f}$ . Under the stated hypothesis, we have that  $\mathbf{a}(f_1 + \Delta_f) = \mathbf{D}(\Delta_f) \mathbf{a}(f_1)$  where  $\mathbf{D}(\Delta_f) = \text{diag}([1 \ e^{2i\pi\Delta_f} \dots e^{2i\pi\Delta_f(M-1)}])$  in the case of ULA. Hence

$$\mathbf{P} = \frac{1}{1 - |c|^2} [\mathbf{a} \mathbf{a}^H + \mathbf{D} \mathbf{a} \mathbf{a}^H \mathbf{D}^H - c \mathbf{a} \mathbf{a}^H \mathbf{D}^H - c^* \mathbf{D} \mathbf{a} \mathbf{a}^H] \quad (6)$$

where, for the sake of notational simplicity, we have temporarily dropped the dependence with respect to  $f_1$  or  $\Delta_f$ , i.e.,  $\mathbf{a} = \mathbf{a}(f_1)$ ,  $c = c(\Delta_f)$ ,  $\mathbf{D} = \mathbf{D}(\Delta_f)$  and  $\mathbf{P} = \mathbf{P}(f_1, \Delta_f)$ . As we are interested in the case where  $\Delta_f \ll 1$ , we can conduct a Taylor expansion of both  $\mathbf{D}$  and  $c$  as

$$\mathbf{D} = \sum \mathbf{D}_k \Delta_f^k; \quad (7)$$

$$c = \sum c_k \Delta_f^k; \quad c_k = \frac{\text{Tr}\{\mathbf{D}_k\}}{M} \quad (8)$$

where  $\text{Tr}\{\cdot\}$  stands for the trace of the matrix between braces. In the case of a ULA,  $\mathbf{D}_k = \frac{(2i\pi)^k}{k!} \text{diag}([0^k \ 1^k \dots (M-1)^k])$ . Substituting these expressions into equation (6) yields

$$\mathbf{P} = \frac{\mathbf{a} \mathbf{a}^H + \sum_{k,l} \Delta_f^{k+l} [\mathbf{D}_k \mathbf{a} \mathbf{a}^H \mathbf{D}_l^H - c_k \mathbf{a} \mathbf{a}^H \mathbf{D}_l^H - c_k^* \mathbf{D}_l \mathbf{a} \mathbf{a}^H]}{1 - \sum_{k,l} \Delta_f^{k+l} c_k c_l^*}. \quad (9)$$

Observing that  $\mathbf{D}_0 = \mathbf{I}$  (since  $f_1 = f_2$  if  $\Delta_f = 0$ ) and that  $c_0 = 1$ , one obtains

$$\mathbf{P} = - \frac{\sum_{n=1} \Delta_f^n \mathbf{M}_n}{\sum_{n=1} \Delta_f^n d_n} \quad (10)$$

with

$$\mathbf{M}_n = \sum_{k=0}^n \mathbf{D}_k \mathbf{a} \mathbf{a}^H \mathbf{D}_{n-k}^H - c_k \mathbf{a} \mathbf{a}^H \mathbf{D}_{n-k}^H - c_k^* \mathbf{D}_{n-k} \mathbf{a} \mathbf{a}^H$$

$$d_n = \sum_{k=0}^n c_k c_{n-k}^*.$$

Since  $c_1^* = -c_1$  (due to pure complex phase terms in the steering vector), it follows that  $\mathbf{M}_1 = \mathbf{0}$  and  $d_1 = 0$ . Therefore,

$$\mathbf{P} = - \frac{\mathbf{M}_2 + \mathbf{M}_3 \Delta_f + \mathbf{M}_4 \Delta_f^2 + O(\Delta_f^3)}{d_2 + d_3 \Delta_f + d_4 \Delta_f^2 + O(\Delta_f^3)}$$

$$\simeq - \frac{1}{d_2} (\mathbf{M}_2 + \mathbf{M}_3 \Delta_f + (\mathbf{M}_4 - \frac{d_4}{d_2} \mathbf{M}_2) \Delta_f^2) + O(\Delta_f^3) \quad (11)$$

where we used the fact that  $d_3 = 0$ . Substituting (11) in (2) (retaining only the terms up to  $\Delta_f^2$ ) and differentiating with respect to  $\Delta_f$ , the following **closed-form expression** of the frequency difference is obtained:

$$\Delta_f^{AML}(f_1) = \frac{\text{Tr}\{\mathbf{M}_3 \hat{\mathbf{R}}\}}{2 \text{Tr}\left\{\left(\frac{d_4}{d_2} \mathbf{M}_2 - \mathbf{M}_4\right) \hat{\mathbf{R}}\right\}} \quad (12)$$

where

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=0}^{N-1} \mathbf{x}_t \mathbf{x}_t^H \quad (13)$$

is the sample covariance matrix. This expression provides an accurate, closed-form expression of the ML estimate of the difference between the two frequencies.

We can now substitute this expression into (2) and solve for  $f_1$  using a 1D grid-based maximum search as

$$\hat{f}_1^{AML} = \arg \max_{f_1} \sum_{t=0}^{N-1} \left\| \mathbf{P}(f_1, \Delta_f^{AML}(f_1)) \mathbf{x}_t \right\|^2. \quad (14)$$

Once  $\hat{f}_1^{AML}$  is obtained, the estimate of  $\Delta_f$  follows from (12) where  $f_1$  is substituted for  $\hat{f}_1^{AML}$ . We can also notice that evaluation of both (2) and (12) can be easily done using fast Fourier algorithms. Indeed, both equations are linear combination of terms of the following type:

$$\text{Tr} \left\{ \mathbf{D}_\ell^H \mathbf{a}(f) \mathbf{a}^H(f) \mathbf{D}_n \hat{\mathbf{R}} \right\} \quad (15)$$

which can be calculated as follows:

$$\begin{aligned} \text{Tr} \left\{ \mathbf{D}_\ell^H \mathbf{a}(f) \mathbf{a}^H(f) \mathbf{D}_n \hat{\mathbf{R}} \right\} &= \sum_{t=0}^{N-1} \mathbf{x}_t^H \mathbf{D}_\ell^H \mathbf{a}(f) \mathbf{a}^H(f) \mathbf{D}_n \mathbf{x}_t \\ &= \sum_{t=0}^{N-1} X_t^\ell(f)^* X_t^n(f) \end{aligned} \quad (16)$$

where  $X_t^n(f) = \mathbf{a}(f)^H \mathbf{D}_n \mathbf{x}_t$  is the Fourier Transform of the weighted version of  $\mathbf{x}_t$  by the diagonal elements of  $\mathbf{D}_n$ . Therefore,  $\Delta_f^{AML}(f_1)$  in (12) can be computed from combinations of  $X_t^n(f)$  for  $n = 0, 1, 2, 3$ .

#### IV. NUMERICAL ILLUSTRATIONS

In this section, we compare the performance of the AML estimator with that of the exact ML estimator based on a 2D grid-search over  $f_1$  and  $f_2$ , as well as to two conventional methods namely ESPRIT and root-MUSIC. The MSE for estimation of the vector  $[f_1 \ f_2]^T$  will serve as the figure of merit and it will be compared to the asymptotic CRB of equation (4).

For all the following simulations we consider a uniformly spaced linear array of  $M = 8$  isotropic sensors. The spatial frequencies of the sources are  $f_1 = 0.1$  and  $f_1 + \Delta_f$  with  $\Delta_f = \frac{1}{10M}$  ( $\Delta_f = \frac{1}{5M}$  in the single snapshot case). The MSE is computed from 1000 Monte-Carlo runs where the Gaussian vectors  $\mathbf{n}_t$  and  $\mathbf{s}_t$  vary in each trial. The signal to noise ratio is defined as

$$SNR = \frac{\text{Tr} \left\{ \mathbf{A}^H \mathbf{A} \mathbf{R}_s \right\}}{\sigma^2 M}. \quad (17)$$

We will consider three scenarios.

##### A. Large sample scenario, uncorrelated sources

In this first scenario, all conditions are met to have an optimal behavior for all DoA estimation procedures. In Fig. 1, the performance comparison is displayed as a function of SNR in order to identify the so-called threshold region where the MSE departs from the CRB. We can first notice that all four methods attain the asymptotic CRB in the asymptotic region (high SNR). More interesting is the threshold region where we

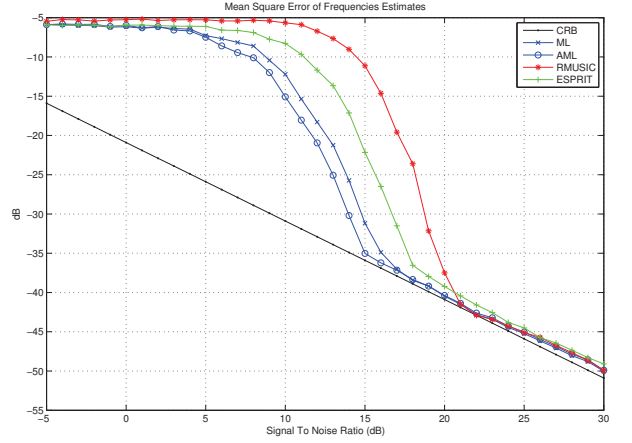


Fig. 1. Large sample scenario, uncorrelated sources, SNR variation

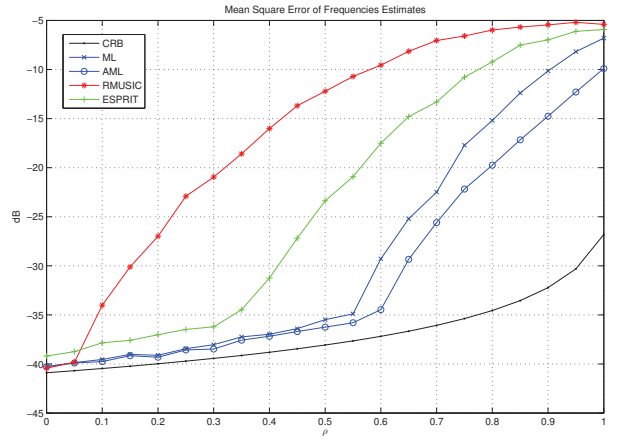


Fig. 2. Large sample scenario, correlated sources

can notice that both the exact MLE and AMLE have a similar behavior and depart from the CRB about 2-3 dB after ESPRIT, and the latter achieves a 2-3 dB gain compared to root-MUSIC. Therefore, the proposed AML estimator has a performance very close to that of the exact MLE in the case of very closely spaced sources, and performs better than ESPRIT and root-MUSIC in this simple scenario. Therefore, we can conclude that the proposed procedure is nearly efficient (for a small source separation) in this ideal scenario. We can even notice a lightly better behavior of the AMLE in the threshold region that suggests a kind of robustness of the proposed method.

##### B. Large sample scenario, correlated sources

We now examine the robustness of the AMLE towards correlation among the two sources. Figure 2 displays the MSE of the four previous algorithms for a correlation coefficient  $\rho$  varying from 0 (previous case) to 1 (coherent sources). The input SNR is 20dB, for which all algorithms achieve the CRB in case of non-correlated sources. We can see that root-MUSIC

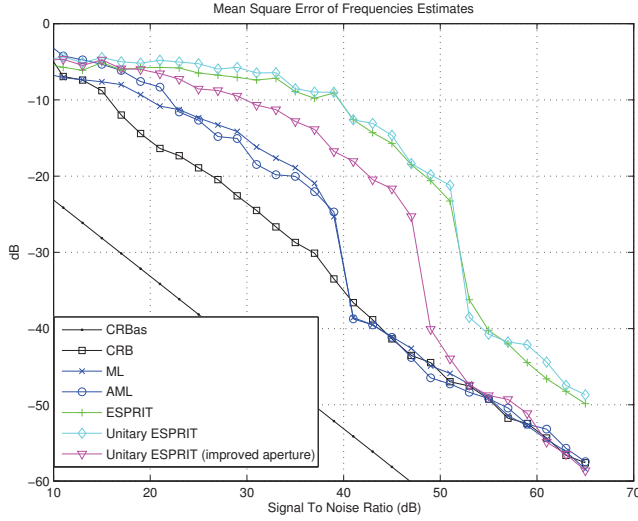


Fig. 3. The single snapshot scenario

departs from the CRB even for small correlation coefficient values. ESPRIT is more robust to correlation as its divergence is more progressive. Both MLE and AMLE are more robust and remain very close to the CRB up to  $\rho = 0.5$ .

### C. The single snapshot scenario

In the single snapshot scenario, neither ESPRIT nor root-MUSIC can be used directly as a rank-two subspace cannot be retrieved from one snapshot. In fact, very few direct high resolution methods can be employed in this case. To still use SCM based techniques, one has to split the actual array into smaller subarrays to conduct the SCM estimation by averaging (Spatial Smoothing technique). The price to be paid will be a reduction of the array aperture and consequently a loss of resolution. The effects of such a procedure will be all the more damaging as the number of sensors is little and the number of assumed sources is high. In our case of interest, we have chosen to compare AMLE and MLE to ESPRIT, Unitary ESPRIT and Unitary ESPRIT with improved aperture [18] procedures based on a covariance matrix estimated from 3 6-sensors subarrays. Figure 3 compares the performances of these estimators. We have added the mean value of the exact CRB of (3). We can first see that in this single snapshot case, all the considered methods do not reach the asymptotic CRB, but that both MLE and AMLE are very close to the exact CRB (nearly efficiency over all SNR values). ESPRIT and Unitary ESPRIT produce almost the same performances that are far from those given by the MLE procedures. Unitary ESPRIT with improved aperture significantly improves the MSE but remains worse than AMLE as the gain difference reaches 10dB over large SNR intervals.

## V. CONCLUSIONS

In this paper, we proposed an approximate ML procedure to estimate the frequencies of two closely spaced sources. The calculation complexity of the proposed procedure is reduced

from a global 2-D maximization, for the exact ML algorithm, to 1-D optimizations than can be easily conducted using Fourier Transforms. The performance of this algorithm is the same as the exact ML for closely-spaced sources and outperforms SCM-based techniques in the case of correlated sources or in the single snapshot scenario.

## REFERENCES

- [1] J. Capon, "High-resolution frequency wavenumber spectrum analysis," *Proceedings IEEE*, vol. 57, pp. 1408–1418, 1969.
- [2] G. Bienvenu and L. Kopp, "Adaptivity to background noise spatial coherence for high resolution passive methods," in *Proceedings ICASSP*, April 1980, pp. 307–310.
- [3] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *RADC Spectral Estimation Workshop*, pp. 243–258, 1979.
- [4] A. J. Barabell, "Improving the resolution performance of eigenstructure-based direction-finding algorithms," in *Proceedings ICASSP*, Boston, MA, May 1983, pp. 336–339.
- [5] B. D. Rao and K. V. S. Hari, "Performance analysis of root-MUSIC," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 37, no. 12, pp. 1939–1949, December 1989.
- [6] R. Roy and T. Kailath, "ESPRIT- estimation of signal parameters via rotational invariance techniques," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 37, no. 7, pp. 984–995, July 1989.
- [7] R. Kumaresan and D. W. Tufts, "Estimating the angles of arrival of multiple plane waves," *IEEE Transactions Aerospace Electronic Systems*, vol. 19, no. 1, pp. 134–139, January 1983.
- [8] M. Pesavento, A. B. Gershman, and M. Haardt, "Unitary root-MUSIC with a real-valued eigendecomposition: A theoretical and experimental study," *IEEE Transactions Signal Processing*, vol. 48, no. 5, pp. 1306–1314, May 2000.
- [9] M. D. Zoltowski, M. Haardt, and C. P. Mathews, "Closed-form 2D angle estimation with rectangular arrays in element space or beamspace via unitary ESPRIT," *IEEE Transactions Signal Processing*, vol. 44, no. 2, pp. 316–328, February 1996.
- [10] P. Stoica and A. Nehorai, "Performance study of conditional and unconditional direction-of-arrival estimation," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 38, no. 10, October 1990.
- [11] T.-J. Shan, M. Wax, and T. Kailath, "On spatial smoothing for direction-of-arrival estimation of coherent signals," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 33, no. 4, pp. 806–811, August 1985.
- [12] I. Ziskind and M. Wax, "Maximum likelihood localization of multiple sources by alternating projection," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 36, no. 10, October 1988.
- [13] A. J. Weiss, A. S. Willsky, and B. C. Levy, "Maximum likelihood array processing for the estimation of superimposed signals," *Proceedings IEEE*, vol. 76, no. 2, pp. 203–204, February 1988.
- [14] A. P. Dempster, N. M. Laird, and D. B. Rubin, "Maximum likelihood from incomplete data via the EM algorithm," *Journal of the Royal Statistical Society*, vol. 39, no. 1, pp. 1–38, 1977.
- [15] Y. Bresler and A. Makovski, "Exact maximum likelihood parameter estimation of superimposed exponential signals in noise," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 34, no. 5, pp. 307–310, October 1986.
- [16] P. Stoica and K. C. Sharman, "Novel eigenanalysis method for direction estimation," *IEEE Proceedings Radar, Sonar and Navigation*, vol. 137, no. 1, February 1990.
- [17] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramér-Rao bound," *IEEE Transactions Acoustics Speech Signal Processing*, vol. 37, no. 5, May 1989.
- [18] K. G. A. Thakre, M. Haardt, "Single snapshot spatial smoothing with improved effective array aperture," *IEEE Signal Processing Letters*, vol. 16, no. 6, pp. 505–508, June 2009.