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Effect of the Generalized Uncertainty Principle on Post-Inflation Preheating

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We examine effects of the Generalized Uncertainty Principle, predicted by various theories of quantum gravity to replace the Heisenberg's uncertainty principle near the Planck scale, on post inflation preheating in cosmology, and show that it can predict either an increase or a decrease in parametric resonance and a corresponding change in particle production. Possible implications are considered.

I. INTRODUCTION

It is generally believed that there was a so-called *preheating* phase near the end of inflation, in which energy was rapidly transferred from the inflaton to matter fields. This was followed by *reheating*, in which thermalization took place, and most of the standard model particles in our universe were produced [1–3]. During preheating, coherent oscillations of the inflaton field $\varphi(t)$, around the minimum of its potential effectively contributed to a time-varying frequency term to the equations of motion of matter fields $\chi(t)$ coupled to it, thereby inducing instabilities by a well understood process known as *parametric resonance* (PR) [4–7]. These in turn resulted in an explosive particle production [8–10]¹ (see also [1] for a recent review). Since the energy density of matter and radiation is exponentially small near the end of inflation, it has been argued that this transfer of energy from inflaton to matter fields must be fast. It was recently shown that (in)homogeneous noise, such as those arising out of quantum fluctuations, indeed increased the instability band, and the rate of particle production [7]. Also it was shown in the past that these resonance effects were sensitive to non-linearities in the equations of motion for χ , which in general were expected to increase the rate of particle production and result in an early termination of

PR [5].

In this paper, we completely analytically study the effect of one such important non-linearity, that predicted from the so-called *Generalized Uncertainty Principle* (GUP). This in turn has been predicted from various approaches to Quantum Gravity, to replace the familiar Heisenberg's Uncertainty Principle near the Planck scale. We observe that, depending on the form of the GUP chosen, and initial conditions, an enhancement of particle production and an early termination of PR can indeed result. Our paper is organized as follows: in Section II we give a brief review of GUP. In Section III, we review relevant aspects of PR. Section IV considers effects of general non-linear terms on PR and Section V the specific ones due to GUP. Section VI generalizes our analysis to an expanding universe. We conclude in Section VII.

II. GENERALIZED UNCERTAINTY PRINCIPLE

Various approaches to quantum gravity, including String Theory, Loop Quantum Gravity, as well as Black Hole Physics predict a minimum measurable length of the order of the Planck length, and a modification of the Heisenberg Uncertainty Principle to a so-called Generalized Uncertainty Principle, or GUP, near the Planck scale [13–19]. Such a form of GUP and the corresponding modified commutators between position and momentum coordinates was suggested earlier by two of the current

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¹ It is also possible PR to result in a particle production which will not be explosive [11, 12]. We thank the anonymous referee for pointing this out to us.

authors [20, 21]

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} [1 + \beta ((\Delta p)^2 + \langle p \rangle^2) + 2\beta (\Delta p_i^2 + \langle p_i \rangle^2)], \quad i = 1, 2, 3 \quad (1)$$

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j) \quad (2)$$

where $p^2 = \sum_{j=1}^3 p_j p_j$, $\beta = \beta_0 / (M_{Pl}c)^2 = \beta_0 \ell_{Pl}^2 / 2\hbar^2$, M_{Pl} = Planck mass, and $M_{Pl}c^2$ = Planck energy $\approx 1.2 \times 10^{19}$ GeV. Note that the new (β -dependent) term is *quadratic* in the Planck length (inverse Planck mass).

Recently, some of us proposed another version of GUP consistent with the above as well as with *Doubly Special Relativity* (DSR) theories [22, 23]. Some phenomenological implications of the above GUP were also examined [22–25] This new version of GUP is of the form

$$\Delta x \Delta p \geq \frac{\hbar}{2} [1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle] \geq \frac{\hbar}{2} \left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right) \Delta p^2 + 4\alpha^2 \langle p \rangle^2 - 2\alpha \sqrt{\langle p^2 \rangle} \right] \quad (3)$$

$$[x_i, p_j] = i\hbar \left[\delta_{ij} - \alpha \left(p \delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right] \quad (4)$$

where $\alpha = \alpha_0 / M_{Pl}c = \alpha_0 \ell_{Pl} / \hbar$. Note that in this case, the new (α -dependent) term is *linear* in the Planck length (inverse Planck mass).

It is also noteworthy that although Eqs. (3) and (4) are not Lorentz covariant, they are DSR covariant [18]. We expect the results of our paper to have similar covariance as well. In addition, since DSR transformations preserve not only the speed of light, but also the Planck momentum and the Planck length, it is not surprising that Eqs. (3) and (4) imply the following minimum measurable length *and* maximum measurable momentum

$$\Delta x \geq (\Delta x)_{min} \approx \alpha_0 \ell_{Pl} \quad (5)$$

$$\Delta p \leq (\Delta p)_{max} \approx \frac{M_{Pl}c}{\alpha_0}. \quad (6)$$

It is evident that the GUP induced terms in both (2) and (4) become important near the Planck scale. Also they both ensure, via the Jacobi identity, that $[x_i, x_j] = 0 = [p_i, p_j]$.

Next, defining

$$x_i = x_{0i}, \quad p_i = p_{0i} (1 + \beta p_0^2) \quad (7)$$

$$\text{and } x_i = x_{0i}, \quad p_i = p_{0i} (1 - \alpha p_0 + 2\alpha^2 p_0^2) \quad (8)$$

where $p_0^2 = \sum_{j=1}^3 p_{0j} p_{0j}$ and x_{0i}, p_{0j} satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar \delta_{ij}$, it is easy to show that equations (2) and (4) are satisfied to order β and α^2 , respectively. Here, p_{0i} can be interpreted as the momentum at low energies (having the standard representation in position space, i.e. $p_{0i} = -i\hbar d/dx_i$), and p_i as that at higher energies.

Finally, employing Eqs. (7) and (8), it is easy to see that a non-relativistic Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r}) \quad [\vec{r} = (x_1, x_3, x_3)] \quad (9)$$

translates to

$$H = H_0 + H_1 + \mathcal{O}(\beta^2) \text{ or } \mathcal{O}(\alpha^2), \quad (10)$$

$$\text{where } H_0 = \frac{p_0^2}{2m} + V(\vec{r})$$

$$\text{and } H_1 = \frac{\beta}{m} p_0^4 \text{ or } H_1 = -\frac{\alpha}{m} p_0^3 \quad (11)$$

for the different versions of GUP defined by equations (2) and (4), respectively. Thus, we see that *any* system with a well defined quantum (or even classical) Hamiltonian H_0 , is perturbed by H_1 , defined above, near the Planck scale. It may be mentioned that it is often assumed that the dimensionless GUP parameters, α_0 and β_0 are of order unity. Also, the momenta p are expected to be much less than $M_{Pl}c/\alpha_0$ or $M_{Pl}c/\sqrt{\beta_0}$, because of the immensity of the Planck scale. Therefore the higher order terms which have been left out are sub-leading and unimportant. As for the leading order terms that have been retained, the resulting effects although small, may be perceptible, which as shall see in later sections, may be further enhanced by various kinematical factors ².

III. PARTICLE PRODUCTION: PARAMETRIC RESONANCE

We review and closely follow the analysis of [4, 7], and consider an oscillating scalar field

$$\varphi(t) = \hbar \cos(\theta t) \quad (12)$$

² Recently, three of the current authors, without assuming the strengths of α_0, β_0 *a priori*, derived bounds on the linear GUP parameter [25] from experimental accuracies of various quantum systems such as the Landau levels, the Simple Harmonic Oscillator, the Lamb Shift, and the tunneling current in a Scanning Tunneling Microscope (STM) etc. From the first three quantum phenomena, the upper bounds on α_0 were found to be 10^{23} , 10^{17} and 10^{10} , respectively. The first bound leads to a length scale bigger than the electroweak length scale, and so although not inconsistent with it, does not lead to any new verifiable scale. The second and third bounds could signal a new and intermediate length scale between the electroweak and the Planck scale. It was also found that even for $\alpha_0 \approx 1$, one might be able to detect quantum gravitational corrections in an STM. Additionally, the bounds on the quadratic GUP parameter [20, 21], for the Lamb Shift, Landau Levels and STM were found 10^{18} , 10^{25} and 10^{10} , respectively. In other words, the bounds derived for the linear GUP parameter are more stringent than those derived for the quadratic GUP parameter. It is possible that suitable astrophysical or cosmological observations may also put some useful bounds on the above parameters.

coupled to another scalar field χ representing the matter, via an interaction

$$\mathcal{L} \propto \frac{1}{2} \varphi \chi^2. \quad (13)$$

Then the evolution equation for χ is nothing but the Mathieu equation, of the form

$$\ddot{\chi} + \omega_0^2 \left(1 + h \cos \left[(2\omega_0 + \epsilon)t \right] \right) \chi = 0 \quad (14)$$

where the argument of the cosine, $\theta \equiv 2\omega_0 + \epsilon$, is so chosen to produce the strongest parametric resonance (PR) via the h -term. We shall assume $h, \epsilon/\omega_0 \ll 1$, and retain terms only to leading order in h, ϵ . Assuming a solution of the form

$$\chi(t) = a_0(t) \cos \left(\frac{\theta}{2} t \right) + b_0(t) \sin \left(\frac{\theta}{2} t \right) \quad (15)$$

with $a_0 \sim e^{s_0 t}$, $b_0 \sim e^{s_0 t}$, substituting in Eq. (14), using $\dot{a}_0 \sim \epsilon a_0$, $\dot{b}_0 \sim \epsilon b_0$ (thereby ignoring $\ddot{a}_0 \sim \epsilon^2 a_0$ and $\ddot{b}_0 \sim \epsilon^2 b_0$ terms), identities such as $\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$ and $\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$, and ignoring weaker resonance terms of the form $\cos(n\frac{\theta}{2}t)$ and $\sin(n\frac{\theta}{2}t)$ ($n \in \mathbb{N} > 1$), we get

$$a_0 s_0 + \frac{b_0}{2} \left(\frac{h\omega_0}{2} + \epsilon \right) = 0 \quad (16)$$

$$b_0 s_0 + \frac{a_0}{2} \left(\frac{h\omega_0}{2} - \epsilon \right) = 0. \quad (17)$$

Solving the above two equations, we obtain

$$\frac{b_0}{a_0} = \sqrt{\frac{\frac{h\omega_0}{2} - \epsilon}{\frac{h\omega_0}{2} + \epsilon}} \equiv R \quad (18)$$

$$s_0 = \frac{1}{2} \sqrt{\left(\frac{1}{2} h\omega_0 \right)^2 - \epsilon^2}. \quad (19)$$

Note that $0 \leq R < \infty$, with $R = 1$ corresponding to $a_0 = b_0$ and $R = 0$ (∞) corresponding to $b_0 = 0$ ($a_0 = 0$). Thus, when $s_0 \in \mathbb{R}$, the solution given in Eq. (15) grows exponentially in the so-called *Instability Region* of $h\omega_0$ given by the parameter range

$$-\frac{1}{2} h\omega_0 < \epsilon < \frac{1}{2} h\omega_0. \quad (20)$$

This is the phenomenon of Parametric Resonance [4].

IV. PARAMETRIC RESONANCE WITH NON-LINEAR TERMS

We introduce a generic non-linearity in the RHS of Eq. (14) of the following form

$$\ddot{\chi} + \omega_0^2 \left(1 + h \cos \left[(2\omega_0 + \epsilon)t \right] \right) \chi = \lambda f(\chi, \dot{\chi}) \quad (21)$$

where $f(\chi, \dot{\chi})$ is an arbitrary non-linear function. As we shall see later, λ is suppressed by powers of Planck mass, and its effects may only show up at very high energies and very small length scales. We will thus treat this term perturbatively. Once again, we assume Eq. (21) has a solution of the form

$$\chi(t) = a(t) \cos \left(\frac{\theta}{2} t \right) + b(t) \sin \left(\frac{\theta}{2} t \right), \quad (22)$$

$$\text{where } a \sim e^{st}, b \sim e^{st} \quad (23)$$

with $a = a_0 + \lambda a_1, b = b_0 + \lambda b_1, s = s_0 + \lambda s_1$.

Thus we can substitute the ‘unperturbed’ solution given by Eq. (15) in the RHS of Eq. (21), simplify again using trigonometrical identities, and retain only the leading order resonance terms to write

$$\begin{aligned} \lambda f(\chi, \dot{\chi}) = & \lambda \omega_0 \sin \left(\frac{\theta}{2} t \right) f_1(a_0, b_0, s_0) \\ & + \lambda \omega_0 \cos \left(\frac{\theta}{2} t \right) f_2(a_0, b_0, s_0). \end{aligned} \quad (24)$$

This modifies Eqs.(16-17) to

$$a s + \frac{b}{2} \left(\frac{h\omega_0}{2} + \epsilon \right) = -\frac{\lambda}{2} f_1 \quad (25)$$

$$b s + \frac{a}{2} \left(\frac{h\omega_0}{2} - \epsilon \right) = \frac{\lambda}{2} f_2. \quad (26)$$

Now we make a further assumption about the smallness of λ , namely that $\lambda \sim \epsilon^{1+p}$, $0 \leq p < 1$ and retain terms up to $\mathcal{O}(\epsilon^2)$ to obtain

$$s^2 = s_0^2 + \frac{\lambda}{4} \left[\frac{f_1}{b_0} \left(\frac{h\omega_0}{2} - \epsilon \right) - \frac{f_2}{a_0} \left(\frac{h\omega_0}{2} + \epsilon \right) \right]. \quad (27)$$

Comparing with the leading order expression $s^2 = s_0^2 + 2\lambda s_0 s_1$ we get

$$2s_0 s_1 = \frac{1}{4} \left[\frac{f_1}{b_0} \left(\frac{h\omega_0}{2} - \epsilon \right) - \frac{f_2}{a_0} \left(\frac{h\omega_0}{2} + \epsilon \right) \right] \quad (28)$$

$$= \frac{1}{4a_0} \left(\frac{h\omega_0}{2} + \epsilon \right) [R f_1 - f_2]. \quad (29)$$

Thus to determine whether PR is enhanced ($s_1 > 0$) or diminished ($s_1 < 0$), one would need to find expressions for f_1, f_2 for specific models of non-linearity.

V. PARAMETRIC RESONANCE WITH GUP

We start with a GUP-modified Hamiltonian in one dimension of the form

$$H = \frac{p^2}{2m} + V(\chi) = \frac{p_0^2}{2m} + V(\chi) + \frac{\kappa \alpha^{n-2}}{m} p_0^n, \quad (30)$$

where $\kappa = \pm 1$. This incorporates the two versions of GUP presented in Section II, with $n = 4$ and $n = 3$,

respectively, (cf. Eq. (11)). The first of the equations of Hamilton, and its inverse, both to $\mathcal{O}(\alpha^{n-2})$ are given by

$$\dot{\chi} = \frac{\partial H}{\partial p_0} = \frac{p_0}{m} + \frac{\kappa n \alpha^{n-2}}{m} p_0^{n-1} \quad (31)$$

$$p_0 = m\dot{\chi} - \kappa n \alpha^{n-2} (m\dot{\chi})^{n-1}. \quad (32)$$

Then the second Hamilton equation $\dot{p}_0 = -\partial H/\partial \chi$ gives

$$m\ddot{\chi} [1 - \kappa n(n-1)\alpha^{n-2}m^{n-2}\dot{\chi}^{n-2}] = -\frac{\partial V}{\partial \chi} \quad (33)$$

$$\text{or, } m\ddot{\chi} + \frac{\partial V}{\partial \chi} + \kappa_1 \frac{\partial V}{\partial \chi} \dot{\chi}^{n-2} = 0 \quad (34)$$

where $\kappa_1 = \kappa n(n-1)\alpha^{n-2}m^{n-2}$. Thus for the time dependent harmonic oscillator, we get from Eq. (34) above

$$\ddot{\chi} + \omega_0^2 \left(1 + h \cos(\theta t)\right) \chi = -\kappa_1 \omega_0^2 \chi \dot{\chi}^{n-2} \quad (35)$$

where we have ignored terms $\mathcal{O}(\kappa_1 h)$. Thus, comparing Eqs. (21) and (35), we obtain $\lambda f = -\kappa_1 \omega_0^2 \chi \dot{\chi}^{n-2}$. Further, we will make the identification $\lambda = \alpha^{n-2}$. Next we write the solution given in Eq. (22) as

$$\chi = ce^{i\frac{\theta}{2}t} + c^* e^{-i\frac{\theta}{2}t} \quad (36)$$

$$\dot{\chi} = Ae^{i\frac{\theta}{2}t} + A^* e^{-i\frac{\theta}{2}t} \quad (37)$$

$$c = \frac{1}{2} \left(a + \frac{b}{i} \right), \quad A = \frac{1}{2} \left[\left(\dot{a} + \frac{\theta}{2}b \right) + \frac{\dot{b} - \frac{\theta}{2}a}{i} \right] \quad (38)$$

from which the quantity $\chi \dot{\chi}^{n-2}$ that appears in the RHS of Eq. (35) reads

$$\chi \dot{\chi}^{n-2} = \left(ce^{i\frac{\theta}{2}t} + c^* e^{-i\frac{\theta}{2}t} \right) \left(Ae^{i\frac{\theta}{2}t} + A^* e^{-i\frac{\theta}{2}t} \right)^{n-2} \quad (39)$$

$$= \sum_{j=0}^{n-2} \binom{n-2}{j} A^j A^{*(n-2-j)} \left(ce^{i(2j-n+3)\frac{\theta}{2}t} + \right. \quad (40)$$

$$\left. c^* e^{i(2j-n+1)\frac{\theta}{2}t} \right). \quad (41)$$

Setting $2j - n + 3 = \pm 1$ and $2j - n + 1 = \pm 1$ to extract the $e^{\pm i\frac{\theta}{2}t}$, i.e. the dominant resonance terms, we see $j = (n-2)/2, (n-4)/2$ and $j = n/2, (n-2)/2$, respectively. It is evident that n must be even and for the rest of the paper, we assume that this is the case³. Collecting these terms, simplifying, using $\dot{a} = sa$, $\dot{b} = sb$, and replacing $\{a, b, s\}$ by $\{a_0, b_0, s_0\}$ by noting that the above term is multiplied by the small parameter λ in the equation of

motion, and finally comparing with Eq. (24) we get

$$f_1 = \ell a_0^3 \left[\frac{n}{n-2} (4s_0^2 + \theta^2) R + R(4s_0^2 - \theta^2) - 4s_0\theta \right] \quad (42)$$

$$f_2 = \ell a_0^3 \left[\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) + 4Rs_0\theta \right] \quad (43)$$

$$\text{where } \ell = \frac{\kappa_2(1+R^2)}{16} \quad (44)$$

$$\text{and } \kappa_2 = -\kappa n(n-1)m^{n-2}\omega_0 \binom{n-2}{n/2} |A|^{n-4} \quad (45)$$

which using Eq. (29), implies in terms of the ratio R ,

$$2s_0s_1 = \frac{\ell a_0^2}{4} \left(\frac{h\omega_0}{2} + \epsilon \right) \left[(R^2 - 1) \left(\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) \right) - 8Rs_0\theta \right]. \quad (46)$$

We use the last equation to summarize the sign of s_1 (Note that s_0 is positive)

	sign of s_1 ($\kappa > 0$)	sign of s_1 ($\kappa < 0$)
$R = 1$	+	-
$R = 0$	+	-
$R = \infty$	+	-
$R \neq 1$	\pm	\pm

Thus, we see that the initial conditions on the matter field (via R) and the GUP one is considering (via the sign of κ) determine whether there is an increase in the exponent of the matter field or not. The various auxiliary variables that were introduced (such as A, C, ℓ etc) do not play any role in it.

A. Instability Region

Now setting $s = 0$ in Eqs. (25) and (26), we see that the PR occurs when the modified instability region is given by

$$-\frac{1}{2}h\omega_0 - \frac{\lambda f_1}{b_0} < \epsilon < \frac{1}{2}h\omega_0 - \frac{\lambda f_2}{a_0} \quad (47)$$

whose width is thus

$$\begin{aligned} \Delta\epsilon &= h\omega_0 - \frac{\lambda}{a_0 R} (Rf_2 - f_1) \\ &= h\omega_0 - \lambda \left[\frac{4(1+R^2)}{R} s_0\theta\ell \right] a_0^2 \\ &= h\omega_0 - 2\lambda h\omega_0\theta\ell a_0^2 \\ &= h\omega_0 \left[1 - 2\lambda\theta\ell a_0^2 \right]. \end{aligned} \quad (48)$$

It follows that parametric resonance is maintained in the presence of GUP. However the region of instability increases when $\kappa > 0$ and vice-versa, i.e. it once again

³ Higher order (weaker) resonance terms can of course arise for n odd.

depends on the GUP under consideration, as well as the amplitude and oscillation frequency of the matter field, which may contribute to further enhancement of this region (but not on R). However, since such an increase or decrease would be proportional to inverse powers of the Planck mass, it may turn out to be too small to have an observable effect at present, although with more accurate experiments and improved observations, it may be detectable in the future. In the next section, we re-do the analysis for an expanding universe.

VI. PARAMETRIC RESONANCE IN AN EXPANDING UNIVERSE WITH NONLINEAR TERMS

We adopt the procedure outlined in [4, 6]. We first write the equation for the matter field in the presence of the inflaton field φ and in an expanding background [1], together with the non-linear terms described by Eq. (30) as

$$\ddot{\chi} + 3H\dot{\chi} + \omega_0^2 [1 + h \cos(2\omega_0 + \epsilon)t] \chi = \lambda f(\chi, \dot{\chi}), \quad (49)$$

where $H = \dot{a}/a$ is the Hubble parameter and $a(t)$ is the scale factor for FRW spacetime. The re-definition

$$Y = a^{3/2} \chi \quad (50)$$

implying

$$\dot{\chi} = a^{-3/2} (\dot{Y} - \frac{3}{2}HY) \quad (51)$$

$$\ddot{\chi} = a^{-3/2} (\ddot{Y} + \frac{9}{4}H^2Y - 3H\dot{Y} - \frac{3}{2}\dot{H}Y) \quad (52)$$

substituted into (49) yields the following equation for the function Y

$$\ddot{Y} + \omega^2 Y = \lambda P[Y, \dot{Y}, a] \quad (53)$$

where

$$\omega^2(t) = \omega_0^2 (1 + h \cos \theta t) - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} \quad (54)$$

with $\theta = 2\omega_0 + \epsilon$,

$$\text{and } P = a^{3/2} f[a^{-3/2}Y, a^{-3/2}(\dot{Y} - \frac{3}{2}HY)]. \quad (55)$$

Next we consider the following cases.

A. Case when $H \neq 0$ and $\lambda = 0$

To gain some insight we first revisit the case where there is no nonlinearity. We assume a solution of the form

$$Y(t) = c_0 \zeta_0(a) \cos\left(\frac{\theta}{2}t\right) + d_0 \zeta_0(a) \sin\left(\frac{\theta}{2}t\right) \quad (56)$$

in which the effects of expansion are included in the scaling function $\zeta_0(a)$. Substituting into Eq. (53) with its RHS set to zero, one obtains

$$(A_1 + A_2) \cos\left(\frac{\theta}{2}t\right) + (B_1 + B_2) \sin\left(\frac{\theta}{2}t\right) + \mathcal{O}(\theta^2, h^2) = 0 \quad (57)$$

with

$$A_1 = (\dot{d}_0\theta + \omega_0^2 c_0 + \frac{\omega_0^2 h}{2} c_0 - \frac{c_0 \theta^2}{4}) \zeta_0 \quad (58)$$

$$B_1 = (-\dot{c}_0\theta + \omega_0^2 d_0 - \frac{\omega_0^2 h}{2} d_0 - \frac{d_0 \theta^2}{4}) \zeta_0 \quad (59)$$

$$A_2 = 2\dot{c}_0\dot{\zeta}_0 + c_0\ddot{\zeta}_0 + d_0\dot{\zeta}_0\theta - \frac{9}{4}H^2 c_0\zeta_0 - \frac{3}{2}\dot{H}c_0\zeta_0 \quad (60)$$

$$B_2 = 2\dot{d}_0\dot{\zeta}_0 + d_0\ddot{\zeta}_0 - c_0\dot{\zeta}_0\theta - \frac{9}{4}H^2 d_0\zeta_0 - \frac{3}{2}\dot{H}d_0\zeta_0 \quad (61)$$

If Eq. (57) is to be justified, the coefficient of the sine and cosine must vanish. In addition, in order to ensure resonant behavior, we further set $A_1, A_2, B_1,$ and B_2 separately equal to zero. Thus, up to order $\mathcal{O}(\epsilon, h)$, the coefficients A_1 and B_1 reduce to the Eqs. (16) and (17), i.e.,

$$A_1 = 2s_0 d_0 + c_0 \left(\frac{h\omega_0}{2} - \epsilon \right) = 0 \quad (62)$$

$$B_1 = -2s_0 c_0 - d_0 \left(\frac{h\omega_0}{2} + \epsilon \right) = 0 \quad (63)$$

where s_0 is the characteristic ('unperturbed') exponent defined in Eq. (19). On the other hand, $A_2 = 0$ and $B_2 = 0$ yield

$$2\dot{c}_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) + c_0 \left(\frac{\ddot{\zeta}_0}{\zeta_0} \right) + d_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) c_0 = 0$$

$$2\dot{d}_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) + d_0 \left(\frac{\ddot{\zeta}_0}{\zeta_0} \right) - c_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) d_0 = 0$$

which can be rewritten as

$$\frac{1}{\zeta_0 c_0} \frac{d}{dt} (c_0^2 \dot{\zeta}_0) + d_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) c_0 = 0 \quad (64)$$

$$\frac{1}{\zeta_0 d_0} \frac{d}{dt} (d_0^2 \dot{\zeta}_0) - c_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) d_0 = 0 \quad (65)$$

The two equations above may be combined to give (setting $c_0 = C_0 e^{s_0 t}$ and $d_0 = D_0 e^{s_0 t}$ in which C_0 and D_0 are constants)

$$\frac{d}{dt} (e^{2s_0 t} \dot{\zeta}_0) - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) \zeta_0 e^{2s_0 t} = 0. \quad (66)$$

This equation may be solved following the procedure given in [6]. For a cosmic-time scale factor of the form $a(t) \propto t^q$, we have⁴

$$\dot{H} + H^2 = q(q-1)t^{-2} = \alpha(q)t^{-2}, \quad H = qt^{-1}, \quad (67)$$

⁴ It is noteworthy that $\alpha(q) \geq 0$ for $q \geq 1$, and that for de Sitter expansion, $\alpha(\infty) = \infty$.

then Eq. (66) becomes

$$\ddot{\zeta}_0 + 2s_0\dot{\zeta}_0 - \beta(q)t^{-2}\zeta_0 = 0 \quad (68)$$

where we have introduced the coefficient $\beta(q) = \frac{3}{4}q^2 + \frac{3}{2}\alpha(q)$. If we use the fact that s_0 is of order ϵ and h , then we are allowed to approximate the above equation to

$$\ddot{\zeta}_0 - \beta(q)t^{-2}\zeta_0 \simeq 0. \quad (69)$$

The solution to this equation takes the form

$$\begin{aligned} \zeta_0(t) &= C_1 t^{\frac{1}{2}(1+\sqrt{1+4\beta})} + C_2 t^{\frac{1}{2}(1-\sqrt{1+4\beta})} \\ &= C_1 t^{\frac{3q}{2}} + C_2 t^{\frac{2-3q}{2}}. \end{aligned} \quad (70)$$

With this solution for $\zeta_0(t)$, the resonant solution reads

$$Y^\pm(t) = C_0 \zeta_0(t) e^{\pm s_0 t} \left[\cos\left(\frac{\theta}{2}t\right) \mp R \sin\left(\frac{\theta}{2}t\right) \right] \quad (71)$$

where R stands for d_0/c_0 .

Following the normalization chosen in [11], C_0 has been determined to be $C_0 = \sqrt{\frac{1}{R\theta}}$. By comparing the solution given by Eq. (70) to the solution associated with the nonresonant case as given in [6], we infer that the second exponent in Eq. (70) must be neglected. Thus, the appropriate exponent for the resonant case is the first one. The requirement that $\zeta_0[a(t)] = 1$ when $a(t) = 1$ implies the normalization $\zeta_0 = a(t)^{3/2}$. Therefore, the full resonant solution in an expanding background behaves as

$$Y^\pm(t) = \sqrt{\frac{1}{R\theta}} e^{s_\pm t} \left[\cos\left(\frac{\theta}{2}t\right) \mp R \sin\left(\frac{\theta}{2}t\right) \right] \quad (72)$$

with the characteristic exponent $s_\pm(t)$ defined by

$$s_\pm(t) = \pm s_0 + \frac{3q}{2t} \ln t. \quad (73)$$

Clearly, the stability band width will decrease after taking the expanding background into account. In particular, following Eq. (20) the bounds of the Instability Region in a flat spacetime are

$$\epsilon_{min} = -\sqrt{\left(\frac{h\omega_0}{2}\right)^2} \quad \text{and} \quad \epsilon_{max} = +\sqrt{\left(\frac{h\omega_0}{2}\right)^2} \quad (74)$$

while when the background is an expanding one as described above the bounds are modified as follows [4, 6, 9]

$$\epsilon_{min} = -\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t} \ln t\right)^2} \quad (75)$$

$$\epsilon_{max} = +\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t} \ln t\right)^2}. \quad (76)$$

Now we are in a position to generalize the above calculation to the nonlinear model governed by Eq. (53), and find the analogue of Eq. (66).

B. Case when $H \neq 0$ and $\lambda \neq 0$

We reconsider equation (53) with the nonlinear terms included

$$\ddot{Y} + \omega^2 Y = \lambda P[Y, \dot{Y}, a] \quad (77)$$

and thus, the solution will now be of the form

$$Y(t) = c(t)\zeta(a) \cos\left(\frac{\theta}{2}t\right) + d(t)\zeta(a) \sin\left(\frac{\theta}{2}t\right) \quad (78)$$

with $c = c_0 + \lambda c_1$, $d = d_0 + \lambda d_1$, $s = s_0 + \lambda s_1$, $\zeta = \zeta_0 + \lambda \zeta_1$. Next we substitute the unperturbed solution given in Eq. (56) in the right-hand side of Eq. (77), and as in the previous sections perform the trigonometrical approximations and retain term of order $\mathcal{O}(h)$. We thus write

$$\begin{aligned} \lambda P[Y, \dot{Y}, a] &= \lambda \omega_0 \sin\left(\frac{\theta}{2}t\right) P_1[c_0, d_0, s_0, \zeta_0, a] + \\ &\quad \lambda \omega_0 \cos\left(\frac{\theta}{2}t\right) P_2[c_0, d_0, s_0, \zeta_0, a]. \end{aligned} \quad (79)$$

Then we obtain up to order $\mathcal{O}(h, \lambda)$

$$\begin{aligned} (B - \lambda \omega_0 P_1) \sin\left(\frac{\theta}{2}t\right) + (A - \lambda \omega_0 P_2) \cos\left(\frac{\theta}{2}t\right) \\ + \mathcal{O}(\theta^2, h^2, \lambda^2) = 0 \end{aligned} \quad (80)$$

with

$$B = B_1 + B_2, \quad A = A_1 + A_2, \quad (81)$$

and the modification of Eqs. (58) - (61) to read

$$B_1 = (-i\theta + \omega_0^2 d - \frac{\omega_0^2 h}{2} d - \frac{d\theta^2}{4}) \zeta \quad (82)$$

$$A_1 = (i\theta + \omega_0^2 c + \frac{\omega_0^2 h}{2} c - \frac{c\theta^2}{4}) \zeta \quad (83)$$

$$B_2 = 2d\dot{\zeta} + d\ddot{\zeta} - c\dot{\zeta}\theta - \frac{9}{4}H^2 d\zeta - \frac{3}{2}\dot{H}d\zeta \quad (84)$$

$$A_2 = 2c\dot{\zeta} + c\ddot{\zeta} + d\dot{\zeta}\theta - \frac{9}{4}H^2 c\zeta - \frac{3}{2}\dot{H}c\zeta. \quad (85)$$

Following the same methodology of the previous section, namely demanding resonant behavior, one can set

$$B_1 = \lambda \omega_0 P_1 \quad (86)$$

$$A_1 = \lambda \omega_0 P_2 \quad (87)$$

$$B_2 = 0 \quad (88)$$

$$A_2 = 0. \quad (89)$$

From the first two equations and up to order $\mathcal{O}(\epsilon, h, \lambda)$ it follows that

$$2sc + d\left(\frac{h\omega_0}{2} + \epsilon\right) = -\lambda\left(\frac{P_1}{\zeta_0}\right), \quad (90)$$

$$2sd + c\left(\frac{h\omega_0}{2} - \epsilon\right) = \lambda\left(\frac{P_2}{\zeta_0}\right). \quad (91)$$

We may now assume that $\lambda \sim \epsilon^{1+p}$ with $0 \leq p < 1$, and keep terms up to order $\mathcal{O}(\epsilon^2)$, we then find

$$2s_0s_1 = \frac{1}{4\zeta_0c_0} \left(\frac{h\omega_0}{2} + \epsilon \right) [RP_1 - P_2] \quad (92)$$

which reduces, for $\zeta_0 = 1$ when $a = 1$ to Eq.(29).

Rewriting equations $A_2 = 0$ and $B_2 = 0$ as in the previous section yields

$$\frac{d}{dt}(e^{2st}\dot{\zeta}) - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) \zeta e^{2st} = 0 \quad (93)$$

where s is defined by

$$s = s_0 + \lambda s_1 \quad (94)$$

and s_1 is given by Eq. (92). At this point, it should be noted that s_1 is time-dependent, so is s . Equation (93) may be written out as

$$\ddot{\zeta} + (2\lambda\dot{s}_1t + 2s)\dot{\zeta} - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) \zeta = 0 \quad (95)$$

giving at order $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(\lambda)$, respectively,

$$\ddot{\zeta}_0 + 2s_0\dot{\zeta}_0 - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) \zeta_0 = 0 \quad (96)$$

$$\ddot{\zeta}_1 + 2s_0\dot{\zeta}_1 - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H} \right) \zeta_1 = -2(\dot{s}_1t + s_1)\dot{\zeta}_0. \quad (97)$$

The solution to the first equation has been determined in the previous section to be

$$\zeta_0(t) = t^{\frac{3q}{2}}. \quad (98)$$

The task now is to substitute Eqs. (98) and (92) into Eq. (97), and ignore terms of order ϵ as they are small compared to the others. We employ Eq. (92) in order to evaluate s_1 but for this purpose we also need to specify the form of functions P_1 and P_2 . This would determine explicitly function ζ_1 and thus the complete resonant solutions in an expanding background where nonlinearities are present will be of the form

$$Y^\pm(t) \propto e^{[\pm s_0 + \frac{3q}{2t} \ln t + \lambda(s_1 + \frac{\zeta_1}{\zeta_0 t})]t} \left[\cos\left(\frac{\theta}{2}t\right) \mp R \sin\left(\frac{\theta}{2}t\right) \right] \quad (99)$$

where s_1 , ζ_0 , and ζ_1 are all known for a given nonlinearity expressed by P_1 and P_2 .

C. Parametric Resonance in an expanding background with GUP

In this subsection we consider the nonlinearity to be the GUP term, namely,

$$\lambda P = -\kappa_1 \omega_0^2 \chi \dot{\chi}^{n-2} \quad (100)$$

with

$$\chi = a^{-3/2}Y, \quad \dot{\chi} = a^{-3/2}(\dot{Y} - \frac{3}{2}HY). \quad (101)$$

In order to find the expressions of P_1 and P_2 defined in Eq. (79), one can follow the steps outlined in section V, and simply make the following replacements

$$a \rightarrow a^{-3/2}c\zeta, \quad b \rightarrow a^{-3/2}d\zeta, \quad c \rightarrow \frac{1}{2}a^{-3/2}(c - id)\zeta, \\ A \rightarrow B = a^{-3/2} \left(\bar{A} - \frac{3}{4}H(c - id)\zeta \right), \quad (102)$$

with $\bar{A} = \frac{1}{2}[(\dot{c}\zeta + c\dot{\zeta} + d\zeta\frac{\theta}{2}) - i(d\dot{\zeta} + d\dot{c} - c\dot{\zeta}\frac{\theta}{2})]$. It should be stressed that the quantity “ a ” that appears on the right-hand side of the above replacements is the scale factor of the expanding universe. Thus the expressions of P_1 and P_2 read

$$P_1 = \ell_0 a^{-9/2} c_0^3 \zeta_0^3 \\ \times \left[\frac{n}{n-2} (4s_0^2 + \theta^2) R + R(4s_0^2 - \theta^2) - 4s_0\theta \right] \quad (103)$$

$$P_2 = \ell_0 a^{-9/2} c_0^3 \zeta_0^3 \\ \times \left[\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) + 4R s_0\theta \right] \quad (104)$$

$$\text{with } R = \frac{d_0}{c_0}, \quad \ell_0 = \frac{\kappa_2^{(0)}(1+R^2)}{16}, \quad (105)$$

$$\text{and } \kappa_2^{(0)} = -\kappa n(n-1)m^{n-2}\omega_0 \binom{n-2}{n/2} |B_0|^{n-4} \quad (106)$$

where B_0 is the quantity B as defined in Eq. (102) but in which c, d, ζ have been replaced with c_0, d_0, ζ_0 , respectively.

Note that these two expressions reduce respectively to Eq. (42) and Eq. (43) when we take the limit $H = 0$, $a(t) = 1$, and $\zeta_0(a) = 1$. Furthermore, the expression of s_1 as given in Eq. (29) now takes the form

$$s_1 = \frac{\ell_0 a^{-9/2} c_0^2 \zeta_0^2}{4s_0} \left(\frac{h\omega_0}{2} + \epsilon \right) \left[(R^2 - 1) \left(\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) \right) - 8R s_0\theta \right] \quad (107)$$

where n can be equal to 4 or 3 depending on the version of GUP under consideration. In Section V was shown that when n is even one obtains the dominant resonance terms, while when n is odd one gets the higher order resonance terms which are weaker. Thus, employing $c_0 = C_0 e^{s_0 t}$, $a(t) = t^q$, $H = qt^{-1}$, $\zeta_0 = t^{\frac{3q}{2}}$, and keeping only terms of order $\mathcal{O}(\epsilon^0)$, Eq. (107) for $n = 4$ becomes

$$s_1 = -\frac{3\kappa m^2 \omega_0^3}{4s_0} (R^4 - 1) \left(\frac{h\omega_0}{2} + \epsilon \right) C_0^2 e^{2t s_0} t^{-\frac{3q}{2}}. \quad (108)$$

Accordingly, Eq. (97) is now written as

$$\ddot{\zeta}_1 - \beta(q)t^{-2}\zeta_1 \simeq \frac{9\kappa m^2 \omega_0^3}{4s_0} q (R^4 - 1) \left(\frac{h\omega_0}{2} + \epsilon \right) \\ \times C_0^2 e^{2t s_0} t^{-1}. \quad (109)$$

The solution of Eq. (109) for ζ_1 together with the expression (108) of s_1 will determine the form of the solution (99).

Once again, from Eq. (108) one can infer about the sign of s_1 , which as in the case of a static background, depends only on the GUP considered (sign of κ) and the initial conditions on the matter field (via R).

	sign of s_1 ($\kappa > 0$)	sign of s_1 ($\kappa < 0$)
$R > 1$	-	+
$R < 1$	+	-

It is evident that $s_1 = 0$ for $R = 1$.

D. Instability region

In this case, the instability region is obtained by setting the exponent s of the complete resonant solutions in an expanding background with nonlinearities (Eq. (99)) to zero. The exponent is

$$s = \pm s_0 + \frac{3q}{2t} \ln t + \lambda \left(s_1 + \frac{\zeta_1}{\zeta_0 t} \right) \quad (110)$$

and setting $s = 0$ and using Eq. (19), one gets

$$\epsilon_{min} = -\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t} \ln t + \lambda\left(s_1 + \frac{\zeta_1}{\zeta_0 t}\right)\right)^2} \quad (111)$$

$$\epsilon_{max} = +\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t} \ln t + \lambda\left(s_1 + \frac{\zeta_1}{\zeta_0 t}\right)\right)^2} \quad (112)$$

with the instability region given as usual by $\Delta\epsilon = \epsilon_{max} - \epsilon_{min}$. A number of comments are in order here. First, it is easily seen that switching off GUP by setting $\lambda = 0$ reduces Eqs. (111) and (112) to the ones relevant for an expanding universe without non-linearities, cf. Eqs. (75) and (76). Similarly, setting $\lambda = 0$ and $q = 0$ reduces the above to ordinary PR, and the corresponding range Eq. (20). Finally, although by setting $q = 0$ one should in principle recover the width given by Eq. (48), (by finding the solution for ζ_1 in Eq. (109) and using the expression for s_1 in Eq. (108)), this is seen much more easily by setting $s = 0$ in Eqs. (90) and (91), leading to the instability band

$$\Delta\epsilon = h\omega_0 - \frac{\lambda}{c_0 R \zeta_0} (R P_2 - P_1) \quad (113)$$

which on using Eqs. (103) and (104) for P_1 and P_2 , respectively, reduces to

$$\begin{aligned} \Delta\epsilon &= h\omega_0 - \frac{\lambda \ell_0 a^{-9/2} c_0^2 \zeta_0^2}{R} \left[4(1 + R^2) s_0 \theta \right] \\ &= h\omega_0 - 2\lambda h\omega_0 \theta \ell_0 c_0^2 \zeta_0^2 a^{-9/2} \\ &= h\omega_0 \left[1 - 2\lambda \theta \ell_0 c_0^2 \zeta_0^2 a^{-9/2} \right]. \end{aligned} \quad (114)$$

In this case too, the increase in the instability region depends on the GUP parameter, as well as parameters related to the expansion of the background, which may further magnify the GUP effect. Still the effect may remain small and unobservable due to powers of inverse Planck mass (via λ), although the situation can change with better experiments and observations in the future. It is evident that setting $a(t) = 1$ and $\zeta_0 = 1$, and also replacing $c_0 \rightarrow a_0$, reduces the above to Eq. (48).

VII. CONCLUDING REMARKS

In this paper, we studied the effect of GUP on parametric resonance in post-inflation preheating in a static as well as an expanding background, and showed that depending on the exact form of the GUP and initial conditions, the phenomenon of parametric resonance and the corresponding instability band can increase, potentially resulting in higher rates of particle production and an early termination of the above. We believe the inclusion of GUP takes into account (at least partially) remnant Planck scale effects on the reheating of the universe. It would be interesting to study the effects of back-reaction of the produced particles in our set-up. It would also be interesting to apply our approach to string-inspired models, for which nonlinearities enter the matter field equation via the Born-Infeld term, *viz*, $\mathcal{L} \sim 1 - \sqrt{1 - \dot{\chi}^2}$. This would perhaps shed some light on the inflationary reheating theory in the context of extended objects, e.g. D-branes. Last, since our matter field Hamiltonian (Eq.(9)) is non-relativistic, it might be worthwhile studying its relativistic generalization (e.g. via a GUP modified Klein-Gordon equation), to have a better understanding of the massless limit among other things. Note that our general formalism, including Eqs.(21) and (49) remain well suited for such generalizations, as well as for a large class on non-linear corrections to parametric resonance and particle production. We hope to report on these elsewhere.

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