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Amicable $T-$matrices and Applications

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ABSTRACT

Amicable $T-$matrices and Applications
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Our main aim in this thesis is to produce new $T-$matrices from the set of existing $T-$matrices. In Theorem 4.3 a multiplication method is introduced to generate new $T-$matrices of order st, provided that there are some specially structured $T-$matrices of orders s and t. A class of *properly amicable* and *double disjoint* $T-$matrices are introduced. A number of properly amicable $T-$matrices are constructed which includes $2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 18, 22$.

To keep the new matrices disjoint an extra condition is imposed on one set of $T-$matrices and named double disjoint $T-$matrices. It is shown that there are some $T-$matrices that are both double disjoint and properly amicable. Using these matrices an infinite family of new $T-$matrices are constructed.

We then turn our attention to the application of $T-$matrices to construct orthogonal designs and complex Hadamard matrices.

Using T-matrices some orthogonal designs constructed from 16 circulant matrices are constructed. It is known that having $T-$matrices of order t and orthogonal designs constructible from 16 circulant matrices lead to an infinite family of orthogonal designs. Using amicable $T-$matrices some complex Hadamard matrices are shown to exist.

ACKNOWLEDGMENTS

Contents

Chapter 1

Introduction and statement of results

In 1893, Jacques Hadamard found two square real matrices of orders 12 and 20 with entries ± 1 satisfying the equality of the so-called *Hadamard inequality*, which states that if A is a $n \times n$ matrix with columns v_i, then

$$|det(A)| \leq \Pi_{i=1}^{n} \|v_i\|, \tag{1.1}$$

where $\|v_i\|$ is the norm of column v_i. In other words, these matrices have the maximum determinant among matrices of the same order with entries in $\{1, -1\}$.

The equality in (1.1) is satisfied if the columns of A are mutually orthogonal or at least one of the columns is 0.

More specifically, if A is a $n \times n$ matrix and the entries a_{ij} are bounded by B, i.e., $|a_{ij}| \leq B$, then

$$|det(A)| \leq B^n n^{\frac{n}{2}}. \tag{1.2}$$

In particular, if $B = 1$, then

$$|det(A)| \leq n^{\frac{n}{2}}. \tag{1.3}$$

Definition 1.1. A square matrix H of order n with entries in $\{1, -1\}$ is called a *Hadamard matrix* if $HH^t = nI_n$, where I_n is the identity matrix of order n. In other

words, a $\{1, -1\}$ matrix H with mutually orthogonal rows is called a Hadamard matrix.

Note that $H^t H = H H^t = n I_n$, so if the rows of H are mutually orthogonal, then the columns of H are also mutually orthogonal.

These matrices were first studied by Sylvester in 1867 under the name of "anallagmatic pavement" 26 years before Hadamard considered them in 1893 [14].

The order of a Hadamard matrix is $1, 2$ or $4k$ for some positive integer k, and it is conjectured that the converse is also true in general.

Conjecture 1.2 (The Hadamard determinant conjecture). *For any positive integer k, there is a Hadamard matrix of order $4k$.*

There are several generalizations and special cases of Hadamard matrices introduced in the mathematical literature. The three main generalizations include weighing matrices, complex Hadamard matrices and orthogonal designs.

An $n \times n$ matrix W with entries in $\{0, 1, -1\}$ satisfying $W W^t = w I_n$, for some positive integer w is called a *weighing matrix* of order n and weight w. If w is equal to the order of the matrix, then the weighing matrix is a Hadamard matrix.

Another generalization is when the entries of an $n \times n$ matrix H are complex numbers of unit modulus and $H H^* = n I_n$, where H^* is the Hermitian transpose of the matrix H. Such an H is called a *complex Hadamard matrix*.

Orthogonal designs are the generalizations of Hadamard matrices when we do not restrict the entries to be 1 or -1, but the rows are still mutually orthogonal.

Definition 1.3. An *orthogonal design* of order n and type $(u_1, u_2, \ldots, u_s)$, u_i positive integers, is an $n \times n$ matrix A, with entries in $\{0, \pm x_1, \ldots, \pm x_s\}$ (x_i's are commuting indeterminates) such that

$$AA^t = \sum_{i=1}^{s} (u_i x_i^2) I_n. \tag{1.4}$$

An orthogonal design of order n and type $(u_1, u_2, \ldots, u_s)$ on variables $x_1, x_2, \ldots, x_s$ is denoted by $OD(n; u_1, u_2, \ldots, u_s)$.

An orthogonal design whose entries are all in $\{1, -1\}$ is a Hadamard matrix.

Definition 1.4. An $n \times n$ matrix $A = (a_{ij})$ is called *circulant* if $a_{ij} = a_{1, j-i+1}$ where $j - i + 1$ is reduced modulo n.

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Definition 1.5. A square matrix $R = [r_{ij}]$ of order n is called *back diagonal* if

$$r_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

One of the most useful methods of construction of orthogonal designs was introduced in [6] and is as follows:

Theorem 1.6 (Goethal-Siedel [6]). *Let A, B, C, D be circulant matrices of order n that satisfy the equation*

$$AA^t + BB^t + CC^t + DD^t = \big(\sum_{i=1}^{k} s_i x_i^2\big) I_n, \tag{1.6}$$

and let R be the back diagonal matrix of order n, then

$$GS = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^t R & -C^t R \\ -CR & -D^t R & A & B^t R \\ -DR & C^t R & -B^t R & A \end{bmatrix} \tag{1.7}$$

is an orthogonal design of order $4n$ and type $(s_1, s_2, \ldots, s_k)$ on $x_1, x_2, \ldots, x_k$.

Definition 1.7. The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same dimension (not necessarily square), denoted by $A * B$, is the entry-wise product $A * B = [a_{ij} b_{ij}]$ which has the same dimension as A and B.

Definition 1.8. Two matrices A and B are called *disjoint* if $A * B = 0$.

Definition 1.9. Four mutually disjoint circulant (or type 1) $\{0, 1, -1\}$ matrices T_i, $i = 1, 2, 3, 4$ of order n which satisfy

$$\sum_{i=1}^{4} T_i T_i^t = qI, \tag{1.8}$$

are called $T-matrices$ *of order n and weight q.* If $q = n$ we say that T_i's are *full* $T-$matrices of order n. In this case, each of the n^2 entries are nonzero for exactly one T_i, $i = 1, 2, 3, 4$.

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Remark 1.10. In this thesis, by $T-$matrices, we always mean full $T-$matrices unless otherwise specified.

Cooper and Wallis [6] proved that the existence of $T-$matrices of order n and weight q implies the existence of orthogonal designs $OD(4n; q, q, q, q)$.

Theorem 1.11 (Cooper-Wallis). *Suppose T_1, T_2, T_3, T_4 are $T-$matrices of order n and weight q, and a, b, c, d are commuting indeterminates. Construct four circulant matrices, A, B, C, D, as follows*

$$
\begin{aligned}
A &= aT_1 + bT_2 + cT_3 + dT_4, \\
B &= -bT_1 + aT_2 - dT_3 + cT_4, \\
C &= -cT_1 + dT_2 + aT_3 - bT_4, \\
D &= -dT_1 - cT_2 + bT_3 + aT_4.
\end{aligned}
\tag{1.9}
$$

Then A, B, C, D can be plugged into Goethal-Seidel array to obtain $OD(4n; q, q, q, q)$.

Definition 1.12. Two square matrices A and B are called *amicable* if $AB^t = BA^t$ and they are called *antiamicable* if $AB^t = -BA^t$.

In [10], the concept of amicability was generalized to the amicable set of matrices.

Definition 1.13. A set $\{A_1, A_2, \ldots, A_{2m}\}$ of real square matrices is called *amicable* with the matching (A_{2i-1}, A_{2i}), $i = 1, \ldots, m$ if

$$
\sum_{i=1}^{2m} (A_{2i-1} A_{2i}^t - A_{2i} A_{2i-1}^t) = 0.
\tag{1.10}
$$

The amicability of $T-$matrices was first defined by Behbahani in his thesis [2] .

Definition 1.14. $T-$matrices T_1, T_2, T_3, T_4 are called *amicable* with the matching (T_1, T_4), (T_2, T_3) if,
$$
T_1 T_4^t - T_4 T_1^t + T_2 T_3^t - T_3 T_2^t = 0.
\tag{1.11}
$$

He also proved that no amicable $T-$matrices of odd order exist for such matching [2]. However, when we consider amicable $T-$matrices with the matching (T_1, T_4^t), (T_2, T_3^t), then there are some amicable $T-$matrices. Note that with such matching,

the involved matrices may not be disjoint. Amicable $T-$matrices are useful in the construction of new $T-$matrices. Due to the importance of these $T-$matrices, we will call them *properly amicable $T-$matrices.*

In chapter 2, we will present some known construction methods to construct Hadamard matrices and orthogonal designs. Many definitions and theorems which will be used throughout the rest of this thesis will be studied in the same chapter.

Chapter 3 covers sequences. Golay, base and $T-$sequences and the interrelationship between these sequences are examined in this chapter.

Definition 1.15. Four disjoint matrices A, B, C, D are called *double disjoint* if

$$(A + B) * (C + D)^t = 0. \tag{1.12}$$

We will present a multiplication theorem to generate new classes of $T-$matrices in chapter 4. Namely,

Theorem 1.16. *Let A_1, A_2, A_3, A_4 are double disjoint $T-$matrices of order t, and B_1, B_2, B_3, B_4 are properly amicable $T-$matrices of order s, then the four matrices*

$$\begin{aligned}
C_1 &= A_1 \otimes B_1 - A_2^t \otimes B_2 + A_3 \otimes B_3^t + A_4^t \otimes B_4^t, \\
C_2 &= A_2 \otimes B_1 + A_1^t \otimes B_2 + A_4 \otimes B_3^t - A_3^t \otimes B_4^t, \\
C_3 &= A_3 \otimes B_1^t + A_4^t \otimes B_2^t - A_1 \otimes B_3 + A_2^t \otimes B_4, \\
C_4 &= A_4 \otimes B_1^t - A_3^t \otimes B_2^t - A_2 \otimes B_3 - A_1^t \otimes B_4
\end{aligned} \tag{1.13}$$

are $T-$matrices of order st.

As a consequence we have the following result.

Theorem 1.17 (Main Result)**.** *There are $T-$matrices of order $(2m + 1)t$, where m is the length of Golay sequences and $t \in \{3, 5, 7, 9, 11, 13\}$.*

We will also prove that there do not exist $T-$matrices of odd order that are both double disjoint and properly amicable, but find $T-$matrices of orders $t = 2, 6, 10, 14, 18, 22$ that satisfy these properties.

It is worthwhile to mention that if A_1, A_2, A_3, A_4 are $T-$matrices of order say, t, which satisfy both double disjointness and properly amicability properties, and

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there are properly amicable $T-$matrices B_1, B_2, B_3, B_4 of order m then there exist $T-$matrices of order $t^a m$, for every nonnegative integer a.

In chapter 5, we will study the application of $T-$matrices to construct new complex Hadamard matrices and orthogonal designs. Our results are summed up in following theorems:

Theorem 1.18. *Let T_1, T_2, T_3, T_4 be four amicable $T-$matrices of order n with the matching $(T_1, T_2), (T_3, T_4)$, and let a, b, c, d be commuting indeterminates. If the four circulant matrices A, B, C, D are constructed using Theorem 1.11. Then A, B, C, D are amicable with the matching (A, B), (C, D) and the matrix H defined by*

$$
H = \begin{bmatrix}
A & B & C & D \\
-B & A & -D & C \\
-C^t & D^t & A^t & -B^t \\
-D^t & -C^t & B^t & A^t
\end{bmatrix}
\tag{1.14}
$$

is an $OD(4n; n, n, n, n)$ constructed from 16 circulant matrices. Further, if M_1, M_2, M_3, M_4 are $T-$matrices of order t, then the matrices

$$
\begin{aligned}
P &= M_1 \otimes A - M_2 \otimes B - M_3 \otimes C^t - M_4 \otimes D^t, \\
Q &= M_1 \otimes B + M_2 \otimes A + M_3 \otimes D^t - M_4 \otimes C^t, \\
R &= M_1 \otimes C - M_2 \otimes D + M_3 \otimes A^t + M_4 \otimes B^t, \\
S &= M_1 \otimes D + M_2 \otimes C - M_3 \otimes B^t + M_4 \otimes A^t
\end{aligned}
\tag{1.15}
$$

can be used to obtain an $OD(4nt; nt, nt, nt, nt)$.

Theorem 1.19. *Let T_1, T_2, T_3, T_4 be properly amicable $T-$matrices of order n that satisfy the double disjoint property. If we let*

$$
\begin{aligned}
A &= (T_1 + T_2) + i(T_4 - T_3)^t, \\
B &= (T_1 - T_2) + i(T_4 + T_3)^t
\end{aligned}
\tag{1.16}
$$

then we have

$$
AA^* + BB^* = 2nI_n.
\tag{1.17}
$$

Hence, the matrix H defined by

$$H = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix} \tag{1.18}$$

is a complex Hadamard matrix of order $2n$.

Chapter 2

Preliminaries

In this chapter, we will provide basic definitions and propositions on Hadamard matrices, orthogonal designs and $T-$matrices which are used throughout this thesis. Our main references for this chapter are [11] and [6].

2.1 Type 1 and Type 2 matrices

Definition 2.1. Let G be an additive abelian group of order t, whose elements have been ordered as $g_1, \cdots, g_t$. Let ψ and ϕ be two functions from G into a commutative ring. We define two matrices, $M = [m_{ij}]$ and $N = [n_{ij}]$, of order t, as follows:

$$m_{ij} = \psi(g_j - g_i), \quad n_{ij} = \phi(g_j + g_i). \tag{2.1}$$

M and N are called *type* 1 and *type* 2 matrices, respectively.

Example 2.2. Consider the field $\mathbb{Z}_5$ and order the elements as

$$g_1 = 0, \ g_2 = 1, \ g_3 = 2, \ g_4 = 3, \ g_5 = 4 \tag{2.2}$$

Let ψ and ϕ be the identity function, then the type 1 and type 2 matrices corresponding to this function are defined by, $M_1 = [m_{ij}]$ and $N_1 = [n_{ij}]$, where $m_{ij} = g_j - g_i$ and

$n_{ij} = g_j + g_i$. Then

$$M_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right) \end{array} \qquad (2.3)$$

and

$$N_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{array}\right) \end{array} \qquad (2.4)$$

Definition 2.3. Let G be a group of order n, with elements ordered as $g_1, g_2, \ldots, g_n$ and suppose that X is a subset of G and $0 \notin X$. If we define two functions ψ, ϕ as follow,

$$\psi(x) = \phi(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in X, \\ -1 & \text{if } x \notin X \end{cases} \qquad (2.5)$$

then the two matrices $M = [m_{ij}]$, $N = [n_{ij}]$ defined by $m_{ij} = \psi(g_j - g_i)$, $n_{ij} = \phi(g_j + g_i)$ are called type 1, respectively type 2, $\{0, 1, -1\}$ incidence matrices generated by X.

Example 2.4. Consider the field $\mathbb{Z}_5$. Order the elements as

$$g_1 = 0, g_2 = 1, g_3 = 2, g_4 = 3, g_5 = 4, \qquad (2.6)$$

and define the set

$$X = \{y : y = g^2 \text{ for some } g \in \mathbb{Z}_5, g \neq 0\} = \{1, 4\}. \qquad (2.7)$$

Then the type 1 and type 2 $\{0, \pm 1\}$ incidence matrices generated by X are given by M_2 and N_2, respectively:

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$$
M_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
\left(\begin{array}{ccccc}
0 & 1 & - & - & 1 \\
1 & 0 & 1 & - & - \\
- & 1 & 0 & 1 & - \\
- & - & 1 & 0 & 1 \\
1 & - & - & 1 & 0
\end{array}\right)
\end{array}
\tag{2.8}
$$

and

$$
N_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
\left(\begin{array}{ccccc}
0 & 1 & - & - & 1 \\
1 & - & - & 1 & 0 \\
- & - & 1 & 0 & 1 \\
- & 1 & 0 & 1 & - \\
1 & 0 & 1 & - & -
\end{array}\right)
\end{array},
\tag{2.9}
$$

where $-$ stands for -1.

Recall from Definition 1.4 that a *circulant matrix* $A = (a_{ij})$ of order n is one for which $a_{ij} = a_{1,j-i+1}$ where $j - i + 1$ is reduced modulus n. For example:

$$
\begin{bmatrix}
a & b & c \\
c & a & b \\
b & c & a
\end{bmatrix}
\tag{2.10}
$$

is a circulant matrix of order 3. Note that the matrices M_1 and M_2 in Examples 2.2 and 2.4 are circulant.

Remark 2.5. Throughout this thesis, if A is a circulant matrix of order n and $a_1, \ldots, a_n$ are the entries on the first row of A, then we denote A by $Circ(a_1 \ldots a_n)$. If $a_i = -1$, for some i, then as usual, we will use $-$ instead.

Definition 2.6. A matrix $A = (a_{ij})$ of order n is called *back circulant* if $a_{ij} = a_{1,i+j-1}$

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where $i + j - 1$ is reduced modulus n. For example:

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \tag{2.11}$$

is a back circulant matrix of order 3. Note that the matrices N_1 and N_2 in Examples 2.2 and 2.4 are back circulant.

Remark 2.7.

$i)$ Any type 1 matrix defined on $\mathbb{Z}_t$ with the standard ordering is circulant since:

$m_{ij} = \psi(j - i) = \psi(j - i + 1 - 1) = m_{1,j-i+1}.$

$ii)$ Any type 2 matrix defined on $\mathbb{Z}_t$ with the standard ordering is back circulant since:

$n_{ij} = \phi(j + i) = \phi(j + i - 1 + 1) = n_{1,i+j-1}.$

$$\tag{2.12}$$

Lemma 2.8. $[6]$ *Suppose A and B are type 1 matrices and C is a type 2 matrix defined on an abelian group G of order t with elements ordered as $g_1, \cdots, g_t$ and $R = [r_{ij}]$ is defined by:*

$$r_{ij} = \begin{cases} 1 & \text{if } g_i + g_j = 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

then:

(i) *type 1 matrices commute with each other, i.e., $AB = BA$,*

(ii) *any type 2 matrix is symmetric, i.e., $C^t = C$,*

(iii) *type 1 matrices are amicable with type 2 matrices, i.e., $AC^t = CA^t$,*

(iv) *A^t and C^t are type 1 and type 2 matrices respectively,*

(v) *$A + B$ and $A - B$ are type 1 matrices,*

(vi) *AR is a type 2 matrix and CR is a type 1 matrix.*

$$\tag{2.14}$$

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ be defined by $a_{ij} = \phi(g_j - g_i)$, $b_{ij} = \psi(g_j - g_i)$ and $c_{ij} = \mu(g_j + g_i)$. Then,

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(i) $(AB)_{ij} = \sum_{h \in G} \phi(h - g_i)\psi(g_j - h)$. Set $z = g_i + g_j - h$. It is clear that as h varies through G so does z, and the last expression is equal to
$\sum_{z \in G} \phi(g_j - z)\psi(z - g_i) = \sum_{z \in G} \psi(z - g_i)\phi(g_j - z) = (BA)_{ij}$.

(ii) $c_{ij} = \mu(g_j + g_i) = \mu(g_i + g_j) = c_{ji}$.

(iii) $(AC^t)_{ij} = \sum_{h \in G} \phi(h - g_i)\mu(g_j + h)$. If $z = g_j - g_i + h$. Then the last expression is equal to $\sum_{z \in G} \phi(z - g_j)\mu(g_i + z) = (CA^t)_{ij}$.

(iv) Define a type 1 matrix $D = (d_{ij})$ using the function τ, where $\tau(x) = \phi(-x)$. Then $d_{ij} = \tau(g_j - g_i) = \phi(g_i - g_j) = a_{ji}$. Thus, A^t is a type 1 matrix.

(v) Define type 1 matrices using functions $\tau_1 + \tau_2$, and $\tau_1 - \tau_2$, where $\tau_1(x) = \phi(-x)$ and $\tau_2(x) = \psi(-x)$ to obtain $A + B$ and $A - B$.

(vi) Let $\tau(x) = \phi(-x)$. Then $(AR)_{ij} = \sum_{h \in G} a_{ih}r_{hj} = a_{il}$, where $a_l + a_j = 0$, and the last expression is equal to $\phi(g_l - g_i) = \phi(-g_j - g_i) = \tau(g_j + g_i)$. So, AR is a type 2 matrix. $\qquad\square$

2.2 Hadamard matrices

In this section, we will study some construction methods for Hadamard matrices.

Recall from Definition 1.1 that an $n \times n$ matrix H with entries in $\{1, -1\}$ is called a Hadamard matrix if $HH^t = nI_n$, where I_n is the identity matrix of order n.

Sylvester [14] showed that if H is a Hadamard matrix of order n, then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}, \tag{2.15}$$

is a Hadamard matrix of order $2n$.

Since there exists a Hadamard matrix of order 2, so there exist Hadamard matrices of orders 2^s, for every positive integer s.

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Definition 2.9. Suppose $M = (m_{ij})$ and $N = (n_{ij})$ are two matrices of orders, $m \times p$ and $n \times q$, respectively. The *Kronecker product* of M and N, denoted by $M \otimes N$ is a matrix of order $mn \times pq$ and is given by

$$M \otimes N = \begin{bmatrix} m_{11}N & m_{12}N & \cdots & m_{1p}N \\ m_{21}N & m_{22}N & \cdots & m_{2p}N \\ \vdots & & & \vdots \\ m_{m1}N & m_{m2}N & \cdots & m_{mp}N \end{bmatrix}. \tag{2.16}$$

Hadamard [8] proved that if H_1 and H_2 are two Hadamard matrices of orders m and n, respectively, then $H_1 \otimes H_2$ is a Hadamard matrix of order mn.

Lemma 2.10. *The Kronecker product of two Hadamard matrices is a Hadamard matrix.*

Proof. Let H_1 and H_2 be two Hadamard matrices of orders m and n respectively, then $H_1 \otimes H_2$ is a ± 1 matrix and we have,

$$(H_1 \otimes H_2)(H_1 \otimes H_2)^t = (H_1 H_1^t) \otimes (H_2 H_2^t) = mI_m \otimes nI_n = mnI_{mn}. \tag{2.17}$$

So $H_1 \otimes H_2$ is a Hadamard matrix of order mn. $\qquad\square$

2.2.1 Paley matrix

Paley, in [13], presented the strongest construction methods to construct new Hadamard matrices.

To present his work, we should first give some definitions and lemmas.

Throughout this thesis, a field $\mathbb{F}$ of order q, is denoted by $\mathbb{F}_q$.

Definition 2.11. An element $x \in \mathbb{F}_q$ is a *quadratic residue* if the equation $x = t^2$ has some solution in $\mathbb{F}_q$. Also, a nonzero element x which is not a quadratic residue is called a *quadratic non-residue*.

Lemma 2.12. *[16] If q is an odd prime power, then half of the nonzero elements in $\mathbb{F}_q$ are quadratic residue.*

Proof. Note that in the field $\mathbb{F}_q$ there are at most $(q-1)/2$ squares since,

$$
\begin{aligned}
1^2 &= (q-1)^2 \\
2^2 &= (q-2)^2 \\
&\;\;\vdots \\
(\frac{q-1}{2})^2 &= (\frac{q+1}{2})^2
\end{aligned}
\tag{2.18}
$$

Further these squares are all distinct, since suppose, $a^2 = b^2$ and say $a < b$, then this implies $(a-b)(a+b) = 0$ and so $a = \pm b$ which is impossible because $1 \le a < b \le (\frac{q-1}{2})$. So exactly half of the nonzero elements in $\mathbb{F}_q$ are quadratic residues. $\qquad\square$

If q is an odd prime power, then the function χ, known as *Legendre symbol*, is defined by

$$
\chi(x) = \begin{cases}
0 & \text{if } x = 0, \\
1 & \text{if } x \text{ is a quadratic residue,} \\
-1 & \text{if } x \text{ is a quadratic non-residue.}
\end{cases}
\tag{2.19}
$$

It is well-known fact that χ is a multiplicative function and $\chi(-1) = (-1)^{\frac{q-1}{2}}$, see [16].

Lemma 2.13. *[16] If q is an odd prime power, then $\sum_{a \in \mathbb{F}_q} \chi(a)\chi(a+b) = -1$ for all $b \in \mathbb{F}_q \setminus \{0\}$.*

Proof. First note that

$$
\chi(a)\chi(a+b) = \chi(a)\chi(a)\chi(1+ba^{-1}) = \chi(1+ba^{-1})
\tag{2.20}
$$

provided that $a \ne 0$. Now, since $b \ne 0$, as a takes all nonzero elements in $\mathbb{F}_q$, $1 + ba^{-1}$ takes all the values in $\mathbb{F}_q$ except for 1. Also, from Lemma 2.12, we have $\sum_{a \in \mathbb{F}_q} \chi(a) = 0$.

Hence,

$$\sum_{a \in \mathbb{F}_q} \chi(a)\chi(a+b) = \sum_{\substack{a \in \mathbb{F}_q \\ a \neq 0}} \chi(1 + ba^{-1})$$

$$= \sum_{\substack{a \in \mathbb{F}_q \\ a \neq 1}} \chi(a) = \sum_{a \in \mathbb{F}_q} \chi(a) - \chi(1) \qquad (2.21)$$

$$= 0 - 1 = -1$$

$\square$

Definition 2.14. Consider the $q \times q$ matrix $Q = (q_{xy})$ on the elements of $\mathbb{F}_q$ with rows and columns indexed by the elements of $\mathbb{F}_q$, defined by

$$q_{xy} = \chi(y - x). \qquad (2.22)$$

Then Q is called *Jacobsthal matrix* [1].

From Lemma 2.12 and Lemma 2.13, it follows that the matrix Q has the following properties:
(i) $QQ^t = qI - J,$
(ii) $QJ = JQ = 0,$
(iii) $Q^t = (-1)^{\frac{1}{2}(q-1)}Q.$

Let W be a matrix of order $q + 1$ defined by

$$W = \begin{bmatrix} 0 & e_q \\ (-1)^{\frac{1}{2}(q-1)}e_q^t & Q \end{bmatrix}, \qquad (2.23)$$

where e_q is a $1 \times q$ vector of all ones.

Theorem 2.15. [13] *Suppose $q \equiv 3 (\mathrm{mod}\, 4)$ is a prime power. Then the matrix*

$$H = W + I_{q+1}, \qquad (2.24)$$

is a Hadamard matrix of order $q + 1$.

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Proof. H is a $(q+1) \times (q+1)$ matrix with elements in $\{1, -1\}$, and we have,

$$
\begin{aligned}
HH^t &= (W + I_{q+1})(W + I_{q+1})^t \\
&= WW^t + W + W^t + I_{q+1} \\
&= WW^t + I_{q+1} \qquad \text{(since } q \equiv 3(\mathrm{mod}\,4), \text{ so } W^t = -W) \qquad (2.25) \\
&= qI_{q+1} + I_{q+1} \\
&= (q+1)I_{q+1}.
\end{aligned}
$$

$\square$

Theorem 2.16. *[13] Suppose $q \equiv 1(\mathrm{mod}\,4)$ is a prime power. Then the matrix*

$$
H = \begin{bmatrix} W + I_{q+1} & W - I_{q+1} \\ W - I_{q+1} & -W - I_{q+1} \end{bmatrix} \qquad (2.26)
$$

is a Hadamard matrix of order $2(q+1)$.

Proof.

$$
\begin{aligned}
HH^t &= \begin{bmatrix} W + I_{q+1} & W - I_{q+1} \\ W - I_{q+1} & -W - I_{q+1} \end{bmatrix} \begin{bmatrix} W^t + I_{q+1} & W^t - I_{q+1} \\ W^t - I_{q+1} & -W^t - I_{q+1} \end{bmatrix} \\
&= \begin{bmatrix} 2(WW^t + I_{q+1}) & 0 \\ 0 & 2(WW^t + I_{q+1}) \end{bmatrix} \qquad (2.27) \\
&= \begin{bmatrix} 2(q+1)I_{q+1} & 0 \\ 0 & 2(q+1)I_{q+1} \end{bmatrix} \\
&= 2(q+1)I_{2(q+1)}.
\end{aligned}
$$

$\square$

2.2.2 Williamson array

In an attempt to construct Hadamard matrices of composite orders, Williamson considered four symmetric circulant matrices.

Theorem 2.17 (Williamson [6]). *Let A, B, C, D be four symmetric circulant $\{1, -1\}$ matrices of order n. Further, suppose that*

$$A^2 + B^2 + C^2 + D^2 = 4nI_n. \tag{2.28}$$

Then

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \tag{2.29}$$

is a Hadamard matrix of order $4n$. H is called a Williamson Hadamard matrix. The four matrices A, B, C, D that satisfy these conditions are called Williamson matrices of order n.

Example 2.18. Let $A = Circ(1\ 1\ 1)$, $B = C = D = Circ(-\ 1\ 1)$, then A, B, C, D are circulant and symmetric and satisfy,

$$AA^t + BB^t + CC^t + DD^t = 12I_3, \tag{2.30}$$

so the matrix

$$
H = \begin{bmatrix}
1 & 1 & 1 & & - & 1 & 1 & & - & 1 & 1 & & - & 1 & 1 \\
1 & 1 & 1 & & 1 & - & 1 & & 1 & - & 1 & & 1 & - & 1 \\
1 & 1 & 1 & & 1 & 1 & - & & 1 & 1 & - & & 1 & 1 & - \\
\\
1 & - & - & & 1 & 1 & 1 & & 1 & - & - & & - & 1 & 1 \\
- & 1 & - & & 1 & 1 & 1 & & - & 1 & - & & 1 & - & 1 \\
- & - & 1 & & 1 & 1 & 1 & & - & - & 1 & & 1 & 1 & - \\
\\
1 & - & - & & - & 1 & 1 & & 1 & 1 & 1 & & 1 & - & - \\
- & 1 & - & & 1 & - & 1 & & 1 & 1 & 1 & & - & 1 & - \\
- & - & 1 & & 1 & 1 & - & & 1 & 1 & 1 & & - & - & 1 \\
\\
1 & - & - & & 1 & - & - & & - & 1 & 1 & & 1 & 1 & 1 \\
- & 1 & - & & - & 1 & - & & 1 & - & 1 & & 1 & 1 & 1 \\
- & - & 1 & & - & - & 1 & & 1 & 1 & - & & 1 & 1 & 1
\end{bmatrix}
\qquad (2.31)
$$

is a Hadamard matrix of order 12.

This class of matrices were introduced by Williamson in 1944, see [17].
Based on Williamson findings and later on developments, the general expectation was that Hadamard matrices of this type exist for all orders which are a multiple of 4.
The existence of an infinite class of these matrices of order $(q + 1)/2$, $q \equiv 1 \pmod 4$ for prime power q, shown by Turyn in [15] added to the excitements. However, by an exhaustive search Doković [3] showed that none exist of order 35. Later computer searches up to the order 59 [9] found only a small number of these matrices.
These matrices, scarce by number though, are very useful in the construction of Hadamard matrices of composite order.

2.3   Orthogonal designs

This section is devoted to orthogonal designs and their basic known construction methods.

Recall from Definition 1.3 that an orthogonal design of order n and type $(u_1, u_2, \ldots, u_s)$, u_i positive integers, is an $n \times n$ matrix A with entries in $\{0, \pm x_1, \ldots, \pm x_s\}$ (x_i's are commuting indeterminate) such that

$$AA^t = \sum_{i=1}^{s} (u_i x_i^2) I_n. \tag{2.32}$$

An orthogonal design of order n and type $(u_1, u_2, \ldots, u_s)$ on variables $x_1, x_2, \ldots, x_s$ is denoted by $OD(n; u_1, u_2, \ldots, u_s)$.

Example 2.19.

$$
\begin{bmatrix} a \end{bmatrix}, \quad
\begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad
\begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}, \quad
\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}
\tag{2.33}
$$

are $OD(1; 1)$, $OD(2; 1, 1)$, $OD(4; 1, 1, 1, 1)$ and $OD(4; 1, 1, 1, 1)$ respectively.

Definition 2.20. The *Radon function* ρ is defined by $\rho(n) = 8q + 2^r$ when $n = 2^k p$, where $p \in \mathbb{Z}^+$ is odd, $k = 4q + r$ and $0 \leq r < 4$.

Remark 2.21. For odd p, $\rho(2^k p)$ depends only on k. The first few values of $\rho(2^k)$ are listed in the following table:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho(2^k)$	1	2	4	8	9	10	12	16	17	18	20	24	25	26	28	32	33

Theorem 2.22. [6] *If there exists an $OD(n; a_1, \cdots, a_s)$, then $s \leq \rho(n)$.*

Theorem 2.23. *If there exists an orthogonal design of order n and type $(u_1, u_2, \cdots, u_s)$, then there exists an orthogonal design of type*

$$
\begin{aligned}
&(i)(u_1, u_2, \cdots, u_{s-1}, u_s, u_s) \text{ in order } 2n \text{ with } s+1 \text{ variables,} \\
&(ii)(u_1, u_2, \cdots, u_{s-1}, u_s, u_s, u_s) \text{ in order } 4n \text{ with } s+2 \text{ variables.}
\end{aligned}
\tag{2.34}
$$

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Proof. In each case, we replace each of the first $s-1$ variables by $x_i I_m$, where $m = 2, 4$, respectively. In cases $(i), (ii)$, the last variable is replaced by,

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \quad \begin{bmatrix} x & y & 0 & -z \\ y & -x & z & 0 \\ 0 & -z & -x & -y \\ z & 0 & -y & x \end{bmatrix}, \tag{2.35}$$

respectively. □

Lemma 2.24. *If A is an orthogonal design of order n and type $(u_1, \cdots, u_s)$ on the variables $x_1, \cdots, x_s$, then there are orthogonal designs of order n and type $(u_1, \cdots, u_i + u_j, \cdots, u_s)$ and $(u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_s)$ on the $s-1$ variables $x_1, \cdots, x_j, \cdots, x_s$.*

Proof. Set the variable $x_i = x_j$ in the first case and $x_j = 0$ in the second. □

Example 2.25.

$$\begin{bmatrix} x & y & 0 & -z \\ y & -x & z & 0 \\ 0 & -z & -x & -y \\ z & 0 & -y & x \end{bmatrix} \tag{2.36}$$

is an orthogonal design of type $(1, 1, 1)$ in order 4. We can make an orthogonal design of type $(1, 2)$ by (for example) setting $z = y$,

$$\begin{bmatrix} x & y & 0 & -y \\ y & -x & y & 0 \\ 0 & -y & -x & -y \\ y & 0 & -y & x \end{bmatrix} \tag{2.37}$$

and of type $(1, 1)$ by (for example) setting $y = 0$.

$$\begin{bmatrix} x & 0 & 0 & -z \\ 0 & -x & z & 0 \\ 0 & -z & -x & 0 \\ z & 0 & 0 & x \end{bmatrix} \tag{2.38}$$

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Lemma 2.26. *If A is an orthogonal design of order n and type $(u_1, \cdots, u_s)$ on $x_1, \cdots, x_s$, then there exists an orthogonal design of order mn and type $(u_1, \cdots, u_s)$ on $x_1, \cdots, x_s$ for any integer $m \geq 1$.*

Proof. Replace each variable x_i of A by $x_i I_m$. □

Definition 2.27. A set of matrices $\{B_1, B_2, \cdots, B_m\}$ of order n with entries in $\{0, \pm x_1, \pm x_2, \cdots, \pm x_k\}$ is said to be of *type* $(s_1, s_2, \cdots, s_k)$ and in variables $x_1, x_2, \cdots, x_k$ if it satisfies the *additive* property,

$$\sum_{i=1}^{m} B_i B_i^t = \sum_{i=1}^{k} (s_i x_i^2) I_n. \tag{2.39}$$

Theorem 1.6, shows that if A, B, C, D are four circulant matrices of order n and type $(s_1, s_2, \ldots, s_k)$ in variables $x_1, x_2, \cdots, x_k$ and R is a back diagonal matrix then there is an orthogonal design $OD(4n; s_1, s_2, \ldots, s_k)$ on $x_1, x_2, \cdots, x_k$.

Definition 2.28. An orthogonal design of type (t, t, t, t) and order $4t$ is called a *Baumert-Hall array of order t* and we denote it by $BH(t)$.

Recall from Theorem 1.11 that if there exist $T-$matrices T_1, T_2, T_3, T_4 of order n and weight q and a, b, c, d be commuting variables, then

$$\begin{aligned}
A &= aT_1 + bT_2 + cT_3 + dT_4, \\
B &= -bT_1 + aT_2 + dT_3 - cT_4, \\
C &= -cT_1 - dT_2 + aT_3 + bT_4, \\
D &= -dT_1 + cT_2 - bT_3 + aT_4
\end{aligned} \tag{2.40}$$

satisfy the additive property,

$$AA^t + BB^t + CC^t + DD^t = q(a^2 + b^2 + c^2 + d^2)I, \tag{2.41}$$

and can be used in the Goethal-Seidel array to obtain an $OD(4n; q, q, q, q)$.

Example 2.29. Let

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ T_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ T_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ T_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.42}$$

Then T_1, T_2, T_3, T_4 are $T-$matrices of order 3, so using the Cooper-Wallis theorem we can construct four circulant matrices A, B, C, D of order 3,

$$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}, \ B = \begin{bmatrix} -b & a & -d \\ -d & -b & a \\ a & -d & -b \end{bmatrix}, \ C = \begin{bmatrix} -c & d & a \\ a & -c & d \\ d & a & -c \end{bmatrix}, \ D = \begin{bmatrix} -d & -c & b \\ b & -d & -c \\ -c & b & -d \end{bmatrix}, \tag{2.43}$$

which satisfy the additive property. Plugging them into Goethal-Siedel array, we get

$$\begin{bmatrix}
a & b & c & -d & a & -b & a & d & -c & b & -c & -d \\
c & a & b & a & -b & -d & d & -c & a & -c & -d & b \\
b & c & a & -b & -d & a & -c & a & d & -d & b & -c \\
\\
d & -a & b & a & b & c & -c & b & -d & -d & -a & c \\
-a & b & d & c & a & b & b & -d & -c & -a & c & -d \\
b & d & -a & b & c & a & -d & -c & b & c & -d & -a \\
\\
-a & -d & c & c & -b & d & a & b & c & -b & -d & a \\
-d & c & -a & -b & d & c & c & a & b & a & -b & -d \\
c & -a & -d & d & c & -b & b & c & a & -d & a & -b \\
\\
-b & c & d & d & a & -c & -a & d & b & a & b & c \\
c & d & -b & a & -c & d & d & b & -a & c & a & b \\
d & -b & c & -c & d & a & b & -a & d & b & c & a
\end{bmatrix} \tag{2.44}$$

which is a Baumert-Hall array $BH(3)$.

Conjecture 2.30. *There exists an $OD(4t; t, t, t, t)$ for every positive integer t.*

Conjecture 2.30 is verified for all values of $t \leq 100$ except for $t = 97$ [11].

Theorem 2.31. *Suppose that there exists a Baumert-Hall array of order t and Williamson matrices of order w, then there is an Hadamard matrix of order $4wt$.*

Proof. Let A be a Baumert-Hall array of order t on commuting indeterminate x_1, x_2, x_3, x_4, and let X_1, X_2, X_3, X_4 be Williamson matrices of order w. Replace the variables x_i by X_i, $i = 1, 2, 3, 4$, in A and call the new matrix H. Since distinct rows in A are orthogonal, by plugging X_i, $i = 1, 2, 3, 4$, in A, the distinct rows in the new matrix are also orthogonal and we have

$$HH^t = t(X_1 X_1^t + X_2 X_2^t + X_3 X_3^t + X_4 X_4^t) \otimes I_{4t} = 4twI_{4tw}. \tag{2.45}$$

$\square$

Example 2.32. In Example 2.29, we listed a Baumert-Hall array of order 3, and in Example 2.18, we showed that there are Williamson type matrices of order 3. Plugging these Williamson type matrices into the Baumert-Hall array, we will get a Hadamard matrix of order 36.

Corollary 2.33. *If there are circulant $T-$matrices of order t and there are Williamson matrices of order w, then there is an Hadamard matrix of order $4tw$. Alternatively, if there are $OD(4t; t, t, t, t)$ and Williamson matrices of order w, there is an Hadamard matrix of order $4tw$.*

Definition 2.34. A Baumert-Hall array of order t constructed from sixteen circulant matrices is called a *Baumert-Hall-Welch* array of order t and is denoted by $BHW(t)$.

2.4 Amicable matrices

Definition 2.35. $T-$matrices T_1, T_2, T_3, T_4 are called *properly amicable* if they are amicable with the matching (T_1, T_4^t), (T_2, T_3^t) ,i.e.,

$$T_1 T_4 - T_1^t T_4^t + T_2 T_3 - T_2^t T_3^t = 0. \tag{2.46}$$

We will show in chapter 4 that properly amicable $T-$matrices exist for many orders including $2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 18, 22$, while Behbahani in [2] showed that there are no amicable $T-$matrices with the matching $(T_1, T_4), (T_2, T_3)$ of odd order.

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Recall from Definition 1.15 that $T-$matrices T_1, T_2, T_3, T_4 are called double disjoint if

$$(T_1 + T_2) * (T_3 + T_4)^t = 0, \qquad (2.47)$$

where $*$ denotes the Hadamard product.

Remark 2.36. The order of T_i's, $i = 1, \ldots, 4$, appearing in (2.47) always follows the order of T_i's, $i = 1, \ldots, 4$, mentioned in Definition 2.35.

Example 2.37. Let $T_1 = Circ(1\ 0\ 0\ 0\ 0)$, $T_2 = Circ(0\ 1\ 1\ 0\ 0)$, $T_3 = Circ(0\ 0\ 0\ 1\ -)$, $T_4 = Circ(0\ 0\ 0\ 0\ 0)$, then T_1, T_2, T_3, T_4, are double disjoint $T-$matrices of order 5.

In chapter 4, we will prove that there are no double disjoint $T-$matrices of odd order which are also properly amicable. However we will show that there are some $T-$matrices of even order which satisfy both of these properties.

Chapter 3

$T-$matrices

In this chapter, we will first introduce Golay and base sequences which are the building blocks of $T-$sequences which will be studied in detail. We conclude the chapter by studying the application of $T-$sequences to construct $T-$matrices. Our main references for this chapter are [6] and [14].

3.1 Sequences

Definition 3.1. Given a sequence $A = \{a_1, \ldots, a_n\}$ of length n, the *non-periodic auto-correlation function*, N_A, of A is defined by:

$$N_A(i) = \sum_{j=1}^{n-i} a_j a_{i+j}, \quad i = 0, 1, \ldots, n-1 \tag{3.1}$$

Consider the following matrix of order n,

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & & a_{1n} \\ 0 & a_{11} & & & a_{1,n-1} \\ \vdots & & & \ddots & \\ 0 & 0 & \ldots & 0 & a_{11} \end{bmatrix}. \tag{3.2}$$

Then $N_A(i)$ is the inner product of rows 1 and $i + 1$.

The non-periodic auto-correlation function can be defined for a family of two or

more sequences.

Definition 3.2. Let $A = \{A_1 = \{a_{11}, \cdots, a_{1n}\}, A_2 = \{a_{21}, \cdots, a_{2n}\}, \cdots, A_m = \{a_{m1}, \cdots, a_{mn}\}\}$ be m sequences of commuting variables of length n. The *non-periodic auto-correlation function for the family of sequences in A* is a function defined by

$$N_A(i) = \sum_{k=1}^{m} N_{A_k}(i). \tag{3.3}$$

Suppose $A = \{a_1, a_2, \ldots, a_m\}$ is a sequence of length m. We associate to this sequence the *Hall polynomial* $A(x) = a_1 + a_2 x + \cdots + a_m x^{m-1}$. The *norm* of A is defined by $N(A) = A(x)A(x^{-1})$. We have

$$N(A) = A(x)A(x^{-1}) = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j x^{i-j} = N_A(0) + \sum_{k=1}^{m-1} N_A(k)(x^k + x^{-k}), \quad x \neq 0. \tag{3.4}$$

Definition 3.3. For the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n, the *reverse sequence*, denoted by A^r, is defined by

$$A^r = \{a_n, a_{n-1}, \ldots, a_1\}. \tag{3.5}$$

Sequence A is called *symmetric* if $A = A^r$.

Proposition 3.4. *For the sequence $A = \{a_1, a_2, \ldots, a_n\}$ we have $A^r(x) = x^{n-1}A(x^{-1})$.*

Proof.

$$\begin{aligned} A^r(x) &= a_n + a_{n-1}x + \cdots + a_1 x^{n-1} \\ &= x^{n-1}(a_1 + \cdots + a_{n-1}x^{-(n-2)} + a_n x^{-(n-1)}) \\ &= x^{n-1}A(x^{-1}). \end{aligned} \tag{3.6}$$

$\square$

Definition 3.5. The *negated sequence* of A, denoted by $-A$, is defined by

$$-A = \{-a_1, -a_2, \ldots, -a_n\}. \tag{3.7}$$

Proposition 3.6. *Let* $A = \{a_1, a_2, \ldots, a_n\}$, *be a sequence of length n then* $N(-A) = N(A^r) = N(A)$.

Proof. Note that from Proposition 3.4 we have $A^r(x) = x^{n-1}A(x^{-1})$, so

$$
\begin{aligned}
N(A^r) &= A^r(x)A^r(x^{-1}) \\
&= x^{n-1}A(x^{-1})x^{-(n-1)}A(x) \\
&= A(x^{-1})A(x) \\
&= N(A)
\end{aligned}
\tag{3.8}
$$

and $N(-A) = (-A(x))(-A(x^{-1})) = A(x)A(x^{-1}) = N(A)$. $\qquad\square$

Definition 3.7. Given two sequences $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_{m-1}\}$, the *interleaving* of two sequences A, B, denoted by A/B is defined by

$$
A/B = \{a_1, b_1, a_2, b_2, \ldots, b_{m-1}, a_m\}.
\tag{3.9}
$$

3.1.1 Golay sequences

Marcel Golay introduced Golay sequences in his article, "Multislit Spectrometry "[7]. These sequences have found various applications in communication theory to separate signals from noises, in radar-signal theory and in surface-acoustic wave devices.

Definition 3.8. Suppose $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\}$ are two sequences where $a_i, b_j \in \{1, -1\}$ and $N_A(j) + N_B(j) = 0$ for $j = 1, \ldots, n-1$, then the sequences A, B are called *Golay sequences of lengths n*.

Let $GS(n)$ denote the set of all Golay sequences of length n. If $GS(n) \neq \emptyset$ we say that n is a *Golay number*.

Example 3.9.

- $1\,1$, $\;1\,-$
- $1\,-\,-\,1\,-\,1\,-\,-\,-\,1$, $\;1\,-\,-\,-\,-\,-\,-\,1\,1\,-$
- $1\,1\,1\,-\,-\,1\,1\,1\,-\,1\,-\,-\,-\,-\,-\,1\,-\,1\,1\,-\,-\,1\,-\,-\,-\,-\,-$,
  $-\,-\,-\,1\,1\,-\,-\,-\,1\,-\,1\,1\,-\,1\,-\,1\,-\,1\,1\,-\,-\,1\,-\,-\,-\,-\,-$

$$\tag{3.10}$$

are Golay sequences of lengths $2, 10, 26$, respectively.

Theorem 3.10. *[14] There are Golay pairs of lengths $2^a 10^b 26^c$ where a, b, c are non-negative integers.*

Conjecture 3.11. *There is a Golay pair of length n if and only if $n = 2^a 10^b 26^c$ where a, b and c are nonnegative integers.*

Remark 3.12. This conjecture is confirmed for all lengths up to 106 [11].

Some existence results on Golay sequences quoted from [14]:

- They do not exist for lengths $2 \cdot 9^c$ (c is a positive integer), or for orders $34, 36, 50, 58$ or 68.
- They do not exist for lengths $2 \cdot 49^c$ (c is a positive integer).
- They do not exist for lengths $2p$ where p has any prime factor $\equiv 3 \pmod 4$.

$$(3.11)$$

3.1.2 Base sequences

Definition 3.13. Four $\{1, -1\}$ sequences A, B, C, D of lengths $m + n, m + n, m, m$, respectively, are called *base sequences* if

$$N_A(i) + N_B(i) + N_C(i) + N_D(i) = \begin{cases} 0, & i = 1, 2, \ldots, m-1, \\ 4m + 2n, & i = 0, \end{cases} \qquad (3.12)$$

$$N_A(i) + N_B(i) = 0, \quad i = m, \ldots, m+n-1.$$

Base sequences of lengths $m + n, m + n, m, m$, are denoted by $BS(m + n, m)$.

Example 3.14. $A = 1\ 1\ 1,\ B = 1\ -\ -,\ C = 1\ -,\ D = 1\ -$ are base sequences of lengths $3, 3, 2, 2$.

Proposition 3.15. *Suppose $A(x), B(x), C(x), D(x)$ are the associated Hall polynomials of the base sequences A, B, C, D of lengths $m + n, m + n, m, m$, then we have*

$$A(x)A(x^{-1}) + B(x)B(x^{-1}) + C(x)C(x^{-1}) + D(x)D(x^{-1}) = 4m + 2n, \quad x \neq 0. \quad (3.13)$$

Proof. Since A, B, C, D are base sequences of lengths $m+n, m+n, m, m$, respectively, using definition of base sequences and equation (3.4), we have

$$
\begin{aligned}
&A(x)A(x^{-1}) + B(x)B(x^{-1}) + C(x)C(x^{-1}) + D(x)D(x^{-1}) \\
&= N_A(0) + N_B(0) + N_C(0) + N_D(0) \\
&\quad + \left(\sum_{k=1}^{m-1} (N_A(k) + N_B(k) + N_C(k) + N_D(k))(x^k + x^{-k}) \right) \\
&\quad + \left(\sum_{k=m}^{m+n-1} (N_A(k) + N_B(k))(x^k + x^{-k}) \right) \\
&= N_A(0) + N_B(0) + N_C(0) + N_D(0) = 4m + 2n.
\end{aligned}
\tag{3.14}
$$

$\square$

If we set $x = 1$ in Proposition 3.15, then we will find

$$
a^2 + b^2 + c^2 + d^2 = 4m + 2n,
\tag{3.15}
$$

where a, b, c, d are the sums of the entries of A, B, C, D, respectively.

In [12], Koukouvinos, Kounias and Sotirakoglou formed further restrictions as follows

Theorem 3.16. *[12] If* $A = \{a_1, \ldots, a_{m+1}\}, B = \{b_1, \ldots, b_{m+1}\}, C = \{c_1, \ldots, c_m\}, D = \{d_1, \ldots, d_m\}$ *are base sequences of lengths* $m+1, m+1, m, m$, *then*

$$
a_k + b_k + a_{m+2-k} + b_{m+2-k} \equiv \begin{cases} 2 \pmod 4 & k = 1, \\ 0 \pmod 4 & k = 2, \ldots, [(m+1)/2] \end{cases}
\tag{3.16}
$$

and $c_k + d_k + c_{m+1-k} + d_{m+1-k} \equiv 0 \pmod 4$.

Also in that article, they showed how one can get new base sequences from the existing base sequences.

Theorem 3.17. *[12] Suppose* A, B, C, D *are base sequences of lengths* $m+1, m+1, m, m$, *and their associated polynomials satisfy*

$$
A(x)C(x^{-1}) + xC(x)A(x^{-1}) = 0, \qquad x \neq 0.
\tag{3.17}
$$

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Then the sequences

$$X = (1, A/C), \ Y = (-1, A/C), \ U = B/D, \ V = B/-D \qquad (3.18)$$

are base sequences of lengths $2m + 2, 2m + 2, 2m + 1, 2m + 1$*, where* $(1, A)$ *means* 1 *followed by sequence* A*.*

Example 3.18. Let $A = 1\ 1\ 1$, $B = 1\ 1\ -$, $C = 1\ -$, $D = 1\ -$, then A, B, C, D, are base sequences of lengths $3, 3, 2, 2$, respectively and their associated polynomials satisfy

$$A(x)C(x^{-1}) + xC(x)A(x^{-1}) = 0, \qquad (3.19)$$

so the sequences $1\ 1\ 1\ 1\ -\ 1$, $-\ 1\ 1\ 1\ -\ 1$, $1\ 1\ 1\ -\ -$, $1\ -\ 1\ 1\ -$ are base sequences of lengths $6, 6, 5, 5$, respectively.

Theorem 3.19. *If* $A = \{a_1, \ldots, a_m\}$*,* $B = \{b_1, \ldots, b_m\}$ *are Golay sequences of lengths* m *then* $(1, A)$*,* $(-1, A)$*,* B*,* B *are base sequences of lengths* $m + 1, m + 1, m, m$ *respectively, where* $(1, A)$ *means* 1 *followed by sequence* A*.*

Proof. Let $X = (1, A)$, $Y = (-1, A)$, $Z = B$, $W = B$, and construct the Hall polynomial associated to these sequences. Then we have,

$$N(X) = X(t)X(t^{-1}) = 1 + N_A(0) + \sum_{k=1}^{m-1} N_A(k)(t^k + t^{-k}) + \sum_{k=1}^{m} a_k(t^k + t^{-k})$$

$$N(Y) = Y(t)Y(t^{-1}) = 1 + N_A(0) + \sum_{k=1}^{m-1} N_A(k)(t^k + t^{-k}) - \sum_{k=1}^{m} a_k(t^k + t^{-k})$$

$$\qquad (3.20)$$

$$N(Z) = Z(t)Z(t^{-1}) = N_B(0) + \sum_{k=1}^{m-1} N_B(k)(t^k + t^{-k})$$

$$N(W) = W(t)W(t^{-1}) = N_B(0) + \sum_{k=1}^{m-1} N_B(k)(t^k + t^{-k})$$

so we have

$$N(X) + N(Y) + N(Z) + N(W) = 2(1 + N_A(0) + N_B(0)) = 4m + 2. \qquad (3.21)$$

Hence X, Y, Z, W are base sequences of lengths $m + 1$, $m + 1$, m, m. $\qquad \square$

Corollary 3.20. *There are base sequences of lengths $m + 1, m + 1, m, m$ for all $m = 2^a 10^b 26^c$, where a, b, c are nonnegative integers.*

Conjecture 3.21 (Doković [4])**.** *There are base sequences of lengths $n + 1, n + 1, n, n$ for all nonnegative integers n.*

The existence of base sequences of lengths $n + 1, n + 1, n, n$ has been verified for all integers $n \leq 38$ (and for all Golay numbers n) [5].

3.1.3 $T-$sequences

Definition 3.22. Four $\{0, 1, -1\}$ sequences A, B, C, D of lengths m are called $T-$sequences if

$$|a_i| + |b_i| + |c_i| + |d_i| = 1, \quad i = 1, \ldots, m,$$

$$N_A(i) + N_B(i) + N_C(i) + N_D(i) = \begin{cases} 0, & i = 1, 2, \ldots, m - 1 \\ m, & i = 0. \end{cases} \quad (3.22)$$

Example 3.23. $A = 1\ 0\ 0, B = 0\ 1\ 0, C = 0\ 0\ 1$ and $D = 0\ 0\ 0$ are $T-$sequences of length 3.

Proposition 3.24. *If $A(x), B(x), C(x), D(x)$ are the associated Hall polynomials to the $T-$sequences A, B, C, D of lengths m, then we have*

$$A(x)A(x^{-1}) + B(x)B(x^{-1}) + C(x)C(x^{-1}) + D(x)D(x^{-1}) = m, \quad x \neq 0. \quad (3.23)$$

Proof. We have

$$\begin{aligned}
&A(x)A(x^{-1}) + B(x)B(x^{-1}) + C(x)C(x^{-1}) + D(x)D(x^{-1}) \\
&= N_A(0) + N_B(0) + N_C(0) + N_D(0) \\
&+ \sum_{k=1}^{m-1} (N_A(k) + N_B(k) + N_C(k) + N_D(k))(x^k + x^{-k}) \\
&= N_A(0) + N_B(0) + N_C(0) + N_D(0) = m.
\end{aligned} \quad (3.24)$$

$\square$

We can also use Golay sequences to construct $T-$sequences.

Theorem 3.25. *Suppose* $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$ *are Golay sequences of lengths* m *and* 0_m *which means* 0 *repeated* m *times. Then* $(1, 0_m)$, $(0, \frac{1}{2}(A + B))$, $(0, \frac{1}{2}(A - B))$, 0_{m+1} *are* $T-$*sequences of lengths* $m + 1$.

Proof. Let $X = (1, 0_m)$, $Y = (0, \frac{1}{2}(A + B))$, $Z = (0, \frac{1}{2}(A - B))$, $W = 0_{m+1}$, and make the associate Hall polynomials

$$N(X) = X(t)X(t^{-1}) = 1$$

$$N(Y) = Y(t)Y(t^{-1}) = \frac{1}{4}\left[\left(\sum_{k=1}^{m}(a_k + b_k)^2\right) + \left(\sum_{k=1}^{m-1} N_{A+B}(k)(t^k + t^{-k})\right)\right]$$

$$N(Z) = Z(t)Z(t^{-1}) = \frac{1}{4}\left[\left(\sum_{k=1}^{m}(a_k - b_k)^2\right) + \left(\sum_{k=1}^{m-1} N_{A-B}(k)(t^k + t^{-k})\right)\right]$$

(3.25)

$$N(W) = 0.$$

Then we have

$$N(X) + N(Y) + N(Z) + N(W)$$

$$= 1 + \frac{1}{4}\left[2(a_1^2 + \cdots + a_m^2 + b_1^2 + \cdots + b_m^2) + \sum_{k=1}^{m-1}(N_{A+B}(k) + N_{A-B}(k))(t^k + t^{-k})\right]$$

$$= 1 + \frac{1}{4}\left[4m + 2(N_A(k) + N_B(k))(t^k + t^{-k})\right]$$

$$= 1 + m.$$

(3.26)

So, X, Y, Z, W are $T-$sequences of lengths $m + 1$. $\qquad\square$

Corollary 3.26. *There are* $T-$*sequences of order* $1 + 2^a 10^b 26^c$, *where* a, b, c *are non-negative integers.*

$T-$sequences can be obtained using base sequences. The proof of the following theorem is similar to the proof of Theorem 3.25.

Theorem 3.27. *Suppose* A, B, C, D *are base sequences of lengths* $m + 1, m + 1, m, m$, *respectively. Then the sequences* $(\frac{1}{2}(A + B), 0_m)$, $(\frac{1}{2}(A - B), 0_m)$, $(0_{m+1}, \frac{1}{2}(C + D))$, $(0_{m+1}, \frac{1}{2}(C - D))$ *are* $T-$*sequences of lengths* $2m + 1$.

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Conjecture 3.28 ($T-$sequence Conjecture). *There are $T-$sequences of order n, for every odd positive integer n.*

Remark 3.29. The $T-$sequence conjecture is verified for all odd positive integers less than 100, with the exception of 97 [11].

We now present a construction method to obtain $T-$matrices from $T-$sequences.

Remark 3.30. $T-$sequences give the first rows of circulant $T-$matrices. If $\{a_{11}, \ldots, a_{1t}\}$, $\{a_{21}, \ldots, a_{2t}\}$, $\{a_{31}, \ldots, a_{3t}\}$, $\{a_{41}, \ldots, a_{4t}\}$ are $T-$sequences of lengths t, then the matrices $T_1 = Circ(a_{11} \ldots a_{1t})$, $T_2 = Circ(a_{21} \ldots a_{2t})$, $T_3 = Circ(a_{31} \ldots a_{3t})$, $T_4 = Circ(a_{41} \ldots a_{4t})$ are $T-$matrices of order t. However, there are circulant $T-$matrices that do not correspond to $T-$sequences in this way.

$T-$matrices are known for all orders $t \leq 200$ with the exception of $t = 97, 103, 109, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 183, 191, 193, 197, 199$, see [11].

Chapter 4

A multiplication theorem on $T-$matrices

In the previous chapter, we studied some methods to construct $T-$sequences and how to obtain $T-$matrices corresponding to these $T-$sequences. There are some other approaches to get new $T-$matrices. In [18], Yang proposed a pioneering approach which is now known as Yang numbers in order to get new $T-$sequences from the existing base sequences. In this chapter, we will study a multiplication theorem to get new $T-$matrices from the existing $T-$matrices.

4.1 Amicable $T-$matrices

Recall from Definition 1.14 that $T-$matrices T_1, T_2, T_3, T_4 are called amicable with the matching (T_1, T_4), (T_2, T_3) if

$$T_1 T_4^t - T_4 T_1^t + T_2 T_3^t - T_3 T_2^t = 0. \tag{4.1}$$

Behbahani [2] proved that amicable $T-$matrices with such a matching of odd order does not exist. However, there are some amicable $T-$matrices with this matching of even order. In Table 4.1 we provide a list of these matrices.

Table 4.1: First rows of circulant amicable $T-$matrices T_1, T_2, T_3, T_4 with the matching (T_1, T_4) , (T_2, T_3)

$t = 2$	0 0
	0 1
	0 0
	1 0

$t = 6$	0 0 0 0 0 0
	0 0 0 0 0 −
	0 1 0 1 0 0
	1 0 1 0 − 0

$t = 10$	0 0 0 0 0 0 0 0 0 0
	0 0 0 0 0 0 0 0 1 0
	0 − 0 1 0 − 0 − 0 −
	1 0 − 0 − 0 1 0 0 0

$t = 14$	0 0 0 0 0 0 0 0 0 0 0 0 0 0
	0 0 0 0 0 0 0 0 0 0 0 − 0 0
	0 1 0 1 0 1 0 1 0 − 0 0 0 −
	1 0 − 0 1 0 1 0 1 0 − 0 1 0

$t = 18$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
	0 0 0 0 0 0 0 0 0 0 0 0 0 − 0 0 0 0
	0 1 0 1 0 1 0 1 0 − 0 1 0 0 0 1 0 −
	− 0 − 0 − 0 1 0 1 0 − 0 1 0 1 0 − 0

The existence of these amicable $T-$matrices inspired us to conjecture that they might exist for every even positive integer.

Conjecture 4.1. *There are amicable $T-$matrices with matching (T_1, T_4), (T_2, T_3) of order t, for every even number t.*

Amicable $T-$matrices with the matching (T_1, T_4^t) , (T_2, T_3^t) are useful in constructing new $T-$matrices and we call them proper amicable $T-$matrices. They exist for some odd integers.

In Table 4.2, we list a number of properly amicable $T-$matrices of orders $3, 5, 7, 9, 11, 13$.

Table 4.2: First rows of circulant properly amicable $T-$matrices T_1, T_2, T_3, T_4

$t = 3$		
1	0	0
0	−	0
0	0	1
0	0	0

$t = 5$				
1	1	0	0	0
0	0	0	0	0
0	0	0	1	−
0	0	1	0	0

$t = 7$						
0	0	0	0	0	0	−
0	1	0	0	1	0	0
0	0	0	1	0	0	0
1	0	1	0	0	−	0

$t = 9$								
0	−	0	0	0	1	−	1	1
0	0	0	−	−	0	0	0	0
1	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

$t = 11$										
0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	0	0	0	1	0	0	0
0	0	1	0	1	−	−	0	0	0	0
0	1	0	0	0	0	0	0	−	0	−

$t = 13$												
0	0	0	0	0	0	1	0	0	0	1	−	−
0	−	−	0	−	1	0	0	0	−	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	−	0	0	0	0

4.2 Double disjoint $T-$matrices

Recall from Definition 1.15 that $T-$matrices T_1, T_2, T_3, T_4 are double disjoint if

$$(T_1 + T_2) * (T_3 + T_4)^t = 0, \tag{4.2}$$

where $*$ denotes the Hadamard product.

In contrast to the difficult task of finding properly amicable $T-$matrices of odd order, it is easy to construct infinite classes of double disjoint $T-$matrices.

Lemma 4.2. *Let A and B be Golay sequences of lengths m. Then the $T-$matrices $T_1 = Circ(1\ 0_{2m})$, $T_2 = Circ(0_{2m+1})$, $T_3 = Circ(0\ 0_m\ B)$, $T_4 = Circ(0\ A\ 0_m)$ form double disjoint $T-$matrices of order $2m + 1$.*

Proof. Note that $T_3 + T_4 = Circ(0\ A\ B)$, so $(T_3 + T_4)^t = Circ(0, B^r, A^r)$ which is a $\{0, \pm 1\}$ matrix and is disjoint from $T_1 + T_2 = Circ(1\ 0_{2m})$. $\qquad\square$

We developed this idea independently, however it turns out that in [19], they proposed this method in an unpublished paper and they called the $T-$matrices T_1, T_2, T_3, T_4 satisfying the relation $(T_1 + T_2) * (T_3 + T_4)^t = 0$, *strongly disjoint $T-$matrices.* In this thesis, we will call them double disjoint as mentioned before.

4.3 Constructing new $T-$matrices

In this section, we will construct new $T-$matrices from the existing $T-$matrices that satisfy certain properties. We developed this idea independently, although we found out that Xia already introduced this idea in an unpublished paper [19] before us. To make this method clear, we first introduce our approach and then we will present their $T-$matrices which are slightly different than our matrices.

Suppose A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are $T-$matrices of orders s and t, respec-

tively. Consider the following matrices

$$C_1 = A_1 \otimes B_1 + A_2 \otimes B_2 + A_3 \otimes B_3 + A_4 \otimes B_4,$$
$$C_2 = A_2 \otimes B_1 - A_1 \otimes B_2 + A_4 \otimes B_3 - A_3 \otimes B_4,$$
$$C_3 = A_3 \otimes B_1 - A_4 \otimes B_2 - A_1 \otimes B_3 + A_2 \otimes B_4,$$
$$C_4 = A_4 \otimes B_1 + A_3 \otimes B_2 - A_2 \otimes B_3 - A_1 \otimes B_4. \tag{4.3}$$

It is easy to see that $C_i, i = 1, \ldots 4$, are disjoint matrices and we have

$$
\begin{aligned}
\sum_{i=1}^{4} C_i C_i^t =& (A_1 A_2^t - A_2 A_1^t - A_3 A_4^t + A_4 A_3^t) \otimes (B_1 B_2^t - B_2 B_1^t - B_3 B_4^t + B_4 B_3^t) \\
&+ (A_1 A_3^t - A_2 A_4^t - A_3 A_1^t + A_4 A_2^t) \otimes (B_1 B_3^t + B_2 B_4^t - B_3 B_1^t - B_4 B_2^t) \\
&+ (A_1 A_4^t - A_2 A_3^t + A_3 A_2^t - A_4 A_1^t) \otimes (B_1 B_4^t - B_2 B_3^t + B_3 B_2^t - B_4 B_1^t) \\
&+ st I_{st}.
\end{aligned}
\tag{4.4}
$$

As we mentioned before, Behbahani in [2] proved that none of the single factors are zero and it seems it is a hard job to find matrices that makes this combination equal to zero. So we devised the following approach.

Change A_2 to A_2^t and A_4 to A_4^t in C_1 and modify C_2, C_3, C_4 in a way to preserve the disjointness property. After applying the necessary changes we get the following matrices:

$$C_1 = A_1 \otimes B_1 + A_2^t \otimes B_2 + A_3 \otimes B_3 + A_4^t \otimes B_4,$$
$$C_2 = A_2 \otimes B_1 - A_1^t \otimes B_2 + A_4 \otimes B_3 - A_3^t \otimes B_4,$$
$$C_3 = A_3 \otimes B_1 - A_4^t \otimes B_2 - A_1 \otimes B_3 + A_2^t \otimes B_4,$$
$$C_4 = A_4 \otimes B_1 + A_3^t \otimes B_2 - A_2 \otimes B_3 - A_1^t \otimes B_4. \tag{4.5}$$

It is not hard to see that C_1, C_2, C_3, C_4 are disjoint and

$$
\begin{aligned}
\sum_{i=1}^{4} C_i C_i^t =& (A_1 A_3^t + A_2 A_4^t - A_3 A_1^t - A_4 A_2^t) \otimes (B_1 B_3^t - B_2 B_4^t - B_3 B_1^t + B_4 B_2^t) \\
&+ st I_{st}.
\end{aligned}
\tag{4.6}
$$

Now, similar to what we did before, modify B_3 to B_3^t and B_4 to B_4^t in C_1 and apply the necessary changes to make C_i, $i = 1, \ldots, 4$, disjoint. However, at this stage, we need to impose extra conditions on A_i to establish the disjointness property, namely,

$$
\begin{aligned}
A_1 * A_4^t = 0, \quad A_1 * A_3^t = 0, \\
A_2 * A_4^t = 0, \quad A_2 * A_3^t = 0,
\end{aligned}
\tag{4.7}
$$

which could be abbreviated as $(A_1 + A_2) * (A_3 + A_4)^t = 0$. Now, the matrices

$$
\begin{aligned}
C_1 &= A_1 \otimes B_1 + A_2^t \otimes B_2 + A_3 \otimes B_3^t + A_4^t \otimes B_4^t, \\
C_2 &= A_2 \otimes B_1 - A_1^t \otimes B_2 + A_4 \otimes B_3^t - A_3^t \otimes B_4^t, \\
C_3 &= A_3 \otimes B_1^t - A_4^t \otimes B_2^t - A_1 \otimes B_3 + A_2^t \otimes B_4, \\
C_4 &= A_4 \otimes B_1^t + A_3^t \otimes B_2^t - A_2 \otimes B_3 - A_1^t \otimes B_4
\end{aligned}
\tag{4.8}
$$

are disjoint and we have

$$
\begin{aligned}
\sum_{i=1}^{4} C_i C_i^t &= (A_1 A_4 - A_2 A_3 - A_1^t A_4^t + A_2^t A_3^t) \otimes (B_1 B_4 + B_2 B_3 - B_1^t B_4^t - B_2^t B_3^t) \\
&\quad + st I_{st}.
\end{aligned}
\tag{4.9}
$$

Changing B_2 to $-B_2$ in all matrices, we obtain the following matrices:

$$
\begin{aligned}
C_1 &= A_1 \otimes B_1 - A_2^t \otimes B_2 + A_3 \otimes B_3^t + A_4^t \otimes B_4^t, \\
C_2 &= A_2 \otimes B_1 + A_1^t \otimes B_2 + A_4 \otimes B_3^t - A_3^t \otimes B_4^t, \\
C_3 &= A_3 \otimes B_1^t + A_4^t \otimes B_2^t - A_1 \otimes B_3 + A_2^t \otimes B_4, \\
C_4 &= A_4 \otimes B_1^t - A_3^t \otimes B_2^t - A_2 \otimes B_3 - A_1^t \otimes B_4.
\end{aligned}
\tag{4.10}
$$

Since A_i and B_i $i = 1, \ldots, 4$, were $T-$matrices, so C_i , $i = 1, \ldots, 4$, are $\{0, \pm 1\}$, type

1 and disjoint matrices that satisfy:

$$\sum_{i=1}^{4} C_i C_i^t = (A_1 A_4 - A_2 A_3 - A_1^t A_4^t + A_2^t A_3^t) \otimes (B_1 B_4 - B_2 B_3 - B_1^t B_4^t + B_2^t B_3^t)$$
$$+ st I_{st}.$$
(4.11)

Provided that B_i $i = 1, \ldots, 4$ are properly amicable $T-$matrices, then C_1, C_2, C_3, C_4 are $T-$matrices of order st. This establishes the following result.

Theorem 4.3 (Multiplication Theorem). *Let A_1, A_2, A_3, A_4 be double disjoint $T-$matrices of order s and B_1, B_2, B_3, B_4 be properly amicable $T-$matrices of order t, then the following matrices are $T-$matrices of order st :*

$$C_1 = A_1 \otimes B_1 - A_2^t \otimes B_2 + A_3 \otimes B_3^t + A_4^t \otimes B_4^t,$$
$$C_2 = A_2 \otimes B_1 + A_1^t \otimes B_2 + A_4 \otimes B_3^t - A_3^t \otimes B_4^t,$$
$$C_3 = A_3 \otimes B_1^t + A_4^t \otimes B_2^t - A_1 \otimes B_3 + A_2^t \otimes B_4,$$
$$C_4 = A_4 \otimes B_1^t - A_3^t \otimes B_2^t - A_2 \otimes B_3 - A_1^t \otimes B_4.$$
(4.12)

Theorem 4.4 (Main Result). *There are $T-$matrices of order $(2m + 1)t$, where m is the length of Golay sequences and $t \in \{3, 5, 7, 9, 11, 13\}$.*

Proof. By a computer search we were able to find properly amicable $T-$matrices of order t, $t \in \{3, 5, 7, 9, 11, 13\}$ which is shown in Table 4.2. Lemma 4.2 shows that the existence of Golay sequences of length m implies the existence of double disjoint $T-$matrices of order $2m+1$. So by applying Theorem 4.3 we have $T-$matrices of order $(2m + 1)t$. □

Corollary 4.5. *There are Hadamard matrices of order $4t(2m + 1)k$, where t is the order of properly amicable $T-$matrices, m is the length of Golay sequence and k the order of Williamson matrices.*

Proof. It follows from Corollary 4.4 and the Cooper-Wallis theorem, Theorem 1.11, that there are $OD(4t(2m + 1); t(2m + 1), t(2m + 1), t(2m + 1), t(2m + 1))$. Replacing the variables by Williamson matrices of order k, we get the Hadamard matrices of order $4t(2m + 1)k$. □

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Throughout this thesis, if we refer to $T-$matrices of multiplicative order, we mean $T-$matrices that we developed in the previous theorem. Next we will present Xia's $T-$matrices.

Theorem 4.6 (Xia, Xia, Zuo). *[19] Suppose A_1, A_2, A_3, A_4 are double disjoint $T-$matrices of order s and B_1, B_2, B_3, B_4 are properly amicable $T-$matrices of order t. Then*

$$\begin{aligned}
C_1 &= A_1 \otimes B_1 + A_2^t \otimes B_2 + A_3 \otimes B_3^t - A_4^t \otimes B_4^t, \\
C_2 &= A_1 \otimes B_3 + A_2^t \otimes B_4 - A_3 \otimes B_1^t + A_4^t \otimes B_2^t, \\
C_3 &= A_2 \otimes B_1^t - A_1^t \otimes B_2 + A_4 \otimes B_3^t + A_3^t \otimes B_4^t, \\
C_4 &= A_2 \otimes B_3 - A_1^t \otimes B_4 - A_4 \otimes B_1^t - A_3^t \otimes B_2^t
\end{aligned} \tag{4.13}$$

are $T-$matrices of order st.

We now prove that it is not possible to impose both conditions on a set of $T-$matrices of odd order.

Theorem 4.7. *There are no properly amicable $T-$matrices of odd order that satisfy the double disjointness property.*

Proof. Suppose there are $T-$matrices T_1, T_2, T_3, T_4 that satisfy both conditions. Consider these matrices module 2, i.e., $A \equiv -A(\mathrm{mod}\, 2)$. Let $U = T_1 + T_4^t$ and $V = T_2 + T_3^t$. Then, using the complementary and amicability conditions, we have

$$\begin{aligned}
UU^t + VV^t &= (T_1 + T_4^t)(T_1^t + T_4) + (T_2 + T_3^t)(T_2^t + T_3) \\
&= T_1 T_1^t + T_2 T_2^t + T_3 T_3^t + T_4 T_4^t \\
&\quad + T_1 T_4 + T_1^t T_4^t + T_2 T_3 + T_2^t T_3^t \\
&\equiv I \qquad (\mathrm{mod}\, 2).
\end{aligned} \tag{4.14}$$

Also, since $(T_1 + T_2) * (T_3 + T_4)^t = 0$, we have $U + V \equiv J \,(\mathrm{mod}\, 2)$, so $V \equiv U + J \,(\mathrm{mod}\, 2)$, and we get

$$I \equiv UU^t + VV^t \equiv UU^t + UU^t + UJ^t + JU^t + JJ^t \equiv J \qquad (\mathrm{mod}\, 2). \tag{4.15}$$

(Note that for a type 1 matrix A, we have $AJ^t = JA^t$.) This contradiction proves that there are no $T-$matrices of odd order satisfying both conditions. $\square$

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Table 4.3: Circulant $T-$matrices T_1, T_2, T_3, T_4 with both conditions for respectively first rows.

t																						
$t=2$	0	1																				
	0	0																				
	0	0																				
	1	0																				
$t=6$	0	0	0	0	0	0																
	0	1	0	1	0	−																
	0	0	0	0	1	0																
	1	0	1	0	0	0																
$t=10$	0	0	0	0	0	0	0	0	0	0												
	0	−	0	−	0	−	0	1	0	−												
	0	0	0	0	0	0	0	0	1	0												
	−	0	1	0	1	0	−	0	0	0												
$t=14$	0	0	0	0	0	0	0	0	0	0	0	0	0	0								
	0	1	0	1	0	−	0	1	0	1	0	1	0	−								
	0	0	0	0	0	0	0	0	0	0	0	0	1	0								
	−	0	1	0	1	0	1	0	1	0	−	0	0	0								
$t=18$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
	0	1	0	1	0	1	0	−	0	−	0	1	0	−	0	−	0	1				
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0				
	1	0	−	0	1	0	1	0	1	0	1	0	−	0	1	0	0	0				
$t=22$	1	0	0	0	0	1	0	1	1	1	−	−	1	−	−	1	−	1	−	−	−	1
	0	−	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

4.4   A family of new $T-$matrices

Unlike the odd order case, it seems that there are plenty of $T-$matrices of even order satisfying both double disjoint and proper amicability properties. Using these matrices, we can generate an infinite class of new $T-$matrices.

Proposition 4.8. *There are properly amicable $T-$matrices of order t, for $t = 2, 6, 10,$ 14, 18, 22 satisfying the double disjoint property.*

Proof. Consider circulant $T-$matrices with the first row given in Table 4.3. $\qquad\square$

Considering matrices defined in Theorem 4.3, it is easy to see that

$$
\begin{aligned}
C_1 C_4 - C_2 C_3 - C_1^t C_4^t + C_2^t C_3^t = {} & t(A_1 A_4 - A_2 A_3 - A_1^t A_4^t + A_2^t A_3^t) \otimes I_t \\
& - sI_s \otimes (B_1 B_4 - B_2 B_3 - B_1^t B_4^t + B_2^t B_3^t).
\end{aligned}
\tag{4.16}
$$

Now if $T-$matrices A_1, A_2, A_3, A_4 are both properly amicable and double disjoint, and B_1, B_2, B_3, B_4 are properly amicable $T-$matrices then we can generate an infinite number of new $T-$matrices, namely, since there are $T-$matrices of order 2, listed in

Table 4.3 that satisfy both conditions and there are properly amicable $T-$matrices of order 3 listed in Table 4.2, we can generate properly amicable $T-$matrices of order 6, and subsequently, we can generate $T-$matrices of orders $12, 24, \ldots$.

Proposition 4.9. *Suppose there exist $T-$matrices A_1, A_2, A_3, A_4 of order s that are both double disjoint and properly amicable and there exist properly amicable $T-$matrices B_1, B_2, B_3, B_4 of order t, then there exist $T-$matrices of order $s^a t$, for every non-negative integer a.*

Proof. Let A_1, A_2, A_3, A_4 be properly amicable $T-$matrices of order s which are also double disjoint and B_1, B_2, B_3, B_4 be properly amicable $T-$matrices of order t. Consider $T-$matrices, C_1, C_2, C_3, C_4 obtained from Theorem 4.3. Using equation (4.16) it is easy to see that C_1, C_2, C_3, C_4 are properly amicable $T-$matrices of order st. Repeating this construction proves the proposition. $\qquad \square$

Chapter 5

Applications of $T-$matrices

In this chapter, we will study the application of $T-$matrices to construct new orthogonal designs and complex Hadamard matrices.

5.1 Constructing orthogonal designs using amicable $T-$matrices

Theorem 5.1. *Let T_1, T_2, T_3, T_4 be $T-$matrices of order n, and let a, b, c, d be commuting indeterminates. Consider the four circulant matrices A, B, C, D obtained from the Cooper-Wallis Theorem 1.11, then the four matrices A, B, C, D are amicable with the matching $(A, B), (C, D)$ if and only if T_1, T_2, T_3, T_4 are amicable with the matching $(T_1, T_2), (T_3, T_4)$.*

Proof. It can be seen that

$$AB^t - BA^t + CD^t - DC^t = (a^2 + b^2 + c^2 + d^2)(T_1 T_2^t - T_2 T_1^t + T_3 T_4^t - T_4 T_3^t). \quad (5.1)$$

□

And thus, we have the following result:

Theorem 5.2. *Suppose T_1, T_2, T_3, T_4 are amicable $T-$matrices of order n with the matching $(T_1, T_2), (T_3, T_4)$, and construct the circulant matrices A, B, C, D in Theorem 5.1. Consider the following matrix constructed from 16 circulant matrices,*

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C^t & D^t & A^t & -B^t \\ -D^t & -C^t & B^t & A^t \end{bmatrix}. \tag{5.2}$$

Then H is an $OD(4n; n, n, n, n)$ constructed from 16 circulant matrices and so H is a $BHW(n)$.

Theorem 5.3. *Consider the Baumert-Hall-Welch array in Theorem 5.2 of order n, constructed from sixteen $n \times n$ circulant matrices in commuting variables a, b, c, d. Further, suppose there are $T-$matrices of order t, then there is a Baumert-Hall array of order nt.*

Proof. It is easy to verify that

$$AA^t + BB^t + CC^t + DD^t = n(a^2 + b^2 + c^2 + d^2)I_n, \tag{5.3}$$

and the distinct rows in H are orthogonal.

Let M_1, M_2, M_3, M_4 be $T-$matrices of order t and form the matrices,

$$\begin{aligned} P &= M_1 \otimes A - M_2 \otimes B - M_3 \otimes C^t - M_4 \otimes D^t, \\ Q &= M_1 \otimes B + M_2 \otimes A + M_3 \otimes D^t - M_4 \otimes C^t, \\ R &= M_1 \otimes C - M_2 \otimes D + M_3 \otimes A^t + M_4 \otimes B^t, \\ S &= M_1 \otimes D + M_2 \otimes C - M_3 \otimes B^t + M_4 \otimes A^t. \end{aligned} \tag{5.4}$$

Then we have,

$$\begin{aligned} PP^t &+ QQ^t + RR^t + SS^t \\ &= (M_1 M_1^t + M_2 M_2^t + M_3 M_3^t + M_4 M_4^t) \otimes (AA^t + BB^t + CC^t + DD^t) \\ &\quad + (M_2 M_1^t - M_1 M_2^t + M_4 M_3^t - M_3 M_4^t) \otimes (AB^t - BA^t + CD^t - DC^t) \\ &= (M_1 M_1^t + M_2 M_2^t + M_3 M_3^t + M_4 M_4^t) \otimes (AA^t + BB^t + CC^t + DD^t) \\ &= nt(a^2 + b^2 + c^2 + d^2)I_{nt}. \end{aligned} \tag{5.5}$$

Since P, Q, R, S are type 1 matrices, they can be plugged into Goethal-Siedel array to

obtain an $OD(4nt; nt, nt, nt, nt)$. □

Example 5.4. Let $T_1 = Circ(0\,0\,0\,0\,0\,0)$, $T_2 = Circ(0\,1\,0\,1\,0\,0)$, $T_3 = Circ(0\,0\,0\,0\,0\,1)$, $T_4 = Circ(1\,0\,-0\,1\,0)$, then T_1, T_2, T_3, T_4 are amicable $T-$matrices with the matching (T_1, T_2), (T_3, T_4). Hence, by Theorem 5.2, the following matrix is a $BHW(6)$. Note that in this matrix $-a$ is denoted by $\bar{a}$, etc.

$$
\begin{bmatrix}
d & b & \bar{d} & b & d & c & c & a & \bar{c} & a & c & \bar{d} & \bar{b} & d & b & d & \bar{b} & a & a & \bar{c} & \bar{a} & \bar{c} & a & b \\
c & d & b & \bar{d} & b & d & \bar{d} & c & a & \bar{c} & a & c & a & \bar{b} & d & b & d & \bar{b} & b & a & \bar{c} & \bar{a} & \bar{c} & a \\
d & c & d & b & \bar{d} & b & c & \bar{d} & c & a & \bar{c} & a & \bar{b} & a & \bar{b} & d & b & d & a & b & a & \bar{c} & \bar{a} & \bar{c} \\
b & d & c & d & b & \bar{d} & a & c & \bar{d} & c & a & \bar{c} & d & \bar{b} & a & \bar{b} & d & b & \bar{c} & a & b & a & \bar{c} & \bar{a} \\
\bar{d} & b & d & c & d & b & \bar{c} & a & c & \bar{d} & c & a & b & d & \bar{b} & a & \bar{b} & d & \bar{a} & \bar{c} & a & b & a & \bar{c} \\
b & \bar{d} & b & d & c & d & a & \bar{c} & a & c & \bar{d} & c & d & b & d & \bar{b} & a & \bar{b} & \bar{c} & \bar{a} & \bar{c} & a & b & a \\
\bar{c} & \bar{a} & c & \bar{a} & \bar{c} & d & d & b & \bar{d} & b & d & c & \bar{a} & c & a & c & \bar{a} & \bar{b} & \bar{b} & d & b & d & \bar{b} & a \\
d & \bar{c} & \bar{a} & c & \bar{a} & \bar{c} & c & d & b & \bar{d} & b & d & \bar{b} & \bar{a} & c & a & c & \bar{a} & a & \bar{b} & d & b & d & \bar{b} \\
\bar{c} & d & \bar{c} & \bar{a} & c & \bar{a} & d & c & d & b & \bar{d} & b & \bar{a} & \bar{b} & \bar{a} & c & a & c & \bar{b} & a & \bar{b} & d & b & d \\
\bar{a} & \bar{c} & d & \bar{c} & \bar{a} & c & b & d & c & d & b & \bar{d} & c & \bar{a} & \bar{b} & \bar{a} & c & a & d & \bar{b} & a & \bar{b} & d & b \\
c & \bar{a} & \bar{c} & d & \bar{c} & \bar{a} & \bar{d} & b & d & c & d & b & a & c & \bar{a} & \bar{b} & \bar{a} & c & b & d & \bar{b} & a & \bar{b} & d \\
\bar{a} & c & \bar{a} & \bar{c} & d & \bar{c} & b & \bar{d} & b & d & c & d & c & a & c & \bar{a} & \bar{b} & \bar{a} & d & b & d & \bar{b} & a & \bar{b} \\
b & \bar{a} & b & \bar{d} & \bar{b} & \bar{d} & a & b & a & \bar{c} & \bar{a} & \bar{c} & d & c & d & b & \bar{d} & b & \bar{c} & d & \bar{c} & \bar{a} & c & \bar{a} \\
\bar{d} & b & \bar{a} & b & \bar{d} & \bar{b} & \bar{c} & a & b & a & \bar{c} & \bar{a} & b & d & c & d & b & \bar{d} & \bar{a} & \bar{c} & d & \bar{c} & \bar{a} & c \\
\bar{b} & \bar{d} & b & \bar{a} & b & \bar{d} & \bar{a} & \bar{c} & a & b & a & \bar{c} & \bar{d} & b & d & c & d & b & c & \bar{a} & \bar{c} & d & \bar{c} & \bar{a} \\
\bar{d} & \bar{b} & \bar{d} & b & \bar{a} & b & \bar{c} & \bar{a} & \bar{c} & a & b & a & b & \bar{d} & b & d & c & d & \bar{a} & c & \bar{a} & \bar{c} & d & \bar{c} \\
b & \bar{d} & \bar{b} & \bar{d} & b & \bar{a} & a & \bar{c} & \bar{a} & \bar{c} & a & b & d & b & \bar{d} & b & d & c & \bar{c} & \bar{a} & c & \bar{a} & \bar{c} & d \\
\bar{a} & b & \bar{d} & \bar{b} & \bar{d} & b & b & a & \bar{c} & \bar{a} & \bar{c} & a & c & d & b & \bar{d} & b & d & d & \bar{c} & \bar{a} & c & \bar{a} & \bar{c} \\
\bar{a} & \bar{b} & \bar{a} & c & a & c & b & \bar{a} & b & \bar{d} & \bar{b} & \bar{d} & c & \bar{d} & c & a & \bar{c} & a & d & c & d & b & \bar{d} & b \\
c & \bar{a} & \bar{b} & \bar{a} & c & a & \bar{d} & b & \bar{a} & b & \bar{d} & \bar{b} & a & c & \bar{d} & c & a & \bar{c} & b & d & c & d & b & \bar{d} \\
a & c & \bar{a} & \bar{b} & \bar{a} & c & b & \bar{d} & b & \bar{a} & b & \bar{d} & \bar{c} & a & c & \bar{d} & c & a & \bar{d} & b & d & c & d & b \\
c & a & c & \bar{a} & \bar{b} & \bar{a} & \bar{d} & \bar{b} & \bar{d} & b & \bar{a} & b & a & \bar{c} & a & c & \bar{d} & c & b & \bar{d} & b & d & c & d \\
\bar{a} & c & a & c & \bar{a} & \bar{b} & b & \bar{d} & \bar{b} & \bar{d} & b & \bar{a} & c & a & \bar{c} & a & c & \bar{d} & d & b & \bar{d} & b & d & c \\
\bar{b} & \bar{a} & c & a & c & \bar{a} & \bar{a} & b & \bar{d} & \bar{b} & \bar{d} & b & \bar{d} & c & a & \bar{c} & a & c & c & d & b & \bar{d} & b & d \\
\end{bmatrix}
$$

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5.2 Constructing complex Hadamard matrices Using $T-$matrices

Recall that $T-$matrices T_1, T_2, T_3, T_4 are called properly amicable if they satisfy the following equation

$$T_1 T_4 - T_2 T_3 - T_1^t T_4^t + T_2^t T_3^t = 0. \tag{5.6}$$

and they are called double disjoint if $(T_1 + T_2) * (T_3 + T_4)^t = 0$, where $*$ is the Hadamard product.

Theorem 5.5. *If there are $T-$matrices of order n which are both properly amicable and double disjoint then there is a complex Hadamard matrix of order $2n$.*

Proof. Let T_1, T_2, T_3, T_4 be properly amicable $T-$matrices of order n that satisfy the double disjointness property. By forming the following two complex matrices,

$$\begin{aligned} A &= (T_1 + T_2) + i(T_4 - T_3)^t \\ B &= (T_1 - T_2) + i(T_4 + T_3)^t \end{aligned} \tag{5.7}$$

we have

$$AA^* + BB^* = 2nI_n, \tag{5.8}$$

where A^* is the Hermitian transpose of matrix A. Hence the matrix H, defined by

$$H = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix} \tag{5.9}$$

is a complex Hadamard matrix of order $2n$. $\qquad \square$

Corollary 5.6. *There are complex Hadamard matrices of orders $4, 12, 20, 28, 36, 44$ which are constructible from two circulant matrices.*

Proof. In table 4.3 we showed that there are $T-$matrices of orders $2, 6, 10, 14, 18, 22$ that are both properly amicable and double disjoint. Using Theorem 5.5 we could generate complex Hadamard matrices of orders $4, 12, 20, 28, 36, 44$. $\qquad \square$

Conjecture 5.7 (Turyn)**.** *There are complex Hadamard matrices of order $2n$, for every positive integer n.*

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