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ASSOCIATION SCHEMES AND MUTUALLY UNBIASED HADAMARD MATRICES

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Abstract

This thesis is an examination of the theory of association schemes. A part of this thesis focuses on the relation between association schemes and different combinatorial objects. Special attention is paid to the notion of Bose-Mesner algebra of an association scheme, which leads us to eigenmatrices, Krein matrices, and intersection matrices. It is shown that if an association scheme is imprimitive, it can be used to generate a quotient association scheme. Different constructions are used to generate association schemes. This thesis explains how to generate a new class of association schemes from mutually unbiased Bush-type Hadamard matrices (abbreviated as MUBH). This class of association schemes leads to an upper bound on the number of mutually unbiased Bush-type Hadamard matrices. Lastly, the existence of this class of association schemes results in the existence of sets of MUBH.

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Chapter 1

Introduction

1.1 Background

This thesis is mainly concerned with association schemes. The concept of association schemes was originally introduced by Bose and Shimamoto [4] in the design of statistical experiments. The theory of association schemes arose in statistics, in the study of group actions, in algebraic graph theory, and in algebraic coding theory, and has seen used in knot theory and numerical integration.

The algebraic theory of the association scheme constitutes a special case of the coherent configuration introduced by Higman [9]. The case of association schemes we are focusing on is called symmetric association schemes, which results in the commutative case. This case has drawn much attention in the last few decades [1, 5, 20, 21].

We studied the concept of association schemes and analysed how this notion is related to distance-regular graphs, symmetric block designs, Hadamard matrices, and Bush-type Hadamard matrices. Holzmann and Kharaghani have shown that Hadamard matrices and Bush-type Hadamard matrices can be used to generate association schemes [11].

As mentioned in [23], Hadamard matrices of order 12 and 20 were found in 1893 [8]. Prior to that in a seminal paper [25] published in 1857, such matrices are found for all orders that are powers of two. Hadamard [8] proved that matrices with entries ± 1 and maximal determinant could exist only for orders 1, 2, and $4t$.

The concept of Bush-type Hadamard matrices is an specific case of regular Hadamard matrices [22]. Bush-type Hadamard matrices have various applications in different areas. Kharaghani and his co-workers [13, 14, 15, 17] showed that these matrices can be used to construct symmetric designs and strongly regular graphs.

There is an interesting combinatorial application for normalized Hadamard matrices of order $2n$ to construct Bush-type Hadamard matrices of order $4n^2$, which is introduced for the first time by Kharaghani [16]. It is open question whether such matrices of order $4n^2$ exist for odd integer n greater than one.

The method introduced by this thesis is to construct 5-class association schemes using mutually unbiased Bush-type Hadamard matrices. Upgrading these 5-class association schemes by evaluating Krein parameters in [18], Kharaghani, Sasani and Suda proved an upper bound for the number of mutually unbiased Bush-type Hadamard matrices of order $4n^2$; they also generated a set of mutually unbiased Bush-type Hadamard matrices from the 5-class association schemes [18].

1.2 Outline

Here we discuss briefly what follows in the succeeding chapters. Section 2.1 is comprised of an overview of the definitions and concepts required for our field of study. In Section 2.2, we describe three definitions of the concept of association schemes that are helpful in providing different aspects of the subject under consideration.

In Section 2.3, we describe how Mathon [21] used symmetric block designs to produce association schemes, and in Section 2.4, we explain how he used Latin squares to generate 3-class association schemes [21].

There is a basis of idempotents for the Bose-Mesner algebra of an association scheme, which is reviewed in Section 3.1. Then, we define the first and second eigenmatrices of an association scheme in Section 3.2. Intersection matrices and Krein matrices of an association scheme appear in Section 3.3.

There are two interesting association schemes called imprimitive association schemes and quotient association schemes. They are related to each other and have crucial properties. We examine them in Sections 3.4 and 3.5, respectively.

We briefly review Hadamard matrices in Section 4.1. In Section 4.2, we define mutually unbiased Hadamard matrices, and in Section 4.3, we explain how they can be used to construct association schemes [11]. In Section 5.1, we study how skew-symmetric Bush-type Hadamard matrices can be used to make 3-class association schemes whose adjacency matrices are non-symmetric [7].

In this thesis, 5-class association schemes derived from mutually unbiased Hadamard matrices are presented. We describe these association schemes in Section 5.2, and in Section 5.3, we find all intersection matrices, Krein matrices, and eigenmatrices of the association schemes for them.

We examine the upper bound, introduced in [18], for the number of mutually unbiased Bush-type Hadamard matrices in Section 5.4 and provide an example in which this upper bound is obtained [12]. In Section 5.5, we explain how in [18], 5-class association schemes are used to generate a set of mutually unbiased Bush-type Hadamard matrices.

Chapter 2

Introduction to Association Schemes

In this chapter, we study the concept of association schemes. In Section 2.1, we review the fundamental concepts required for association schemes which can be defined from three different points of view. In Section 2.2, we review these three different equivalent concepts. In Section 2.3, we describe how Mathon [21] used symmetric block designs to produce 3-class association schemes. In Section 2.4, we explain how to use Latin squares to create 3-class association schemes [21].

2.1 Preliminary

Let A and B be two algebras over a field F, a map $K : A \to B$ is called a *homomorphism* if for all a in F and all $x, y \in A$ we have

1. $K(ax) = aK(x)$;

2. $K(x+y) = K(x) + K(y)$; and

3. $K(xy) = K(x)K(y)$.

Distance-regular graphs and strongly regular graphs play crucial roles in the study of association schemes, accordingly we give review on basic definitions from graph theory. Some definitions are elementary, however, reviewing them helps us to clarify the symbols we are using. Also see [3] for relevant definitions.

Definition 2.1. A *graph* G is an ordered pair (V,E) such that E is a subset of the set of the unordered pairs of V. Each element of V is called a *vertex* and each element of E is called an *edge*. The edge $\{x,y\}$ joining x and y is denoted by xy, and it is said that x and y are *incident* with xy, and x and y are *adjacent* or *neighbours* of each other. The *degree* of a vertex is the number of its neighbours. The graph G is called *complete* if we have $xy \in E$, for any two vertices x and y in V.

Definition 2.2. A *partition* on a set X of points is a set of disjoint subsets of X such that their union is equal to the whole set X. The characteristic matrix $P = P(\pi)$ of a partition $\pi = \{C_1, \dots, C_k\}$ of a set X with n elements is the n by k matrix defined as follows:

$$P_{ij} = \begin{cases} 1 & \text{if } i \in C_j; \\ 0 & \text{otherwise.} \end{cases}$$

A partition $\pi = (C_1, \dots, C_k)$ of P is *equitable* if, for all i and j, the number of neighbours which a vertex in C_i has in the cell C_j is independent of the choice of vertex in C_i [6].

A graph $G' = (V', E')$ is called a subgraph of the graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph Q of G with r vertices is called r-clique if all its vertices are adjacent.

Definition 2.3. For a given graph G, if degrees of all of its vertices are equal to a constant k, it is called a *k-regular graph* or a *regular graph* of order k.

A path between two vertices x and y of a graph is a sequence of edges $\{xx_0, x_0x_1, \dots, x_ny\}$ and it is called an *xy-path*. The *length* of a path is the number of edges it includes. For any two vertices x and y in V, their distance $d(x, y)$ is

$$min\{length\ of\ Q : Q\ is\ a\ xy\text{-}path\}.$$

The *diameter* of a graph G is defined to be $max_{x,y}d(x, y)$ and is denoted by *diam* G. For a vertex x of a given graph $G = (V, E)$, let $G_i(x) = \{u \in V | d(x, u) = i\}$.

Definition 2.4. A k-regular graph $G = (V, E)$ with diameter d is a *distance regular graph* if for all $x \in V$ and for $i = 1, \dots, d$, each $u \in G_i(x)$ has a_i neighbours in $G_{i-1}(x)$, b_i neighbours in $G_i(x)$, and c_i neighbours in $G_{i+1}(x)$.

Definition 2.5. A graph G is *strongly regular* with parameter (v,k,λ,μ) if

1. G is a k-regular graph with v vertices;
2. Any two adjacent vertices have λ common neighbours;
3. Any two non-adjacent vertices have μ common neighbours.

A graph of this type will be denoted by $SRG(v,k,\lambda,\mu)$.

Note that any distance-regular graph with diameter 2 and v vertices is a strongly regular graph $SRG(v,k,b_1,a_2)$. To get familiar with the theory of block designs, we restate the definitions of a symmetric block design and a system of linked symmetric block designs from [21] as follows:

Definition 2.6. A *symmetric block design* (v,k,λ) is a set V of v points and a set B of v k-subsets of V, called blocks, such that:

1. Any point of V belongs to k blocks;
2. Any two blocks have λ common elements;
3. Any two points belong to λ blocks.

To form an *incidence graph* of a symmetric block design (v,k,λ), $V \cup B$ is considered as the set of vertices, and two vertices are adjacent if and only if one of them is a block and the other is a point of the block [10].

Definition 2.7. A *system of linked symmetric block designs* $(v,k,\lambda;l,\alpha,\beta)$ is a collection $\{V_1,\dots,V_l\}$ of sets satisfying the following conditions:

1. Any pair $\{V_i,V_j\}$ forms a symmetric block design (v,k,λ);
2. For any three sets, V_i, V_j, and V_k, the number of elements $x \in V_i$ incident with both $y \in V_j$ and $z \in V_k$ is equal to α, if y and z are incident, and is equal to β otherwise.

Next we consider Latin squares and orthogonal Latin squares; Refer to [12] for more details.

Definition 2.8. A *Latin square* is an n by n array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

The following is an example of a Latin square.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}$$

Let $L = (L_{ij})$ and $K = (K_{ij})$ be two Latin squares. The array $R = (R_{ij})$, where $R_{ij} = (L_{ij}, K_{ij})$ is called L and K superimposed and it is said that one Latin square is *superimposed* on the other.

Definition 2.9. A pair of Latin squares A and B of order n is *orthogonal* if, when superimposed, each of the possible n^2 ordered pairs occurs exactly once.

For example, the following two Latin squares of order 3 are orthogonal .

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array} \qquad \begin{array}{ccc} (1,1) & (2,2) & (3,3) \\ (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \end{array}$$

A B A and B superimposed

Definition 2.10. A set of Latin squares is called *Mutually Orthogonal Latin Squares* if every pair of Latin squares in the set is orthogonal. They are also called *hypergraecolatin* squares and abbreviated as MOLS.

Here, we define suitable Latin square and mutually suitable Latin squares.

Definition 2.11. Two Latin squares L and K of size n on the symbol set $\{0,1,\ldots,n-1\}$ are called *suitable* if every superimposition of each row of L on each row of K results in only one element of the form (a,a).

Definition 2.12. A set of Latin squares is called *Mutually Suitable Latin Squares* abbreviated as MSLS, if every pair of Latin squares in the set is suitable.

Example 2.13. Three mutually suitable Latin squares of size 4 are:

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

The following lemma shows the numbers of MOLS and MSLS are equal.

Lemma 2.14. *[12]. There are m MOLS of size n if and only if there are m MSLS of size n.*

Proof. Let L and K be two orthogonal Latin squares on $\{0,1,\ldots,n-1\}$ both having their rows and columns labeled by the set. Let $((i,j),k)$ denotes the entry at (i,j) position of a Latin square which is equal to k. Then the transformation $((i,j),k) \Rightarrow ((k,j),i)$ results in a pair of suitable Latin squares. The transformation $((k,j),i) \Rightarrow ((i,j),k)$ implies the reverse implication. □

It is well known that there are $q-1$ MOLS of size q if q is a power of a prime. Moreover, $q-1$ MOLS of size q can be found by taking the non-zero element a in the field with q elements F_q and letting the a-th square have cell at the intersection of the row x and the column y given by $ax+y$. We have the following:

Lemma 2.15. *Let q be a prime power. There are $q-1$ mutually suitable Latin squares of size q.*

2.2 Equivalent Definitions for an Association Scheme

There are three different versions for definition of an association scheme which provide various standpoints of association schemes. These three definitions are based on partitions, graphs, and matrices respectively. In [1], the definition of association schemes are phrased as follows:

Definition 2.16. The pair (X,S) is called a d-class association scheme if X is a set of n points, and $S = \{R_0, \ldots, R_d\}$ is a partition of $X \times X$ such that

1. $(x,y) \in R_0$ if and only if $x = y$;
2. $R_i^t = R_{i'}$, for some $i' = 1, \ldots, d$, where $R_i^t = \{(x,y) | (y,x) \in R_i\}$;
3. For all i, j, k in $\{0, 1, \ldots, d\}$ and for all $(x,z) \in R_k$, we have

$$|\{y \in X : (x,y) \in R_i \text{ and } (y,z) \in R_j)\}| = p_{ij}^{(k)}.$$

If R_i is symmetric for all $i = 1, \ldots, d$, the association scheme is called *symmetric*. We mostly deal with symmetric association schemes. In this context, association schemes are symmetric unless it is explicitly mentioned. The parameters $p_{ij}^{(k)}$, $0 \leq i, j, k \leq d$ are called the *intersection numbers* of the association scheme.

In Section 5.1, we provide an example of an association scheme whose relations are not symmetric.

The definition of an association scheme implies that $p_{ij}^{(0)} = 0$ if $i \neq j$, and $p_{ij}^{(k)} = p_{ji}^{(k)}$. If $(x,y) \in R_i$, x is called *i-th associate* of y and y is also i-th associate of x. Condition (3) implies that each element of X has $a_i = p_{ii}^{(0)}$ i-th associates. The parameter a_i is called the *valency* of i-th associate class. If one or more of i,j, and k is equal to zero, the value of $p_{ij}^{(k)}$ is trivial. Hence, we just focus on $p_{ij}^{(k)}$ when $i, j, k \neq 0$.

Example 2.17. [1]. Let X be a rectangle with p rows and q columns, where $p,\ q \geq 2$. The classes of the 3-class association scheme are defined as follows:

$R_0 = \{(x,y) : x = y\}$

$R_1 = \{(x,y) : x,\ y \textit{ are in the same row but } x \neq y\}$

$R_2 = \{(x,y) : x,\ y \textit{ are in the same column but } x \neq y\}$

$R_3 = \{(x,y) : x,\ y \textit{ are in different rows and columns}\}.$

This association scheme has $\nu = pq$ objects and the intersection numbers are as appeared in Table 2.1.

Table 2.1: Intersection numbers of the rectangular association scheme

a_i	$p_{ij}^{(1)}$			$p_{ij}^{(2)}$		
$q-1$	$q-2$	0	0	0	0	$q-1$
$p-1$	0	0	$p-1$	0	$p-2$	0
$(p-1)(q-1)$	0	$p-1$	$(p-2)(q-1)$	$q-1$	0	$(p-1)(q-2)$

$p_{ij}^{(3)}$		
0	1	$q-2$
1	0	$p-2$
$q-2$	$p-2$	$(p-2)(q-2)$

This kind of association scheme is called a *rectangular association scheme* and is denoted by $R(p,q)$.

We review two more equivalent definitions of an association scheme provided in [1]. It is useful to work with graphs to generate association schemes or use association schemes to make different kind of graphs. Moreover, any association scheme can be considered as an edge-coloured complete undirected graph.

Definition 2.18. A d-class association scheme (X,S) is an edge-coloured complete undirected graph with a finite vertex set X and d colours such that

1. Every colour is used at least once;

Table 2.2: Intersection numbers of the association scheme derived from a cube

p_{ij}^{yellow}			p_{ij}^{red}			p_{ij}^{black}		
0	0	2	0	0	3	2	1	0
0	0	1	0	0	0	1	0	0
2	1	0	3	0	0	0	0	2

2. For all i, j, k in $\{1, \ldots, d\}$ there is an integer $p_{ij}^{(k)}$ such that, whenever xz is an edge of colour k

$$|\{y \in X : xy \; has \; colour \; i \; and \; yz \; has \; colour \, j\}| = p_{ij}^{(k)};$$

3. For each $i \in \{1, \ldots, d\}$ there is an integer a_i such that each vertex belongs to exactly a_i edges of colour i.

As noted in [24], any distance regular graph G with diameter d is equivalent to a d-class association scheme with relations $\{R_0, R_1, \ldots, R_d\}$, where $(x, y) \in R_i$ if $d(x, y) = i$.

Example 2.19. [1] Let X be the eight vertices of a cube. Consider the graph whose edges are the edges, main diagonals, and face diagonals of the cube. Suppose the colours of edges, main diagonals, and face diagonals are yellow, red, and black, respectively. This association scheme has 8 points. The intersection numbers of this association scheme are given in Table 2.2. Note that $a_{yellow} = 3$, $a_{red} = 1$, and $a_{black} = 3$.

The following definition of an association scheme is based on matrices. The matrix of all one of order n is shown by J_n.

Definition 2.20. A set of non-zero $(0, 1)$-matrices $A_0, A_1, \ldots, A_d$ in $R^{n \times n}$ is an association scheme with d classes on a finite set X of order n if

1. $A_0 = I_n$;

2. A_i is symmetric for $i = 1, \ldots, d$;

3. $A_i A_j$ is a linear combination of $A_0, \ldots, A_d$, for all i, j in $\{1, \ldots, d\}$;

4. $\sum_{i=0}^{d} A_i = J_n$.

In fact, if we consider an association scheme (X,S) with relation $\{R_0,\ldots,R_d\}$ (Definition 2.16), the matrix $A_k = (a_{ij}^{(k)})$ in the set $\{A_0,A_1,\ldots,A_d\}$ is defined as follows:

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } (i,j) \in R_k, \\ 0 & \text{otherwise} \end{cases}$$

for $k = \{1,2,\ldots,d\}$, and the set $\{A_0,A_1,\ldots,A_d\}$ is called *the set of adjacency matrices*.

Since each matrix in this definition is symmetric, the association scheme is commutative. The real vector space spanned by $\{A_0,A_1,\ldots,A_d\}$ is called the *Bose-Mesner algebra* of the association scheme (X,S) and is denoted by $\mathcal{A}$. The following example provides a set of matrices that forms a 3-class association scheme.

Example 2.21. The following matrices are the adjacency matrices of a 3-class association scheme on 8 elements.

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 2.3: Intersection numbers of the association scheme derived from a symmetric block design

a_i	$p_{ij}^{(1)}$			$p_{ij}^{(2)}$			$p_{ij}^{(3)}$		
k	0	0	$k-1$	0	0	k	λ	$k-\lambda$	0
$v-1$	$k-1$	$v-k$	0	k	$v-k-1$	0	0	0	$v-2$
$v-k$	0	0	$v-k$	0	0	$v-k-1$	$k-\lambda$	$v-2k+\lambda$	0

The linear combinations of their products are as follows:

$$A_0A_0 = A_0,\quad A_0A_1 = A_1,\quad A_0A_2 = A_2,\quad A_0A_3 = A_3,$$

$$A_1A_0 = A_1,\quad A_1A_1 = 3A_0 + 2A_1,\quad A_1A_2 = 2A_2 + 3A_3,\quad A_1A_3 = A_2,$$

$$A_2A_0 = A_2,\quad A_2A_1 = 2A_2 + 3A_3,\quad A_2A_2 = 3A_0 + 2A_1,\quad A_2A_3 = A_1,$$

$$A_3A_0 = A_3,\quad A_3A_1 = A_2,\quad A_3A_2 = A_1,\quad A_3A_3 = A_0.$$

2.3 A system of Linked Symmetric Block Designs

As it is mentioned in [19], an incidence graph of a symmetric block design (v,k,λ), defined in Page 6, is a distance-regular graph with diameter 3 which is equivalent to a 3-class association scheme. This association scheme has $2v$ points and its intersection numbers are shown in Table 2.3.

In the following example [21], the incidence graph of a $(16,6,2)$-design generates a 3-class association scheme:

Example 2.22. Let $(16,6,2)$ be a block design with $V = \{1,2,\ldots,16\}$ and $B = \{B_1,\ldots,B_{16}\}$, where

$B_1 = \{2,3,4,5,9,13\}$, $B_2 = \{1,3,4,6,10,14\}$, $B_3 = \{1,2,4,7,11,15\}$,

$B_4 = \{1,2,3,8,12,16\}$, $B_5 = \{6,7,8,1,9,13\}$, $B_6 = \{5,7,8,2,10,14\}$,

$B_7 = \{5,6,8,3,11,15\}$,$B_8 = \{5,6,7,4,12,16\}$, $B_9 = \{10,11,12,1,5,13\}$,

$B_{10} = \{9,11,12,2,6,14\}$, $B_{11} = \{9,10,12,3,7,15\}$, $B_{12} = \{9,10,11,4,8,16\}$,

Table 2.4: Intersection numbers of the association scheme derived from a system of linked symmetric block designs

a_i	$p_{ij}^{(1)}$			$p_{ij}^{(2)}$		
$v-1$	$v-2$	0	0	0	$k-1$	k
kl_1	0	λl_1	$(k-\lambda)l_1$	$k-1$	αl_2	$(k-\alpha)l_2$
$(v-k)l_1$	0	$(k-\lambda)l_1$	$(v-2k+\lambda)l_1$	$v-k$	$(k-\alpha)l_2$	$(v-2k+\alpha)l_2$

$p_{ij}^{(3)}$		
0	k	$v-k-1$
k	βl_2	$(k-\beta)l_2$
$v-k-1$	$(k-\beta)l_2$	$(v-2k+\beta)l_2$

$B_{13} = \{1,5,9,14,15,16\}$, $B_{14} = \{2,6,10,13,15,16\}$, $B_{15} = \{3,7,11,13,14,16\}$,

and $B_{16} = \{4,8,12,13,14,15\}$.

The set of vertices of the incidence graph G of this design is $V \cup B = \{1,2,\ldots,16,B_1,\ldots,B_{16}\}$. In this graph, 1 and B_2 are adjacent since B_2 is a block and 1 is a point of B_2, 1 and 2 are not adjacent since both of them are points, B_1 and B_2 are not adjacent since both of them are block, and 1 and B_1 are not adjacent since 1 does not belong to B_1.

The graph G is a distance-regular graph with diameter 3 and therefore, it is a 3-class association scheme (X,S) with relation $\{R_0,R_1,R_2,R_3\}$, where $(x,y) \in R_i$ if $d(x,y) = i$, $0 \leq i \leq 3$.

In [21], linked symmetric designs are used to derive 3-class association schemes. Any linked symmetric designs $(v,k,\lambda;l,\alpha,\beta)$ with collection $\{V_1,\ldots,V_l\}$ is equivalent to a 3-class association scheme with relation $\{R_0,R_1,R_2,R_3\}$, where $(x,y) \in R_1$ if $\{x,y\} \subseteq V_i$ for some $1 \leqslant i \leqslant l$; $(x,y) \in R_2$ if x and y belong to different sets and are adjacent; and $(x,y) \in R_3$ if x and y belong to different sets and are not adjacent. This association scheme has vl points and the intersection numbers are shown in Table 2.4 where $l_1 = l-1$ and $l_2 = l-2$.

The following example shows how Mathon [21] used 7 linked symmetric block designs $(16,6,2;7,1,3)$ to construct a 3-class association scheme.

Table 2.5: Squares used to generate a system of linked symmetric block designs

1				2				3				4			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	6	5	8	7	7	8	5	6	8	7	6	5
9	10	11	12	12	11	10	9	10	9	12	11	11	12	9	10
13	14	15	16	15	16	13	14	16	15	14	13	14	13	16	15

5				6				7			
1	2	5	6	1	2	5	6	1	2	9	10
4	3	8	7	3	4	7	8	3	4	11	12
10	9	14	13	14	13	10	9	6	5	14	13
11	12	15	16	16	15	12	11	8	7	16	15

Example 2.23. The 7 linked symmetric block designs $(16,6,2;7,1,3)$ can be generated by the squares of order 4 provided in Table 2.5. For every $1 \leqslant i \leqslant 7$, we have $V_i = \{1,2,\ldots,16\}$. If we want to make the block $[4(i-1)+j]$, where $1 \leq i,j \leq 4$, of the kth design $1 \leq k \leq 7$, we consider the 6 entries in the i-th row and j-th column not equal to the component (i,j) in the k-th square.

In Example 2.22, we use the first square to make the blocks. The system expressed in this example is a linked symmetric block designs. Any pair $\{V_i,V_j\}$ is a 2-$(16,6,2)$ block design, which means any two blocks from distinct designs have either 1 or 3 common elements. This structure, denoted by κ is isomorphic to a block design $(16,6,2)$, in the way that the blocks of one design are points of κ and the blocks of the other design are the blocks of κ. Here, incident means coming from different designs and having one common element.

By considering any three sets $\{V_i,V_j,V_k\}$, the number of elements $x \in V_k$ incident with both $y \in V_i$ and $z \in V_j$ is one if two points are incident and three otherwise. Note that points here means blocks, and incident means coming from different designs and having one common element.

If we consider all 7 designs, the association scheme has 112 points which are blocks of all the designs. Then two points (or blocks of the designs) are in the first associate if they belong to the same design, and they are second or third associate if they belong to different designs and are or are not incident respectively.

If we choose any number $1 \leq b \leq 7$ of designs, still we can form an association scheme with $16b$ points.

2.4 Mutually Orthogonal Latin Squares

In [21], Mathon used MOLS to create 3-class association schemes. In this Section, we discuss his method and then we provide an example. Consider $s < n$ mutually orthogonal Latin squares of order n . This system is equivalent to the strongly regular graph

$$SRG(n^2,(s+2)(n-1),(n-2)+s(s+1),(s+2)(s+1))$$

also called a Latin square graph $L_{s+2}(n)$. The construction is as follows:

- The vertices are $\{1,2,\ldots,n^2\}$. They are set in an n by n array, called square of arrangement, as followed:

$$\begin{pmatrix} 1 & \ldots & n \\ n+1 & \ldots & 2n \\ \vdots & \ddots & \vdots \\ n^2-n+1 & \ldots & n^2 \end{pmatrix}$$

- Two vertices are adjacent if they are in the same column or the same row of the square of arrangement or they have the same symbol in one of the Latin squares. Suppose we have 25 vertices and L is one of the Latin square we consider.

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}$$

Then the square of arrangement is as followed:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{matrix}$$

Consider the positions of 17 and 21 in the square of arrangement, the symbols appearing in these two position in L are the same and equal to 4. Hence, two vertices 17 and 21 are adjacent.

Given $s = i + j - 2$ mutually orthogonal Latin squares of order n, $1 \leq i < j \leq k = n - i - j + 1$, it is possible to partition $L_{s+2}(n)$ into the graphs $L_i(n)$ and $L_j(n)$ consisting of i and j sets of n disjoint n-cliques K_n, respectively. Then a 3-class association scheme can be formed. The objects are the n^2 nodes of $L_{s+2}(n)$. Two objects are first or second associate if they are adjacent in $L_i(n)$ or $L_j(n)$, respectively, otherwise they are 3-rd associate.

It is well known that the maximum number of MOLS of order n is $n-1$ and the maximum number for MOLS occurs if n is prime or prime power.

Example 2.24. [21]. In this example, we use some of the following four mutually orthogonal Latin squares to generate 3-class association schemes.

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	2	3	4	0	1	3	4	0	1	2	4	0	1	2	3
2	3	4	0	1	4	0	1	2	3	1	2	3	4	0	3	4	0	1	2
3	4	0	1	2	1	2	3	4	0	4	0	1	2	3	2	3	4	0	1
4	0	1	2	3	3	4	0	1	2	2	3	4	0	1	1	2	3	4	0

Using all four MOLS, we obtain a $SRG(25,24,23,30)$ which is a complete graph and gives us a 2-class association scheme. Since we want to generate a 3-class association scheme, we use just the first two MOLS. Then $s=2$ and $i+j=4$. Let $i=2$ and $j=2$.

We wish to construct the graph:

$$G \simeq L_{s+2}(n) \simeq SRG(n^2,(s+2)(n-1),(n-2)+s(s+1),(s+2)(s+1)),$$

where $s+2=4$ and $n=5$, that is, $G \simeq L_4(5) \simeq SRG(25,16,9,12)$. Let the vertices of the graph be $\{1,2,\ldots,25\}$ in the following square of arrangement A:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Then two vertices are adjacent if they are in the same row or column of A, or they have the same number in one of the Latin squares, we have,

$$l_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}, l_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{pmatrix}$$

In the graph G, for example the vertices 1 and 2 are adjacent because they are in the same row of A; the vertices 1 and 6 are adjacent because they are in the same column of A; and the vertices 1 and 10 are adjacent because if we transfer their position from A to the Latin square l_1, both of them appear with the same symbol 0. This graph is shown in Figure 2.1.

We have $L_i(n)$ and $L_j(n)$ both consist of two sets of 5 disjoint 5-cliques. There are different ways to define $L_i(n)$ and $L_j(n)$ based on cliques. We explain here one of the possibilities. Let $L_i(n)$ consists of each edge pq such that p and q are in the same row of A, or they have a same symbol in the first Latin square l_1. In the similar way, let $L_j(n)$ consists of each edge pq such that p and q are in the same column of A, or they have same symbol in the second Latin square l_2.

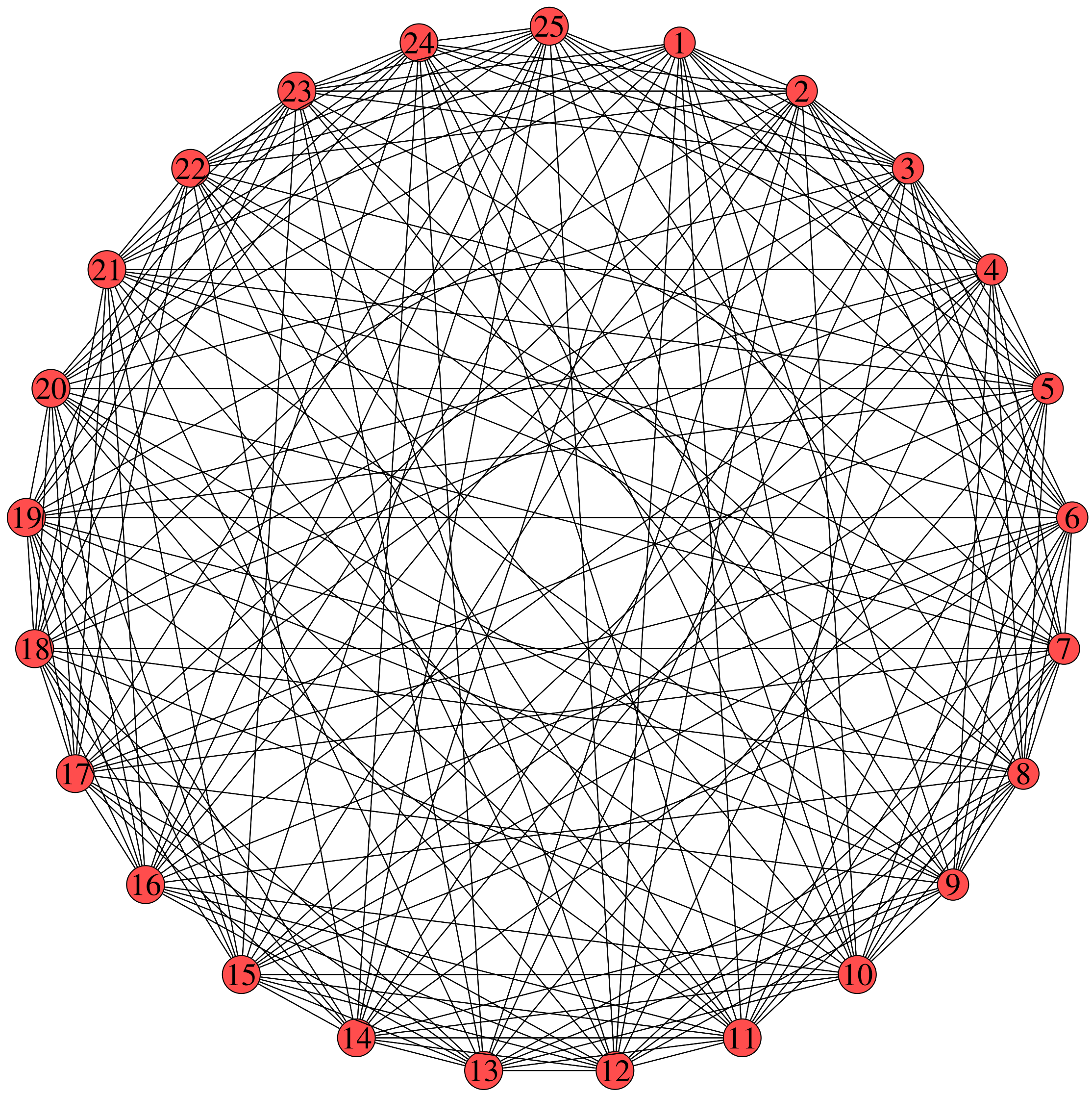

Figure 2.1: $SRG(25, 16, 9, 12)$

Chapter 3

More on Association Schemes

We study the theory of association schemes more in depth in this chapter. We provide a review on the Bose-Mesner algebra $\mathcal{A}$ of an association scheme and a review on a basis of primitive idempotents for $\mathcal{A}$ in Section 3.1. In Section 3.2, we mention the definition of eigenmatrices of an association scheme, and in Section 3.3, we continue by discussing the concept of intersection and Krein matrices of association schemes. There are also surveys on imprimitive association schemes and quotient association schemes in Section 3.4 and 3.5, respectively.

3.1 Idempotent Bases

Now, we study the Bose-Mesner algebra of an association scheme. Then, we provide two bases for this algebra. Some definitions required for the subject are reviewed below.

The Hadamard product of two matrices $A=(a_{ij})$ and $B=(b_{ij})$ is denoted by $A\circ B$ and is defined by $A\circ B=(a_{ij}b_{ij})$. Two matrices are *disjoint* if their Hadamard product is equal to the zero matrix.

If $A=(a_{ij})$ is an n by m matrix and $B=(b_{ij})$ is a r by s matrix, then the Kronecker product of A and B, denoted by $A\otimes B$, is a mr by ns block matrix defined

$$\begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$$

For a given association scheme (X,S) with d classes and adjacency matrices $A_0,\dots,A_d$, suppose $\mathcal{A}$ is the Bose-Mesner algebra of this scheme. By Definition of Bose-Mesner algebra in Page 12, we know $\mathcal{A} = span(A_0, \ldots, A_d)$. From the Definition 2.20, we conclude A_i and A_j are disjoint, when $0 \leq i \neq j \leq d$. This means the set $\{A_0, \ldots, A_d\}$ is a set of linearly independent matrices because suppose $a_0A_0 + a_1A_1 + \cdots + a_dA_d = 0$, for some real number $a_0, \ldots, a_d$. If we multiply A_i from right using the Hadamard product, we have $a_iA_i \circ A_i = a_iA_i = 0$; this shows $a_i = 0$. Since i was arbitrary, the set $\{A_0, \ldots, A_d\}$ is a set of linearly independent matrices and form a basis for the algebra $\mathcal{A}$.

It is well known that an square matrix B of order n is called *positive semi-definite* if for any column vector $w \in \mathcal{R}^n$, we have $w^TBw \geq 0$. A *principal submatrix* is a submatrix in which the set of row indices that remain is the same as the set of column indices that remain.

It is easy to show that a principal submatrix of a positive semi-definite matrix is positive semi-definite. Suppose C of order k is a principal submatrix of a matrix B of order n, and suppose w is a column vector in $\mathcal{R}^k$. Let z be the column vector in $\mathcal{R}^n$ obtained by extending w by putting zero in the positions that are removed from B to gain C. Then, we have $w^TCw = z^TBz$, and since B is positive semi-definite, $w^TCw \geq 0$. Hence, C is also positive semi-definite.

Definition 3.1. A square matrix B is idempotent if $B^2 = B$, using matrix multiplication.

There is a basis of mutually orthogonal idempotent matrices for the Bose-Mesner algebra of an association scheme. To construct the basis, we start with a review following [6].

The transpose of the conjugate of a complex matrix B is denoted by B^*. If B is a real matrix, the transpose of B is denoted by B^T. The matrix is *hermitian* if $B^* = B$ and is *unitary* if $BB^* = I$. Therefore, a real matrix is hermitian iff it is symmetric and it is unitary iff it is orthogonal. If $B = -B^T$ for a real matrix B, it is called a *skew-symmetric* matrix. We call B *normal* if $BB^* = B^*B$. If B is normal then its eigenvectors span $\mathbb{C}^n$, and eigenvectors in different eigenspaces are orthogonal. Suppose B is a normal matrix. For each eigenvalue

λ of B, let U_λ be a matrix whose columns form an orthogonal basis for the eigenspace belonging to λ and set $E_\lambda := U_\lambda U_\lambda^*$. The matrices E_λ are called the principal idempotents of B. The set of eigenvalues of B is denoted by $ev(B)$.

It is noteworthy that E_λ is the matrix representing orthogonal projection onto the column space of U_λ, i.e., onto the eigenspace belonging to λ. Here are two fundamental theorems. See [6] for details.

Theorem 3.2. (Spectral decomposition). *Let B be a hermitian matrix of order n with principal idempotents E_λ, where $\lambda \in ev(B)$. Then the followings hold:*

1. $E_\lambda^2 = E_\lambda$ *and* $E_\lambda E_\theta = 0$ *if* $\lambda \neq \theta$;
2. $BE_\lambda = \lambda E_\lambda$;
3. $\sum_{\lambda \in ev(B)} E_\lambda = I$;
4. *If $f(B)$ is a rational function which is defined at each eigenvalue of B then* $f(B) = \sum_{\lambda \in ev(B)} f(\lambda) E_\lambda$.

Theorem 3.3. *For a given association scheme (X,S) with $|X| = n$, d classes, and adjacency matrices $A_0, \ldots, A_d$, suppose $\mathcal{A}$ is the Bose-Mesner algebra. Then there is a set of pairwise orthogonal idempotent matrices $E_0, \ldots, E_d$ and real numbers p_{ij} such that*

1. $\sum_{j=0}^{d} E_j = I$;
2. $A_i E_j = p_{ji} E_j$, *for all i, j in* $\{1, \ldots, d\}$;
3. $E_0 = \frac{1}{n} J$;
4. $\{E_0, \ldots, E_d\}$ *is a basis for* $\mathcal{A}$.

The matrices E_j are called *primitive idempotents* or *principal idempotents* of the association scheme (X,S).

3.2 Eigenmatrices

We continue the subject of association schemes by introducing eigenmatrices, intersection numbers, and Krein parameters. Our main reference in this Section is [6]. The column space of E_j is a common eigenspace for the matrices in the Bose-Mesner algebra $\mathcal{A}$. From the condition (2) of Theorem 3.3, the eigenvalue of A_i on this eigenspace is p_{ji}, and the number p_{ji} are known as eigenvalues of (X,S). We have $p_{0j} = 1$ for all j because $A_0 = I$, and as $E_0 = \frac{1}{n}J$, we have p_{i0} is equal to the number of non zero entries in any row of A_i or valency a_i of A_i. We denote the dimension of the eigenspace belonging to E_j by m_j, and we call the numbers $m_0,\ldots,m_d$ the multiplicities of (X,S). Since the set $A_0,\ldots,A_d$ forms a basis for the algebra $\mathcal{A}$, there are numbers q_{ij} such that

$$E_i = \frac{1}{n}\sum_{k=0}^{d} q_{ki}A_k. \tag{3.1}$$

By multiplying both sides of the equation 3.1 from right by A_j using Hadamard multiplication, we find $E_i \circ A_j = \frac{1}{n}q_{ji}A_j$, which is similar to condition (2) in the Theorem 3.3.

The matrices $P = (p_{ij})$ and $Q = (q_{ij})$ are called the first and second *eigenmatrices* of the association scheme, respectively. By using equation 3.1 to substitute for E_j in $A_i = A_iI = A_i\sum_j E_j = \sum_j p_{ji}E_j$, we find

$$\begin{aligned} A_i &= \frac{1}{n}\sum_j p_{ji}\sum_r q_{rj}A_r \\ &= \frac{1}{n}\sum_j\sum_r p_{ji}q_{rj}A_r. \end{aligned}$$

Therefore, we have $\sum_{k=0}^{d} p_{ki}q_{jk} = n\delta_{ij}$; i.e., $QP = nI$.

Consider the matrix A_iA_j. The value of the (r,s)-entry of the matrix A_iA_j is equal to the multiplication of r-th row of A_i and s-th column of A_j. Since A_i and A_j are $\{0,1\}$-matrices, the multiplication of r-th row of A_i and s-th column of A_j is equal to the number of ones matched in this row and column. In fact, this number is equal to

$$|\{z|(r,z)\text{-entry of } A_i \text{ and } (z,s)\text{-entry of } A_j \text{ both are nonzero}\}| =$$
$$|\{z|(r,z) \in R_i \text{ and } (z,s) \in R_j\}| = p_{ij}^{(k)}.$$

This leads us to the fact that $A_iA_j = \sum_{k=0}^{d} p_{ij}^{(k)} A^k$. Since $E_i \circ E_j$ is in the algebra $\mathcal{A}$, there are numbers $q_{ij}^{(k)}$ such that $E_i \circ E_j = \frac{1}{n}\sum_{r=0}^{d} q_{ij}^{(r)} E_r$. The numbers $q_{ij}^{(k)}$ are called the *krein parameters* of the association scheme. Since A_i are $\{0,1\}$-matrices, the intersection numbers are non-negative.

Lemma 3.4. *[6]* (The Krein condition). *The Krein parameters $q_{ij}^{(k)}$ of an association scheme are non-negative.*

Proof. We have $E_i \circ E_j = \frac{1}{n}\sum_{k=0}^{d} q_{ij}^{(k)} E_k$. Multiply both sides of this equality by E_l for some $l = 0, 1, \ldots, d$, using matrix multiplication. Then, we have

$$\begin{aligned}(E_i \circ E_j)E_l &= (\frac{1}{n}\sum_{k=0}^{d} q_{ij}^{(k)} E_k)E_l \\ &= \frac{1}{n}\sum_{k=0}^{d} q_{ij}^{(k)} E_k E_l \\ &= \frac{1}{n} q_{ij}^{(l)} E_l.\end{aligned}$$

This shows that $\frac{1}{n} q_{ij}^{(l)}$ is the eigenvalue of $E_i \circ E_j$ on the column space of E_l. Moreover,

$$(E_i \otimes E_j)^2 = E_i^2 \otimes E_j^2 = E_i \otimes E_j$$

and so the eigenvalues of $E_i \otimes E_j$ are 0 or 1. The matrix $E_i \otimes E_j$ is symmetric because the matrices E_i and E_j are symmetric. A symmetric matrix with non-negative eigenvalues is a positive semi-definite matrix. Hence, $E_i \otimes E_j$ is positive semi-definite. This implies that the matrix $E_i \circ E_j$ is also positive semi-definite because $E_i \circ E_j$ is a principal submatrix of $E_i \otimes E_j$. So all eigenvalues of $E_i \circ E_j$ are non-negative. Hence, all $q_{ij}^{(k)}$, for $0 \le i, j, k \le d$

are non-negative. □

By multiplying A_r, $r = 0, 1, \ldots, d$, on the left side of the equality $A_i = \sum_i p_{ji} E_j$, $0 \leq i, j \leq d$, we have $A_i A_r = \sum_i p_{li} E_l A_r = \sum_i p_{li} E_l (\sum_r p_{sr} E_s) = \sum_l p_{li} p_{lr} E_l$. On the other hand, we have

$$\begin{aligned} A_i A_r &= \sum_k p_{ir}^{(k)} A_k \\ &= \sum_k p_{ir}^{(k)} (\sum_l p_{lk} E_l) \\ &= \sum_l (\sum_k p_{ir}^{(k)}) p_{lk} E_l. \end{aligned}$$

This implies for all $0 \leq i, j, r \leq d$,

$$\sum_k p_{lk} p_{ir}^{(k)} = p_{li} p_{lr}. \tag{3.2}$$

In a similar way, we have

$$\sum_k q_{lk} q_{ir}^{(k)} = q_{li} q_{lr}. \tag{3.3}$$

Therefore, by having the eigenmatrices of an association scheme, we are able to evaluate all intersection numbers and Krein parameters of the association scheme.

3.3 Intersection Matrices and Krein Matrices

In this Section, we evaluate intersection and Krein matrices, which are really practical to evaluate the entries of eigenmatrices. We do a review on some definitions phrased in [2].

Let (X, S) be an association scheme with adjacency matrices $A_0, A_1, \ldots, A_d$, intersection numbers $p_{ij}^{(k)}$, $0 \leq i, j, k \leq d$, and Krein parameters $q_{ij}^{(k)}$, $0 \leq i, j, k \leq d$. Then, the matrix $P_i = (p_{ij}^{(k)})$ is called i-th *intersection matrix* of (X, S), and the matrix $B_i^* = (q_{ij}^{(k)})$ is called the i-th *Krein matrix* of (X, S).

Let $\mathcal{A}$ be the Bose-Mesner algebra, and $M_n(\mathbb{C})$ be full matrix algebra of degree n over $\mathbb{C}$. Regarding the left multiplication of $\mathcal{A}$ as a linear transformation on $\mathcal{A}$ and expressing them in matrix form with respect to the basis $\{A_0, A_1, \ldots, A_d\}$, we have an algebra homomorphism from $\mathcal{A}$ to $M_n(\mathbb{C})$.

This is called *left regular representation* of $\mathcal{A}$ with respect to the basis $\{A_0, A_1, \ldots, A_d\}$. Since we have $A_iA_j = \sum_{r=0}^{d} p_{ij}^{(k)} A_r$, the image of A_i is P_i^T. Hence, the correspondence $A_i \to P_i^T$ is an isomorphism for $\mathcal{A}$. Since $\mathcal{A}$ is a commutative algebra, $P_i \to P_i^T$ also gives us an isomorphism. Hence, the correspondence $A_i \to P_i$ is an isomorphism of $\mathcal{A}$. This shows A_i and P_i have the same minimal polynomial, and therefore same eigenvalues with different multiplicities.

3.4 Imprimitive Association Schemes

Here we study the concept of imprimitive association schemes. An association scheme (X, S) with adjacency matrices $A_0, \ldots, A_d$ is *primitive* if all graphs $G_1, \ldots, G_d$ are connected, where the graph G_i is a graph made with adjacency matrix A_i; otherwise, it is called *imprimitive*. It is also relevant to consider $G_1, \ldots, G_d$ as the classes of the scheme, and then the Bose-Mesner algebra of matrices can convert to the Bose-Mesner algebra of graphs.

Let $G_1, \ldots, G_d$ be the classes of an association scheme. A partition π of an association scheme (X, S) is equitable if it is an equitable partition for each graph G_i. In [6], important properties of an imprimitive association scheme are developed in the form of the following theorem:

Theorem 3.5. *Let (X, S) be an imprimitive association scheme with classes $G_1, \ldots, G_d$. Let $|X| = n$ and G be the disconnected graph which makes the association scheme imprimitive with m components. Let σ be the partition formed by the components of G. Then σ is equitable and all its cells have the same cardinality. If S is the characteristic matrix of σ then SS^T is in the Bose-Mesner algebra $\mathcal{A}$ of the scheme.*

Proof. Let A be the adjacency matrix of G with exactly m components. As G is regular its components are also regular. Let k be the valency of G. Suppose $\mathbf{j}$ is the column vector of all ones. Since $A\mathbf{j} = k\mathbf{j}$, k is the eigenvalue of A with multiplicity m. Therefore, the characteristic vectors of the components of G, which are column of S, form a basis for the eigenspace belonging to k. Denote this eigenspace by U. Hence, U is the column space of S. If $G_i \in \mathcal{A}$ and A_i is the adjacency matrix of G_i, then $AA_i = A_iA$. If $\mathbf{x} \in U$ then

$$AA_i\mathbf{x} = A_iA\mathbf{x} = A_i(k\mathbf{x}) = kA_i\mathbf{x}.$$

This shows that $A_i\mathbf{x} \in U$ if $\mathbf{x} \in U$. Since each column of S is a member of U, we have the columns of matrix A_iS are also in U and therefore are linear combinations of columns of S, and consequently, they are constant on each component. On the other hand, we have

$$(A_iS)_{pq} = \sum_{k=0}^{n} (A_i)_{pk} S_{kq} = \sum_{p\sim k} S_{kq},$$

where $p \sim k$ means pk is an edge in A_i. The last sum is equal to the number of neighbours of vertex p contained in the q-th component of σ which depends on the component of σ in which i lies, and therefore σ is equitable for each graph G_i in $\mathcal{A}$.

It remains to prove that all components of σ or G have the same cardinality. Two vertices v and u lie in the same component of σ if and only if $(A^r)_{uv} \neq 0$ for some non-negative integer r, or equivalently if and only if $(A+I)^n_{uv} \neq 0$. If the edge uv lies in G_i then $(A^r)_{uv} \neq 0$ if and only if $A^r \circ A_i$ is a non-zero multiple of A_i. Define the subset $\mathfrak{I}$ of the set $\{0,1,\ldots,d\}$ by

$$\mathfrak{I} = \{i : (A+I)^n \circ A_i \neq 0\}$$

and then let $H = \cup_{i\in\mathfrak{I}} G_i$. Then H has the same components as G and each component is a complete graph. Each of these complete graphs has the same cardinality since H is a regular graph. Therefore, all cells of σ have the same cardinality. If D is the adjacency matrix of H, we have $D = SS^T - I$. Hence, $SS^T \in \mathcal{A}$. □

3.5 Quotient Association Schemes

As it is appeared in [6], an imprimitive association scheme is made from two smaller schemes. Let (X,S) be an association scheme with Bose-Mesner algebra $\mathcal{A}$ and with classes $G_1,\dots,G_d$, where G_1 is not connected. Consider one of components of G_1 with vertex set C. Then C induces a subgraph of G_i for $i=1,\dots,d$. Some of the subgraphs are empty, however, it can be proved that the non-empty subgraphs form an association scheme with vertex set C.

Next association scheme made from imprimitive association scheme is called *quotient*. The vertex set of the quotient association scheme includes the components of G_1. Let S be the characteristic matrix of the partition made by the components of G_1. Then SS^T is a $(0,1)$-matrix in $\mathcal{A}$ and $SS^T = I_q \otimes J_r$, for some positive integer q and r. Let $\mathfrak{I}$ denote the subset of $\{0,1,\dots,d\}$ such that

$$SS^T = \sum_{i\in\mathfrak{I}} A_i.$$

Then we have

$$(SS^T)^2 = (I_q\otimes J_r)(I_q\otimes J_r) = I_q\otimes rJ_r = r(I_q\otimes J_r) = rSS^T = \sum_{i,j\in\mathfrak{I}} A_iA_j,$$

which implies that if i and j belong to $\mathfrak{I}$ and $p_{ij}^{(k)} \neq 0$ then $k\in\mathfrak{I}$. Now, define a relation $\approx$ on $\{0,1,\dots,d\}$ such that $a\approx b$ if $p_{ia}^{(b)}\neq 0$ for some $i\in\mathfrak{I}$. This relation is a symmetric relation because we have $p_{ia}^{(b)}\neq 0$ if and only if $p_{ib}^{(a)}\neq 0$. Now, we show why this relation is an equivalence relation. Let $a\approx b$ and $b\approx c$. Then for some $i,j\in\mathfrak{I}$ both $p_{ia}^{(b)}$ and $p_{jb}^{(c)}$ are non-zero. We show $a\approx c$.

By $p_{ia}^{(b)}\neq 0$, there are vertices x, y, and w such that xw and wy are edges in G_a and G_b respectively, and xy is an edge in G_i. Since $p_{jb}^{(c)}\neq 0$, we have $p_{cj}^{(b)}\neq 0$. Therefore, there is a vertex z such that wz is an edge in G_c and yz is an edge in G_j. The existence of y implies $p_{ij}^{(l)}\neq 0$, for some $l=0,1,\dots,d$. Since $i,j\in\mathfrak{I}$, we conclude $l\in\mathfrak{I}$. So $p_{ac}^{(l)}\neq 0$, and hence $p_{la}^{(c)}\neq 0$. This implies $a\approx c$.

Let $\mathfrak{I}_i$ be the subset of elements of $\{0,1,\ldots,d\}$ equivalent to i and $\mathfrak{I}_0 := \mathfrak{I}$. For each equivalence class $\mathfrak{I}_i$ there is a matrix B_i such that

$$\sum_{j\in\mathfrak{I}_i} A_j = B_i \otimes J_r. \tag{3.4}$$

We have already seen this for $i=0$. For the case of $i\neq 0$, let C and D be distinct components of G_1. Assume that u and v are vertices of C and D respectively, joined by an edge from G_a. Let x and y be a second pair of vertices chosen from C and D respectively, joined by an edge from G_b. Since u and x belong to C, there is $i\in\mathfrak{I}$ such that ux is an edge in G_i. Therefore, if xv is an edge in G_c, we have $c\approx a$. In the similar way, we have $b\approx c$ because v and y both lie in D. Consequently $b\approx a$. Define the matrix

$$M = \sum_{i\in\mathfrak{I}_a} A_i,$$

where $a\neq 0$. Therefore, if u and v are vertices in distinct components of G_1 such that $M_{uv}\neq 0$ then $M_{xy}=1$ for any vertices x and y such that x is in the same component as u and y is in the same component as v, respectively. Hence, $M = B\otimes J_r$ for some $\{0,1\}$-matrix B. This implies there are matrices B_i such that 3.4 holds for each equivalence class $\mathfrak{I}_i$.

It follows from 3.4 that B_i is a symmetric $\{0,1\}$-matrix and that the sum of all the matrices B_i is J_q. We also have

$$(B_i\otimes J_r)(B_{i'}\otimes J_r) = (B_iB_{i'}\otimes rJ_r),$$

which implies that $B_iB_{i'}\otimes J_r\in\mathcal{A}$, and therefore the matrices B_i also satisfy the condition 3 of the Definition 2.20. Hence, the matrices B_i form the *quotient* association scheme.

In the following example, we construct the quotient association scheme of the imprimitive association scheme which appeared in Example 2.21.

Example 3.6. Consider the association scheme on 8 points of example 2.21. This association scheme is imprimitive because the graph whose adjacency matrix is A_1, consists of 2 disjoint 4-cliques and is disconnected. Then the set $\mathfrak{I}$ defined in the proof of the theorem 3.5 for this association scheme is $\{0,1\}$, and we have

$$\sum_{i\in\mathfrak{I}} A_i = A_0 + A_1 = I_2 \otimes J_4 = \begin{pmatrix} 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1 \end{pmatrix}.$$

The equivalence classes are $\mathfrak{I}_0 = \mathfrak{I}, \mathfrak{I}_1 = \{2,3\}$. We have $2 \approx 3$ because $p_{12}^{(3)} = 3 \neq 0$ and $1 \in \mathfrak{I}$. We have

$$\sum_{i\in\mathfrak{I}_1} A_i = A_2 + A_3 = (J_2 - I_2) \otimes J_4 = \begin{pmatrix} 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix}.$$

This implies that the adjacency matrices of the quotient association scheme are $B_0 = I_2$ and $B_1 = J_2 - I_2$.

Here we review the definition of fiber and uniform association scheme [18]. For an imprimitive association scheme (X,S) with adjacency matrices $A_0, A_1, \ldots, A_d$, there is a set $\mathfrak{I}$ such that $\sum_{i\in\mathfrak{I}} A_i = I_p \otimes J_r$, for some positive integer p and r. Therefore, the set X is partitioned into p subsets each one of which is called a *fiber*. For fibers U and V, let $I(U,V)$ denote the set of indices of adjacency matrices that have an edge between U and V. We define a $\{0,1\}$-matrix A_i^{UV} by

$$(A_i^{UV})_{xy} = \begin{cases} 1 & \text{if } (A_i)_{xy} = 1,\, x \in U,\, y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.7. An imprimitive association scheme is called *uniform* if it has a quotient scheme with one class and there exists integers a_{ij}^k such that for all fibers U, V, W and $i \in I(U,V)$, $j \in I(V,W)$, we have

$$A_i^{UV} A_j^{VW} = \sum_k a_{ij}^k A_k^{UW}.$$

Chapter 4

Hadamard Matrices

In this chapter, we study a method of using sets of mutually unbiased regular Hadamard matrices to generate 3-class association schemes. This method, introduced in [11], is given in Section 4.3. Prior to that, we briefly review Hadamard matrices in the next Section while in Section 4.2, we study mutually unbiased Hadamard matrices.

4.1 Definitions

A Hadamard matrix H is a $\{-1,1\}$-matrix of order n with orthogonal rows, (under the usual inner product on $\mathbb{R}^n$); that is, $HH^T = nI_n$.

Definition 4.1. A Hadamard matrix of order n for which the row sums and column sums are all the same is called *regular*.

Note that in a regular Hadamard matrix of order n the row or column sums must be $\sqrt{n}$. Suppose H is a Hadamard matrix of order n and suppose c_i is the i-th column sum. Let $\mathbf{j}$ denote the column of all ones and $\mathbf{y}$ denote the column with c_i as its i-th element. Then we have

$$\mathbf{j}^T HH^T \mathbf{j} = \mathbf{y}^T \mathbf{y} = c_1^2 + c_2^2 + \cdots + c_n^2.$$

On the other hand, since $HH^T = I_n$, we have

$$\mathbf{j}^T HH^T \mathbf{j} = \mathbf{j}^T I_n \mathbf{j} = n^2.$$

Now, apply the Cauchy-Schwarz inequality giving

$$c_1 + \cdots + c_n \leq \sqrt{n}\sqrt{c_1^2 + \cdots + c_n^2}. \tag{4.1}$$

If H is a regular Hadamard matrix, then $\mathbf{y}$ is a multiple of $\mathbf{j}$, and we have equality in the Cauchy-Schwarz inequality. Let the column sums of H be equal to m. Then from 4.1, we have

$$mn = n\sqrt{n},$$

which implies that $m = \sqrt{n}$. In the same way, we can prove that the row sum of H is also equal to $\sqrt{n}$. A Hadamard matrix is *normalized* if all the elements of the first row and first column are 1. The following is a normalized Hadamard matrix of order 4.

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix}.$$

4.2 Unbiased Hadamard Matrices and Bases for $\mathcal{R}^n$

Mutually unbiased regular Hadamard matrices are important objects for generating association schemes. In this Section, we provide the definition of mutually unbiased bases for $\mathbb{R}^{4n^2}$ derived from mutually unbiased regular Hadamard matrices. The three definitions, expressed below, appeared in [11].

Definition 4.2. Two Hadamard matrices H and K of order n are called *unbiased* if $HK^T = \sqrt{n}L$ for some Hadamard matrix L.

Since the elements of a Hadamard matrix are 1 or -1, the elements of the multiplication of two Hadamard matrices are integral. This implies that the order of unbiased Hadamard matrices must be a perfect square.

Definition 4.3. Two orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{R}^n$ are called *mutually unbiased real bases* if $\langle u,v\rangle \in \{\frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}\}$ for all $u \in \mathcal{B}_1$ and $v \in \mathcal{B}_2$, where $\langle , \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$.

Definition 4.4. A set of Hadamard matrices $\{H_1, \dots, H_k\}$ is called *mutually unbiased* if every pair of Hadamard matrices in this set is unbiased.

Let $\{H_1, \dots, H_k\}$ be a set of mutually unbiased Hadamard (abbreviated as MUH) matrices of order $4n^2$. Then the set

$$\{I_{4n^2}, \frac{1}{2n}H_1, \dots, \frac{1}{2n}H_k\}$$

forms a set of mutually unbiased bases (abbreviated as MUB) in R^{4n^2}.

If the elements of a set of MUH matrices are regular, they can be used to generate association schemes. Mutually unbiased bases are called mutually unbiased regular bases (abbreviated MURB) if they are derived from mutually unbiased regular Hadamard matrices. Here, we state two lemmas about regular Hadamard matrices [11].

Lemma 4.5. *Let H and K be two unbiased regular Hadamard matrices of order $4n^2$. Then $\frac{1}{2n}HK^T$ is also a regular Hadamard matrix.*

Proof. Let $C = \frac{1}{2n}HK^T$. Then

$$CC^T = (\frac{1}{2n}HK^T)(\frac{1}{2n}HK^T)^T = \frac{1}{4n^2}(HK^T)(HK^T)^T = \frac{1}{4n^2}H(K^TK)H^T =$$

$$\frac{4n^2}{4n^2}H(I_{4n^2})H^T = HH^T = 4n^2 I_{4n^2}.$$

Hence, $\frac{1}{2n}HK^T$ is a Hadamard matrix. Let $\mathbf{j}$ be the column vector of all ones, then we have

$$C\mathbf{j} = (\frac{1}{2n}HK^T)\mathbf{j} = (\frac{1}{2n}H)(2n\mathbf{j}) = 2n\mathbf{j}.$$

Similarly, we have $\mathbf{j}^T C = 2n\mathbf{j}^T$. Therefore, $\frac{1}{2n}HK^T$ is a regular Hadamard matrix. □

Lemma 4.6. *There is a set of mutually unbiased regular Hadamard (abbreviated MURH) matrices of order $4n^2$ with m elements if we have a set of MUH matrices of order $4n^2$ with $m+1$, $m \geq 2$ elements.*

Proof. Let $\{H_1, H_2, \ldots, H_{m+1}\}$ with $m \geq 2$ be a set of MUH matrices of order $4n^2$. If $H_1 = [h_{ij}]$ and $D = [h_{1j}\delta_{ij}]$, where δ_{ij} is the Kronecker delta function defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

then the set $\{H_1D, H_2D, \ldots, H_{m+1}D\}$ is a set of MUH matrices. By definition of D, we have the first row of H_1D is all 1 entries. Since the elements of $\{H_1D, H_2D, \ldots, H_{m+1}D\}$ are unbiased, the row sums of all other matrices are in $\{-2n, 2n\}$. By negating all rows with row sum $-2n$, all matrices in the set $\{H_1D, H_2D, \ldots, H_{m+1}D\}$ except H_1D are regular Hadamard matrices. This implies that the set $\{H_2D, H_3D, \ldots, H_{m+1}D\}$ is a set of mutually unbiased regular Hadamard matrices with m elements. □

If $\{I, \frac{1}{2n}H_1, \frac{1}{2n}H_2, \ldots, \frac{1}{2n}H_m\}$ is a MUB for $\mathbb{R}^{4n^2}$, the *Gramian* matrix is as follows:

$$M = \begin{bmatrix} I \\ H_1/2n \\ \cdots \\ H_m/2n \end{bmatrix} \begin{bmatrix} I & H_1^T/2n & \ldots & H_m^T/2n \end{bmatrix}. \tag{4.2}$$

Now, we mention a lemma from [11] which provides important properties of the Gramian matrix.

Lemma 4.7. *Let M be the Gramian of a MURB for $\mathbb{R}^{4n^2}$. Suppose $N = 2n(M - I)$. Then $M^2 = (m+1)M$, M and N are symmetric, and N is a $\{0, -1, 1\}$-matrix with*

$$\begin{pmatrix} 0 & L_{12} & \ldots & L_{1(m+1)} \\ L_{21} & 0 & \ldots & L_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{(m+1)1} & L_{(m+1)2} & \ldots & 0 \end{pmatrix},$$

where L_{ij} are regular Hadamard matrices.

Proof. The matrix M is symmetric because of its structure. Let $F_0 = I$ and $F_i = \frac{1}{2n}H_i$, for $1 \le i \le m$. Then we have

$$M^2 = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_m \end{bmatrix} \begin{bmatrix} F_0^T & F_1^T & \dots & F_m^T \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_m \end{bmatrix} \begin{bmatrix} F_0^T & F_1^T & \dots & F_m^T \end{bmatrix}$$

$$= \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_m \end{bmatrix} (m+1)I \begin{bmatrix} F_0^T & F_1^T & \dots & F_m^T \end{bmatrix} = (m+1)M.$$

Hence, we have $M^2 = (m+1)M$. If we expand M, we have

$$M = \begin{pmatrix} I & H_1^T/2n & \dots & H_m^T/2n \\ H_1/2n & I & \dots & H_1H_m/4n^2 \\ \vdots & \vdots & \ddots & \vdots \\ H_m/2n & H_mH_1/4n^2 & \dots & I \end{pmatrix}.$$

Therefore, by letting $N = 2n(M-I)$, we have

$$N = \begin{pmatrix} 0 & H_1^T & \dots & H_m^T \\ H_1 & 0 & \dots & H_1H_m/2n \\ \vdots & \vdots & \ddots & \vdots \\ H_m & H_mH_1/2n & \dots & 0 \end{pmatrix}.$$

Since each pair H_i and H_j is unbiased, each of $\frac{1}{2n}H_iH_j^T$ are $\{-1,1\}$-matrices. By setting $\frac{1}{2n}H_iH_j^T = L_{ij}$ and using the Lemma 4.5, we see that the matrix N has the desired form. □

4.3 3-class Association Schemes

Association schemes derived from a set of MURB have interesting properties and they can be generalized to 5-class association schemes. In [11], the following theorem is used to present a 3-class association scheme constructed from a set of MURB. Then, we determine the intersection matrices, Krein matrices, and first and second eigenmatrices of this association scheme.

Note that if A is a $\{0,1,-1\}$-matrix, we can write $A = A^+ - A^-$, where A^+ and A^- are disjoint $\{0,1\}$-matrices.

Theorem 4.8. *Let M be the Gramian of a MURB of size $m+1$, $m \geq 2$, in $\mathbb{R}^{4n^2}$ as given in 4.2 and let $N = 2n(M-I)$. Then $I_{4n^2(m+1)}$, N^+, N^- and $I_{m+1} \otimes J_{4n^2} - I_{4n^2(m+1)}$ form an imprimitive 3-class association scheme.*

Proof. Let $p = m+1$ and $q = 4n^2$ and set $D = (I_p \otimes J_q) - I_{pq}$. Note $N^+ - N^- = N$ and $N^+ + N^- = J_{pq} - (I_p \otimes J_q)$. It suffices to prove that multiplication of any two pair of our matrices is a linear combination of all four matrices. By Lemma 4.7, we have $M^2 = (m+1)M$. Hence, we have

$$\begin{aligned} N^2 &= 4n^2(M^2 - 2M + I_{pq}) = 4n^2((m+1)M + I_{pq}) \\ &= 2n(m-1)N + 4n^2 m I_{pq}. \end{aligned} \tag{4.3}$$

From Lemma 4.7, we have

$$N^+ = \begin{pmatrix} 0 & L^+_{12} & \cdots & L^+_{1(m-1)} \\ L^+_{21} & 0 & \cdots & L^+_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ L^+_{(m+1)1} & L^+_{(m+1)2} & \cdots & 0 \end{pmatrix}, \; N^- = \begin{pmatrix} 0 & L^-_{12} & \cdots & L^-_{1(m-1)} \\ L^-_{21} & 0 & \cdots & L^-_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ L^-_{(m+1)1} & L^-_{(m+1)2} & \cdots & 0 \end{pmatrix}.$$

The regularity of L_{ij}'s implies

$$J_q L^{\pm}_{ij} = L^{\pm}_{ij} J_q = (2n^2 \pm n) J_q, \tag{4.4}$$

Therefore

$$J_{pq} N^{\pm} = N^{\pm} J_{pq} = (2n^2 \pm n) m J_{pq}. \tag{4.5}$$

Set

$$\begin{aligned} S &= (m-1)J_{pq} + I_p \otimes J_q \\ &= (m-1)J_{pq} + D + I_{pq} \\ &= mI_{pq} + (m-1)N^+ + (m-1)N^- + mD. \end{aligned}$$

From 4.4 and 4.5, we have

$$\begin{aligned} (N^+ + N^-)N^+ &= [J_{pq} - (I_p \otimes J_q)]N^+ \\ &= (2n^2 + n)S, \\ N^+(N^+ + N^-) &= N^+[J_{pq} - (I_p \otimes J_q)] \\ &= (2n^2 + n)S. \end{aligned} \tag{4.6}$$

These imply that N^+ and N^- commute. In the same way, we have

$$(N^+ + N^-)N^- = (2n^2 - n)S. \tag{4.7}$$

Adding 4.6 and 4.7 gives

$$(N^+ + N^-)^2 = 4n^2 S. \tag{4.8}$$

From 4.3 and 4.8

$$\begin{aligned} N^+N^- = N^-N^+ &= (N^+ + N^-)^2/4 - (N^+ - N^-)^2/4 \\ &= n^2 S - \frac{n}{2}(m-1)N - n^2 m I_{pq} \qquad (4.9) \\ &= (n^2 - \frac{n}{2})(m-1)N^+ + (n^2 + \frac{n}{2})(m-1)N^- + n^2 mD. \qquad (4.10) \end{aligned}$$

Subtracting 4.9 from 4.6, respectively 4.7, gives

$$\begin{aligned} (N^\pm)^2 &= (n^2 \pm n)S + \frac{n}{2}(m-1)N + n^2 m I_{pq} \\ &= (2n^2 \pm n) m I_{pq} + (n^2 + \frac{n}{2} \pm n)(m-1)N^+ \\ &+ (n^2 - \frac{n}{2} \pm n)(m-1)N^- + (n^2 \pm n)mD. \qquad (4.11) \end{aligned}$$

Next

$$\begin{aligned} D^2 &= (I_p \otimes J_q)^2 - 2(I_p \otimes J_q) + I_{pq} \\ &= (4n^2 - 2)(I_p \otimes J_q) + I_{pq} \\ &= (4n^2 - 2)D + (4n^2 - 1)I_{pq}. \qquad (4.12) \end{aligned}$$

From 4.5

$$\begin{aligned}N^+D = DN^+ &= (I_p \otimes J_q)N^+ - N^+\\&= (2n^2+n)[J_{pq} - (I_p \otimes J_q)] - N^+\\&= (2n^2+n)[N^+ + N^-] - N^+\\&= (2n^2+n-1)N^+ + (2n^2+n)N^-. \end{aligned} \tag{4.13}$$

In the same way, we have

$$\begin{aligned}N^-D &= DN^-\\&= (2n^2-n)N^+ + (2n^2-n-1)N^-. \end{aligned} \tag{4.14}$$

Therefore, matrices $I_{4n^2(m+1)}$, D, N^+, and N^- form an association scheme. Moreover, this association scheme is imprimitive because the graph of D consists of $m+1$ disjoint $4n^2$-cliques and hence, it is disconnected. □

By reviewing the proof of the Theorem 4.8, we evaluate the intersection matrices of the association scheme shown below:

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 4n^2-1 & 4n^2-2 & 0 & 0\\ 0 & 0 & 2n^2+n-1 & 2n^2+n\\ 0 & 0 & 2n^2-n & 2n^2-n-1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2n^2+n-1 & 2n^2+n \\ (2n^2+n)m & (n^2+n)m & (n^2+\frac{n}{2}+n)(m-1) & (n^2-\frac{n}{2}+n)(m-1) \\ 0 & n^2m & (n^2-\frac{n}{2})(m-1) & (n^2+\frac{n}{2})(m-1) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2n^2-n & 2n^2-n-1 \\ 0 & n^2m & (n^2-\frac{n}{2})(m-1) & (n^2+\frac{n}{2})(m-1) \\ (2n^2-n)m & (n^2-n)m & (n^2+\frac{n}{2}-n)(m-1) & (n^2-\frac{n}{2}-n)(m-1) \end{pmatrix}.$$

Evaluating the eigenvalues of these matrices, we have the first eigenmatrix P:

$$P = \begin{pmatrix} 1 & n(2n+1)m & n(2n-1)m & 4n^2-1 \\ 1 & nm & -nm & -1 \\ 1 & -n & n & -1 \\ 1 & -n(2n+1) & -n(2n-1) & 4n^2-1 \end{pmatrix}.$$

From the matrix $\frac{1}{3}P^{-1}$, we find

$$Q = \begin{pmatrix} 1 & 4n^2-1 & (4n^2-1)m & m \\ 1 & 2n-1 & -2n+1 & -1 \\ 1 & -2n-1 & 2n+1 & -1 \\ 1 & -1 & -m & m \end{pmatrix}.$$

Equation 3.3 gives us all four Krein matrices

$$B_0^* = I_4,$$

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4n^2-1 & \frac{2(2n^2-m-1)}{m+1} & \frac{4n^2}{m+1} & 0 \\ 0 & \frac{4n^2m}{m+1} & \frac{(4n^2-2)m-2}{m+1} & 4n^2-1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{4n^2m}{m+1} & \frac{(4n^2-2)m-2}{m+1} & 4n^2-1 \\ 4mn^2-m & \frac{2(2m^2n^2-m^2-m)}{(m+1)} & \frac{2(2m^2n^2-m^2+1)}{(m+1)} & 4(m-1)n^2-m+1 \\ 0 & m & m-1 & 0 \end{pmatrix},$$

$$B_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & m & m-1 & 0 \\ m & 0 & 0 & m-1 \end{pmatrix}.$$

Specifically the following system of linear equations gives us the first row of B_1^*:

$$\begin{aligned} q_{01}^{(0)} + (4n^2-1)q_{01}^{(1)} + m(4n^2-1)q_{01}^{(2)} + mq_{01}^{(3)} &= 4n^2-1 \\ q_{01}^{(0)} + (2n-1)q_{01}^{(1)} + (-2n+1)q_{01}^{(2)} - q_{01}^{(3)} &= 2n-1 \\ q_{01}^{(0)} - (2n+1)q_{01}^{(1)} + (2n+1)q_{01}^{(2)} - q_{01}^{(3)} &= -2n-1 \\ q_{01}^{(0)} - q_{01}^{(1)} - mq_{01}^{(2)} + mq_{01}^{(3)} &= -1 \end{aligned}$$

All systems of linear equations that we need to evaluate $B_0^*, \ldots, B_3^*$, are stated in Appendix B.

Chapter 5

Bush-type Hadamard Matrices

The theory of Bush-type Hadamard matrices is a special case of regular Hadamard matrices. In Theorem 4.8 in Section 4.3, we gave the construction of a class of 3-class association schemes derived from mutually unbiased regular Hadamard matrices. If we use mutually unbiased Bush-type Hadamard (MUBH) matrices instead, we have the 3-class association schemes of the Theorem 4.8, and moreover we can generalize this association scheme to 5-class association schemes. We describe this process in Section 5.2, and in Section 5.3, we evaluate all intersection matrices, Krein matrices, and eigenmatrices of the 5-class association scheme.

In [7], it is explained how Bush-type Hadamard matrices of skew-type form 3-class association schemes whose relations are not symmetric. In the next Section, a different method, suggested by my supervisor, is presented to show a similar result that has appeared in [7]. In Section 5.4, we study the upper bound for the number of mutually unbiased Bush-type Hadamard matrices of order $4n^2$ [18], and then provide an example in which this upper bound is attained [12]. In Section 5.5, we study how to generate sets of mutually unbiased Bush-type Hadamard matrices from association schemes [18].

Definition 5.1. [13] A *Bush-type* Hadamard matrix is a Hadamard matrix $H = [H_{ij}]$ of order n^2 with sub blocks H_{ij} of size n such that $H_{ii} = J_n$ and $H_{ij}J_n = J_nH_{ij} = 0$, $1 \leq i \neq j \leq n$.

5.1 Bush-type Matrices of skew-type and Association Schemes

All matrices in the definition of an association scheme provided in this thesis are symmetric. There are also (non-symmetric) association schemes where the adjacency matrices are not required to be symmetric. However, the transpose of each of these matrices must be in the set of these matrices. From Bush-type Hadamard matrices of skew-type, One can construct non-symmetric association schemes whose adjacency matrices commute with each other [7]. We now present a method, suggested by my supervisor, to show a similar result that has appeared in [7].

Let $L = (l_{ij})$ be a symmetric Latin square of order $n = 3(4^k)$, $k \geq 1$, with 1 on diagonal and $H = (h_{ij})$ be a Hadamard matrix of order n. Consider the matrices $C_i = r_i^T r_i$, where r_i is the i-th row of H, $1 \leq i \leq 3(4^k)$.

For each i, the matrix $C_i = (c_{rs}^{(i)})$ is symmetric because we have

$$c_{rs}^{(i)} = h_{ir}h_{is} = h_{is}h_{ir} = c_{sr}^{(i)}.$$

Suppose $l_{rs} = i$. If $s \geq r$, we replace i of the matrix L with C_i, and if $r > s$, we replace i of the matrix L with $-C_i$ to obtain matrix O of order n^2. Since H is normalized and all elements on diagonal of L are 1, O is a Bush-type Hadamard matrix of skew-type. Let $K = O - I_n \otimes J_n$. Hence, $K = (J_n - I_n) \otimes K_{ij}$, where K_{ij}'s have row sums and column sums zero. So $(I_n \otimes J_n)K = K(I_n \otimes J_n) = 0$.

Theorem 5.2. *The matrix K gives a non-symmetric commutative 3-class association scheme, with classes A_0, A_1, A_2, and A_3 defined as follows:*

1. $A_0 = I_{n^2}$;
2. $A_1 = I_n \otimes J_n - I_{n^2}$;
3. $A_2 = K^+$;
4. $A_3 = K^-$.

Proof. We prove this by showing that the multiplication of any pairs of elements of $\{A_0, A_1, A_2, A_3\}$ is a linear combination of A_0, A_1, A_2, and A_3. Cases involving A_0 are trivial. We also have $A_1^2 = (n-1)A_0 + (n-2)A_1$. To prove the other cases, we note that $A_2 + A_3 = (J_n - I_n) \otimes J_n$,

$A_2 = (J_n - I_n) \otimes D_{ij}$, and $A_3 = (J_n - I_n) \otimes R_{ij}$, where D_{ij}'s and R_{ij}'s are regular matrices with regularity $n/2$. This implies

$$J_n D_{ij} = D_{ij} J_n = J_n R_{ij} = R_{ij} J_n = \frac{n}{2} J_n.$$

Therefore,

$$\begin{aligned}(I_n \otimes J_n)A_2 = A_2(I_n \otimes J_n) &= (I_n \otimes J_n)A_3 = A_3(I_n \otimes J_n) \\ &= \frac{n}{2}(I_n \otimes J_n) = \frac{n}{2}(A_2 + A_3). \qquad (5.1)\end{aligned}$$

On the other hand, we have $I_n \otimes J_n = A_0 + A_1$. Hence

$$\begin{aligned}A_1A_2 = A_2A_1 &= (\frac{n}{2} - 1)A_2 + \frac{n}{2}A_3, \\ A_1A_3 = A_3A_1 &= \frac{n}{2}A_2 + (\frac{n}{2} - 1)A_3.\end{aligned}$$

We also know

$$\begin{aligned}(A_2 + A_3)A_2 &= ((J_n - I_n) \otimes J_n)((J_n - I_n) \otimes D_{ij}) \\ &= (I_n + (n-2)J_n) \otimes \frac{n}{2}J_n, \qquad (5.2) \\ (A_2 + A_3)A_3 &= ((J_n - I_n) \otimes J_n)((J_n - I_n) \otimes R_{ij}) \\ &= (I_n + (n-2)J_n) \otimes \frac{n}{2}J_n, \qquad (5.3)\end{aligned}$$

which implies A_2 and A_3 commute. Adding 5.2 and 5.3 gives

$$(A_2 + A_3)^2 = 2((I_n + (n-2)J_n) \otimes \frac{n}{2}J_n).$$

Next, we evaluate $(A_2 - A_3)^2$. Since

$$A_2 - A_3 = K = O - I_n \otimes J_n,$$

it follows

$$\begin{aligned}(A_2 - A_3)^2 &= (O - I_n \otimes J_n)(O - I_n \otimes J_n)\\ &= OO - O(I_n \otimes J_n) - (I_n \otimes J_n)O + (I_n \otimes J_n)(I_n \otimes J_n)\\ &= -n^2 I_{n^2} + n(I_n \otimes J_n).\end{aligned}$$

Hence,

$$\begin{aligned}A_2A_3 = A_3A_2 &= \frac{(A_2 + A_3)^2}{4} - \frac{(A_2 - A_3)^2}{4}\\ &= \frac{1}{2}((I_n + (n-2)J_n) \otimes \frac{n}{2}J_n) - \frac{1}{4}(-n^2 I_{n^2} + n(I_n \otimes J_n))\\ &= (\frac{n^2}{2} - \frac{n}{2})A_0 + \frac{n}{4}(n-2)(A_1 + A_2 + A_3).\end{aligned}$$

This implies

$$A_2^2 = A_3^2 = \frac{n^2}{4}A_1 + (\frac{n^2}{4} - \frac{n}{2})(A_2 + A_3).$$

□

5.2 Bush-type Hadamard Matrices and Association Schemes

Association schemes of class five which are constructed from MUBH matrices are important in the sense that their Krein parameters give an upper bound on the number of mutually unbiased Bush-type Hadamard matrices. In this Section, we generalize the association scheme of Theorem 4.8 to a 5-class association scheme by using a set of MUBH matrices. To develop this idea, we need to mention the following lemma from [18].

Lemma 5.3. *The multiplication of two unbiased Bush-type Hadamard matrices H and K of order $4n^2$ is of the form of $HK^T = 2nL$, where L is a Bush-type Hadamard matrix.*

Proof. If $D = I_{2n} \otimes J_{2n}$, then we have L is of Bush-type if and only if $LD = DL = 2nD$. On the other hand, we have

$$LD = \frac{1}{2n} HK^T D = \frac{1}{2n} H(2nD) = HD = 2nD.$$

In the same way, we have $DL = 2nD$. This implies L is a Bush-type Hadamard matrix. □

In the following theorem, we generalize the association scheme of the Theorem 4.8 to a 5-class association scheme:

Theorem 5.4. *Let N^+ and N^- be the matrices given in Theorem 4.8. Define*

$A_0 = I_{4n^2(m+1)}$,

$A_1 = I_{m+1} \otimes I_{2n} \otimes (J_{2n} - I_{2n})$,

$A_2 = I_{m+1} \otimes (J_{2n} - I_{2n}) \otimes J_{2n}$,

$A_3 = (J_{m+1} - I_{m+1}) \otimes I_{2n} \otimes J_{2n}$,

$A_4 = N^+ - A_3$, *and*

$A_5 = N^-$.

Then $A_0, A_1, \ldots, A_5$ form an imprimitive 5-class association scheme.

Proof. From Theorem 4.8, we know

$$A_5^2 = (N^-)^2 = (2n^2 - n)mI_{pq} + (n^2 + \frac{n}{2} - n)(m-1)(A_3 + A_4) + (n^2 - \frac{n}{2} - n)(m-1)A_5 + (n^2 - n)m(A_1 + A_2). \quad (5.4)$$

Also

$$A_0 + A_1 = I_{m+1} \otimes I_{2n} \otimes J_{2n};$$

$$A_4 + A_5 = (J_{m+1} - I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes J_{2n};$$

$$A_1 + A_2 = D;$$

$$A_4 + A_3 = N^+,$$

where $D = I_{m+1} \otimes J_{4n^2} - I_{(m+1)4n^2}$. The case A_0 is obvious. For other cases, we have

$$\begin{aligned}
A_1A_1 &= (I_{m+1} \otimes I_{2n} \otimes (J_{2n} - I_{2n}))(I_{m+1} \otimes I_{2n} \otimes (J_{2n} - I_{2n})) \\
&= I_{m+1} \otimes I_{2n} \otimes ((2n-2)J_{2n} + I_{2n}) \\
&= I_{m+1} \otimes I_{2n} \otimes ((2n-2)J_{2n} - (2n-2)I_{2n} + (2n-2)I_{2n} + I_{2n}) \\
&= (I_{m+1} \otimes I_{2n} \otimes (2n-2)(J_{2n} - I_{2n})) + (I_{m+1} \otimes I_{2n} \otimes (2n-1)I_{2n}) \\
&= (2n-2)(I_{m+1} \otimes I_{2n} \otimes (J_{2n} - I_{2n})) + (2n-1)(I_{m+1} \otimes I_{2n} \otimes I_{2n}) \\
&= (2n-1)A_0 + (2n-2)A_1.
\end{aligned}$$

$$\begin{aligned}
A_2A_2 &= (I_{m+1} \otimes (J_{2n} - I_{2n}) \otimes J_{2n})(I_{m+1} \otimes (J_{2n} - I_{2n}) \otimes J_{2n}) \\
&= I_{m+1} \otimes ((2n-2)J_{2n} + I_{2n}) \otimes 2nJ_{2n} \\
&= I_{m+1} \otimes ((2n-2)J_{2n} - (2n-2)I_{2n} + (2n-2)I_{2n} + I_{2n}) \otimes 2nJ_{2n} \\
&= (I_{m+1} \otimes (2n-2)(J_{2n} - I_{2n}) \otimes 2nJ_{2n}) + (I_{m+1} \otimes (2n-1)I_{2n} \otimes 2nJ_{2n}) \\
&= 2n(2n-2)(I_{m+1} \otimes (J_{2n} - I_{2n}) \otimes J_{2n}) + 2n(2n-1)(I_{m+1} \otimes I_{2n} \otimes J_{2n}) \\
&= 2n(2n-2)A_2 + 2n(2n-1)(A_0 + A_1).
\end{aligned}$$

$$\begin{aligned}
A_3A_3 &= ((J_{m+1}-I_{m+1})\otimes I_{2n}\otimes J_{2n})((J_{m+1}-I_{m+1})\otimes I_{2n}\otimes J_{2n}) \\
&= ((m-1)J_{m+1}+I_{m+1})\otimes I_{2n}\otimes 2nJ_{2n} \\
&= ((m-1)J_{m+1}-(m-1)I_{m+1}+(m-1)I_{m+1}+I_{m+1})\otimes I_{2n}\otimes 2nJ_{2n} \\
&= ((m-1)(J_{m+1}-I_{m+1})\otimes I_{2n}\otimes 2nJ_{2n})+(mI_{m+1}\otimes I_{2n}\otimes 2nJ_{2n}) \\
&= 2n(m-1)A_3+2mn(A_0+A_1).
\end{aligned}$$

$$\begin{aligned}
A_1A_2 = A_2A_1 &= (I_{m+1}\otimes I_{2n}\otimes (J_{2n}-I_{2n}))(I_{m+1}\otimes (J_{2n}-I_{2n})\otimes J_{2n}) \\
&= I_{m+1}\otimes (J_{2n}-I_{2n})\otimes (2n-1)J_{2n} \\
&= (2n-1)A_2.
\end{aligned}$$

$$\begin{aligned}
A_1A_3 = A_3A_1 &= (I_{m+1}\otimes I_{2n}\otimes (J_{2n}-I_{2n}))((J_{m+1}-I_{m+1})\otimes I_{2n}\otimes J_{2n}) \\
&= (J_{m+1}-I_{m+1})\otimes I_{2n}\otimes (2n-1)J_{2n} \\
&= (2n-1)A_3.
\end{aligned}$$

$$\begin{aligned}
A_2A_3 = A_3A_2 &= (I_{m+1}\otimes (J_{2n}-I_{2n})\otimes J_{2n})((J_{m+1}-I_{m+1})\otimes I_{2n}\otimes J_{2n}) \\
&= (J_{m+1}-I_{m+1})\otimes (J_{2n}-I_{2n})\otimes 2nJ_{2n} \\
&= 2n(A_4+A_5).
\end{aligned}$$

Since $N^+N^- = N^-N^+ = (n^2-n/2)(m-1)N^+ + (n^2+n/2)(m-1)N^- + n^2mD$, $A_5 = N^-$, and $A_4+A_3 = N^+$, we have

$$A_5(A_4+A_3) = (n^2-\frac{n}{2})(m-1)(A_3+A_4)+(n^2+\frac{n}{2})(m-1)A_5+n^2m(A_1+A_2). \quad (5.5)$$

On the other hand, we have

$$A_5 = (J_{m+1} - I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes O_{ij},$$

and

$$A_4 = (J_{m+1} - I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes T_{ij},$$

where O_{ij}'s and T_{ij}'s are blocks of order $2n$ satisfying $O_{ij}J_{2n} = n$ and $T_{ij}J_{2n} = n$. Therefore,

$$\begin{aligned}
A_5A_3 = A_3A_5 &= ((J_{m+1} - I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes O_{ij})((J_{m+1} - I_{m+1}) \otimes I_{2n} \otimes J_{2n}) \\
&= ((m-1)J_{m+1} + I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes nJ_{2n} \\
&= ((m-1)J_{m+1} - (m-1)I_{m+1} + (m-1)I_{m+1} + I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes nJ_{2n} \\
&= ((m-1)(J_{m+1} - I_{m+1}) \otimes (J_{2n} - I_{2n}) \otimes nJ_{2n}) + (mI_{m+1} \otimes (J_{2n} - I_{2n}) \otimes nJ_{2n}) \\
&= (m-1)n(A_4 + A_5) + mnA_2.
\end{aligned}$$

In the same way, we find

$$A_4A_3 = A_3A_4 = (m-1)n(A_4 + A_5) + mnA_2.$$

By 5.5, we have

$$\begin{aligned}
A_5A_4 = n^2mA_1 + m(n^2 - n)A_2 + (n^2 - \frac{n}{2})(m-1)A_3 \\
+ (n^2 - \frac{n}{2} - n)(m-1)A_4 + (n^2 + \frac{n}{2} - n)(m-1)A_5.
\end{aligned}$$

Since we have

$$\begin{aligned}
N^+N^+ &= (2n^2+n)mI + (n^2+\frac{n}{2}+n)(m-1)N^+ \\
&\quad + (n^2-\frac{n}{2}+n)(m-1)N^- + (n^2+n)mD, \\
N^+ &= A_4 + A_3,
\end{aligned}$$

and $D = A_1 + A_2$, we imply

$$\begin{aligned}
(A_4+A_3)(A_4+A_3) &= (2n^2+n)mI + (n^2+\frac{n}{2}+n)(m-1)(A_4+A_3) \\
&\quad + (n^2-\frac{n}{2}+n)(m-1)A_5 + (n^2+n)m(A_2+A_1).
\end{aligned}$$

Since $A_4A_4 = (A_4+A_3)(A_4+A_3) - 2A_3A_4 - A_3A_3$, we evaluate

$$\begin{aligned}
A_4A_4 &= (2n^2-n)mI + (n^2-n)mA_1 + (n^2-n)mA_2 + (n^2+\frac{n}{2}-n)(m-1)A_3 \\
&\quad + (n^2+\frac{n}{2}-n)(m-1)A_4 + (n^2-\frac{n}{2}-n)(m-1)A_5.
\end{aligned}$$

We also evaluate

$$\begin{aligned}
A_5A_2 = A_2A_5 &= ((J_{m+1}-I_{m+1})\otimes(J_{2n}-I_{2n})\otimes O_{ij})(I_{m+1}\otimes(J_{2n}-I_{2n})\otimes J_{2n}) \\
&= (J_{m+1}-I_{m+1})\otimes((2n-2)J_{2n}+I_{2n})\otimes nJ_{2n} \\
&= (J_{m+1}-I_{m+1})\otimes((2n-2)J_{2n}-(2n-2)I_{2n}+(2n-2)I_{2n}+I_{2n})\otimes nJ_{2n} \\
&= (2n-2)n(A_4+A_5) + (2n-1)nA_3.
\end{aligned}$$

This implies

$$A_4A_2 = A_2A_4 = (2n-2)n(A_4+A_5) + (2n-1)nA_3.$$

Since

$$DN^+ = N^+D = (2n^2+n-1)N^+ + (2n^2+n)N^-,$$

we have

$$(A_1+A_2)(A_3+A_4) = (2n^2+n-1)(A_3+A_4) + (2n^2+n)A_5.$$

Since

$$A_1A_4 = A_4A_1 = (A_1+A_2)(A_3+A_4) - A_1A_3 - A_2A_3 - A_2A_4,$$

we have

$$A_1A_4 = A_4A_1 = (n-1)A_4 + nA_5.$$

Since we have

$$DN^- = N^-D = (2n^2-n)N^+ + (2n^2-n-1)N^-,$$

then

$$(A_2+A_1)A_5 = (2n^2-n)(A_3+A_4) + (2n^2-n-1)A_5.$$

Finally, we have

$$A_5A_1 = A_1A_5 = (A_2+A_1)A_5 - A_2A_5.$$

In a summary, we have

$$
\begin{aligned}
A_1A_1 &= (2n-1)A_0 + (2n-2)A_1,\\
A_1A_2 &= A_2A_1 = (2n-1)A_2,\\
A_1A_3 &= A_3A_1 = (2n-1)A_3,\\
A_1A_4 &= A_4A_1 = (n-1)A_4 + nA_5,\\
A_1A_5 &= A_5A_1 = nA_4 + (n-1)A_5,\\
A_2A_2 &= 2n(2n-1)I + 2n(2n-1)A_1 + 2n(2n-2)A_2,\\
A_2A_3 &= A_3A_2 = 2n(A_4 + A_5),\\
A_2A_4 &= A_4A_2 = (2n-1)nA_3 + (2n-2)n(A_4 + A_5),\\
A_2A_5 &= A_5A_2 = (2n-1)nA_3 + (2n-2)n(A_4 + A_5),\\
A_3A_3 &= 2mn(I + A_1) + 2n(m-1)A_3,\\
A_3A_4 &= A_4A_3 = mnA_2 + (m-1)n(A_4 + A_5),\\
A_3A_5 &= A_5A_3 = mnA_2 + (m-1)n(A_4 + A_5),\\
A_4A_4 &= (2n^2-n)mI + (n^2-n)mA_1 + (n^2-n)mA_2 + (n^2+\frac{n}{2}-n)(m-1)A_3\\
&\quad + (n^2-\frac{n}{2})(m-1)A_4 + (n^2-\frac{3n}{2})(m-1)A_5,\\
A_4A_5 &= A_5A_4 = n^2mA_1 + m(n^2-n)A_2 + (n^2-\frac{n}{2})(m-1)A_3\\
&\quad + (n^2-\frac{3n}{2})(m-1)A_4 + (n^2-\frac{n}{2})(m-1)A_5, and\\
A_5A_5 &= (2n^2-n)(m)I + (n^2-n)(m)(A_1 + A_2)\\
&\quad + (n^2-\frac{n}{2})(m-1)(A_3 + A_4) + (n^2-\frac{3n}{2})(m-1)A_5.
\end{aligned}
$$

□

5.3 Eigenmatrices, Intersection Matrices, and Krein Matrices

The first and second eigenmatrices of an association scheme are really important, indeed, they are the change of bases matrices of two bases of the Bose-Mesner algebra. In this Section, we study the association schemes of the Theorem 5.4 by evaluating the first and second eigenmatrices. When the linear combinations of the product of adjacency matrices of an association scheme are available, we can easily evaluate all intersection matrices of the scheme. The intersection matrices of the association scheme of Theorem 5.4 are shown below.

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2n-1 & 2n-2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2n-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2n-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-1 & n \\ 0 & 0 & 0 & 0 & n & n-1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2n-1 & 0 & 0 & 0 \\ 2n(2n-1) & 2n(2n-1) & 2n(2n-1) & 0 & 0 & \\ 0 & 0 & 0 & 0 & 2n & 2n \\ 0 & 0 & 0 & n(2n-1) & n(2n-2) & n(2n-2) \\ 0 & 0 & 0 & n(2n-1) & n(2n-2) & n(2n-2) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2n-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2n & 2n \\ 2mn & 2mn & 0 & 2n(m-1) & 0 & 0 \\ 0 & 0 & mn & 0 & n(m-1) & n(m-1) \\ 0 & 0 & mn & 0 & n(m-1) & n(m-1) \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & n-1 & n \\ 0 & 0 & 0 & n(2n-1) & n(2n-2) & n(2n-2) \\ 0 & 0 & mn & 0 & n(m-1) & n(m-1) \\ m(2n^2-n) & m(n^2-n) & m(n^2-n) & (m-1)(n^2+\frac{n}{2}-n) & (m-1)(n^2-\frac{n}{2}) & (m-1)(n^2-\frac{3n}{2}) \\ 0 & mn^2 & m(n^2-n) & (m-1)(n^2-\frac{n}{2}) & (m-1)(n^2-\frac{3n}{2}) & (m-1)(n^2-\frac{n}{2}) \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & n & n-1 \\ 0 & 0 & 0 & n(2n-1) & n(2n-2) & n(2n-2) \\ 0 & 0 & mn & 0 & n(m-1) & n(m-1) \\ 0 & mn^2 & m(n^2-n) & (m-1)(n^2-\frac{n}{2}) & (m-1)(n^2-\frac{3n}{2}) & (m-1)(n^2-\frac{n}{2}) \\ m(2n^2-n) & m(n^2-n) & m(n^2-n) & (m-1)(n^2-\frac{n}{2}) & (m-1)(n^2-\frac{n}{2}) & (m-1)(n^2-\frac{3n}{2}) \end{pmatrix}.$$

By evaluating the eigenvalues of these matrices, we have eigenmatrix

$$P = \begin{pmatrix} 1 & 2n-1 & 2n(2n-1) & 2nm & nm(2n-1) & nm(2n-1) \\ 1 & -1 & 0 & 0 & nm & -nm \\ 1 & 2n-1 & -2n & 2nm & -nm & -nm \\ 1 & 2n-1 & -2n & -2n & n & n \\ 1 & -1 & 0 & 0 & -n & n \\ 1 & 2n-1 & 2n(2n-1) & -2n & -n(2n-1) & -n(2n-1) \end{pmatrix}.$$

Computing $\frac{1}{6}P^{-1}$ gives

$$Q = \begin{pmatrix} 1 & 2n(2n-1) & 2n-1 & m(2n-1) & 2nm(2n-1) & m \\ 1 & -2n & 2n-1 & m(2n-1) & -2nm & m \\ 1 & 0 & -1 & -m & 0 & m \\ 1 & 0 & 2n-1 & -2n+1 & 0 & -1 \\ 1 & 2n & -1 & 1 & -2n & -1 \\ 1 & -2n & -1 & 1 & 2n & -1 \end{pmatrix}.$$

Then, Equation 3.3 gives all six Krein matrices.

$$B_0^* = I_6,$$

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ (4n^2-2n) & \frac{4(n^2-n)}{(m+1)} & \frac{-2((m+1)n-2n^2)}{(m+1)} & \frac{4n^2}{(m+1)} & \frac{4(n^2-n)}{(m+1)} & 0 \\ 0 & \frac{-(m-2n+1)}{(m+1)} & 0 & 0 & \frac{2n}{(m+1)} & 0 \\ 0 & \frac{2mn}{(m+1)} & 0 & 0 & \frac{(2mn-m-1)}{(m+1)} & 0 \\ 0 & \frac{4(mn^2-mn)}{(m+1)} & \frac{4mn^2}{(m+1)} & \frac{2(2mn^2-(m+1)*n)}{(m+1)} & \frac{4(mn^2-mn)}{(m+1)} & (4n^2-2n) \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-(m-2n+1)}{(m+1)} & 0 & 0 & \frac{2n}{(m+1)} & 0 \\ (2n-1) & 0 & (2n-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & (2n-2) & 0 & (2n-1) \\ 0 & \frac{2mn}{(m+1)} & 0 & 0 & \frac{(2mn-m-1)}{(m+1)} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{2mn}{(m+1)} & 0 & 0 & \frac{(2mn-m-1)}{(m+1)} & 0 \\ 0 & 0 & 0 & (2n-2) & 0 & (2n-1) \\ 2mn-m & 0 & 2mn-2m & 2(m-1)n-2m+2 & 0 & 2(m-1)n-m+1 \\ 0 & \frac{(2m^2n-m^2-m)}{(m+1)} & 0 & 0 & \frac{(2m^2n-m^2+1)}{(m+1)} & 0 \\ 0 & 0 & m & m-1 & 0 & 0 \end{pmatrix},$$

$$B_4^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{4(mn^2-mn)}{(m+1)} & \frac{4mn^2}{(m+1)} & \frac{2(2mn^2-(m+1)n)}{(m+1)} & \frac{4(mn^2-mn)}{(m+1)} & (4n^2-2n) \\ 0 & \frac{2mn}{(m+1)} & 0 & 0 & \frac{(2mn-m-1)}{(m+1)} & 0 \\ 0 & \frac{(2m^2n-m^2-m)}{(m+1)} & 0 & 0 & \frac{(2m^2n-m^2+1)}{(m+1)} & 0 \\ 4mn^2-2mn & \frac{4(m^2n^2-m^2n)}{(m+1)} & \frac{2(2m^2n^2-(m^2+m)n)}{(m+1)} & \frac{2(2m^2n^2-(m^2-1)n)}{(m+1)} & \frac{4(m^2n^2-m^2n)}{(m+1)} & a \\ 0 & m & 0 & 0 & m-1 & 0 \end{pmatrix},$$

where $a = 4(m-1)n^2 - 2(m-1)n$; and

$$B_5^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & m & m-1 & 0 & 0 \\ 0 & m & 0 & 0 & m-1 & 0 \\ m & 0 & 0 & 0 & 0 & m-1 \end{pmatrix}.$$

For instance, the following system of linear equations allows us to determine the first row of B_5^*.

$$q_{05}^{(0)} + (2n)(2n-1)q_{05}^{(1)} + (2n-1)q_{05}^{(2)} + m(2n-1)q_{05}^{(3)} + m(2n)(2n-1)q_{05}^{(4)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} - 2nq_{05}^{(1)} + (2n-1)q_{05}^{(2)} + m(2n-1)q_{05}^{(3)} - 2mnq_{05}^{(4)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} - q_{05}^{(2)} - mq_{05}^{(3)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} + (2n-1)q_{05}^{(2)} - (-2n+1)q_{05}^{(3)} - q_{05}^{(5)} = -1$$
$$q_{05}^{(0)} + 2nq_{05}^{(1)} - q_{05}^{(2)} + q_{05}^{(3)} - 2nq_{05}^{(4)} - q_{05}^{(5)} = -1$$
$$q_{05}^{(0)} - 2nq_{05}^{(1)} - q_{05}^{(2)} + q_{05}^{(3)} + 2nq_{05}^{(4)} - q_{05}^{(5)} = -1$$

All systems of linear equations that we need to evaluate $B_0^*, \ldots, B_5^*$, appear in Appendix C.

Example 5.5. [12] In this example, we use a Gramian of a MURB to make a 5-class association scheme. First we construct the MURB. We start with the following Hadamard matrix of order 4.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix}$$

Let $C_i = r_i^T r_i$, where r_i is the i-th row of H. Then we have

$$C_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 1 & - & - \\ 1 & 1 & - & - \\ - & - & 1 & 1 \\ - & - & 1 & 1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 1 & - & 1 & - \\ - & 1 & - & 1 \\ 1 & - & 1 & - \\ - & 1 & - & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 1 & - & - & 1 \\ - & 1 & 1 & - \\ - & 1 & 1 & - \\ 1 & - & - & 1 \end{pmatrix}.$$

For the next step, we consider three following *MSLS*:

$$MSLS_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, MSLS_2 = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, MSLS_3 = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

By replacing i with C_i in $MSLS_j$, $1 \leq i \leq 4$ and $1 \leq j \leq 3$, we make a MURH whose elements $\{H_1, H_2, H_3\}$ are of order 16 as shown below:

$$
H_1 = \left(\begin{array}{cccc@{\quad}cccc@{\quad}cccc@{\quad}cccc}
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 \\
\\
1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 \\
\\
1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
- & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
\\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
$$

$$H_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 \\
 & & & & & & & & & & & & & & & \\
1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - \\
- & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - \\
- & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 \\
 & & & & & & & & & & & & & & & \\
1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - \\
- & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 \\
 & & & & & & & & & & & & & & & \\
1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 \\
1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

$$H_3 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 \\
 & & & & & & & & & & & & & & & \\
1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - \\
1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - \\
- & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
 & & & & & & & & & & & & & & & \\
1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 \\
- & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 \\
 & & & & & & & & & & & & & & & \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 \\
1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 \\
- & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 \\
- & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

Then the set $\{I, \frac{1}{4}H_1, \frac{1}{4}H_2, \frac{1}{4}H_3\}$ is a set of MURB. The Gramian matrix M of $\{I, \frac{1}{4}H_1, \frac{1}{4}H_2, \frac{1}{4}H_3\}$ gives a 5 class association scheme with the classes explained in Theorem 5.4. The adjacency matrices of this association scheme and the elements of the basis of principal idempotents of the Bose-Mesner algebra are of order 64. All these 12 matrices are listed in Appendix A.

We study this association scheme by providing the parameters and matrices related to it. We have $n = 2$ and $m = 3$. Hence, the first and second eigenmatrices of scheme are:

$$P = \begin{pmatrix} 1 & 3 & 12 & 12 & 18 & 18 \\ 1 & -1 & 0 & 0 & 6 & -6 \\ 1 & 3 & -4 & 12 & -6 & -6 \\ 1 & 3 & -4 & -4 & 2 & 2 \\ 1 & -1 & 0 & 0 & -2 & 2 \\ 1 & 3 & 12 & -4 & -6 & -6 \end{pmatrix}, \; Q = \begin{pmatrix} 1 & 12 & 3 & 9 & 36 & 3 \\ 1 & -4 & 3 & 9 & -12 & 3 \\ 1 & 0 & -1 & -3 & 0 & 3 \\ 1 & 0 & 3 & -3 & 0 & -1 \\ 1 & 4 & -1 & 1 & -4 & -1 \\ 1 & -4 & -1 & 1 & 4 & -1 \end{pmatrix}$$

and the six Krein matrices are as follows:

$$B_0^* = I_6,$$

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 12 & 2 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 6 & 12 & 8 & 6 & 12 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; B_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 9 & 0 & 6 & 4 & 0 & 6 \\ 0 & 6 & 0 & 0 & 7 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \end{pmatrix}, B_4^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 6 & 12 & 8 & 6 & 12 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 6 & 0 & 0 & 7 & 0 \\ 36 & 18 & 24 & 28 & 18 & 24 \\ 0 & 3 & 0 & 0 & 2 & 0 \end{pmatrix},$$

$$B_5^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The multiplications among elements of the set $\{A_0, A_1, \ldots, A_5\}$ are shown below.

$A_0A_0 = I$, $A_0A_1 = A_1$, $A_0A_2 = A_2$, $A_0A_3 = A_3$,$A_0A_4 = A_4$, $A_0A_5 = A_5$,

$A_1A_1 = 3I + 2A_1$, $A_1A_2 = 3A_2$, $A_1A_3 = 3A_3$, $A_1A_4 = A_4 + 2A_5$, $A_1A_5 = 2A_4 + A_5$,

$A_2A_2 = 12I + 12A_1 + 8A_2$, $A_2A_3 = 4A_4 + 4A_5$, $A_2A_4 = 6A_3 + 4A_4 + 4A_5$,

$A_2A_5 = 6A_3 + 4A_4 + 4A_5$, $A_3A_3 = 12I + 12A_1 + 8A_3$, $A_3A_4 = 6A_2 + 4A_4 + 4A_5$,

$A_3A_5 = 6A_2 + 4A_4 + 4A_5$, $A_4A_4 = 18I + 6A_1 + 6A_2 + 6A_3 + 6A_4 + 2A_5$,

$A_4A_5 = 12A_1 + 6A_2 + 6A_3 + 2A_4 + 6A_5$, $A_5A_5 = 18I + 6A_1 + 6A_2 + 6A_3 + 6A_4 + 2A_5$.

This association scheme is imprimitive because the graph whose adjacency matrix is A_1 is disconnected. Indeed, it includes 16 disjoint cliques of size 4. The choice of $\mathfrak{I}$, satisfying $\sum_{i\in\mathfrak{I}} A_i = I_p \otimes J_q$ for some p and q, is not unique. We choose $\mathfrak{I} = \{0,1,2\}$, so $\sum_{i\in\mathfrak{I}} A_i = I_4 \otimes J_{16}$. The equivalence classes are $\mathfrak{I}_0 = \mathfrak{I} = \{0,1,2\}$ and $\mathfrak{I}_1 = \{3,4,5\}$ with the property that $\sum_{i\in\mathfrak{I}_1} A_i = (J_4 - I_4)\otimes J_{16}$. Since we have

$$\sum_{i\in\mathfrak{I}_0} A_i = I_4 \otimes J_{16}$$

and

$$\sum_{i\in\mathfrak{I}_1} A_i = (J_4 - I_4)\otimes J_{16},$$

the matrices $B_0 = I_4$ and $B_1 = J_4 - I_4$ form the quotient association scheme.

5.4 Upper Bound on the Number of MUBH Matrices

In this Section, we study a method of evaluating an upper bound for the number of MUBH matrices of order $4n^2$ [18]. Then, an example from [12] is provided in which a set of MUBH matrices is constructed whose cardinality is equal to the upper bound.

Kharaghani and Suda [18] determined the upper bound on the number of MUBH matrices as follows:

Theorem 5.6. *The number of MUBH matrices of order* $4n^2$ *is at most* $2n-1$.

Proof. Suppose m is the number of MUBH matrices of order $4n^2$. By Theorem 5.4, we have a 5-class association scheme made by this MUBH. By looking at B_1^* of this association scheme appeared in page 60, we have $q_{12}^2 = \frac{-(m-2n+1)}{(m+1)}$. Lemma 3.4 shows that the Krein parameters of an association scheme are non-negative. Therefore, $2n-m-1$ must be non-negative. Hence, we conclude $m \leq 2n-1$. □

So far, every item is more or less a quotation [12].

Theorem 5.7. *If there are m mutually suitable Latin squares of size* $2n$ *with all one entries on diagonal, where* $2n$ *is the order of a Hadamard matrix, then there are m mutually unbiased Bush-type Hadamard matrices of order* $4n^2$.

Proof. Let $C_0,\ldots,C_{2n-1}$ be the matrices corresponding to the normalized Hadamard matrix of order $2n$. We may assume that all Latin squares are on the set $\{0,1,\ldots,2n-1\}$ and their rows and columns are all labeled by the set. Replacing the entry i in each of the Latin squares by the matrix C_i, $0 \leq i \leq 2n-1$ will result in m mutually unbiased Bush-type Hadamard matrices of order $4n^2$. □

Example 5.8. [12] In the first part of the Example 5.5, we made a set of MUBH matrices of order 16 with three elements. By Theorem 5.6, we reached the upper bound on the number of mutually unbiased Bush-type Hadamard matrices of order 16. The matrices H_1, H_2, H_3 are mutually unbiased Bush-type Hadamard matrices which all are symmetric and the set

$$\{I_{16}, \frac{1}{4}H_1, \frac{1}{4}H_2, \frac{1}{4}H_3\}$$

forms a group under matrix multiplication. We note that H_1, H_2, H_3 have the following form.

$$H_1 = \begin{pmatrix} C_0 & C_2 & C_3 & C_1 \\ C_2 & C_0 & C_1 & C_3 \\ C_3 & C_1 & C_0 & C_2 \\ C_1 & C_3 & C_2 & C_0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} C_0 & C_3 & C_1 & C_2 \\ C_3 & C_0 & C_2 & C_1 \\ C_1 & C_2 & C_0 & C_3 \\ C_2 & C_1 & C_3 & C_0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ C_1 & C_0 & C_3 & C_2 \\ C_2 & C_3 & C_0 & C_1 \\ C_3 & C_2 & C_1 & C_0 \end{pmatrix}$$

5.5 A Set of MUBH Matrices Derived from Association Schemes

Using the association scheme of Theorem 5.4, we can construct a set of mutually unbiased Bush-type Hadamard matrices. In this Section, we study the structure of this set by giving the following theorem from [18]:

Theorem 5.9. *Suppose (X,R) is an association scheme with 5 classes $A_0, \ldots, A_5$ and with the same eigenmatrices of the association scheme of Theorem 5.4. Then there exists a set of MUBH matrices $\{H_1, \ldots, H_m\}$ of order $4n^2$.*

Proof. Let $B_0 = A_0$, $B_1 = A_3 + A_4$, $B_2 = A_5$ and $B_3 = A_1 + A_2$. We rearrange the vertices so that $B_3 = (I_{m+1} \otimes J_{4n^2}) - I_{4n^2(m+1)}$.

We first determine the form of A_3. Since A_1 is the adjacency matrix of an imprimitive strongly regular graph with eigenvalues $2n-1$ and -1 with multiplicities $2n(m+1)$ and $2n(2n-1)(m+1)$ respectively, A_1 is equal to $I_{2n(m+1)} \otimes (J_{2n} - I_{2n})$ after rearranging the vertices. Since $B_3 = (I_{m+1} \otimes J_{4n^2}) - I_{4n^2(m+1)} = A_1 + A_2$, the matrix A_2 has the form of $I_{m+1} \otimes (J_{2n} - I_{2n}) \otimes J_{2n}$. Since B_3 and A_3 are disjoint and $A_2A_3 = 2n(A_4 + A_5)$, we have $A_3 = (J_{m+1} - I_{m+1}) \otimes I_{2n} \otimes J_{2n}$.

Letting $G = (m+1)(E_0 + E_1 + E_2)$, we observe that

$$\begin{aligned} G &= (m+1)(E_0 + E_1 + E_2) \\ &= \frac{1}{4n^2} \sum_{i=0}^{5} (q_{i0} + q_{i1} + q_{i2}) A_i \\ &= A_0 + \frac{1}{2n} A_3 + \frac{1}{2n} A_4 - \frac{1}{2n} A_5. \end{aligned}$$

Since $A_3 + A_4 + A_4 = (J_{m+1} - I_{m+1}) \otimes I_{2n} \otimes J_{2n}$, G has the following form

$$G = \begin{pmatrix} I_{2n} & \frac{1}{2n} H_{1,2} & \cdots & \frac{1}{2n} H_{1,m+1} \\ \frac{1}{2n} H_{2,1} & I_{2n} & \cdots & \frac{1}{2n} H_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n} H_{m+1,1} & \frac{1}{2n} H_{m+1,2} & \cdots & I_{2n} \end{pmatrix},$$

where $H_{i,j}$, for $i \neq j$, is a $\{1, -1\}$-matrix.

Let $H_k := H_{k+1,1}$, for each $1 \leq k \leq m$. Now, we want to prove that the set $\{H_1, H_2, \ldots, H_m\}$ is a set of mutually unbiased Bush-type Hadamard matrices. Consider the principal submatrix $\overline{G}$ of the matrix G obtained by restricting to the indices on the first and $(k+1)$-th blocks, that is,

$$\overline{G} = \begin{pmatrix} I_{2n} & \frac{1}{2n} H_k^T \\ \frac{1}{2n} H_k & I_{2n} \end{pmatrix}.$$

Since the association scheme is uniform, restricting to indices on the first and second blocks yields an association scheme with the eigenmatrix $\overline{P} = (\overline{p}_{ij})$ obtained by putting $m = 1$. Since $\frac{m+1}{2}\overline{E}_0$, $\frac{m+1}{2}\overline{E}_1$, and $\frac{m+1}{2}\overline{E}_2$ are principal idempotents of the subscheme, we have $\overline{G}^2 = 2\overline{G}$. On the other hand, by matrix multiplication we have

$$\overline{G}^2 = \begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^T \\ \frac{1}{2n}H_k & I_{2n} \end{pmatrix} \begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^T \\ \frac{1}{2n}H_k & I_{2n} \end{pmatrix} = \begin{pmatrix} I_{2n} + \frac{1}{4n^2}H_k^T H_k & \frac{1}{n}H_k^T \\ \frac{1}{n}H_k & I_{2n} + \frac{1}{4n^2}H_k H_k^T \end{pmatrix}.$$

Hence, the equality

$$\begin{pmatrix} 2I_{2n} & \frac{1}{n}H_k^T \\ \frac{1}{n}H_k & 2I_{2n} \end{pmatrix} = \begin{pmatrix} I_{2n} + \frac{1}{4n^2}H_k^T H_k & \frac{1}{n}H_k^T \\ \frac{1}{n}H_k & I_{2n} + \frac{1}{4n^2}H_k H_k^T \end{pmatrix}$$

implies that H_k is a Hadamard matrix of order $4n^2$.

The next step is to prove that the Hadamard matrix H_k is Bush-type. For this, we evaluate $\overline{A}_3\overline{G}$ in two ways, and the final equality gives us the desired result. First, we have

$$\begin{aligned}
\overline{A}_3\overline{G} &= (m+1)\overline{A}_3(\overline{E}_0 + \overline{E}_1 + \overline{E}_2) \\
&= (m+1)(\sum_{i=0}^{5} \overline{p}_{i3}\overline{E}_i)(\overline{E}_0 + \overline{E}_1 + \overline{E}_2) \\
&= (m+1)(\sum_{i=0}^{2} \overline{p}_{i3}\overline{E}_i) \\
&= 2n(m+1)(\overline{E}_0 + \overline{E}_2) \\
&= 2n(m+1)\left(\frac{1}{4n^2(m+1)}\sum_{i=0}^{5}(\overline{q}_{i0} + \overline{q}_{i2})\overline{A}_i\right) \\
&= \overline{A}_0 + \overline{A}_1 + \overline{A}_2 \\
&= \begin{pmatrix} I_{2n} \otimes J_{2n} & I_{2n} \otimes J_{2n} \\ I_{2n} \otimes J_{2n} & I_{2n} \otimes J_{2n} \end{pmatrix}.
\end{aligned}$$

On the other hand

$$\overline{A}_3\overline{G} = \begin{pmatrix} 0 & I_{2n}\otimes J_{2n} \\ I_{2n}\otimes J_{2n} & 0 \end{pmatrix} \begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^T \\ \frac{1}{2n}H_k & I_{2n} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2n}(I_{2n}\otimes J_{2n})H_k & I_{2n}\otimes J_{2n} \\ I_{2n}\otimes J_{2n} & \frac{1}{2n}(I_{2n}\otimes J_{2n})H_k^T \end{pmatrix}$$

Therefore, we obtain the following equality

$$(I_{2n}\otimes J_{2n})H_k = (I_{2n}\otimes J_{2n})H_k^T = 2n(I_{2n}\otimes J_{2n}),$$

which implies H_k is Bush-type.

As the final step, we prove $H_1,\dots,H_m$ are unbiased. Let k,l be integer satisfying $1\le k<l\le m$. Now, consider the principal submatrix $\overline{G}$ of the matrix G obtained by restricting to the indices on the first, $(k+1)$-th, and $(l+1)$-th blocks, that is

$$\overline{G} = \begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^T & \frac{1}{2n}H_l^T \\ \frac{1}{2n}H_k & I_{2n} & \frac{1}{2n}H_{k+1,l+1} \\ \frac{1}{2n}H_l & \frac{1}{2n}H_{l+1,k+1} & I_{2n} \end{pmatrix}.$$

Since $\frac{m+1}{3}\overline{E}_0$, $\frac{m+1}{3}\overline{E}_1$, and $\frac{m+1}{3}\overline{E}_2$ are principal idempotents of the subscheme, we have $\overline{G}^2 = 3\overline{G}$. On the other hand, by matrix multiplication we have

$$\overline{G}^2 = \begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^T & \frac{1}{2n}H_l^T \\ \frac{1}{2n}H_k & I_{2n} & \frac{1}{2n}H_{k+1,l+1} \\ \frac{1}{2n}H_l & \frac{1}{2n}H_{l+1,k+1} & I_{2n} \end{pmatrix} \begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^T & \frac{1}{2n}H_l^T \\ \frac{1}{2n}H_k & I_{2n} & \frac{1}{2n}H_{k+1,l+1} \\ \frac{1}{2n}H_l & \frac{1}{2n}H_{l+1,k+1} & I_{2n} \end{pmatrix} =$$

$$\begin{pmatrix} I_{2n}+\frac{1}{4n^2}H_k^TH_k+\frac{1}{4n^2}H_l^TH_l & \frac{1}{n}H_k^T+\frac{1}{4n^2}H_l^TH_{l+1,k+1} & \frac{1}{n}H_l^T+\frac{1}{4n^2}H_k^TH_{k+1,l+1} \\ \frac{1}{n}H_k+\frac{1}{4n^2}H_{k+1,l+1}H_l & \frac{1}{4n^2}H_kH_k^T+I_{2n}+\frac{1}{4n^2}H_{k+1,l+1}H_{l+1,k+1} & \frac{1}{4n^2}H_kH_l^T+\frac{1}{n}H_{k+1,l+1} \\ \frac{1}{n}H_l+\frac{1}{4n^2}H_{l+1,k+1}H_k & \frac{1}{4n^2}H_lH_k^T+\frac{1}{n}H_{l+1,k+1} & C \end{pmatrix},$$

where the matrix C is equal to $\frac{1}{4n^2}H_lH_l^T+\frac{1}{4n^2}H_{l+1,k+1}H_{k+1,l+1}+I_{2n}$. On the other hand, we have $\overline{G}^2=3\overline{G}$. Hence, we have

$$\begin{pmatrix} 3I_{2n} & \frac{3}{2n}H_k^T & \frac{3}{2n}H_l^T \\ \frac{3}{2n}H_k & 3I_{2n} & \frac{3}{2n}H_{k+1,l+1} \\ \frac{3}{2n}H_l & \frac{3}{2n}H_{l+1,k+1} & 3I_{2n} \end{pmatrix} =$$

$$\begin{pmatrix} I_{2n}+\frac{1}{4n^2}H_k^TH_k+\frac{1}{4n^2}H_l^TH_l & \frac{1}{n}H_k^T+\frac{1}{4n^2}H_l^TH_{l+1,k+1} & \frac{1}{n}H_l^T+\frac{1}{4n^2}H_k^TH_{k+1,l+1} \\ \frac{1}{n}H_k+\frac{1}{4n^2}H_{k+1,l+1}H_l & \frac{1}{4n^2}H_kH_k^T+I_{2n}+\frac{1}{4n^2}H_{k+1,l+1}H_{l+1,k+1} & \frac{1}{4n^2}H_kH_l^T+\frac{1}{n}H_{k+1,l+1} \\ \frac{1}{n}H_l+\frac{1}{4n^2}H_{l+1,k+1}H_k & \frac{1}{4n^2}H_lH_k^T+\frac{1}{n}H_{l+1,k+1} & C \end{pmatrix},$$

Comparing the $(2,3)$-block of two matrices in the last equation, we have

$$\frac{1}{4n^2}H_kH_l^T+\frac{2}{2n}H_{k+1,l+1}=\frac{3}{2n}H_{k+1,l+1}$$

which implies $H_kH_l^T=2nH_{k+1,l+1}$. Since $H_{k+1,l+1}$ is a $\{1,-1\}$-matrix, H_k and H_l are unbiased. Therefore, the set of matrices derived from the association scheme is a set of MUBH matrices. □

5.6 Conclusion

There is a relation between association schemes and Hadamard matrices in the sense that one can be used to generate the other. Kharaghani and his co-workers worked on this relation from many aspects. The fundamental part of this thesis was concerned about how Hadamard matrices and association schemes interact with each other. More generally, the theory of association schemes deals with many combinatorial notions. In this thesis, a discussion on association schemes derived from practical concepts such as graphs and designs were provided. Moreover, studying the Bose-Mesner algebra of an association scheme provides us with eigenmatrices of the association scheme and also other crucial matrices, such as intersection matrices and Krein matrices.

We devoted a part of this thesis to study these matrices and review a practical way to evaluate them. We also discussed how imprimitive association schemes and quotient association schemes are related. The diversity of applications of association schemes persuaded us to study the concept of association schemes more in depth. It is hoped that this will lead to a better understanding of this area of combinatorics and lead to further avenues of research. The open questions on this area are about the existence of association schemes of some specific orders. One can continue this research by using combinatorial concepts to generate new classes of association schemes. Conversely, researchers may use association schemes to generate different combinatorial objects.

Bibliography

[1] R. A. Bailey. *Association schemes*, volume 84 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004. Designed experiments, algebra and combinatorics.

[2] Eiichi Bannai and Tatsuro Ito. *Algebraic combinatorics. I*. The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984. Association schemes.

[3] Béla Bollobás. *Graph theory*, volume 63 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. An introductory course.

[4] R. C. Bose and T. Shimamoto. Classification and analysis of partially balanced incomplete block designs with two associate classes. *J. Amer. Statist. Assoc.*, 47:151–184, 1952.

[5] P. Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, (10):vi+97, 1973.

[6] C. D. Godsil. *Algebraic combinatorics*. Chapman and Hall Mathematics Series. Chapman & Hall, New York, 1993.

[7] R. W. Goldbach and H. L. Claasen. 3-class association schemes and Hadamard matrices of a certain block form. *European J. Combin.*, 19(8):943–951, 1998.

[8] Jacques Hadamard. Resolution d'une question relative aux determinants. volume 2, pages 240–248. Bull. Sci. Math., 1893.

[9] D. G. Higman. *Combinatorial considerations about permutation groups*. Mathematical Institute, Oxford University, Oxford, 1972. Lectures given in 1971.

[10] A. J. Hoffman and D. K. Ray-Chaudhuri. On the line graph of a symmetric balanced incomplete block design. *Trans. Amer. Math. Soc.*, 116:238–252, 1965.

[11] W. Holzmann and H. Kharaghani. Mutually unbiased Hadamard matrices and association schemes, unpublished manuscript, 2010.

[12] W. H. Holzmann, H. Kharaghani, and W. Orrick. On the real unbiased Hadamard matrices. In *Combinatorics and graphs*, volume 531 of *Contemp. Math.*, pages 243–250. Amer. Math. Soc., Providence, RI, 2010.

[13] Zvonimir Janko. The existence of a Bush-type Hadamard matrix of order 36 and two new infinite classes of symmetric designs. *J. Combin. Theory Ser. A*, 95(2):360–364, 2001.

[14] Zvonimir Janko, Hadi Kharaghani, and Vladimir D. Tonchev. The existence of a Bush-type Hadamard matrix of order 324 and two new infinite classes of symmetric designs. *Des. Codes Cryptogr.*, 24(2):225–232, 2001.

[15] Dieter Jungnickel and H. Kharaghani. Balanced generalized weighing matrices and their applications. *Matematiche (Catania)*, 59(1-2):225–261 (2006), 2004.

[16] H. Kharaghani. New classes of weighing matrices. *Ars Combin.*, 19:69–72, 1985.

[17] H. Kharaghani. On the twin designs with the Ionin-type parameters. *Electron. J. Combin.*, 7:Research Paper 1, 11 pp. (electronic), 2000.

[18] H. Kharaghani, S. Sasani, and S. Suda. Mutually unbiased Bush-type Hadamard matrices and association schemes. *Electron. J. Combin.*, 22, 2015.

[19] William J. Martin. Symmetric designs, sets with two intersection numbers and Krein parameters of incidence graphs. *J. Combin. Math. Combin. Comput.*, 38:185–196, 2001.

[20] William J. Martin and Hajime Tanaka. Commutative association schemes. *European J. Combin.*, 30(6):1497–1525, 2009.

[21] Rudolf Mathon. 3-class association schemes. In *Proceedings of the Conference on Algebraic Aspects of Combinatorics (Univ. Toronto, Toronto, Ont., 1975)*, pages 123–155. Congressus Numerantium, No. XIII. Utilitas Math., Winnipeg, Man., 1975.

[22] Mikhail Muzychuk and Qing Xiang. Symmetric Bush-type Hadamard matrices of order $4m^4$ exist for all odd m. *Proc. Amer. Math. Soc.*, 134(8):2197–2204, 2006.

[23] Jennifer Seberry and Mieko Yamada. Hadamard matrices, sequences, and block designs. In *Contemporary design theory*, Wiley-Intersci. Ser. Discrete Math. Optim., pages 431–560. Wiley, New York, 1992.

[24] J. J. Seidel. Introduction to association schemes. In *Séminaire Lotharingien de Combinatoire (Thurnau, 1991)*, volume 476 of *Publ. Inst. Rech. Math. Av.*, pages 77–91. Univ. Louis Pasteur, Strasbourg, 1992.

[25] J. J. Sylvester. Thoughts on Iinverse Orthogonal Matrices, Simultaneous Sign Successions, and Tesselated Pavements in two or more Colours, with Applications to Newton's Rule, Ornamental Tile-work, and the Theory of Numbers. volume 34, pages 461–475. Phil. Mag., 1867.

Appendix A

Adjacency Matrices

The adjacency matrices $\{A_0, \ldots, A_5\}$ of the association scheme mentioned in the example 5.5 and the elements of basis of principal idempotents $\{E_0, E_1, \ldots, E_5\}$ of the Bose-Mesner algebra are provided here. Examining them helps us to understand the theoretical part of this thesis better.

For convenience, we present $6E_i$ instead of E_i because the elements of $6E_i$ are all integers. If a is an integer, we denote $-a$ by $\overline{a}$. Moreover, we have $c = 12$ and $z = 36$.

$$A_0 = I_{64},$$
$$A_1 = I_{16} \otimes J_4,$$
$$6E_0 = J.$$

$$A_2 = \begin{pmatrix}
0000111111111111000000000000000000000000000000000000000000000000 \\
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\end{pmatrix}$$

$$A_3 = \begin{pmatrix}
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0000000000001111000000000000111100000000000011110000000000000000
\end{pmatrix}$$

$$
A_4 = \begin{pmatrix}
0000000000000000000011001010100100001001110010100000101010011100 \\
0000000000000000000011000101011000000110110001010000010101101100 \\
0000000000000000000000111010011000000110001110100000101001100011 \\
0000000000000000000000110101100100001001001101010000010110010011 \\
0000000000000000110000001001101010010000101011001010000011001001 \\
0000000000000000110000000110010101100000010111000101000011000110 \\
0000000000000000001100000110101001100000101000111010000000110110 \\
0000000000000000001100001001010110010000010100110101000000111001 \\
0000000000000000101010010000110011001010000010011001110000001010 \\
0000000000000000010101100000110011000101000001100110110000000101 \\
0000000000000000101001100000001100111010000001100110001100001010 \\
0000000000000000010110010000001100110101000010011001001100000101 \\
0000000000000000100110101100000010101100100100001100100110100000 \\
0000000000000000011001011100000001011100011000001100011001010000 \\
0000000000000000011010100011000010100011011000000011011010100000 \\
0000000000000000100101010011000001010011100100000011100101010000 \\
0000110010101001000000000000000000001010100111000000100111001010 \\
0000110001010110000000000000000000000101011011000000011011000101 \\
0000001110100110000000000000000000001010011000110000011000111010 \\
0000001101011001000000000000000000000101100100110000100100110101 \\
1100000010011010000000000000000010100000110010011001000010101100 \\
1100000001100101000000000000000001010000110001100110000001011100 \\
0011000001101010000000000000000010100000001101100110000010100011 \\
0011000010010101000000000000000001010000001110011001000001010011 \\
1010100100001100000000000000000010011100000010101100101000001001 \\
0101011000001100000000000000000001101100000001011100010100000110 \\
1010011000000011000000000000000001100011000010100011101000000110 \\
0101100100000011000000000000000010010011000001010011010100001001 \\
1001101011000000000000000000000011001001101000001010110010010000 \\
0110010111000000000000000000000011000110010100000101110001100000 \\
0110101000110000000000000000000000110110101000001010001101100000 \\
1001010100110000000000000000000000111001010100000101001110010000 \\
0000100111001010000010101001110000000000000000000000110010101001 \\
0000011011000101000001010110110000000000000000000000110001010110 \\
0000011000111010000010100110001100000000000000000000001110100110 \\
0000100100110101000001011001001100000000000000000000001101011001 \\
1001000010101100101000001100100100000000000000001100000010011010 \\
0110000001011100010100001100011000000000000000001100000001100101 \\
0110000010100011101000000011011000000000000000000011000001101010 \\
1001000001010011010100000011100100000000000000000011000010010101 \\
1100101000001001100111000000101000000000000000001010100100001100 \\
1100010100000110011011000000010100000000000000000101011000001100 \\
0011101000000110011000110000101000000000000000001010011000000011 \\
0011010100001001100100110000010100000000000000000101100100000011 \\
1010110010010000110010011010000000000000000000001001101011000000 \\
0101110001100000110001100101000000000000000000000110010111000000 \\
1010001101100000001101101010000000000000000000000110101000110000 \\
0101001110010000001110010101000000000000000000001001010100110000 \\
0000101010011100000010011100101000001100101010010000000000000000 \\
0000010101101100000001101100010100001100010101100000000000000000 \\
0000101001100011000001100011101000000011101001100000000000000000 \\
0000010110010011000010010011010100000011010110010000000000000000 \\
1010000011001001100100001010110011000000100110100000000000000000 \\
0101000011000110011000000101110011000000011001010000000000000000 \\
1010000000110110011000001010001100110000011010100000000000000000 \\
0101000000111001100100000101001100110000100101010000000000000000 \\
1001110000001010110010100000100110101001000011000000000000000000 \\
0110110000000101110001010000011001010110000011000000000000000000 \\
0110001100001010001110100000011010100110000000110000000000000000 \\
1001001100000101001101010000100101011001000000110000000000000000 \\
1100100110100000101011001001000010011010110000000000000000000000 \\
1100011001010000010111000110000001100101110000000000000000000000 \\
0011011010100000101000110110000001101010001100000000000000000000 \\
0011100101010000010100111001000010010101001100000000000000000000
\end{pmatrix}
$$

$$A_5 = \begin{pmatrix}
0000000000000000000000110101011000000110001101010000010101100011 \\
0000000000000000000000111010100100001001001110100000101010010011 \\
0000000000000000000011000101100100001001110001010000010110011100 \\
0000000000000000000011001010011000000110110010100000101001101100 \\
0000000000000000001100000110010101100000010100110101000000110110 \\
0000000000000000001100001001101010010000101000111010000000111001 \\
0000000000000000110000001001010110010000010111000101000011001001 \\
0000000000000000110000000110101001100000101011001010000011000110 \\
0000000000000000010101100000001100110101000001100110001100000101 \\
0000000000000000101010010000001100111010000010011001001100001010 \\
0000000000000000010110010000110011000101000010011001110000000101 \\
0000000000000000101001100000110011001010000001100110110000001010 \\
0000000000000000011001010011000001010011011000000011011001010000 \\
0000000000000000100110100011000010100011100100000011100110100000 \\
0000000000000000100101011100000001011100100100001100100101010000 \\
0000000000000000011010101100000010101100011000001100011010100000 \\
0000001101010110000000000000000000000101011000110000011000110101 \\
0000001110101001000000000000000000001010100100110000100100111010 \\
0000110001011001000000000000000000000101100111000000100111000101 \\
0000110010100110000000000000000000001010011011000000011011001010 \\
0011000001100101000000000000000001010000001101100110000001010011 \\
0011000010011010000000000000000010100000001110011001000010100011 \\
1100000010010101000000000000000001010000110010011001000001011100 \\
1100000001101010000000000000000010100000110001100110000010101100 \\
0101011000000011000000000000000001100011000001010011010100000110 \\
1010100100000011000000000000000010010011000010100011101000001001 \\
0101100100001100000000000000000010011100000001011100010100001001 \\
1010011000001100000000000000000001101100000010101100101000000110 \\
0110010100110000000000000000000000110110010100000101001101100000 \\
1001101000110000000000000000000000111001101000001010001110010000 \\
1001010111000000000000000000000011001001010100000101110010010000 \\
0110101011000000000000000000000011000110101000001010110001100000 \\
0000011000110101000001010110001100000000000000000000001101010110 \\
0000100100111010000010101001001100000000000000000000001110101001 \\
0000100111000101000001011001110000000000000000000000110001011001 \\
0000011011001010000010100110110000000000000000000000110010100110 \\
0110000001010011010100000011011000000000000000000011000001100101 \\
1001000010100011101000000011100100000000000000000011000010011010 \\
1001000001011100010100001100100100000000000000001100000010010101 \\
0110000010101100101000001100011000000000000000001100000001101010 \\
0011010100000110011000110000010100000000000000000101011000000011 \\
0011101000001001100100110000101000000000000000001010100100000011 \\
1100010100001001100111000000010100000000000000000101100100001100 \\
1100101000000110011011000000101000000000000000001010011000001100 \\
0101001101100000001101100101000000000000000000000110010100110000 \\
1010001110010000001110011010000000000000000000001001101000110000 \\
0101110010010000110010010101000000000000000000001001010111000000 \\
1010110001100000110001101010000000000000000000000110101011000000 \\
0000010101100011000001100011010100000011010101100000000000000000 \\
0000101010010011000010010011101000000011101010010000000000000000 \\
0000010110011100000010011100010100001100010110010000000000000000 \\
0000101001101100000001101100101000001100101001100000000000000000 \\
0101000000110110011000000101001100110000011001010000000000000000 \\
1010000000111001100100001010001100110000100110100000000000000000 \\
0101000011001001100100000101110011000000100101010000000000000000 \\
1010000011000110011000001010110011000000011010100000000000000000 \\
0110001100000101001101010000011001010110000000110000000000000000 \\
1001001100001010001110100000100110101001000000110000000000000000 \\
1001110000000101110001010000100101011001000011000000000000000000 \\
0110110000001010110010100000011010100110000011000000000000000000 \\
0011011001010000010100110110000001100101001100000000000000000000 \\
0011100110100000101000111001000010011010001100000000000000000000 \\
1100100101010000010111001001000010010101110000000000000000000000 \\
1100011010100000101011000110000001101010110000000000000000000000
\end{pmatrix}$$

$$6E_1 = \left(\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc}
c&4&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4\\
4&c&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4\\
4&4&c&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4\\
4&4&4&c&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4\\
0&0&0&0&c&4&4&4&0&0&0&0&0&0&0&0&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4\\
0&0&0&0&4&c&4&4&0&0&0&0&0&0&0&0&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4\\
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0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&c&4&4&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&4&c&4&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&4&4&c&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&0&0&0&0&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&c&0&0&0&0&0&0&0&0&0&0&0&0\\
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\end{array}\right)$$

$$6E_2 = \begin{pmatrix}
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
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\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1} \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3 \\
\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{1}&3&3&3&3
\end{pmatrix}$$

$$
6E_3 = \begin{pmatrix}
9 & 9 & 9 & 9 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
9 & 9 & 9 & 9 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
9 & 9 & 9 & 9 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
9 & 9 & 9 & 9 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar3 & \bar3 & \bar3 & \bar3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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\end{pmatrix}
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$$6E_4 = \left(\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc}
z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&4\\
\bar{c}&z&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&4&\bar{4}&4&\bar{4}&4\\
\bar{c}&\bar{c}&z&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&\bar{4}\\
\bar{c}&\bar{c}&\bar{c}&z&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&4&\bar{4}&0&0&0&0&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&\bar{4}\\
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\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&0&0&0&0\\
4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&0&0&0&0\\
4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}&4&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&0&0&0&0\\
\bar{4}&4&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z&4&\bar{4}&4&4&4&\bar{4}&4&4&4&\bar{4}&4&\bar{4}&0&0&0&0&4&4&\bar{4}&4&4&\bar{4}&4&4&4&\bar{4}&4&4&0&0&0&0\\
0&0&0&0&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}\\
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\bar{4}&4&4&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&\bar{4}&\bar{4}&4&4&0&0&0&0&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&4&4\\
4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&4&4&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4\\
4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}&0&0&0&0&0&0&0&0&4&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&4\\
\bar{4}&4&4&\bar{4}&0&0&0&0&4&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&0&0&0&0&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z&0&0&0&0&0&0&0&0&4&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&\bar{4}\\
\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&4&4\\
\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&4&\bar{4}&4&4&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}&0&0&0&0&4&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&4&4\\
4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}&0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&0&0&0&0&4&4&4&\bar{4}\\
4&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&0&0&0&0&\bar{4}&4&4&4&\bar{4}&4&4&\bar{4}&4&4&4&\bar{4}&0&0&0&0&4&4&\bar{4}&4&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z&0&0&0&0&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&4&0&0&0&0&4&4&4&\bar{4}\\
\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&4&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0\\
\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&0&0&0&0\\
\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&0&0&0&0&4&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&0&0&0&0\\
4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&0&0&0&0&4&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z&4&4&\bar{4}&4&\bar{4}&4&4&4&4&4&\bar{4}&4&0&0&0&0\\
0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&4&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{c}&z&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&4&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{c}&\bar{c}&z&\bar{c}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&4&0&0&0&0&4&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&\bar{4}&\bar{c}&\bar{c}&\bar{c}&z&0&0&0&0&0&0&0&0&0&0&0&0\\
\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0\\
4&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&4&4&4&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&4&0&0&0&0&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&\bar{4}&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}&0&0&0&0&0&0&0&0\\
\bar{4}&4&\bar{4}&4&0&0&0&0&4&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&4&4&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}&0&0&0&0&0&0&0&0\\
4&\bar{4}&4&\bar{4}&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&0&0&0&0&4&\bar{4}&4&4&4&\bar{4}&\bar{4}&\bar{4}&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z&0&0&0&0&0&0&0&0\\
\bar{4}&\bar{4}&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&4&0&0&0&0&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}&0&0&0&0\\
4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&4&0&0&0&0&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&4&\bar{4}&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&4&0&0&0&0&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}&0&0&0&0\\
4&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&0&0&0&0&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&4&0&0&0&0&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&4&4&4&0&0&0&0&4&4&\bar{4}&\bar{4}&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}&0&0&0&0\\
\bar{4}&4&4&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&4&0&0&0&0&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&4&4&0&0&0&0&4&4&\bar{4}&\bar{4}&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z&0&0&0&0\\
\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&0&0&0&0&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&4&4&\bar{4}&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&\bar{4}&4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&z&\bar{c}&\bar{c}&\bar{c}\\
\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&4&0&0&0&0&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&z&\bar{c}&\bar{c}\\
4&4&\bar{4}&\bar{4}&4&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&\bar{4}&\bar{4}&4&\bar{4}&4&0&0&0&0&\bar{4}&\bar{4}&4&\bar{4}&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&\bar{4}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&z&\bar{c}\\
4&4&\bar{4}&\bar{4}&\bar{4}&4&4&\bar{4}&\bar{4}&4&4&\bar{4}&0&0&0&0&4&\bar{4}&\bar{4}&4&4&4&\bar{4}&\bar{4}&4&4&4&\bar{4}&0&0&0&0&\bar{4}&4&4&\bar{4}&4&4&\bar{4}&4&\bar{4}&4&4&\bar{4}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{c}&\bar{c}&\bar{c}&z
\end{array}\right)$$

$$6E_5 = \begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
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\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} \\
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\end{pmatrix}$$

Appendix B

Systems of Linear Equations for Theorem 4.8

We provide all systems of linear equations we need to evaluate $B_0^*, \ldots, B_3^*$ of association scheme of the Theorem 4.8.

The following system of linear equations is for evaluating the first row of B_0^*.

$$q_{00}^{(0)} + (4n^2-1)q_{00}^{(1)} + m(4n^2-1)q_{00}^{(2)} + mq_{00}^{(3)} = 1$$
$$q_{00}^{(0)} + (2n-1)q_{00}^{(1)} + (-2n+1)q_{00}^{(2)} - q_{00}^{(3)} = 1$$
$$q_{00}^{(0)} - (2n+1)q_{00}^{(1)} + (2n+1)q_{00}^{(2)} - q_{00}^{(3)} = 1$$
$$q_{00}^{(0)} - q_{00}^{(1)} - mq_{00}^{(2)} + mq_{00}^{(3)} = 1$$

The following system of linear equations is for evaluating the second row of B_0^*.

$$q_{10}^{(0)} + (4n^2-1)q_{10}^{(1)} + m(4n^2-1)q_{10}^{(2)} + mq_{10}^{(3)} = (4n^2-1)$$
$$q_{10}^{(0)} + (2n-1)q_{10}^{(1)} + (-2n+1)q_{10}^{(2)} - q_{10}^{(3)} = (2n-1)$$
$$q_{10}^{(0)} - (2n+1)q_{10}^{(1)} + (2n+1)q_{10}^{(2)} - q_{10}^{(3)} = (-2n-1)$$
$$q_{10}^{(0)} - q_{10}^{(1)} - mq_{10}^{(2)} + mq_{10}^{(3)} = -1$$

The following system of linear equations is for evaluating the third row of B_0^*.

$$q_{20}^{(0)} + (4n^2-1)q_{20}^{(1)} + m(4n^2-1)q_{20}^{(2)} + mq_{20}^{(3)} = m(4n^2-1)$$
$$q_{20}^{(0)} + (2n-1)q_{20}^{(1)} + (-2n+1)q_{20}^{(2)} - q_{20}^{(3)} = -2n+1$$
$$q_{20}^{(0)} - (2n+1)q_{20}^{(1)} + (2n+1)q_{20}^{(2)} - q_{20}^{(3)} = 2n+1$$
$$q_{20}^{(0)} - q_{20}^{(1)} - mq_{20}^{(2)} + mq_{20}^{(3)} = -m$$

The following system of linear equations is for evaluating the fourth row of B_0^*.

$$q_{30}^{(0)} + (4n^2-1)q_{30}^{(1)} + m(4n^2-1)q_{30}^{(2)} + mq_{30}^{(3)} = m$$
$$q_{30}^{(0)} + (2n-1)q_{30}^{(1)} + (-2n+1)q_{30}^{(2)} - q_{30}^{(3)} = -1$$
$$q_{30}^{(0)} - (2n+1)q_{30}^{(1)} + (2n+1)q_{30}^{(2)} - q_{30}^{(3)} = -1$$
$$q_{30}^{(0)} - q_{30}^{(1)} - mq_{30}^{(2)} + mq_{30}^{(3)} = m$$

The following system of linear equations is for evaluating the first row of B_1^*.

$$\begin{aligned}
&q_{01}^{(0)} + (4n^2-1)q_{01}^{(1)} + m(4n^2-1)q_{01}^{(2)} + mq_{01}^{(3)} = 4n^2-1\\
&q_{01}^{(0)} + (2n-1)q_{01}^{(1)} + (-2n+1)q_{01}^{(2)} - q_{01}^{(3)} = 2n-1\\
&q_{01}^{(0)} - (2n+1)q_{01}^{(1)} + (2n+1)q_{01}^{(2)} - q_{01}^{(3)} = -2n-1\\
&q_{01}^{(0)} - q_{01}^{(1)} - mq_{01}^{(2)} + mq_{01}^{(3)} = -1
\end{aligned}$$

The following system of linear equations is for evaluating the second row of B_1^*.

$$\begin{aligned}
&q_{11}^{(0)} + (4n^2-1)q_{11}^{(1)} + m(4n^2-1)q_{11}^{(2)} + mq_{11}^{(3)} = (4n^2-1)^2\\
&q_{11}^{(0)} + (2n-1)q_{11}^{(1)} + (-2n+1)q_{11}^{(2)} - q_{11}^{(3)} = (2n-1)^2\\
&q_{11}^{(0)} - (2n+1)q_{11}^{(1)} + (2n+1)q_{11}^{(2)} - q_{11}^{(3)} = (-2n-1)^2\\
&q_{11}^{(0)} - q_{11}^{(1)} - mq_{11}^{(2)} + mq_{11}^{(3)} = 1
\end{aligned}$$

The following system of linear equations is for evaluating the third row of B_1^*.

$$\begin{aligned}
&q_{21}^{(0)} + (4n^2-1)q_{21}^{(1)} + m(4n^2-1)q_{21}^{(2)} + mq_{21}^{(3)} = m(4n^2-1)^2\\
&q_{21}^{(0)} + (2n-1)q_{21}^{(1)} + (-2n+1)q_{21}^{(2)} - q_{21}^{(3)} = -(-2n+1)^2\\
&q_{21}^{(0)} - (2n+1)q_{21}^{(1)} + (2n+1)q_{21}^{(2)} - q_{21}^{(3)} = -(2n+1)^2\\
&q_{21}^{(0)} - q_{21}^{(1)} - mq_{21}^{(2)} + mq_{21}^{(3)} = m
\end{aligned}$$

The following system of linear equations is for evaluating the fourth row of B_1^*.

$$\begin{aligned}
&q_{31}^{(0)} + (4n^2-1)q_{31}^{(1)} + m(4n^2-1)q_{31}^{(2)} + mq_{31}^{(3)} = m(4n^2-1)\\
&q_{31}^{(0)} + (2n-1)q_{31}^{(1)} + (-2n+1)q_{31}^{(2)} - q_{31}^{(3)} = -2n+1\\
&q_{31}^{(0)} - (2n+1)q_{31}^{(1)} + (2n+1)q_{31}^{(2)} - q_{31}^{(3)} = 2n+1\\
&q_{31}^{(0)} - q_{31}^{(1)} - mq_{31}^{(2)} + mq_{31}^{(3)} = -m
\end{aligned}$$

The following system of linear equations is for evaluating the first row of B_2^*.

$$\begin{aligned}
&q_{02}^{(0)} + (4n^2-1)q_{02}^{(1)} + m(4n^2-1)q_{02}^{(2)} + mq_{02}^{(3)} = m(4n^2-1)\\
&q_{02}^{(0)} + (2n-1)q_{02}^{(1)} + (-2n+1)q_{02}^{(2)} - q_{02}^{(3)} = -2n+1\\
&q_{02}^{(0)} - (2n+1)q_{02}^{(1)} + (2n+1)q_{02}^{(2)} - q_{02}^{(3)} = 2n+1\\
&q_{02}^{(0)} - q_{02}^{(1)} - mq_{02}^{(2)} + mq_{02}^{(3)} = -m
\end{aligned}$$

The following system of linear equations is for evaluating the second row of B_2^*.

$$q_{12}^{(0)} + (4n^2-1)q_{12}^{(1)} + m(4n^2-1)q_{12}^{(2)} + mq_{12}^{(3)} = m(4n^2-1)^2$$
$$q_{12}^{(0)} + (2n-1)q_{12}^{(1)} + (-2n+1)q_{12}^{(2)} - q_{12}^{(3)} = -(2n-1)^2$$
$$q_{12}^{(0)} - (2n+1)q_{12}^{(1)} + (2n+1)q_{12}^{(2)} - q_{12}^{(3)} = -(2n+1)^2$$
$$q_{12}^{(0)} - q_{12}^{(1)} - mq_{12}^{(2)} + mq_{12}^{(3)} = m$$

The following system of linear equations is for evaluating the third row of B_2^*.

$$q_{22}^{(0)} + (4n^2-1)q_{22}^{(1)} + m(4n^2-1)q_{22}^{(2)} + mq_{22}^{(3)} = (m(4n^2-1))^2$$
$$q_{22}^{(0)} + (2n-1)q_{22}^{(1)} + (-2n+1)q_{22}^{(2)} - q_{22}^{(3)} = (-2n+1)^2$$
$$q_{22}^{(0)} - (2n+1)q_{22}^{(1)} + (2n+1)q_{22}^{(2)} - q_{22}^{(3)} = (2n+1)^2$$
$$q_{22}^{(0)} - q_{22}^{(1)} - mq_{22}^{(2)} + mq_{22}^{(3)} = m^2$$

The following system of linear equations is for evaluating the fourth row of B_2^*.

$$q_{32}^{(0)} + (4n^2-1)q_{32}^{(1)} + m(4n^2-1)q_{32}^{(2)} + mq_{32}^{(3)} = m^2(4n^2-1)$$
$$q_{32}^{(0)} + (2n-1)q_{32}^{(1)} + (-2n+1)q_{32}^{(2)} - q_{32}^{(3)} = -(-2n+1)$$
$$q_{32}^{(0)} - (2n+1)q_{32}^{(1)} + (2n+1)q_{32}^{(2)} - q_{32}^{(3)} = -(2n+1)$$
$$q_{32}^{(0)} - q_{32}^{(1)} - mq_{32}^{(2)} + mq_{32}^{(3)} = -m^2$$

The following system of linear equations is for evaluating the first row of B_3^*.

$$q_{03}^{(0)} + (4n^2-1)q_{03}^{(1)} + m(4n^2-1)q_{03}^{(2)} + mq_{03}^{(3)} = m$$
$$q_{03}^{(0)} + (2n-1)q_{03}^{(1)} + (-2n+1)q_{03}^{(2)} - q_{03}^{(3)} = -1$$
$$q_{03}^{(0)} - (2n+1)q_{03}^{(1)} + (2n+1)q_{03}^{(2)} - q_{03}^{(3)} = -1$$
$$q_{03}^{(0)} - q_{03}^{(1)} - mq_{03}^{(2)} + mq_{03}^{(3)} = m$$

The following system of linear equations is for evaluating the second row of B_3^*.

$$q_{13}^{(0)} + (4n^2-1)q_{13}^{(1)} + m(4n^2-1)q_{13}^{(2)} + mq_{13}^{(3)} = m(4n^2-1)$$
$$q_{13}^{(0)} + (2n-1)q_{13}^{(1)} + (-2n+1)q_{13}^{(2)} - q_{13}^{(3)} = -(2n-1)$$
$$q_{13}^{(0)} - (2n+1)q_{13}^{(1)} + (2n+1)q_{13}^{(2)} - q_{13}^{(3)} = 2n+1$$
$$q_{13}^{(0)} - q_{13}^{(1)} - mq_{13}^{(2)} + mq_{13}^{(3)} = -m$$

The following system of linear equations is for evaluating the third row of B_3^*.

$$
\begin{aligned}
&q_{23}^{(0)} + (4n^2-1)q_{23}^{(1)} + m(4n^2-1)q_{23}^{(2)} + mq_{23}^{(3)} = m^2(4n^2-1)\\
&q_{23}^{(0)} + (2n-1)q_{23}^{(1)} + (-2n+1)q_{23}^{(2)} - q_{23}^{(3)} = -(-2n+1)\\
&q_{23}^{(0)} - (2n+1)q_{23}^{(1)} + (2n+1)q_{23}^{(2)} - q_{23}^{(3)} = -(2n+1)\\
&q_{23}^{(0)} - q_{23}^{(1)} - mq_{23}^{(2)} + mq_{23}^{(3)} = -m^2
\end{aligned}
$$

The following system of linear equations is for evaluating the fourth row of B_3^*.

$$
\begin{aligned}
&q_{33}^{(0)} + (4n^2-1)q_{33}^{(1)} + m(4n^2-1)q_{33}^{(2)} + mq_{33}^{(3)} = m^2\\
&q_{33}^{(0)} + (2n-1)q_{33}^{(1)} + (-2n+1)q_{33}^{(2)} - q_{33}^{(3)} = 1\\
&q_{33}^{(0)} - (2n+1)q_{33}^{(1)} + (2n+1)q_{33}^{(2)} - q_{33}^{(3)} = 1\\
&q_{33}^{(0)} - q_{33}^{(1)} - mq_{33}^{(2)} + mq_{33}^{(3)} = m^2
\end{aligned}
$$

Appendix C

System of Linear Equations for Theorem 5.4

We provide all systems of linear equations we need to evaluate $B_0^*, \ldots, B_5^*$ of association scheme of the Theorem 5.4.

The following system of linear equations gives us the first row of B_0^*.

$$
\begin{aligned}
&q_{00}^{(0)} + (2n)(2n-1)q_{00}^{(1)} + (2n-1)q_{00}^{(2)} + m(2n-1)q_{00}^{(3)} + 2nm(2n-1)q_{00}^{(4)} + mq_{00}^{(5)} = 1\\
&q_{00}^{(0)} - 2nq_{00}^{(1)} + (2n-1)q_{00}^{(2)} + m(2n-1)q_{00}^{(3)} - 2mnq_{00}^{(4)} + mq_{00}^{(5)} = 1\\
&q_{00}^{(0)} - q_{00}^{(2)} - mq_{00}^{(3)} + mq_{00}^{(5)} = 1\\
&q_{00}^{(0)} + (2n-1)q_{00}^{(2)} + (-2n+1)q_{00}^{(3)} - q_{00}^{(5)} = 1\\
&q_{00}^{(0)} + 2nq_{00}^{(1)} - q_{00}^{(2)} + q_{00}^{(3)} - 2nq_{00}^{(4)} - q_{00}^{(5)} = 1\\
&q_{00}^{(0)} - 2nq_{00}^{(1)} - q_{00}^{(2)} + q_{00}^{(3)} + 2nq_{00}^{(4)} - q_{00}^{(5)} = 1
\end{aligned}
$$

The following system of linear equations gives us the second row of B_0^*.

$$
\begin{aligned}
&q_{01}^{(0)} + (2n)(2n-1)q_{01}^{(1)} + (2n-1)q_{01}^{(2)} + m(2n-1)q_{01}^{(3)} + 2nm(2n-1)q_{01}^{(4)} + mq_{01}^{(5)} = 2n(2n-1)\\
&q_{01}^{(0)} - 2nq_{01}^{(1)} + (2n-1)q_{01}^{(2)} + m(2n-1)q_{01}^{(3)} - 2mnq_{01}^{(4)} + mq_{01}^{(5)} = -2n\\
&q_{01}^{(0)} - q_{01}^{(2)} - mq_{01}^{(3)} + mq_{01}^{(5)} = 0\\
&q_{01}^{(0)} + (2n-1)q_{01}^{(2)} + (-2n+1)q_{01}^{(3)} - q_{01}^{(5)} = 0\\
&q_{01}^{(0)} + 2nq_{01}^{(1)} - q_{01}^{(2)} + q_{01}^{(3)} - 2nq_{01}^{(4)} - q_{01}^{(5)} = 2n\\
&q_{01}^{(0)} - 2nq_{01}^{(1)} - q_{01}^{(2)} + q_{01}^{(3)} + 2nq_{01}^{(4)} - q_{01}^{(5)} = -2n
\end{aligned}
$$

The following system of linear equations gives us the third row of B_0^*.

$$
\begin{aligned}
&q_{02}^{(0)} + (2n)(2n-1)q_{02}^{(1)} + (2n-1)q_{02}^{(2)} + m(2n-1)q_{02}^{(3)} + 2nm(2n-1)q_{02}^{(4)} + mq_{02}^{(5)} = 2n-1\\
&q_{02}^{(0)} - 2nq_{02}^{(1)} + (2n-1)q_{02}^{(2)} + m(2n-1)q_{02}^{(3)} - 2mnq_{02}^{(4)} + mq_{02}^{(5)} = 2n-1\\
&q_{02}^{(0)} - q_{02}^{(2)} - mq_{02}^{(3)} + mq_{02}^{(5)} = -1\\
&q_{02}^{(0)} + (2n-1)q_{02}^{(2)} + (-2n+1)q_{02}^{(3)} - q_{02}^{(5)} = 2n-1\\
&q_{02}^{(0)} + 2nq_{02}^{(1)} - q_{02}^{(2)} + q_{02}^{(3)} - 2nq_{02}^{(4)} - q_{02}^{(5)} = -1\\
&q_{02}^{(0)} - 2nq_{02}^{(1)} - q_{02}^{(2)} + q_{02}^{(3)} + 2nq_{02}^{(4)} - q_{02}^{(5)} = -1
\end{aligned}
$$

The following system of linear equations gives us the forth row of B_0^*.

$$q_{03}^{(0)} + (2n)(2n-1)q_{03}^{(1)} + (2n-1)q_{03}^{(2)} + m(2n-1)q_{03}^{(3)} + 2nm(2n-1)q_{03}^{(4)} + mq_{03}^{(5)} = m(2n-1)$$
$$q_{03}^{(0)} - 2nq_{03}^{(1)} + (2n-1)q_{03}^{(2)} + m(2n-1)q_{03}^{(3)} - 2mnq_{03}^{(4)} + mq_{03}^{(5)} = m(2n-1)$$
$$q_{03}^{(0)} - q_{03}^{(2)} - mq_{03}^{(3)} + mq_{03}^{(5)} = -m$$
$$q_{03}^{(0)} + (2n-1)q_{03}^{(2)} + (-2n+1)q_{03}^{(3)} - q_{03}^{(5)} = -2n+1$$
$$q_{03}^{(0)} + 2nq_{03}^{(1)} - q_{03}^{(2)} + q_{03}^{(3)} - 2nq_{03}^{(4)} - q_{03}^{(5)} = 1$$
$$q_{03}^{(0)} - 2nq_{03}^{(1)} - q_{03}^{(2)} + q_{03}^{(3)} + 2nq_{03}^{(4)} - q_{03}^{(5)} = 1$$

The following system of linear equations gives us the fifth row of B_0^*.

$$q_{04}^{(0)} + (2n)(2n-1)q_{04}^{(1)} + (2n-1)q_{04}^{(2)} + m(2n-1)q_{04}^{(3)} + 2nm(2n-1)q_{04}^{(4)} + mq_{04}^{(5)} = 2nm(2n-1)$$
$$q_{04}^{(0)} - 2nq_{04}^{(1)} + (2n-1)q_{04}^{(2)} + m(2n-1)q_{04}^{(3)} - 2mnq_{04}^{(4)} + mq_{04}^{(5)} = -2mn$$
$$q_{04}^{(0)} - q_{04}^{(2)} - mq_{04}^{(3)} + mq_{04}^{(5)} = 0$$
$$q_{04}^{(0)} + (2n-1)q_{04}^{(2)} + (-2n+1)q_{04}^{(3)} - q_{04}^{(5)} = 0$$
$$q_{04}^{(0)} + 2nq_{04}^{(1)} - q_{04}^{(2)} + q_{04}^{(3)} - 2nq_{04}^{(4)} - q_{04}^{(5)} = -2n$$
$$q_{04}^{(0)} - 2nq_{04}^{(1)} - q_{04}^{(2)} + q_{04}^{(3)} + 2nq_{04}^{(4)} - q_{04}^{(5)} = 2n$$

The following system of linear equations gives us the sixth row of B_0^*.

$$q_{05}^{(0)} + (2n)(2n-1)q_{05}^{(1)} + (2n-1)q_{05}^{(2)} + m(2n-1)q_{05}^{(3)} + 2nm(2n-1)q_{05}^{(4)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} - 2nq_{05}^{(1)} + (2n-1)q_{05}^{(2)} + m(2n-1)q_{05}^{(3)} - 2mnq_{05}^{(4)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} - q_{05}^{(2)} - mq_{05}^{(3)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} + (2n-1)q_{05}^{(2)} + (-2n+1)q_{05}^{(3)} - q_{05}^{(5)} = -1$$
$$q_{05}^{(0)} + 2nq_{05}^{(1)} - q_{05}^{(2)} + q_{05}^{(3)} - 2nq_{05}^{(4)} - q_{05}^{(5)} = -1$$
$$q_{05}^{(0)} - 2nq_{05}^{(1)} - q_{05}^{(2)} + q_{05}^{(3)} + 2nq_{05}^{(4)} - q_{05}^{(5)} = -1$$

The following system of linear equations gives us the first row of B_1^*.

$$q_{01}^{(0)} + (2n)(2n-1)q_{01}^{(1)} + (2n-1)q_{01}^{(2)} + m(2n-1)q_{01}^{(3)} + 2nm(2n-1)q_{01}^{(4)} + mq_{01}^{(5)} = 2n(2n-1)$$
$$q_{01}^{(0)} - 2nq_{01}^{(1)} + (2n-1)q_{01}^{(2)} + m(2n-1)q_{01}^{(3)} - 2mnq_{01}^{(4)} + mq_{01}^{(5)} = -2n$$
$$q_{01}^{(0)} - q_{01}^{(2)} - mq_{01}^{(3)} + mq_{01}^{(5)} = 0$$
$$q_{01}^{(0)} + (2n-1)q_{01}^{(2)} + (-2n+1)q_{01}^{(3)} - q_{01}^{(5)} = 0$$
$$q_{01}^{(0)} + 2nq_{01}^{(1)} - q_{01}^{(2)} + q_{01}^{(3)} - 2nq_{01}^{(4)} - q_{01}^{(5)} = 2n$$
$$q_{01}^{(0)} - 2nq_{01}^{(1)} - q_{01}^{(2)} + q_{01}^{(3)} + 2nq_{01}^{(4)} - q_{01}^{(5)} = -2n$$

The following system of linear equations gives us the second row of B_1^*.

$$q_{11}^{(0)} + (2n)(2n-1)q_{11}^{(1)} + (2n-1)q_{11}^{(2)} + m(2n-1)q_{11}^{(3)} + 2nm(2n-1)q_{11}^{(4)} + mq_{11}^{(5)} = (2n-1)^2$$
$$q_{11}^{(0)} - 2nq_{11}^{(1)} + (2n-1)q_{11}^{(2)} + m(2n-1)q_{11}^{(3)} - 2mnq_{11}^{(4)} + mq_{11}^{(5)} = (2n)^2$$
$$q_{11}^{(0)} - q_{11}^{(2)} - mq_{11}^{(3)} + mq_{11}^{(5)} = 0$$
$$q_{11}^{(0)} + (2n-1)q_{11}^{(2)} + (-2n+1)q_{11}^{(3)} - q_{11}^{(5)} = 0$$
$$q_{11}^{(0)} + 2nq_{11}^{(1)} - q_{11}^{(2)} + q_{11}^{(3)} - 2nq_{11}^{(4)} - q_{11}^{(5)} = (2n)^2$$
$$q_{11}^{(0)} - 2nq_{11}^{(1)} - q_{11}^{(2)} + q_{11}^{(3)} + 2nq_{11}^{(4)} - q_{11}^{(5)} = (2n)^2$$

The following system of linear equations gives us the third row of B_1^*.

$$q_{21}^{(0)} + (2n)(2n-1)q_{21}^{(1)} + (2n-1)q_{21}^{(2)} + m(2n-1)q_{21}^{(3)} + 2nm(2n-1)q_{21}^{(4)} + mq_{21}^{(5)} = 2n(2n-1)^2$$
$$q_{21}^{(0)} - 2nq_{21}^{(1)} + (2n-1)q_{21}^{(2)} + m(2n-1)q_{21}^{(3)} - 2mnq_{21}^{(4)} + mq_{21}^{(5)} = -2n(2n-1)$$
$$q_{21}^{(0)} - q_{21}^{(2)} - mq_{21}^{(3)} + mq_{21}^{(5)} = 0$$
$$q_{21}^{(0)} + (2n-1)q_{21}^{(2)} + (-2n+1)q_{21}^{(3)} - q_{21}^{(5)} = 0$$
$$q_{21}^{(0)} + 2nq_{21}^{(1)} - q_{21}^{(2)} + q_{21}^{(3)} - 2nq_{21}^{(4)} - q_{21}^{(5)} = -2n$$
$$q_{21}^{(0)} - 2nq_{21}^{(1)} - q_{21}^{(2)} + q_{21}^{(3)} + 2nq_{21}^{(4)} - q_{21}^{(5)} = 2n$$

The following system of linear equations gives us the forth row of B_1^*.

$$q_{31}^{(0)} + (2n)(2n-1)q_{31}^{(1)} + (2n-1)q_{31}^{(2)} + m(2n-1)q_{31}^{(3)} + 2nm(2n-1)q_{31}^{(4)} + mq_{31}^{(5)} = 2nm(2n-1)^2$$
$$q_{31}^{(0)} - 2nq_{31}^{(1)} + (2n-1)q_{31}^{(2)} + m(2n-1)q_{31}^{(3)} - 2mnq_{31}^{(4)} + mq_{31}^{(5)} = -2mn(2n-1)$$
$$q_{31}^{(0)} - q_{31}^{(2)} - mq_{31}^{(3)} + mq_{31}^{(5)} = 0$$
$$q_{31}^{(0)} + (2n-1)q_{31}^{(2)} + (-2n+1)q_{31}^{(3)} - q_{31}^{(5)} = 0$$
$$q_{31}^{(0)} + 2nq_{31}^{(1)} - q_{31}^{(2)} + q_{31}^{(3)} - 2nq_{31}^{(4)} - q_{31}^{(5)} = 2n$$
$$q_{31}^{(0)} - 2nq_{31}^{(1)} - q_{31}^{(2)} + q_{31}^{(3)} + 2nq_{31}^{(4)} - q_{31}^{(5)} = -2n$$

The following system of linear equations gives us the fifth row of B_1^*.

$$q_{41}^{(0)} + (2n)(2n-1)q_{41}^{(1)} + (2n-1)q_{41}^{(2)} + m(2n-1)q_{41}^{(3)} + 2nm(2n-1)q_{41}^{(4)} + mq_{41}^{(5)} = m(2n(2n-1))^2$$
$$q_{41}^{(0)} - 2nq_{41}^{(1)} + (2n-1)q_{41}^{(2)} + m(2n-1)q_{41}^{(3)} - 2mnq_{41}^{(4)} + mq_{41}^{(5)} = 4mn^2$$
$$q_{41}^{(0)} - q_{41}^{(2)} - mq_{41}^{(3)} + mq_{41}^{(5)} = 0$$
$$q_{41}^{(0)} + (2n-1)q_{41}^{(2)} + (-2n+1)q_{41}^{(3)} - q_{41}^{(5)} = 0$$
$$q_{41}^{(0)} + 2nq_{41}^{(1)} - q_{41}^{(2)} + q_{41}^{(3)} - 2nq_{41}^{(4)} - q_{41}^{(5)} = -(2n)^2$$
$$q_{41}^{(0)} - 2nq_{41}^{(1)} - q_{41}^{(2)} + q_{41}^{(3)} + 2nq_{41}^{(4)} - q_{41}^{(5)} = -(2n)^2$$

The following system of linear equations gives us the sixth row of B_1^*.

$$q_{51}^{(0)}+(2n)(2n-1)q_{51}^{(1)}+(2n-1)q_{51}^{(2)}+m(2n-1)q_{51}^{(3)}+2nm(2n-1)q_{51}^{(4)}+mq_{51}^{(5)}=2nm(2n-1)$$
$$q_{51}^{(0)}-2nq_{51}^{(1)}+(2n-1)q_{51}^{(2)}+m(2n-1)q_{51}^{(3)}-2mnq_{51}^{(4)}+mq_{51}^{(5)}=-2mn$$
$$q_{51}^{(0)}-q_{51}^{(2)}-mq_{51}^{(3)}+mq_{51}^{(5)}=0$$
$$q_{51}^{(0)}+(2n-1)q_{51}^{(2)}+(-2n+1)q_{51}^{(3)}-q_{51}^{(5)}=0$$
$$q_{51}^{(0)}+2nq_{51}^{(1)}-q_{51}^{(2)}+q_{51}^{(3)}-2nq_{51}^{(4)}-q_{51}^{(5)}=-2n$$
$$q_{51}^{(0)}-2nq_{51}^{(1)}-q_{51}^{(2)}+q_{51}^{(3)}+2nq_{51}^{(4)}-q_{51}^{(5)}=2n$$

The following system of linear equations gives us the first row of B_2^*.

$$q_{02}^{(0)}+(2n)(2n-1)q_{02}^{(1)}+(2n-1)q_{02}^{(2)}+m(2n-1)q_{02}^{(3)}+2nm(2n-1)q_{02}^{(4)}+mq_{02}^{(5)}=2n-1$$
$$q_{02}^{(0)}-2nq_{02}^{(1)}+(2n-1)q_{02}^{(2)}+m(2n-1)q_{02}^{(3)}-2mnq_{02}^{(4)}+mq_{02}^{(5)}=2n-1$$
$$q_{02}^{(0)}-q_{02}^{(2)}-mq_{02}^{(3)}+mq_{02}^{(5)}=-1$$
$$q_{02}^{(0)}+(2n-1)q_{02}^{(2)}+(-2n+1)q_{02}^{(3)}-q_{02}^{(5)}=2n-1$$
$$q_{02}^{(0)}+2nq_{02}^{(1)}-q_{02}^{(2)}+q_{02}^{(3)}-2nq_{02}^{(4)}-q_{02}^{(5)}=-1$$
$$q_{02}^{(0)}-2nq_{02}^{(1)}-q_{02}^{(2)}+q_{02}^{(3)}+2nq_{02}^{(4)}-q_{02}^{(5)}=-1$$

The following system of linear equations gives us the second row of B_2^*.

$$q_{12}^{(0)}+(2n)(2n-1)q_{12}^{(1)}+(2n-1)q_{12}^{(2)}+m(2n-1)q_{12}^{(3)}+2nm(2n-1)q_{12}^{(4)}+mq_{12}^{(5)}=2n(2n-1)^2$$
$$q_{12}^{(0)}-2nq_{12}^{(1)}+(2n-1)q_{12}^{(2)}+m(2n-1)q_{12}^{(3)}-2mnq_{12}^{(4)}+mq_{12}^{(5)}=-2n(2n-1)$$
$$q_{12}^{(0)}-q_{12}^{(2)}-mq_{12}^{(3)}+mq_{12}^{(5)}=0$$
$$q_{12}^{(0)}+(2n-1)q_{12}^{(2)}+(-2n+1)q_{12}^{(3)}-q_{12}^{(5)}=0$$
$$q_{12}^{(0)}+2nq_{12}^{(1)}-q_{12}^{(2)}+q_{12}^{(3)}-2nq_{12}^{(4)}-q_{12}^{(5)}=-2n$$
$$q_{12}^{(0)}-2nq_{12}^{(1)}-q_{12}^{(2)}+q_{12}^{(3)}+2nq_{12}^{(4)}-q_{12}^{(5)}=2n$$

The following system of linear equations gives us the third row of B_2^*.

$$q_{22}^{(0)}+(2n)(2n-1)q_{22}^{(1)}+(2n-1)q_{22}^{(2)}+m(2n-1)q_{22}^{(3)}+2nm(2n-1)q_{22}^{(4)}+mq_{22}^{(5)}=(2n-1)^2$$
$$q_{22}^{(0)}-2nq_{22}^{(1)}+(2n-1)q_{22}^{(2)}+m(2n-1)q_{22}^{(3)}-2mnq_{22}^{(4)}+mq_{22}^{(5)}=(2n-1)^2$$
$$q_{22}^{(0)}-q_{22}^{(2)}-mq_{22}^{(3)}+mq_{22}^{(5)}=1$$
$$q_{22}^{(0)}+(2n-1)q_{22}^{(2)}+(-2n+1)q_{22}^{(3)}-q_{22}^{(5)}=(2n-1)^2$$
$$q_{22}^{(0)}+2nq_{22}^{(1)}-q_{22}^{(2)}+q_{22}^{(3)}-2nq_{22}^{(4)}-q_{22}^{(5)}=1$$
$$q_{22}^{(0)}-2nq_{22}^{(1)}-q_{22}^{(2)}+q_{22}^{(3)}+2nq_{22}^{(4)}-q_{22}^{(5)}=1$$

The following system of linear equations gives us the forth row of B_2^*.

$$q_{32}^{(0)} + (2n)(2n-1)q_{32}^{(1)} + (2n-1)q_{32}^{(2)} + m(2n-1)q_{32}^{(3)} + 2nm(2n-1)q_{32}^{(4)} + mq_{32}^{(5)} = m(2n-1)^2$$
$$q_{32}^{(0)} - 2nq_{32}^{(1)} + (2n-1)q_{32}^{(2)} + m(2n-1)q_{32}^{(3)} - 2mnq_{32}^{(4)} + mq_{32}^{(5)} = -2mn(2n-1)$$
$$q_{32}^{(0)} - q_{32}^{(2)} - mq_{32}^{(3)} + mq_{32}^{(5)} = 0$$
$$q_{32}^{(0)} + (2n-1)q_{32}^{(2)} + (-2n+1)q_{32}^{(3)} - q_{32}^{(5)} = 0$$
$$q_{32}^{(0)} + 2nq_{32}^{(1)} - q_{32}^{(2)} + q_{32}^{(3)} - 2nq_{32}^{(4)} - q_{32}^{(5)} = 2n$$
$$q_{32}^{(0)} - 2nq_{32}^{(1)} - q_{32}^{(2)} + q_{32}^{(3)} + 2nq_{32}^{(4)} - q_{32}^{(5)} = -2n$$

The following system of linear equations gives us the fifth row of B_2^*.

$$q_{42}^{(0)} + (2n)(2n-1)q_{42}^{(1)} + (2n-1)q_{42}^{(2)} + m(2n-1)q_{42}^{(3)} + 2nm(2n-1)q_{42}^{(4)} + mq_{42}^{(5)} = 2nm(2n-1)^2$$
$$q_{42}^{(0)} - 2nq_{42}^{(1)} + (2n-1)q_{42}^{(2)} + m(2n-1)q_{42}^{(3)} - 2mnq_{42}^{(4)} + mq_{42}^{(5)} = -2mn(2n-1)$$
$$q_{42}^{(0)} - q_{42}^{(2)} - mq_{42}^{(3)} + mq_{42}^{(5)} = 0$$
$$q_{42}^{(0)} + (2n-1)q_{42}^{(2)} + (-2n+1)q_{42}^{(3)} - q_{42}^{(5)} = 0$$
$$q_{42}^{(0)} + 2nq_{42}^{(1)} - q_{42}^{(2)} + q_{42}^{(3)} - 2nq_{42}^{(4)} - q_{42}^{(5)} = 2n$$
$$q_{42}^{(0)} - 2nq_{42}^{(1)} - q_{42}^{(2)} + q_{42}^{(3)} + 2nq_{42}^{(4)} - q_{42}^{(5)} = -2n$$

The following system of linear equations gives us the sixth row of B_2^*.

$$q_{52}^{(0)} + (2n)(2n-1)q_{52}^{(1)} + (2n-1)q_{52}^{(2)} + m(2n-1)q_{52}^{(3)} + 2nm(2n-1)q_{52}^{(4)} + mq_{52}^{(5)} = m(2n-1)$$
$$q_{52}^{(0)} - 2nq_{52}^{(1)} + (2n-1)q_{52}^{(2)} + m(2n-1)q_{52}^{(3)} - 2mnq_{52}^{(4)} + mq_{52}^{(5)} = m(2n-1)$$
$$q_{52}^{(0)} - q_{52}^{(2)} - mq_{52}^{(3)} + mq_{52}^{(5)} = -1$$
$$q_{52}^{(0)} + (2n-1)q_{52}^{(2)} + (-2n+1)q_{52}^{(3)} - q_{52}^{(5)} = -(2n-1)$$
$$q_{52}^{(0)} + 2nq_{52}^{(1)} - q_{52}^{(2)} + q_{52}^{(3)} - 2nq_{52}^{(4)} - q_{52}^{(5)} = 1$$
$$q_{52}^{(0)} - 2nq_{52}^{(1)} - q_{52}^{(2)} + q_{52}^{(3)} + 2nq_{52}^{(4)} - q_{52}^{(5)} = 1$$

The following system of linear equations gives us the first row of B_3^*.

$$q_{03}^{(0)} + (2n)(2n-1)q_{03}^{(1)} + (2n-1)q_{03}^{(2)} + m(2n-1)q_{03}^{(3)} + 2nm(2n-1)q_{03}^{(4)} + mq_{03}^{(5)} = m(2n-1)$$
$$q_{03}^{(0)} - 2nq_{03}^{(1)} + (2n-1)q_{03}^{(2)} + m(2n-1)q_{03}^{(3)} - 2mnq_{03}^{(4)} + mq_{03}^{(5)} = m(2n-1)$$
$$q_{03}^{(0)} - q_{03}^{(2)} - mq_{03}^{(3)} + mq_{03}^{(5)} = -m$$
$$q_{03}^{(0)} + (2n-1)q_{03}^{(2)} + (-2n+1)q_{03}^{(3)} - q_{03}^{(5)} = -2n+1$$
$$q_{03}^{(0)} + 2nq_{03}^{(1)} - q_{03}^{(2)} + q_{03}^{(3)} - 2nq_{03}^{(4)} - q_{03}^{(5)} = 1$$
$$q_{03}^{(0)} - 2nq_{03}^{(1)} - q_{03}^{(2)} + q_{03}^{(3)} + 2nq_{03}^{(4)} - q_{03}^{(5)} = 1$$

The following system of linear equations gives us the second row of B_3^*.

$$q_{13}^{(0)} + (2n)(2n-1)q_{13}^{(1)} + (2n-1)q_{13}^{(2)} + m(2n-1)q_{13}^{(3)} + 2nm(2n-1)q_{13}^{(4)} + mq_{13}^{(5)} = 2nm(2n-1)^2$$
$$q_{13}^{(0)} - 2nq_{13}^{(1)} + (2n-1)q_{13}^{(2)} + m(2n-1)q_{13}^{(3)} - 2mnq_{13}^{(4)} + mq_{13}^{(5)} = -2nm(2n-1)$$
$$q_{13}^{(0)} - q_{13}^{(2)} - mq_{13}^{(3)} + mq_{13}^{(5)} = 0$$
$$q_{13}^{(0)} + (2n-1)q_{13}^{(2)} + (-2n+1)q_{13}^{(3)} - q_{13}^{(5)} = 0$$
$$q_{13}^{(0)} + 2nq_{13}^{(1)} - q_{13}^{(2)} + q_{13}^{(3)} - 2nq_{13}^{(4)} - q_{13}^{(5)} = 2n$$
$$q_{13}^{(0)} - 2nq_{13}^{(1)} - q_{13}^{(2)} + q_{13}^{(3)} + 2nq_{13}^{(4)} - q_{13}^{(5)} = -2n$$

The following system of linear equations gives us the third row of B_3^*.

$$q_{23}^{(0)} + (2n)(2n-1)q_{23}^{(1)} + (2n-1)q_{23}^{(2)} + m(2n-1)q_{23}^{(3)} + 2nm(2n-1)q_{23}^{(4)} + mq_{23}^{(5)} = m(2n-1)^2$$
$$q_{23}^{(0)} - 2nq_{23}^{(1)} + (2n-1)q_{23}^{(2)} + m(2n-1)q_{23}^{(3)} - 2mnq_{23}^{(4)} + mq_{23}^{(5)} = m(2n-1)^2$$
$$q_{23}^{(0)} - q_{23}^{(2)} - mq_{23}^{(3)} + mq_{23}^{(5)} = -m$$
$$q_{23}^{(0)} + (2n-1)q_{23}^{(2)} + (-2n+1)q_{23}^{(3)} - q_{23}^{(5)} = -(2n-1)^2$$
$$q_{23}^{(0)} + 2nq_{23}^{(1)} - q_{23}^{(2)} + q_{23}^{(3)} - 2nq_{23}^{(4)} - q_{23}^{(5)} = -1$$
$$q_{23}^{(0)} - 2nq_{23}^{(1)} - q_{23}^{(2)} + q_{23}^{(3)} + 2nq_{23}^{(4)} - q_{23}^{(5)} = -1$$

The following system of linear equations gives us the forth row of B_3^*.

$$q_{33}^{(0)} + (2n)(2n-1)q_{33}^{(1)} + (2n-1)q_{33}^{(2)} + m(2n-1)q_{33}^{(3)} + 2nm(2n-1)q_{33}^{(4)} + mq_{33}^{(5)} = (m(2n-1))^2$$
$$q_{33}^{(0)} - 2nq_{33}^{(1)} + (2n-1)q_{33}^{(2)} + m(2n-1)q_{33}^{(3)} - 2mnq_{33}^{(4)} + mq_{33}^{(5)} = (m(2n-1))^2$$
$$q_{33}^{(0)} - q_{33}^{(2)} - mq_{33}^{(3)} + mq_{33}^{(5)} = m^2$$
$$q_{33}^{(0)} + (2n-1)q_{33}^{(2)} + (-2n+1)q_{33}^{(3)} - q_{33}^{(5)} = (-2n+1)^2$$
$$q_{33}^{(0)} + 2nq_{33}^{(1)} - q_{33}^{(2)} + q_{33}^{(3)} - 2nq_{33}^{(4)} - q_{33}^{(5)} = 1$$
$$q_{33}^{(0)} - 2nq_{33}^{(1)} - q_{33}^{(2)} + q_{33}^{(3)} + 2nq_{33}^{(4)} - q_{33}^{(5)} = 1$$

The following system of linear equations gives us the fifth row of B_3^*.

$$q_{43}^{(0)} + (2n)(2n-1)q_{43}^{(1)} + (2n-1)q_{43}^{(2)} + m(2n-1)q_{43}^{(3)} + 2nm(2n-1)q_{43}^{(4)} + mq_{43}^{(5)} = 2n(m(2n-1))^2$$
$$q_{43}^{(0)} - 2nq_{43}^{(1)} + (2n-1)q_{43}^{(2)} + m(2n-1)q_{43}^{(3)} - 2mnq_{43}^{(4)} + mq_{43}^{(5)} = -2nm^2(2n-1)$$
$$q_{43}^{(0)} - q_{43}^{(2)} - mq_{43}^{(3)} + mq_{43}^{(5)} = 0$$
$$q_{43}^{(0)} + (2n-1)q_{43}^{(2)} + (-2n+1)q_{43}^{(3)} - q_{43}^{(5)} = 0$$
$$q_{43}^{(0)} + 2nq_{43}^{(1)} - q_{43}^{(2)} + q_{43}^{(3)} - 2nq_{43}^{(4)} - q_{43}^{(5)} = -2n$$
$$q_{43}^{(0)} - 2nq_{43}^{(1)} - q_{43}^{(2)} + q_{43}^{(3)} + 2nq_{43}^{(4)} - q_{43}^{(5)} = 2n$$

The following system of linear equations gives us the sixth row of B_3^*.

$$q_{53}^{(0)} + (2n)(2n-1)q_{53}^{(1)} + (2n-1)q_{53}^{(2)} + m(2n-1)q_{53}^{(3)} + 2nm(2n-1)q_{53}^{(4)} + mq_{53}^{(5)} = m^2(2n-1)$$
$$q_{53}^{(0)} - 2nq_{53}^{(1)} + (2n-1)q_{53}^{(2)} + m(2n-1)q_{53}^{(3)} - 2mnq_{53}^{(4)} + mq_{53}^{(5)} = m^2(2n-1)$$
$$q_{53}^{(0)} - q_{53}^{(2)} - mq_{53}^{(3)} + mq_{53}^{(5)} = -m^2$$
$$q_{53}^{(0)} + (2n-1)q_{53}^{(2)} + (-2n+1)q_{53}^{(3)} - q_{53}^{(5)} = 2n-1$$
$$q_{53}^{(0)} + 2nq_{53}^{(1)} - q_{53}^{(2)} + q_{53}^{(3)} - 2nq_{53}^{(4)} - q_{53}^{(5)} = -1$$
$$q_{53}^{(0)} - 2nq_{53}^{(1)} - q_{53}^{(2)} + q_{53}^{(3)} + 2nq_{53}^{(4)} - q_{53}^{(5)} = -1$$

The following system of linear equations gives us the first row of B_4^*.

$$q_{04}^{(0)} + (2n)(2n-1)q_{04}^{(1)} + (2n-1)q_{04}^{(2)} + m(2n-1)q_{04}^{(3)} + 2nm(2n-1)q_{04}^{(4)} + mq_{04}^{(5)} = 2nm(2n-1)$$
$$q_{04}^{(0)} - 2nq_{04}^{(1)} + (2n-1)q_{04}^{(2)} + m(2n-1)q_{04}^{(3)} - 2mnq_{04}^{(4)} + mq_{04}^{(5)} = -2mn$$
$$q_{04}^{(0)} - q_{04}^{(2)} - mq_{04}^{(3)} + mq_{04}^{(5)} = 0$$
$$q_{04}^{(0)} + (2n-1)q_{04}^{(2)} + (-2n+1)q_{04}^{(3)} - q_{04}^{(5)} = 0$$
$$q_{04}^{(0)} + 2nq_{04}^{(1)} - q_{04}^{(2)} + q_{04}^{(3)} - 2nq_{04}^{(4)} - q_{04}^{(5)} = -2n$$
$$q_{04}^{(0)} - 2nq_{04}^{(1)} - q_{04}^{(2)} + q_{04}^{(3)} + 2nq_{04}^{(4)} - q_{04}^{(5)} = 2n$$

The following system of linear equations gives us the second row of B_4^*.

$$q_{14}^{(0)} + (2n)(2n-1)q_{14}^{(1)} + (2n-1)q_{14}^{(2)} + m(2n-1)q_{14}^{(3)} + 2nm(2n-1)q_{14}^{(4)} + mq_{14}^{(5)} = m(2n(2n-1))^2$$
$$q_{14}^{(0)} - 2nq_{14}^{(1)} + (2n-1)q_{14}^{(2)} + m(2n-1)q_{14}^{(3)} - 2mnq_{14}^{(4)} + mq_{14}^{(5)} = m(2n)^2$$
$$q_{14}^{(0)} - q_{14}^{(2)} - mq_{14}^{(3)} + mq_{14}^{(5)} = 0$$
$$q_{14}^{(0)} + (2n-1)q_{14}^{(2)} + (-2n+1)q_{14}^{(3)} - q_{14}^{(5)} = 0$$
$$q_{14}^{(0)} + 2nq_{14}^{(1)} - q_{14}^{(2)} + q_{14}^{(3)} - 2nq_{14}^{(4)} - q_{14}^{(5)} = -(2n)^2$$
$$q_{14}^{(0)} - 2nq_{14}^{(1)} - q_{14}^{(2)} + q_{14}^{(3)} + 2nq_{14}^{(4)} - q_{14}^{(5)} = -(2n)^2$$

The following system of linear equations gives us the third row of B_4^*.

$$q_{24}^{(0)} + (2n)(2n-1)q_{24}^{(1)} + (2n-1)q_{24}^{(2)} + m(2n-1)q_{24}^{(3)} + 2nm(2n-1)q_{24}^{(4)} + mq_{24}^{(5)} = 2nm(2n-1)^2$$
$$q_{24}^{(0)} - 2nq_{24}^{(1)} + (2n-1)q_{24}^{(2)} + m(2n-1)q_{24}^{(3)} - 2mnq_{24}^{(4)} + mq_{24}^{(5)} = -2mn(2n-1)$$
$$q_{24}^{(0)} - q_{24}^{(2)} - mq_{24}^{(3)} + mq_{24}^{(5)} = 0$$
$$q_{24}^{(0)} + (2n-1)q_{24}^{(2)} + (-2n+1)q_{24}^{(3)} - q_{24}^{(5)} = 0$$
$$q_{24}^{(0)} + 2nq_{24}^{(1)} - q_{24}^{(2)} + q_{24}^{(3)} - 2nq_{24}^{(4)} - q_{24}^{(5)} = 2n$$
$$q_{24}^{(0)} - 2nq_{24}^{(1)} - q_{24}^{(2)} + q_{24}^{(3)} + 2nq_{24}^{(4)} - q_{24}^{(5)} = -2n$$

The following system of linear equations gives us the forth row of B_4^*.

$$q_{34}^{(0)} + (2n)(2n-1)q_{34}^{(1)} + (2n-1)q_{34}^{(2)} + m(2n-1)q_{34}^{(3)} + 2nm(2n-1)q_{34}^{(4)} + mq_{34}^{(5)} = 2n(m(2n-1))^2$$
$$q_{34}^{(0)} - 2nq_{34}^{(1)} + (2n-1)q_{34}^{(2)} + m(2n-1)q_{34}^{(3)} - 2mnq_{34}^{(4)} + mq_{34}^{(5)} = -2m^2n(2n-1)$$
$$q_{34}^{(0)} - q_{34}^{(2)} - mq_{34}^{(3)} + mq_{34}^{(5)} = 0$$
$$q_{34}^{(0)} + (2n-1)q_{34}^{(2)} + (-2n+1)q_{34}^{(3)} - q_{34}^{(5)} = 0$$
$$q_{34}^{(0)} + 2nq_{34}^{(1)} - q_{34}^{(2)} + q_{34}^{(3)} - 2nq_{34}^{(4)} - q_{34}^{(5)} = -2n$$
$$q_{34}^{(0)} - 2nq_{34}^{(1)} - q_{34}^{(2)} + q_{34}^{(3)} + 2nq_{34}^{(4)} - q_{34}^{(5)} = 2n$$

The following system of linear equations gives us the fifth row of B_4^*.

$$q_{44}^{(0)} + (2n)(2n-1)q_{44}^{(1)} + (2n-1)q_{44}^{(2)} + m(2n-1)q_{44}^{(3)} + 2nm(2n-1)q_{44}^{(4)} + mq_{44}^{(5)} = (2nm(2n-1))^2$$
$$q_{44}^{(0)} - 2nq_{44}^{(1)} + (2n-1)q_{44}^{(2)} + m(2n-1)q_{44}^{(3)} - 2mnq_{44}^{(4)} + mq_{44}^{(5)} = (2mn)^2$$
$$q_{44}^{(0)} - q_{44}^{(2)} - mq_{44}^{(3)} + mq_{44}^{(5)} = 0$$
$$q_{44}^{(0)} + (2n-1)q_{44}^{(2)} + (-2n+1)q_{44}^{(3)} - q_{44}^{(5)} = 0$$
$$q_{44}^{(0)} + 2nq_{44}^{(1)} - q_{44}^{(2)} + q_{44}^{(3)} - 2nq_{44}^{(4)} - q_{44}^{(5)} = (2n)^2$$
$$q_{44}^{(0)} - 2nq_{44}^{(1)} - q_{44}^{(2)} + q_{44}^{(3)} + 2nq_{44}^{(4)} - q_{44}^{(5)} = (2n)^2$$

The following system of linear equations gives us the sixth row of B_4^*.

$$q_{54}^{(0)} + (2n)(2n-1)q_{54}^{(1)} + (2n-1)q_{54}^{(2)} + m(2n-1)q_{54}^{(3)} + 2nm(2n-1)q_{54}^{(4)} + mq_{54}^{(5)} = 2nm^2(2n-1)$$
$$q_{54}^{(0)} - 2nq_{54}^{(1)} + (2n-1)q_{54}^{(2)} + m(2n-1)q_{54}^{(3)} - 2mnq_{54}^{(4)} + mq_{54}^{(5)} = -2m^2n$$
$$q_{54}^{(0)} - q_{54}^{(2)} - mq_{54}^{(3)} + mq_{54}^{(5)} = 0$$
$$q_{54}^{(0)} + (2n-1)q_{54}^{(2)} + (-2n+1)q_{54}^{(3)} - q_{54}^{(5)} = 0$$
$$q_{54}^{(0)} + 2nq_{54}^{(1)} - q_{54}^{(2)} + q_{54}^{(3)} - 2nq_{54}^{(4)} - q_{54}^{(5)} = 2n$$
$$q_{54}^{(0)} - 2nq_{54}^{(1)} - q_{54}^{(2)} + q_{54}^{(3)} + 2nq_{54}^{(4)} - q_{54}^{(5)} = -2n$$

The following system of linear equations gives us the first row of B_5^*.

$$q_{05}^{(0)} + (2n)(2n-1)q_{05}^{(1)} + (2n-1)q_{05}^{(2)} + m(2n-1)q_{05}^{(3)} + m(2n)(2n-1)q_{05}^{(4)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} - 2nq_{05}^{(1)} + (2n-1)q_{05}^{(2)} + m(2n-1)q_{05}^{(3)} - 2mnq_{05}^{(4)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} - q_{05}^{(2)} - mq_{05}^{(3)} + mq_{05}^{(5)} = m$$
$$q_{05}^{(0)} + (2n-1)q_{05}^{(2)} - (-2n+1)q_{05}^{(3)} - q_{05}^{(5)} = -1$$
$$q_{05}^{(0)} + 2nq_{05}^{(1)} - q_{05}^{(2)} + q_{05}^{(3)} - 2nq_{05}^{(4)} - q_{05}^{(5)} = -1$$
$$q_{05}^{(0)} - 2nq_{05}^{(1)} - q_{05}^{(2)} + q_{05}^{(3)} + 2nq_{05}^{(4)} - q_{05}^{(5)} = -1$$

The following system of linear equations gives us the second row of B_5^*.

$$q_{15}^{(0)} + (2n)(2n-1)q_{15}^{(1)} + (2n-1)q_{15}^{(2)} + m(2n-1)q_{15}^{(3)} + 2nm(2n-1)q_{15}^{(4)} + mq_{15}^{(5)} = 2nm(2n-1)$$
$$q_{15}^{(0)} - 2nq_{15}^{(1)} + (2n-1)q_{15}^{(2)} + m(2n-1)q_{15}^{(3)} - 2mnq_{15}^{(4)} + mq_{15}^{(5)} = -2mn$$
$$q_{15}^{(0)} - q_{15}^{(2)} - mq_{15}^{(3)} + mq_{15}^{(5)} = 0$$
$$q_{15}^{(0)} + (2n-1)q_{15}^{(2)} + (-2n+1)q_{15}^{(3)} - q_{15}^{(5)} = 0$$
$$q_{15}^{(0)} + 2nq_{15}^{(1)} - q_{15}^{(2)} + q_{15}^{(3)} - 2nq_{15}^{(4)} - q_{15}^{(5)} = -2n$$
$$q_{15}^{(0)} - 2nq_{15}^{(1)} - q_{15}^{(2)} + q_{15}^{(3)} + 2nq_{15}^{(4)} - q_{15}^{(5)} = 2n$$

The following system of linear equations gives us the third row of B_5^*.

$$q_{25}^{(0)} + 2n(2n-1)q_{25}^{(1)} + (2n-1)q_{25}^{(2)} + m(2n-1)q_{25}^{(3)} + 2mn(2n-1)q_{25}^{(4)} + mq_{25}^{(5)} = m(2n-1)$$
$$q_{25}^{(0)} - 2nq_{25}^{(1)} + (2n-1)q_{25}^{(2)} + m(2n-1)q_{25}^{(3)} - 2mnq_{25}^{(4)} + mq_{25}^{(5)} = m(2n-1)$$
$$q_{25}^{(0)} - q_{25}^{(2)} - mq_{25}^{(3)} + mq_{25}^{(5)} = -m$$
$$q_{25}^{(0)} + (2n-1)q_{25}^{(2)} + (-2n+1)q_{25}^{(3)} - q_{25}^{(5)} = 1-2n$$
$$q_{25}^{(0)} + 2nq_{25}^{(1)} - q_{25}^{(2)} + q_{25}^{(3)} - 2nq_{25}^{(4)} - q_{25}^{(5)} = 1$$
$$q_{25}^{(0)} - 2nq_{25}^{(1)} - q_{25}^{(2)} + q_{25}^{(3)} + 2nq_{25}^{(4)} - q_{25}^{(5)} = 1$$

The following system of linear equations gives us the forth row of B_5^*.

$$q_{35}^{(0)} + 2n(2n-1)q_{35}^{(1)} + (2n-1)q_{35}^{(2)} + m(2n-1)q_{35}^{(3)} + 2nm(2n-1)q_{35}^{(4)} + mq_{35}^{(5)} = m^2(2n-1)$$
$$q_{35}^{(0)} - 2nq_{35}^{(1)} + (2n-1)q_{35}^{(2)} + m(2n-1)q_{35}^{(3)} - 2mnq_{35}^{(4)} + mq_{35}^{(5)} = m^2(2n-1)$$
$$q_{35}^{(0)} - q_{35}^{(2)} - mq_{35}^{(3)} + mq_{35}^{(5)} = -m^2$$
$$q_{35}^{(0)} + (2n-1)q_{35}^{(2)} + (-2n+1)q_{35}^{(3)} - q_{35}^{(5)} = 2n-1$$
$$q_{35}^{(0)} + 2nq_{35}^{(1)} - q_{35}^{(2)} + q_{35}^{(3)} - 2nq_{35}^{(4)} - q_{35}^{(5)} = -1$$
$$q_{35}^{(0)} - 2nq_{35}^{(1)} - q_{35}^{(2)} + q_{35}^{(3)} + 2nq_{35}^{(4)} - q_{35}^{(5)} = -1$$

The following system of linear equations gives us the fifth row of B_5^*.

$$q_{45}^{(0)} + 2n(2n-1)q_{45}^{(1)} + (2n-1)q_{45}^{(2)} + m(2n-1)q_{45}^{(3)} + m(2n)(2n-1)q_{45}^{(4)} + mq_{45}^{(5)} = m^2(2n-1)$$
$$q_{45}^{(0)} - 2nq_{45}^{(1)} + (2n-1)q_{45}^{(2)} + m(2n-1)q_{45}^{(3)} - 2mnq_{45}^{(4)} + mq_{45}^{(5)} = -2nm^2$$
$$q_{45}^{(0)} - q_{45}^{(2)} - mq_{45}^{(3)} + mq_{45}^{(5)} = 0$$
$$q_{45}^{(0)} + (2n-1)q_{45}^{(2)} + (-2n+1)q_{45}^{(3)} - q_{45}^{(5)} = 0$$
$$q_{45}^{(0)} + 2nq_{45}^{(1)} - q_{45}^{(2)} + q_{45}^{(3)} - 2nq_{45}^{(4)} - q_{45}^{(5)} = 2n$$
$$q_{45}^{(0)} - 2nq_{45}^{(1)} - q_{45}^{(2)} + q_{45}^{(3)} + 2nq_{45}^{(4)} - q_{45}^{(5)} = -2n$$

The following system of linear equations gives us the sixth row of B_5^*.

$$q_{55}^{(0)} + 2n(2n-1)q_{55}^{(1)} + (2n-1)q_{55}^{(2)} + m(2n-1)q_{55}^{(3)} + m(2n)(2n-1)q_{55}^{(4)} + mq_{55}^{(5)} = m^2$$
$$q_{55}^{(0)} - 2nq_{55}^{(1)} + (2n-1)q_{55}^{(2)} + m(2n-1)q_{55}^{(3)} - 2mnq_{55}^{(4)} + mq_{55}^{(5)} = m^2$$
$$q_{55}^{(0)} - q_{55}^{(2)} - mq_{55}^{(3)} + mq_{55}^{(5)} = m^2$$
$$q_{55}^{(0)} + (2n-1)q_{55}^{(2)} + (-2n+1)q_{55}^{(3)} - q_{55}^{(5)} = 1$$
$$q_{55}^{(0)} + 2nq_{55}^{(1)} - q_{55}^{(2)} + q_{55}^{(3)} - 2nq_{55}^{(4)} - q_{55}^{(5)} = 1$$
$$q_{55}^{(0)} - 2nq_{55}^{(1)} - q_{55}^{(2)} + q_{55}^{(3)} + 2nq_{55}^{(4)} - q_{55}^{(5)} = 1$$