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Biangular vectors

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Lethbridge, Alta. : University of Lethbridge, Dept. of Mathematics and Computer Science, 2013

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BIANGULAR VECTORS

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Bachelor of Science, University of Lethbridge, 2011

A Thesis
Submitted to the School of Graduate Studies
of the University of Lethbridge
in Partial Fulfillment of the
Requirements for the Degree

MASTER OF SCIENCE

Department of Mathematics and Computer Science
University of Lethbridge
LETHBRIDGE, ALBERTA, CANADA

Dedication

To all those who pretended to

know what I was talking about.

And to the few that actually understood.

Abstract

This thesis introduces *unit weighing matrices*, a generalization of Hadamard matrices. When dealing with unit weighing matrices, a lot of the structure that is held by Hadamard matrices is lost, but this loss of rigidity allows these matrices to be used in the construction of certain combinatorial objects. We are able to fully classify these matrices for many small values by defining equivalence classes analogous to those found with Hadamard matrices. We then proceed to introduce an extension to mutually unbiased bases, called *mutually unbiased weighing matrices*, by allowing for different subsets of vectors to be orthogonal. The bounds on the size of these sets of matrices, both lower and upper, are examined. In many situations, we are able to show that these bounds are sharp. Finally, we show how these sets of matrices can be used to generate combinatorial objects such as strongly regular graphs and association schemes.

Acknowledgments

This journey started nearly five years ago when a professor was handing out our calculus final exams. As he handed me the exam, he spoke to me for the first time and said "You're working with me this summer." I can never thank Dr. Hadi Kharaghani enough for taking a chance on me and teaching me what I know today, so I will leave it at *Thank you*.

I also wish to thank another professor at the University of Lethbridge, Dr. Howard Cheng, for teaching me almost everything that I know about computer programming – all of the good habits, and the bad.

A lot of the ground work in this thesis was laid out while I was working with Hugh Ramp. I need to thank him for keeping me in line and for the many great insights. You are a great friend.

There are way too many people to thank, so I will use their initials and leave it to the reader as an exercise to figure out exactly who they are. ☺

As always, my family drives me forward:

BB, KB, TB, BS and MS.

There are many people inside and outside the university that have guided my way through this process. I will list some here – I'm sure that I will miss someone, and for that, I am sorry.

AA, JB, ATF, BF, wG, KG, BH, TH, WH, AL, SL, TT, MW, VW and SY.

Thank you.

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Chapter 1

Introduction

'Obvious' is the most dangerous

word in mathematics.

– E. T. Bell

This thesis is a combination of many novel ideas that have been studied in the past few years. The bulk of the study has been centred around the idea of biangular line-sets where we impose certain conditions in order to obtain specific combinatorial objects. The work found within is a combination of published work, [4, 6], submitted work, [5], and forthcoming publications.

Hadamard matrices have garnered the interest of many mathematicians and physicists over the past century. With their impeccable structure, it is no surprise that these objects appear in many seemingly unrelated areas (see [22, 32, 37]). At their historical roots, Hadamard matrices were studied by James Sylvester in 1867, who focused on a specialized infinite family of Hadamard matrices [35].

Nearly 25 years later, Jacques Hadamard constructed the first two Hadamard matrices that did not fit into Sylvester's specialized case [18]. Furthermore, Hadamard gave an infinite family of his own. Soon after, a very famous conjecture was formuated: that there is a Hadamard matrix for every order that is a multiple of 4. This hypothesis has come to be known as the *Hadamard conjecture*.

It is now more than a century later, and many more examples of Hadamard matrices have been found. There have been many steps towards a resolution of the Hadamard conjecture. However, while we are edging towards a resolution of this conjecture, we are still lacking the key insight that is needed to finally put a pin in it.

Many generalizations of Hadamard matrices have emerged over the years: orthogonal designs, weighing matrices [16] and unit Hadamard matrices [12] to name a few. In this thesis, we introduce another extension of Hadamard matrices, unit weighing matrices, and classify them for many small orders and weights. These matrices give us most of the structure that is held by Hadamard matrices, as well as the extra flexibility needed to solve certain problems.

We then utilize these matrices by introducing yet another topic: mutually unbiased weighing matrices. These are an extension of the well-known mutually unbiased bases [13]. Once again, we lose a little structure by dealing with weighing matrices instead of Hadamard matrices, but this loss of rigidity allows us to solve some problems that cannot be done with Hadamard matrices.

Majority of the content in the thesis will be used directly or indirectly to solve problems related to sets of vectors which have *nice* pairwise inner products. To be more specific, the inner product of any two vectors in the set must have a particular absolute value. There is a well-known upper bound on the size of these sets [9], which we use as the ground work for our searches. In many small cases, the upper bound can be obtained by vectors which are taken directly from the objects created in the first few chapters.

In the final chapter, we use these *nice* sets to generate combinatorial objects. Many of these objects were previously unknown. For the objects which were already known, the methods to provide them are novel.

We will split our time in this thesis between the real and the complex case. The reader is urged to keep this in mind as they progress through this thesis, since many theorems come in two forms: the real case and the complex case. When it is not specified, it is assumed that the theorem is true in the complex case (and thus, the real case as well).

1.1 A Note on Notation

Mathematicians are notorious for generating acronyms for subject matter. In this thesis, we will refrain from utilizing these acronyms as most of them will look too similar and are likely to cause headaches (e.g., MUBs, MUHM, MUCH, MUH, MOLS, MSLS, MUWM, MUCWM, MUUWM, etc.). With that being said, when these objects are introduced, we will specify the acronym for any reader who wishes to read other articles within the field where these acronyms are used heavily.

Any variable which utilizes a capital letter is a matrix. I_n is the identity matrix of order n and J_n is the square all-ones matrix of order n. For I_n, the n will be dropped when it can be inferred from context. You may also assume, without fault, that any H or W in this thesis represents a Hadamard matrix or a weighing matrix, respectively. For any matrix X, its transpose, entry-wise conjugation and Hermitian transpose are denoted by X^T, $\overline{X}$ and X^*, respectively. When matrices are explicitly written, any blank entries are zeroes. The indices of matrices will be 0-based.

The set of unimodular numbers, i.e., complex numbers with an absolute value of 1, will be denoted $\mathbb{T}$. Furthermore, $\mathbb{T}_0$ will be used to denote $\mathbb{T} \cup \{0\}$.

When a "-1" is to appear in a matrix, the shortened "$-$" within the matrix will be used. For example, instead of writing

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we will instead use

$$H = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}.$$

We would also like to warn the reader that ω is used in this thesis to mean different values at different portions of the thesis. You may assume, however, that it will represent some root of unity.

Finally, we would also like to point out that Jacques Hadamard was a French mathematician, meaning that the ‘H’ at the beginning of his last name is silent. However, for this thesis, we will use the anglicized version of his name by saying ‘*a Hadamard matrix*’ in lieu of the correct ‘*an Hadamard matrix*’.

Chapter 2

Background

The shortest path between two truths

in the real domain passes

through the complex domain.

– J. Hadamard

We begin our campaign by giving a definition, which will lay the foundation for the entire thesis.

Definition 2.1. A *real Hadamard matrix* (usually referred to as just a *Hadamard matrix* or shortened to be an H-matrix) is an $n \times n$ matrix consisting of entries in $\{\pm 1\}$ such that $HH^T = nI_n$.

Example 2.2. Here are three Hadamard matrices of orders 2, 4 and 8.

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}, \; H_4 = \begin{pmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{pmatrix} \text{ and}$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & - & - & - & - & 1 \\ 1 & 1 & 1 & - & 1 & 1 & 1 & - \\ 1 & 1 & - & 1 & 1 & 1 & - & 1 \\ 1 & - & 1 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & - & 1 & - & - & 1 & - \\ - & 1 & 1 & 1 & 1 & - & - & - \\ - & 1 & 1 & 1 & - & 1 & 1 & 1 \\ 1 & - & 1 & 1 & - & 1 & - & - \end{pmatrix}.$$

For the sake of this thesis, it is important to notice that we can view the rows of a Hadamard matrix as a collection of n vectors in $\{\pm 1\}^n$ that are pairwise orthogonal. This idea of deconstructing matrices into vectors will be revisited throughout the thesis.

2.1 Equivalence of Hadamard Matrices

At first glance, the locations of the positives and negatives in a Hadamard matrix seem quite random. We will soon see that we may alter the way that these matrices look to give us a better sense of the underlying structure of the matrices.

Proposition 2.3. *If H is a Hadamard matrix, then so are H^T and PHQ, where P and Q are signed-permutation matrices.*

Proof. We will prove both claims directly from the definition of Hadamard matrices. By the definition of a Hadamard matrix, we have that $H^{-1} = \frac{1}{n}H^T$. This implies

$$H^T\left(H^T\right)^T = nH^{-1}H = nI.$$

Secondly, $(PHQ)(PHQ)^T = PHQQ^TH^TP^T = nI$ since P and Q are orthogonal matrices. □

With this in our pocket, we will introduce the following.

Definition 2.4. Two Hadamard matrices, H_1 and H_2, are said to be *equivalent* if there exist two signed-permutation matrices, say P and Q, such that $H_1 = PH_2Q$. Equivalence is denoted by $H_1 \cong H_2$.

In a more direct sense, this means that we may permute or negate the rows and the columns of our matrix without affecting which equivalence class the matrix is in. It is important to note that the transpose of H is not included in Definition 2.4; some authors do include H^T as part of the equivalence, but we do not.

Definition 2.5. A Hadamard matrix is *dephased* if the first row and first column contain only ones.

Which immediately leads us to the following.

Lemma 2.6. *Every Hadamard matrix is equivalent to a dephased Hadamard matrix.*

Proof. For each column, look at the first entry. If it is -1, then negate that column. Then repeat the process for the rows. The resulting equivalent matrix will be dephased. □

Determining the number of inequivalent Hadamard matrices is a very laborious task. The classification of Hadamard matrices has been an ongoing process over the past few decades. In the next section, we will see that Hadamard matrices can only exist if $n \leq 2$ or n is a multiple of 4. A very simple program can be written to determine the number of inequivalent Hadamard matrices of order $n \leq 24$. For $n = 28$, decades were needed to fully classify all 487 Hadamard matrices of order 28 [26]. In 2013, Kharaghani and Tayfeh-Rezaie finished the classification of Hadamard matrices of order 32 [25]. The number of inequivalent Hadamard matrices can be found in Table 2.1.

Table 2.1: Number of inequivalent Hadamard matrices of order n ($n \leq 32$)

n	# Matrices
1	1
2	1
4	1
8	1
12	1
16	5
20	3
24	60
28	487
32	13710027

2.2 Existence of Hadamard Matrices

A few observations can be made about Hadamard matrices. It is immediate to note that the order of a Hadamard matrix must be even, but it takes a little closer inspection to note that the order must be small or a multiple of four.

Lemma 2.7. *If $n > 2$ is the order of a Hadamard matrix, then n is a multiple of four.*

Proof. Let H be a Hadamard matrix of order $n > 2$. By Lemma 2.6, we know that H is equivalent to a dephased Hadamard matrix, say H'. Let's examine the first three rows of H'. We may permute the columns of this submatrix in such a way that we arrive at the following

$$\begin{pmatrix} \overbrace{1\ 1\ \cdots\ 1\ 1}^{a \text{ columns}} & \overbrace{1\ 1\ \cdots\ 1\ 1}^{b \text{ columns}} & \overbrace{1\ 1\ \cdots\ 1\ 1}^{c \text{ columns}} & \overbrace{1\ 1\ \cdots\ 1\ 1}^{d \text{ columns}} \\ 1\ 1\ \cdots\ 1\ 1 & 1\ 1\ \cdots\ 1\ 1 & -\ -\ \cdots\ -\ - & -\ -\ \cdots\ -\ - \\ 1\ 1\ \cdots\ 1\ 1 & -\ -\ \cdots\ -\ - & 1\ 1\ \cdots\ 1\ 1 & -\ -\ \cdots\ -\ - \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

That is, we permute the columns in such a way that the the first a columns have exactly three ones in the first three rows, the next b columns have a one in the first two rows and a

-1 in the third row, the next c columns' first three rows are $1,-1,1$, respectively and the last d columns have $1,-1,-1$ in their first three rows.

From the orthogonality of each of the three pairs of rows, as well as imposing that the order of the matrix is n, we have

$$\begin{cases} a+b+c+d=n \\ a+b-c-d=0 \\ a-b+c-d=0 \\ a-b-c+d=0 \end{cases}$$

which has the unique solution of $a=b=c=d=n/4$. Since a,b,c,d and n are all integers, we have that n must be a multiple of 4. □

It is a common belief that this is the only obstacle that must be overcome. In fact, we have the following famous conjecture.

Conjecture 2.8 (Hadamard Conjecture)**.** If $n=4k$ for some $k\geq 1$, then there exists a Hadamard matrix of order n.

Prior to 2004, Conjecture 2.8 had been verified for all $n<428$. In 2004, Kharaghani and Tayfeh-Rezaie found a Hadamard matrix of order 428 [24], leaving $n=668$ as the smallest order for with the Hadamard conjecture has not been verified.

2.3 Construction of Hadamard Matrices

The study of Hadamard matrices started nearly a century and a half ago when James Sylvester constructed an infinite family of matrices which satisfied the condition laid out in Definition 2.1 (even though they were not called "Hadamard matrices" at the time).

Theorem 2.9 (Sylvester, [35]). *If H is a Hadamard matrix, then*

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix}$$

is also a Hadamard matrix.

Proof. This can easily be verified straight from the definition. □

The observation in Theorem 2.9 was crucial to the generation of the following infinite class of Hadamard matrices which are today called *Sylvester matrices*.

Corollary 2.10 (Sylvester, [35]). *There exists a Hadamard matrix of order 2^k for any $k \geq 0$.*

Proof. $H = \begin{pmatrix} 1 \end{pmatrix}$ is a Hadamard matrix. Apply Theorem 2.9 k times to H. □

Jacques Hadamard was the next mathematician to examine these matrices in detail. He was looking for examples of matrices whose determinants attained the following upper bound.

Theorem 2.11 (Hadamard, [18]). *If $\{v_0, \ldots, v_{n-1}\}$ are the rows of a square matrix A, then*

$$|\det(A)| \leq \prod_{i=0}^{n-1} ||v_i||. \tag{2.1}$$

Moreover, if every entry's absolute value is at most $B \in \mathbb{R}$, then

$$|\det(A)| \leq B^n n^{n/2}. \tag{2.2}$$

It is natural to study the case where $B = 1$ in Theorem 2.11 since all matrices may be scaled to this value. With this appropriate scaling, Hadamard showed that the bound (2.2) is realized in the real case if and only if A is a Hadamard matrix [18]. In the same article, Hadamard constructed Hadamard matrices of order 12 and 20. These are the smallest order that do not fit into Sylvester's infinite class. Hadamard also gave a more generalized version of Sylvester's construction which can most easily be described through use of the Kronecker product.

Definition 2.12. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ and $p \times q$ matrices, respectively. The *Kronecker product* of A and B, denoted $A \otimes B$, is the following $mp \times nq$ matrix

$$\begin{pmatrix} a_{0,0}B & \cdots & a_{0,n-1}B \\ \vdots & \ddots & \vdots \\ a_{m-1,1}B & \cdots & a_{m-1,n-1}B \end{pmatrix}.$$

The following is an immediate way to construct new Hadamard matrices from old ones.

Lemma 2.13 (Hadamard, [18])**.** *If H_1 and H_2 are Hadamard matrices of order m and n, respectively, then $H_1 \otimes H_2$ is a Hadamard matrix of order mn.*

Proof.

$$(H_1 \otimes H_2)(H_1 \otimes H_2)^T = H_1 H_1^T \otimes H_2 H_2^T = mI_m \otimes nI_n = mnI_{mn}$$

□

This new construction method means that when any new Hadamard matrix is formed, we may use the Kronecker product to give us many (possibly new) Hadamard matrices. But unfortunately, this construction comes with an inherent problem. Creating Hadamard matrices of order $4n$ where n is even is quite a bit easier than when n is odd. With the method above, Hadamard matrices of order $4a$ and $4b$ will be able to construct a new Hadamard matrix of order $16ab$. The following construction allows us to create a Hadamard matrix of half of that order.

Theorem 2.14 ([1])**.** *If there exist Hadamard matrices of order $4a$ and $4b$, then there exists a Hadamard matrix of order $8ab$.*

Proof. Let H_1 be a Hadamard matrix of order $4a$ and H_2 be a Hadamard matrix of order $4b$. Let A and B be $4a \times 2a$ matrices and let C and D be $2b \times 4b$ matrices such that

$$H_1 = \begin{pmatrix} A & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} C \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ D \end{pmatrix}.$$

We then form

$$H = \frac{1}{2}(A+B) \otimes C + \frac{1}{2}(A-B) \otimes D.$$

It is important to note that if we examine corresponding entries in $A+B$ and $A-B$, then exactly one of them is zero and the other is either 2 or -2. Thus, each entry in H is either 1 or -1. Next, let us examine the following product.

$$\begin{aligned}
HH^T &= \left(\tfrac{1}{2}(A+B)\otimes C+\tfrac{1}{2}(A-B)\otimes D\right)\left(\tfrac{1}{2}(A+B)\otimes C+\tfrac{1}{2}(A-B)\otimes D\right)^T \\
&= \left(\tfrac{1}{2}(A+B)\otimes C\right)\left(\tfrac{1}{2}(A+B)\otimes C\right)^T+\left(\tfrac{1}{2}(A-B)\otimes D\right)\left(\tfrac{1}{2}(A-B)\otimes D\right)^T \\
&\quad +\left(\tfrac{1}{2}(A+B)\otimes C\right)\left(\tfrac{1}{2}(A-B)\otimes D\right)^T+\left(\tfrac{1}{2}(A-B)\otimes D\right)\left(\tfrac{1}{2}(A+B)\otimes C\right)^T \\
&= \left(\tfrac{1}{4}(A+B)(A+B)^T\right)\otimes CC^T+\left(\tfrac{1}{4}(A-B)(A-B)^T\right)\otimes DD^T \\
&\quad +\left(\tfrac{1}{4}(A+B)(A-B)^T\right)\otimes CD^T+\left(\tfrac{1}{4}(A-B)(A+B)^T\right)\otimes DC^T \\
&= \left(\tfrac{1}{4}(A+B)(A+B)^T\right)\otimes 4bI_{2b}+\left(\tfrac{1}{4}(A-B)(A-B)^T\right)\otimes 4bI_{2b} \\
&\quad +\left(\tfrac{1}{4}(A+B)(A-B)^T\right)\otimes 0_{2b}+\left(\tfrac{1}{4}(A-B)(A+B)^T\right)\otimes 0_{2b} \\
&= \left[\left(\tfrac{1}{4}(A+B)(A+B)^T\right)+\left(\tfrac{1}{4}(A-B)(A-B)^T\right)\right]\otimes 4bI_{2b} \\
&= \left[\tfrac{1}{2}\left(AA^T+BB^T\right)\right]\otimes 4bI_{2b} \\
&= \tfrac{1}{2}(4aI_{4a})\otimes 4bI_{2b} \\
&= 8abI_{8ab}
\end{aligned}$$

which implies, from Definition 2.1, that H is a Hadamard matrix. □

The first infinite class of Hadamard matrices of order $4n$ which includes many values for which n is odd is attributed to Paley. Before stating the result, we must first take a small detour through some field theory results.

Definition 2.15. Let $\mathbb{F}_p$ be a finite field of order p and let a be a nonzero element of $\mathbb{F}_p$. a is a *quadratic residue mod p* if there exists $b \in \mathbb{F}_p$ such that $a \equiv b^2 \pmod{p}$. The *Legendre symbol mod p*, $\chi : \mathbb{F}_p \to \mathbb{Z}$, is defined as

$$\chi(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \text{ is a quadratic residue mod } p, \\ -1 & \text{otherwise.} \end{cases}$$

Lemma 2.16. *If p is an odd prime number, then exactly $(p-1)/2$ elements in $\mathbb{F}_p$ are quadratic residues.*

Proof. First, we note that for all $c \in \mathbb{F}_p$, $c^2 = (-c)^2$, so there are at most $(p-1)/2$ quadratic residues. To avoid these trivial collisions, we will examine $a,b \in \mathbb{F}_p$ such that $1 \leq a,b \leq (p-1)/2$. If we also assume that $a^2 = b^2$, then we have

$$(a+b)(a-b) = a^2 - b^2 = 0.$$

Since we are in a field, either $a+b=0$ or $a-b=0$. The first equality cannot hold since $a+b \in \{2,3,\ldots,p-1\}$. The second equality is true if and only if $a=b$. So for any pair (a,b) such that $1 \leq a \neq b \leq (p-1)/2$, we have that $a^2 \neq b^2$. Thus, there are at least $(p-1)/2$ quadratic residues mod p, and the result follows. □

Definition 2.17. Let p be an odd prime number. Then we define the *Jacobsthal matrix* to be the $p \times p$ matrix, $Q_p = [q_{ij}]$, such that

$$q_{ij} = \chi(i-j),$$

where $i-j$ is reduced mod p.

Theorem 2.18. *Let p be an odd prime number. Then $Q_p Q_p^T = pI - J$.*

Proof. Let v_i and v_j be the i^{th} and j^{th} rows of Q_p. If $i = j$, then $\langle v_i, v_j \rangle = p-1$ since there are exactly $p-1$ nonzeroes per row (each of which is ± 1). Now, assume that $i \neq j$, then we have that

$$\langle v_i, v_j \rangle = \sum_{a \in \mathbb{F}_p} \chi(i-a)\chi(j-a) = \sum_{b \in \mathbb{F}_p} \chi(b)\chi(b+(j-i)).$$

From here, we note that when $b = 0$, $\chi(b) = 0$, so we have

$$\langle v_i, v_j \rangle = \sum_{b \in \mathbb{F}_p \setminus \{0\}} \chi(b)\chi(b+(j-i)).$$

Next, we will use the fact that χ is a multiplicative function to see that

$$\langle v_i, v_j \rangle = \sum_{b \in \mathbb{F}_p \setminus \{0\}} \chi(b)\chi(b)\chi(1+b^{-1}(j-i)) = \sum_{b \in \mathbb{F}_p \setminus \{0\}} \chi(1+b^{-1}(j-i))$$

Since $j-i$ is fixed and nonzero, $1+b^{-1}(j-i)$ will run through each element in $\mathbb{F}_p$ except 1. By using Lemma 2.16 and the fact that $\chi(1) = 1$ for all p,

$$\langle v_i, v_j \rangle = \sum_{c \in \mathbb{F}_p \setminus \{1\}} \chi(c) = \left(\sum_{c \in \mathbb{F}_p} \chi(c) \right) - \chi(1) = 0 - 1 = -1.$$

Thus, the inner product of two distinct rows of Q_p are -1, and the result follows. □

Theorem 2.19 (Paley, [29]). *Let $p \equiv 3 \pmod 4$ be an odd prime number. Then*

$$H = \begin{pmatrix} 1 & 1_p \\ 1_p^T & Q_p - I \end{pmatrix}$$

is a Hadamard matrix of order $p+1$, where 1_p is a row vector of p ones.

Proof.

$$
\begin{aligned}
HH^T &= \begin{pmatrix} 1 & 1_p \\ 1_p^T & Q_p - I \end{pmatrix} \begin{pmatrix} 1 & 1_p \\ 1_p^T & Q_p - I \end{pmatrix}^T \\
&= \begin{pmatrix} p+1 & 0_p \\ 0_p^T & J + (Q_p - I)(Q_p - I)^T \end{pmatrix} \\
&= \begin{pmatrix} p+1 & 0_p \\ 0_p^T & J + Q_p Q_p^T - Q_p - Q_p^T + I \end{pmatrix} \\
&= \begin{pmatrix} p+1 & 0_p \\ 0_p^T & J + Q_p Q_p^T + I \end{pmatrix} && \big(\text{since } Q_p = -Q_p^T \text{ if } p \equiv 3 \pmod 4\big) \\
&= \begin{pmatrix} p+1 & 0_p \\ 0_p^T & J + (pI - J) + I \end{pmatrix} && (\text{by Theorem 2.18}) \\
&= \begin{pmatrix} p+1 & 0_p \\ 0_p^T & (p+1)I_p \end{pmatrix} \\
&= (p+1)I_{p+1}
\end{aligned}
$$

□

Theorem 2.20 (Paley, [29]). *Let $p \equiv 1 \pmod 4$ be an odd prime number. Then*

$$H = \left(\begin{array}{cc|cc} 1 & 1_p & -1 & 1_p \\ 1_p^T & Q_p + I & 1_p^T & Q_p - I \\ \hline -1 & 1_p & -1 & -1_p \\ 1_p^T & Q_p - I & -1_p^T & -Q_p - I \end{array}\right)$$

is a Hadamard matrix of order $2(p+1)$, where 1_p is a row vector of p ones.

Proof.

$$
\begin{aligned}
HH^T &= \left(\begin{array}{cc|cc} 1 & 1_p & -1 & 1_p \\ 1_p^T & Q_p+I & 1_p^T & Q_p-I \\ \hline -1 & 1_p & -1 & -1_p \\ 1_p^T & Q_p-I & -1_p^T & -Q_p-I \end{array}\right)\left(\begin{array}{cc|cc} 1 & 1_p & -1 & 1_p \\ 1_p^T & Q_p+I & 1_p^T & Q_p-I \\ \hline -1 & 1_p & -1 & -1_p \\ 1_p^T & Q_p-I & -1_p^T & -Q_p-I \end{array}\right)^T \\
&= \begin{pmatrix} 2(p+1) & 0 & 0 & 0 \\ 0 & 2(J+Q_pQ_p^T+I) & 0 & 2(Q_p^T-Q_p) \\ 0 & 0 & 2(p+1) & 0 \\ 0 & 2(Q_p-Q_p^T) & 0 & 2(J+Q_pQ_p^T+I) \end{pmatrix} \\
&= \begin{pmatrix} 2(p+1) & 0 & 0 & 0 \\ 0 & 2(J+(pI-J)+I) & 0 & 0 \\ 0 & 0 & 2(p+1) & 0 \\ 0 & 0 & 0 & 2(J+(pI-J)+I) \end{pmatrix} \\
&= \begin{pmatrix} 2(p+1) & 0 & 0 & 0 \\ 0 & 2(p+1)I & 0 & 0 \\ 0 & 0 & 2(p+1) & 0 \\ 0 & 0 & 0 & 2(p+1)I \end{pmatrix} \\
&= 2(p+1)I_{2(p+1)},
\end{aligned}
$$

where the third last equality is true since $Q_p = Q_p^T$ when $p \equiv 1 \pmod 4$. □

Later, a similar idea was used to show that p can be any odd prime power in the previous two lemmas through the use of finite fields. In 1944, Williamson introduced a new class of Hadamard matrices which needs the idea of circulant matrices.

Definition 2.21. Given a vector of n elements, $(v_0, v_1, \ldots, v_{n-1})$, a *circulant matrix*, $A = [a_{ij}]$, is a matrix defined as $a_{ij} = v_{i-j}$ where $i-j$ is reduced modulo n. Circulant matrices can be represented by their first row by $A = \mathrm{Circ}(v_0, \ldots, v_{n-1})$.

From these, Williamson gave the following.

Theorem 2.22 ([16]). *Let A, B, C and D be four symmetric circulant matrices of order n which satisfy*

$$A^2 + B^2 + C^2 + D^2 = 4nI_n.$$

Then

$$\begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}$$

is a Hadamard matrix of order $4n$*.*

Proof. This can easily be verified by multiplying out the matrix with its transpose and utilizing our assumption. □

Example 2.23. For example, $A = \mathrm{Circ}(1\ 1\ 1)$, $B = C = D = \mathrm{Circ}(-\ 1\ 1)$ satisfies the conditions laid out in Theorem 2.22, and so we may create a 12×12 Hadamard matrix.

These constructions account for a large portion of the Hadamard matrices that are currently known, especially for smaller values. There are many other constructions for Hadamard matrices, most of which are out of the scope of this thesis (see [16] for more constructions).

Chapter 3

Generalizations of Hadamard Matrices

Curiosity.

Not good for cats,

but very good for scientists.

– L. Fleinhardt

(This chapter is based on published work, [6].)

In this chapter, we introduce the idea of unit weighing matrices. These matrices are a generalization of Hadamard matrices. We fully classify the matrices for small orders and then proceed to show their usefulness in Chapter 4.

When Hadamard matrices were originally studied, they were square matrices with entries in $\{\pm 1\}$ which contained mutually orthogonal rows (as defined in Definition 2.1). There have been quite a few generalizations of Hadamard matrices that have appeared over the years. We will examine two of these generalizations which have been explored extensively in the literature.

3.1 Weighing Matrices

First, we will remove the restriction that the entries must be in the set $\{\pm 1\}$ by allowing a third entry, 0.

Definition 3.1. An $n \times n$ matrix, W, with entries in $\{0, \pm 1\}$ such that $WW^T = wI$ for some w is called a *weighing matrix* of order n and weight w. Weighing matrices are often denoted

$W(n,w)$.

Example 3.2. W_7 is a weighing matrix of order 7 and weight 4, a $W(7,4)$.

$$W_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}$$

We note that a $W(n,n)$ is a Hadamard matrix. Unlike Hadamard matrices, however, the order of weighing matrices are not restricted nearly as much. For example, the matrix given in Example 3.2 is of order 7, which is not a multiple of 4. We have the following extension of the Hadamard conjecture which states that a weighing matrix should exist for every weight of specific orders.

Conjecture 3.3 ([27])**.** Let n be any integer multiple of four. Then there exists a weighing matrix of the type $W(n,w)$ for all $w \leq n$.

We may define equivalence between weighing matrices in the same way that we define the equivalence of Hadamard matrices (i.e., row and column permutations and negations do not change which equivalence class you are in). Using this notion, Chan *et al.* classified all weighing matrices with weights smaller than 6 in 1986 [10]. However, while working on the matrices in the remaining portion of this section, we found a mistake in their classification of matrices of weight 5. Independently, Harada and Munemasa also discovered the error. We refer the reader to [19] for the full details of where the matrices were missed. The classification of real weighing matrices of weights 1, 2, 3 and 4 will be dealt with in Subsection 3.5.1, Theorem 3.30, Corollary 3.34 and Corollary 3.38, respectively. For this reason, we will jump right to the classification of weight 5 matrices and will deal with the first four later.

In order to classify these matrices, we must first introduce the direct sum.

Definition 3.4. Let A and B be two matrices of dimension $m \times n$ and $p \times q$, respectively. Then the *direct sum* of A and B, denoted $A \oplus B$, is the $(m+p) \times (n+q)$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Theorem 3.5 ([10, 19])**.** *Any $W(n,5)$ is equivalent to the direct sum of a specific set of weighing matrices. The set includes 5 sporadic cases and two infinite families.*

The set of matrices that make up Theorem 3.5 (including the two that were originally missed) are in Appendix C.1. Attempts have been made to classify weighing matrices for larger weights, but even in the next case, $w = 6$, the complexity involved with the case analysis is immense. However, I feel that this classification could be completed within the next few years with a lot of elbow grease.

3.1.1 Circulant Weighing Matrices

One very particular type of weighing matrices are those that are generated from circulating the first row.

Definition 3.6. A *circulant weighing matrix* is a circulant matrix (see Definition 2.21) that is also a weighing matrix. In literature, circulant weighing matrices of order n and weight w are denoted by $CW(n,w)$.

Example 3.7. Here are two examples of circulant weighing matrices. H_4 is the lone non-trivial example of a circulant Hadamard matrix, while W_7 is a $CW(7,4)$.

$$H_4 = \text{Circ}(\, - \;\; 1 \;\; 1 \;\; 1 \,) = \begin{pmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix}$$

$$W_7 = \text{Circ}(\, 1 \;\; 0 \;\; 1 \;\; 1 \;\; - \;\; 0 \;\; 0 \,) = \begin{pmatrix} 1 & 0 & 1 & 1 & - & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & - & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & - \\ - & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & - & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & - & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & - & 0 & 0 & 1 \end{pmatrix}$$

Definition 3.8. Given a circulant weighing matrix, $W = \text{Circ}(a_0, a_1, \ldots, a_{n-1})$, the associated *Hall polynomial* is defined as $f(t) := \sum_{i=0}^{n-1} a_i t^i$.

Lemma 3.9. *Let W be a circulant weighing matrix of order n and weight w. If $f(t) = a_0 + \cdots a_{n-1}t^{n-1}$ is the associated Hall polynomial of W, then $f(\omega)f(\overline{\omega}) = w$ for all n^{th} roots of unity, ω.*

Proof.

$$\begin{aligned} f(\omega)f(\overline{\omega}) &= (a_0 + a_1\omega + \cdots + a_{n-1}\omega^{n-1})(a_0 + a_1\overline{\omega} + \cdots + a_{n-1}\overline{\omega^{n-1}}) \\ &= \sum_{d=0}^{n-1} \left(\omega^d \left[\sum_{i=0}^{n-1} a_i a_{i-d} \right] \right), \end{aligned}$$

where the indices are being reduced mod n. In the sum above, the inner summation represents the inner products of two rows of W. Thus, we have that

$$f(\omega)f(\overline{\omega}) = w + \sum_{d=1}^{n-1} \left(\omega^d [0]\right) = w.$$

□

Theorem 3.10. *Let W be a circulant weighing matrix of order n and weight w. Then w must be a perfect square.*

Proof. Let W be a circulant weighing matrix and let $f(t)$ be the associated Hall polynomial. By Lemma 3.9, we have that $f(1)^2 = w$. Since $f(1)$ is simply the sum of the entries in any given row of W, it must be an integer, so w is a perfect square. □

A lot of interest has been shown in circulant weighing matrices, but most of the results are outside the scope of this thesis. For further details that are not provided here, we recommend [30] and the references therein for general knowledge and [2, 14, 33, 34] for the classification of small circulant weighing matrices.

3.2 Unit Hadamard Matrices

Another generalization of Hadamard matrices is to remove the restriction that the matrices must be real.

Definition 3.11. An $n \times n$ matrix, H, with all unimodular entries such that $HH^* = nI_n$ is called a *unit Hadamard matrix*.

Other names for unit Hadamard matrices are *generalized Hadamard matrices* and *complex Hadamard matrices*[1].

Unit Hadamard matrices are the one and only type of matrix that we will discuss that have no restrictions on the order. In fact, we have the following strong theorem.

Theorem 3.12. *There exists a unit Hadamard matrix for any order $n \geq 1$.*

[1]Be warned that the term "complex Hadamard matrices" has been used by different authors to mean different things.

Proof. The proof is immediate by noting that the Fourier matrix of order n, $F = [f_{jk}]$, is a unit Hadamard matrix where

$$f_{jk} = e^{jk \cdot 2\pi i/n}$$

for all $0 \leq j,k < n$. $\square$

Example 3.13. Here are the first three Fourier matrices, where $\omega = e^{2\pi i/3}$.

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}.$$

For a comprehensive look at unit Hadamard matrices, we refer the reader to Szöllősi's Ph.D. thesis [36].

3.3 Unit Weighing Matrices

In this thesis, we combine the previous two sections to introduce a new class of matrices.

Definition 3.14. An $n \times n$ matrix, W, with entries in $\mathbb{T}_0$ such that $WW^* = wI_n$ for some w is called a *unit weighing matrix*. Unit weighing matrices are denoted $UW(n,w)$.

These matrices take a bit of weighing matrices and a bit of unit Hadamard matrices and fuse them together. By doing this, we open the door to solving many problems related to line-sets (these problems will be discussed further in Chapters 4 and 5).

In the rest of this chapter, we will start the classification process for unit weighing matrices. The classification of unit weighing matrices is much more difficult than the classification of real weighing matrices due to the fact that each entry of a unit weighing matrix has infinitely many choices.

3.4 Equivalence of Unit Weighing Matrices

The following proposition will serve as an analogue of Proposition 2.3.

Proposition 3.15. *For a given unit weighing matrix, applying any of the following operations will result in a unit weighing matrix:*

(T1) Permuting the rows or columns.

(T2) Multiplying any row or column of the matrix by a number in $\mathbb{T}$.

(T3) Taking the Hermitian transpose.

(T4) Conjugating every entry in the matrix.

Proof. Each of these can be easily verified. □

Note that by applying $(T3)$ followed by $(T4)$, we have that the transpose of a unit weighing matrix is also a unit weighing matrix. From Proposition 3.15, we may give a definition of equivalence similar to that of Definition 2.4.

Definition 3.16. Two unit weighing matrices, W_1 and W_2, are *equivalent* if one can be obtained from the other by performing a finite number of operations $(T1)$ and $(T2)$ to it.

Though Definitions 2.4 and 3.16 are stated differently, they are essentially the same if one introduces the notion of a *unimodular permutation matrix*. Note that $(T3)$ and $(T4)$ are excluded from the definition in order to maintain consistency between the definitions of equivalence for Hadamard matrices and unit weighing matrices.

The first large difference between real weighing matrices and unit weighing matrices is the number of equivalence classes of matrices. In the real case, there are clearly only a finite number of classes. However, when we are dealing with unit weighing matrices, there may be infinitely many . In fact, there may be uncountably many, as we will see in the Example 3.19. But first, we must introduce the following definition.

Definition 3.17 (Haagerup's Invariant)**.** Let $W = [w_{ij}]$ be a unit weighing matrix. We define the following multiset[2]

$$\Lambda(W) = \left\{ w_{ij}\overline{w_{kj}}w_{kl}\overline{w_{il}} : 0 \leq i,j,k,l < n \right\}.$$

With this, we can give the following strong theorem.

Theorem 3.18. *If two unit weighing matrices, W_1 and W_2, are equivalent, then $\Lambda(W_1) = \Lambda(W_2)$.*

Proof. We note that the four operations in Proposition 3.15 do not affect this multiset, and so the result follows. □

It is the contrapositive of this statement that is typically used. This has been used quite heavily to determine the inequivalence of unit Hadamard matrices. By using Theorem 3.18, we can see that there can be infinitely many inequivalent unit weighing matrices.

Example 3.19. There are infinitely many $UW(4,4)$.

Proof. Let

$$W_4(x) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & x & -x \\ 1 & - & -x & x \end{pmatrix},$$

which is a family of $UW(4,4)$ with one parameter. Then we have that

$$\Lambda\left(W_4(x)\right) = \{(1,148),(-1,44),(x,12),(-x,20),(\bar{x},12),(-\bar{x},20)\}.$$

From this, we have that $\Lambda\left(W_4(e^{i\theta})\right) \neq \Lambda\left(W_4(e^{i\phi})\right)$ for $0 < \theta, \phi < \pi/2$ and $\theta \neq \phi$. By using Theorem 3.18, we have infinitely many $UW(4,4)$s. □

[2]When we refer to multisets, we will represent them as a set of ordered pairs of the form "(value,count)". For example, $\{1,1,2,3,4,4,4\}$ would be represented as $\{(1,2),(2,1),(3,1),(4,3)\}$.

Example 3.19 shows us that we must find a different way to classify unit weighing matrices.

Definition 3.20. A *family* of unit weighing matrices is a set of unit weighing matrices that can be parameterized in such a way that any unimodular number may be substituted for any given variable and the matrix is still a unit weighing matrix. When a matrix has no parameters, we call the matrix a *sporadic* case.

$W_4(x)$ is an example of a family of weighing matrices.

Definition 3.21. Two weighing matrices, $W_1(x_1, x_2, \ldots, x_n)$ and $W_2(y_1, y_2, \ldots, y_m)$, are in the *same family* (or *family equivalent*) if there exists $w_1, w_2, \ldots, w_n, z_1, z_2, \ldots, z_m \in \mathbb{T}$ such that $W_1(w_1, \ldots, w_n)$ is equivalent to $W_2(z_1, \ldots, z_m)$. We denote the family equivalence by $W_1 \cong W_2$.

From this point on, when we state that two unit weighing matrices are "equivalent", we mean that they are in the same family.

In order to study the number of inequivalent unit weighing matrices (i.e., the number of distinct families), we define the following ordering, $\prec$, on the elements of $\mathbb{T}_0$.

1. $e^{i\theta} \prec 0$ for all θ
2. $e^{i\theta} \prec e^{i\phi} \iff 0 \leq \theta < \phi < 2\pi$

Definition 3.22. We say that a unit weighing matrix, W, of order n and weight w is in *standard form* if the following conditions apply:

(S1) The first nonzero entry in each row is 1.

(S2) The first nonzero entry in each column is 1.

(S3) The first row is w ones followed by $n - w$ zeroes.

(S4) The rows are in lexicographical order according to $\prec$.

To clarify the ordering in $(S4)$ (say we are interested in row i and row j), we denote row i by $R_i = (a_0, a_1, \dots, a_{n-1})$ and row j by $R_j = (b_0, b_1, \dots, b_{n-1})$ and let k be the smallest index such that $a_k \neq b_k$. Then $R_i < R_j \iff a_k \prec b_k$. Definition 3.22 is the analogue to Definition 2.5.

Theorem 3.23. *Every unit weighing matrix is equivalent to a unit weighing matrix that is in standard form.*

Proof. Let W be a unit weighing matrix of weight w. Let $r_i \in \mathbb{T}$ be the first nonzero entry in row i. Multiply each row i by $\overline{r_i} \in \mathbb{T}$, so that the condition $(S1)$ holds. For column j, let $c_j \in \mathbb{T}$ be the first nonzero entry in the transformed matrix. Multiply each column j by $\overline{c_j} \in \mathbb{T}$, which satisfies condition $(S2)$. Permute the columns so that the first row has w nonzeroes (each of which must be one since $(S2)$ is satisfied) followed by $n-w$ zeroes, which satisfies $(S3)$. Finally, sort the rows of the matrix lexicographically with the ordering $\prec$. Note that the first row will not move since it is the least lexicographic row in the matrix. The transformed matrix now satisfies condition $(S4)$, and hence, is in standard form. □

It is important to note that two matrices that have different standard forms may be equivalent to one another. Studying the number of standardized weighing matrices will lead to an upper bound on the number of inequivalent unit weighing matrices.

3.5 Existence of Unit Weighing Matrices

In this section, we will study the existence of unit weighing matrices with small weights. In all cases, we will describe an upper bound on the number of inequivalent weighing matrices by studying the number of weighing matrices in standard form (see Definition 3.22). Before we start the analysis of the existence and non-existence of specific types of unit weighing matrices, we will give a definition which will be used heavily throughout the remainder of this chapter.

Definition 3.24. Let $S \subset \mathbb{T}$. S is said to have *m-orthogonality* if there are $a_1, \ldots, a_m \in S$ and $b_1, \ldots, b_m \in S$ such that $\sum_{i=1}^{m} c_i = 0$, where $c_i = a_i\overline{b_i}$.

We will be using the following results for a few small values of m in this thesis.

Proposition 3.25. *Let $S \subset \mathbb{T}$ and $a, b, c, d \in \mathbb{T}$. Then (up to a relabelling of variables),*

(a) S has 0-orthogonality.

(b) S does not have 1-orthogonality.

(c) $a + b = 0 \iff a = -b$.

(d) $a + b + c = 0 \iff a = e^{i\theta}, b = e^{i\theta + \frac{2\pi i}{3}}$ and $c = e^{i\theta - \frac{2\pi i}{3}}$.

(e) $a + b + c + d = 0 \iff a = -b$ and $c = -d$.

Proof. Each of these statements can be easily proven geometrically by viewing the unimodular numbers as unit vectors in $\mathbb{R}^2$. □

Note that a set S may have m-orthogonality for many values of m. For example, the set of all third roots of unity has m-orthogonality for all multiples of 3, whereas the set of all sixth roots of unity has m-orthogonality for all $m \neq 1$.

m-orthogonality will be used in a very particular way in this thesis. If we examine two distinct rows of a unit weighing matrix, then we know that their complex inner product is 0. So the set of entries in our weighing matrix is what we are interested in. The value of m is the number of columns that contribute a nonzero amount to the inner product of the two rows (i.e., both rows contain nonzero entries in those columns).

Example 3.26. Let

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & a & b & c & 0 & 0 \\ 1 & d & 0 & 0 & 1 & 1 \\ 1 & e & 0 & 0 & f & g \\ 0 & 0 & 1 & h & i & j \\ 0 & 0 & 1 & k & l & m \end{pmatrix}$$

be a partially filled unit weighing matrix (with all variables in $\mathbb{T}$). Then we know that since the inner product of the first and second row must be 0, the set $\{1,a,b,c\}$ must have 4-orthogonality. Moreover, since the inner product of the second and third row must be 0, the set $\{1,a,d\}$ must have 2-orthogonality.

As a shorthand, the phrases "*since rows 1 and 2 have 4-orthogonality*" or "*by the 2-orthogonality of rows 2 and 3*" will be used in lieu of the full statements given in Example 3.26.

We begin by extending a result of [16, Proposition 2.5] to unit weighing matrices.

Lemma 3.27. *If there is a $UW(n,w)$ and $n > z^2 - z + 1$, where $z = n - w$ is the number of zeroes in each row of the matrix, then there is a set that has $(n-2z)$-orthogonality.*

Proof. First, note that the cases where $z \le 1$ are straightforward. Now assume $z \ge 2$. Through appropriate row and column permutations, we may assume that the first z entries in the first row and first column are 0.

- Let $Z(i,j)$ be the number of zeroes in the first j rows of the i^{th} column.
- Let $E(k)$ be the row that contains the last 0 in column k (*i.e.*, $Z(k,j) = w$ for all $j \ge E(k)$ and $Z(k,j) < w$ for all $j < E(k)$).

By construction, $E(1) = z$. We know that $1 \le Z(2,E(1)) \le z$, so by appropriate row permutations, the next $z-Z(2,E(1))$ rows will have a zero in the second column. This implies

$$E(2) = E(1) + (z - Z(2,E(1))) = 2z - Z(2,E(1)) \leq 2z - 1.$$

Furthermore, $1 \leq Z(3,E(2)) \leq z$. We once again perform row permutations so that the next $z-Z(3,E(2))$ rows have a zero in the third column, so

$$E(3) = E(2) + (z - Z(3,E(2))) \leq (2z-1) + (z - Z(3,E(2))) \leq 3z - 2.$$

In general, following this process, we have

$$E(k) \leq kz - (k-1)$$

for $k \leq z$. So this gives $E(j) \leq z^2 - (z-1)$ for $j \leq z$. Thus, if we examine row $z^2 - z + 2$, we know that the first z columns already have z zeroes in them, and thus, all z zeroes must appear in the last $n-z$ columns of that row. The set of entries in this row and the first row has $(n-2z)$-orthogonality. It is noteworthy to mention that $n > z^2 - z + 1$ implies that $n - 2z \geq 0$. □

Corollary 3.28 (Geramita-Geramita-Wallis, [16])**.** *For odd n, a necessary condition that a $W(n,w)$ exists is that $n \leq (n-w)^2 - (n-w) + 1$.*

Proof. Let $z = n - w$. For odd n, $n-2z$ is odd, but $\{\pm 1\}$ does not have $(n-2z)$-orthogonality. The result follows from Lemma 3.27. □

With these in our hand, we will begin the classification of unit weighing matrices.

3.5.1 Weight 1

Any weighing matrix of weight 1 is equivalent to the identity matrix. Thus, $UW(n,1)$ exists for every $n \in \mathbb{N}$.

3.5.2 Weight 2

We begin with the non-existence of a certain type of unit weighing matrices.

Lemma 3.29. *There is no* $UW(3,2)$.

Proof. By Lemma 3.27, the existence of a $UW(3,2)$ would imply the existence of a set having 1-orthogonality, which would contradict Proposition 3.25(b). □

This leads us to the following theorem.

Theorem 3.30. *A* $UW(n,2)$ *exists if and only if n is even. Moreover, there is exactly one equivalent class of* $UW(n,2)$ *for each even n and it contains a real weighing matrix.*

Proof. Let W be a $UW(n,2)$. By Theorem 3.23, we may transform W into a weighing matrix in standard form (we will call this matrix W'). Thus, the first two entries of the first column and first row are ones. The second entry in the second row must then be -1 by 2-orthogonality of the first two rows. So we have that

$$W' = \left(\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{0} \\ 1 & - & \\ \hline \multicolumn{2}{c|}{0} & W'' \end{array}\right)$$

where W'' is a $UW(n-2,2)$. We may now use the same process on W'' and continue until we arrive at the bottom right corner. If n is even, then we can complete the matrix. However, if n is odd, then the process would end with a 3×3 block which must be a $UW(3,2)$. But we know from Lemma 3.29 that it does not exist. Thus, there is no $UW(n,2)$ for n odd. Since the number of equivalence classes of weighing matrices is bounded above by the number of standardized matrices and there is only one standardized matrix, every weighing matrix of order n and weight 2, for n even, is equivalent to

$$\begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes I_{n/2}.$$

□

3.5.3 Weight 3

Weight 3 is the first example where unit weighing matrices differ from real weighing matrices.

Lemma 3.31. *Every $UW(n,3)$ is equivalent to a weighing matrix whose top leftmost submatrix is either a $UW(3,3)$ or a $UW(4,3)$.*

Proof. This proof can be found in Appendix A.1. □

Theorem 3.32. *Every $UW(n,3)$ is equivalent to a matrix of the following form:*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$.

Proof. Let W be a $UW(n,3)$. From the proof of Lemma 3.31, we have that W can be transformed in such a way that the top leftmost block is either a $UW(3,3)$ or $UW(4,3)$. So the first 3 (or 4) rows and columns of the matrix are complete (i.e., no more nonzero entries can be added to those rows or columns), and as such, are trivially orthogonal with the remainder of the matrix. Thus, the lower $(n-3)\times(n-3)$ submatrix (or $(n-4)\times(n-4)$ submatrix) is a $UW(n-3,3)$ (or $UW(n-4,3)$). The top left submatrix will also be of the desired form (see the proof of Lemma 3.31). Continue inductively. The blocks may then be permuted such that all of the $UW(3,3)$ submatrices appear above the $UW(4,3)$ submatrices, and the result follows. □

Corollary 3.33. *For $n \geq 3$, there is a $UW(n,3)$ if and only if $n \neq 5$. The number of equivalence classes is bounded above by the number of distinct decompositions of n into sums of non-negative multiples of 3 and 4.*

Proof. By Theorem 3.32, we have the general structure of a unit weighing matrix of weight 3. A simple induction can show that for any $n \in \{m | m \geq 3 \text{ and } m \neq 5\}$, n can be written as the sum of threes and fours. We then take one such representation (say $n = 3a + 4b$) and construct

$$W = \underbrace{W_3 \oplus \cdots \oplus W_3}_{a \text{ items}} \oplus \underbrace{W_4 \oplus \cdots \oplus W_4}_{b \text{ items}}.$$

The second assertion is immediate. □

Note that an alternate way to show that $UW(5,3)$ does not exist is to use Lemma 3.27.

Corollary 3.34. *There is a $W(n,3)$ if and only if n is a multiple of 4. Moreover, there is only one class of equivalent matrices.*

Proof. We may use a proof similar to Corollary 3.33, except we may only use W_4, and not W_3. □

3.5.4 Weight 4

Similar to $UW(n,3)$, any $UW(n,4)$ can be decomposed as blocks along the main diagonal.

Lemma 3.35. *Each $UW(n,4)$ are equivalent to a $UW(n,4)$ with diagonal blocks consisting of the following matrices: W_5, W_6, W_7, W_8 and $E_{2m}(x)$ where $2 \leq m \leq \frac{n}{2}$ and $x \in \mathbb{T}$.*

$$W_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix}, W_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 & 0 \\ 1 & \overline{\omega} & \omega & 0 & 0 & 1 \\ 1 & 0 & 0 & - & - & - \\ 0 & 1 & 0 & - & -\overline{\omega} & -\omega \\ 0 & 0 & 1 & - & -\omega & -\overline{\omega} \end{pmatrix} \mathit{for}\ \omega = e^{\frac{2\pi i}{3}},$$

$$W_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}, W_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & - & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & - \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 1 & - \end{pmatrix} \mathit{and}$$

$$
E_{2m}(x) = \left(\begin{array}{ccccccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & & & & \\
1 & 1 & - & - & & & & & & & & & & & \\
1 & - & 0 & 0 & 1 & 1 & & & & & & & & & \\
1 & - & 0 & 0 & - & - & & & & & & & & & \\
 & & 1 & - & 0 & 0 & 1 & 1 & & & & & & & \\
 & & 1 & - & 0 & 0 & - & - & & & & & & & \\
 & & & & 1 & - & 0 & 0 & & & & & & & \\
 & & & & 1 & - & 0 & 0 & & & & & & & \\
 & & & & & & & & \ddots & & & & & & \\
 & & & & & & & & & 0 & 0 & 1 & 1 & & \\
 & & & & & & & & & 0 & 0 & - & - & & \\
 & & & & & & & & & 1 & - & 0 & 0 & 1 & 1 \\
 & & & & & & & & & 1 & - & 0 & 0 & - & - \\
 & & & & & & & & & & & 1 & - & x & -x \\
 & & & & & & & & & & & 1 & - & -x & x
\end{array}\right).
$$

Proof. The case analysis for this is quite long. As such, the full proof can be found in Appendix A.2. □

Example 3.36. To give a better understanding of the E_{2m}, here are the first three orders:

$$E_4(x) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & x & -x \\ 1 & - & -x & x \end{pmatrix}, E_6(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 \\ 1 & - & 0 & 0 & - & - \\ 0 & 0 & 1 & - & x & -x \\ 0 & 0 & 1 & - & -x & x \end{pmatrix} \text{ and}$$

$$E_8(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 & - & - & 0 & 0 \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - & 0 & 0 & - & - \\ 0 & 0 & 0 & 0 & 1 & - & x & -x \\ 0 & 0 & 0 & 0 & 1 & - & -x & x \end{pmatrix}.$$

Corollary 3.37. *There is a $UW(n,4)$ for every $n \geq 4$. The number of equivalence classes is bounded above by the number of distinct decompositions of n into sums of non-negative multiples of $5,6,7,8$ and $2m$, $m \geq 2$. (See Appendix C.2 for the different combinations available for all $n \leq 14$.)*

Proof. Similar to Corollary 3.33 □

Corollary 3.38. *Every real weighing matrix of weight four is comprised of blocks of W_7, W_8 and $E_{2m}(1)$ along the main diagonal.*

Proof. Let W be a real weighing matrix of order n and weight 4. Clearly, $\Lambda(W)$ contains only real entries (in particular, only 0 and ± 1). Since $\Lambda(W_5)$ and $\Lambda(W_6)$ contain third roots of unity, W cannot contain W_5 or W_6 as submatrices. Moreover, if $x_0 \notin \mathbb{R}$, then $\Lambda(E_{2m}(x_0))$ will contain a non-real entry, and so W cannot contain $E_{2m}(x_0)$ as a submatrix either. Combining this with Lemma 3.35, we have that we may only use W_7, W_8, $E_{2m}(1)$ and $E_{2m}(-1)$ for real weighing matrices. Note that $E_{2m}(1)$ is equivalent to $E_{2m}(-1)$ by swapping the last and second last columns. This implies that $W(n,4)$ exists for any $n \neq 5,9$. Moreover, the number of equivalence classes of $W(n,4)$ is bounded above by the number of decompositions of n into sums of non-negative multiples of $7,8$, and $2m$, $m \geq 2$. □

3.5.5 Weight 5

For weight 5, only a partial classification has been completed. In the following pages, we will show the results for $n \leq 7$.

UW(5,5)

Haagerup [17] found that the only unit Hadamard matrix of order five is the Fourier matrix F_5 given here:

$$F_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega & \omega^4 & \omega^3 \\ 1 & \omega^3 & \omega^4 & \omega & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{5}}$.

UW(6,5)

Lemma 3.39. *Every $UW(6,5)$ is equivalent to the following matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & a & 0 & b & c \\ 1 & -y & 0 & d & f & g \\ 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{pmatrix}$$

where all variables represent unimodular numbers.

Proof. Let W an arbitrary $UW(6,5)$. By appropriate row and column permutations, we may place the zeroes (one per row and column) along the back diagonal. We may then multiply each of the first five rows by the multiplicative inverse of the first entry in that row. We repeat this process on each of the columns, followed by the sixth row to arrive at the following matrix (note that the variables listed here will be relabelled in the final step of the proof to match the labels given in the statement).

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & a & b & c & 0 & 1 \\ 1 & d & f & 0 & g & h \\ 1 & j & 0 & k & l & m \\ 1 & 0 & n & p & q & r \\ 0 & 1 & s & t & u & v \end{pmatrix}$$

By 4-orthogonality with row 1, we have that at least one of a, b or c equals -1 and the other two are negations of one another. In all three cases, we can transform the matrix in such a way that $a = -1$.

Case 1: $a = -1$. This is in the desired form.

Case 2: $b = -1$. This can be transformed into the desired form by swapping columns 2,3, swapping rows 4,5 and multiplying row 6 by $\bar{s}$.

Case 3: $c = -1$. This can be transformed into the desired form by swapping columns 2,4, swapping rows 3,5 and multiplying row 6 by $\bar{t}$.

Now, since column 1 and column 2 must be orthogonal, we have that $d = -j$. By appropriate relabelling, we have our result. □

Lemma 3.40. *There are at most 7 distinct equivalence classes of* $UW(6,5)$.

Proof. The proof of this Lemma is a large amount of tedious case analysis. For this reason, the full details are available in Appendix A.3. The cases reveal that all $UW(6,5)$ are equivalent to at least one of the seven matrices listed in Table 3.1.

□

We will continue chopping away at the number of equivalence classes of weighing matrices with the following lemma, which shows that all of the non-sporadic families found in Lemma 3.40 are equivalent.

Lemma 3.41. *Let*

$$W_1(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & - & - & 0 & 1 \\ 1 & - & x & 0 & -x & x \\ 1 & - & 0 & -x & x & -x \\ 1 & 0 & -x & x & - & - \\ 0 & 1 & x & -x & - & - \end{pmatrix}.$$

Then $W_1 \cong T_1 \cong T_2 \cong T_3 \cong T_4 \cong T_5$.

Proof. To show that two families of unit weighing matrices are equivalent, we must give the permutation matrices that transform one matrix into another. Moreover, since we are

Table 3.1: List of standardized $UW(6,5)$.

$$T_1(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & - & -x & 0 & x & - \\ 1 & 1 & 0 & - & - & -\bar{x} \\ 1 & 0 & - & x & -x & \bar{x} \\ 0 & 1 & - & -x & x & \bar{x} \end{pmatrix} \quad T_2(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & - & 1 & 0 & 1 \\ 1 & x & -x & 0 & - & - \\ 1 & -x & 0 & - & x & -x \\ 1 & 0 & x & - & -x & x \\ 0 & 1 & - & -\bar{x} & \bar{x} & \bar{x} \end{pmatrix}$$

$$T_3(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & -\bar{x} & - & 0 & \bar{x} & - \\ 1 & \bar{x} & 0 & - & -\bar{x} & - \\ 1 & 0 & -x & x & - & 1 \\ 0 & 1 & - & - & 1 & 1 \end{pmatrix} \quad T_4(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & 1 & - & 0 & 1 \\ 1 & x & - & 0 & -x & x \\ 1 & -x & 0 & x & - & - \\ 1 & 0 & - & -x & x & -x \\ 0 & 1 & \bar{x} & - & -\bar{x} & -\bar{x} \end{pmatrix}$$

$$T_5(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & x & -x & 0 & - & x \\ 1 & -x & 0 & x & - & -x \\ 1 & 0 & - & - & 1 & - \\ 0 & 1 & x & -x & - & - \end{pmatrix} \quad T_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & i & -i & 0 & 1 \\ 1 & i & - & 0 & -i & - \\ 1 & -i & 0 & i & - & -i \\ 1 & 0 & -i & - & i & i \\ 0 & 1 & - & -i & i & -i \end{pmatrix}$$

$$T_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & -i & i & 0 & 1 \\ 1 & -i & - & 0 & i & - \\ 1 & i & 0 & -i & - & i \\ 1 & 0 & i & - & -i & -i \\ 0 & 1 & - & i & -i & i \end{pmatrix}$$

dealing with families of matrices, we must also provide the variable transformation needed to arrive at the desired matrix. In particular, for each case, we shall show that $W_1(x) = P_i(y_i)T_i(y_i)Q_i(y_i)$ by giving P_i, Q_i and y_i for all $1 \leq i \leq 5$. These values can be found in Table 3.2.

□

Lemma 3.42. *There are at least two inequivalent $UW(6,5)$.*

Proof. Through a lengthy computation, we have that

$$\Lambda(W_1(x)) = \{(0,666),(1,294),(-1,144),(x,48),(-x,48),(\bar{x},48),(-\bar{x},48)\}$$

and

$$\Lambda(T_6) = \Lambda(T_7) = \{(0,666),(1,270),(-1,120),(i,120),(-i,120)\}.$$

By Theorem 3.18, $W_1(x) \ncong T_6$ and $W_1(x) \ncong T_7$ for any $x \in \mathbb{T}$. □

Thus, in determining the total number of equivalence classes of unit weighing matrices of order 6 and weight 5, we are down to determining whether or not T_6 is equivalent to T_7.

Lemma 3.43. $T_6 \ncong T_7$

Proof. Our goal will be to alter the look of T_7 in an attempt to make it look more like T_6. When any row permutation is applied to T_7, then there is a unique column permutation that must be applied to the matrix that places the zeroes along the back diagonal (since there is exactly one zero in each column). At this point, we must simply make the first entry in each row and column a one by appropriate row and column multiplications. If T_6 and T_7 are equivalent, then one of these permutations must result in the same matrix. Thus, there are only 6! matrices to examine. A quick computer computation determines that these two matrices are not equivalent. □

Table 3.2: List of P_i,Q_i and y_i such that $W_1(x) = P_i(y_i)T_i(y_i)Q_i(y_i)$.

i	y_i	$P_i(x)$	$Q_i(x)$
1	$-x$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$
2	$\bar{x}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\bar{x} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{x} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & - \end{pmatrix}$
3	$-\bar{x}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - \\ 0 & \bar{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\bar{x} & 0 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - \end{pmatrix}$
4	x	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
5	$\bar{x}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 \\ 0 & \bar{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{x} \\ 0 & 0 & -\bar{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{x} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Theorem 3.44. *There are exactly 3 inequivalent unit weighing matrices of order 6 and weight 5:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & - & - & 0 & 1 \\ 1 & - & x & 0 & -x & x \\ 1 & - & 0 & -x & x & -x \\ 1 & 0 & -x & x & - & - \\ 0 & 1 & x & -x & - & - \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & i & j & 0 & 1 \\ 1 & i & - & 0 & j & - \\ 1 & j & 0 & i & - & j \\ 1 & 0 & j & - & i & i \\ 0 & 1 & - & j & i & j \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & i & j & 0 & 1 \\ 1 & i & j & 0 & - & i \\ 1 & j & 0 & - & i & - \\ 1 & 0 & - & i & j & j \\ 0 & 1 & i & - & j & i \end{pmatrix}.$$

$W_1(x)$ is given in Lemma 3.41 and the two sporadic cases given in Table 3.1.

Proof. From Lemma 3.40 and Lemma 3.41, we know that there are at most 3 inequivalent matrices. Lemma 3.42 and Lemma 3.43 show that those three matrices are not equivalent. □

Note that if we consider conjugation and transposition as part of the equivalence relation, then T_6 is equivalent to T_7 since $T_6 = T_7^*$.

Proposition 3.45. *Every $UW(6,5)$ is equivalent to a symmetric matrix and every non-sporadic $UW(6,5)$ is equivalent to a Hermitian matrix.*

Proof. Let W be a $UW(6,5)$. We know by Theorem 3.44 that W is equivalent to exactly one of three matrices. The first statement is true since the three matrices given in Theorem 3.44 are symmetric. The second statement is true by noting that $W_1 \cong T_3$ (T_3 is given in the proof of Lemma 3.40 and is the matrix in the second row, first column of Table 3.1). If we swap rows 3 and 4 of T_3, then we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & \bar{x} & 0 & - & -\bar{x} & - \\ 1 & -\bar{x} & - & 0 & \bar{x} & - \\ 1 & 0 & -x & x & - & 1 \\ 0 & 1 & - & - & 1 & 1 \end{pmatrix}$$

which is Hermitian symmetric. □

Lemma 3.46. *There is exactly one real $W(6,5)$ (up to equivalence).*

Proof. Obviously, the two sporadic cases in Theorem 3.44 are not real, so we may use an argument similar to the proof Corollary 3.38 to see that the only hope for a real $W(6,5)$ is $W_1(1)$ and $W_1(-1)$. By swapping columns 2 and 3 and rows 2 and 3 of $W_1(1)$, we note that these two matrices are equivalent. □

UW(7,5)

Lemma 3.47. *Any $UW(7,5)$ must include the following rows (after appropriate column permutations):*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & a & b & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & c & d & f & g \\ 0 & 0 & 1 & h & k & m & n \end{pmatrix}$$

Proof. To prove this condition, we show that three rows must exist with disjoint zeroes (two rows have disjoint zeroes if for every column, there is at most one zero between the two rows).

Let W be a $UW(7,5)$. We can begin by assuming the standard starting row of five ones and two zeroes. Permute the rows such that the second row is not disjoint from row 1. Two cases may occur from this: there is an overlap of either one or two zeroes between the first and second rows. If there is an overlap of two zeroes, then the third row must be disjoint from both the first and second rows. If there is single overlap, then permute the rows so that the third row has one overlap with the first. Then the fourth row must be disjoint from the first row since the last two columns are complete. Thus, in either case, there are at least two disjoint rows.

From here, we can easily show that there must be three rows which are mutually disjoint. To do this, we assume that the first two rows are disjoint (say their zeroes are in columns $1-4$). We may only put one more zero in each of those 4 columns, but we have 5 rows left, so at least one row must have no zeroes in columns $1-4$. So this row, along with the first two, are mutually disjoint.

□

Theorem 3.48. *There is no* $UW(7,5)$.

Proof. Any $UW(7,5)$ must contain the submatrix given in Lemma 3.47. We will show that the rows cannot be mutually orthogonal.

Taking the pairwise standard complex inner product of the rows, we obtain the following system of equations:

$$\begin{cases} 1+a+b & = & 0 \\ 1+c+d & = & 0 \\ 1+h+k & = & 0 \\ 1+f+g & = & 0 \end{cases}$$

This implies $a,b,c,d,f,g,h,k \in \{e^{\pm 2\pi i/3}\}$ where a,c,h,f are the conjugates of b,d,k,g, respectively. We will now rewrite the submatrix.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & a & \overline{a} & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & c & \overline{c} & f & \overline{f} \\ 0 & 0 & 1 & h & \overline{h} & m & n \end{pmatrix}$$

Now let's consider the inner product of the second and fourth vectors: $a+m+n=0$. Since $a \in \{e^{\pm i\frac{2\pi}{3}}\}$, we have that $m,n \in \{1,\overline{a}\}$ where $m \neq n$, by Proposition 3.25 (d). The inner product of rows 3 and 4 is now the sum of 4 third roots of unity, which cannot be zero. Thus, no $UW(7,5)$ can exist.

□

Chapter 4

Unbiasedness

Take chances,

make mistakes,

get messy!

– V. F. Frizzle

(This chapter is based on published work, [5].)

In this chapter, we introduce the idea of mutually unbiased unit weighing matrices. This utilizes the matrices that we introduced in the previous chapter.

Unbiasedness is a topic that has been studied in a variety of different settings. The roots of unbiasedness can be traced to physics, [23, 31, 37].

We start with the definition of unbiased bases.

Definition 4.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two orthonormal bases in $\mathbb{C}^n$. $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *unbiased* if

$$\forall u \in \mathcal{B}_1, v \in \mathcal{B}_2, |\langle u, v\rangle| = \frac{1}{\sqrt{n}}.$$

When we put a number of these bases together, we have the following fundamental definition.

Definition 4.2. Let $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ be a set of orthonormal bases in $\mathbb{C}^n$. $\mathcal{B}$ is called *mutually unbiased* if for all $1 \leq i, j \leq k$, $i \neq j$, $\mathcal{B}_i$ is unbiased with $\mathcal{B}_j$. These are often called "MUBs".

To put mutually unbiased bases in different terminology, we are looking for orthonormal bases whose basis vectors from different bases meet at a specific angle. However, rather than looking at these objects as bases, we will instead focus on a slightly different set of objects.

Definition 4.3. Two unit Hadamard matrices, H_1 and H_2, are *unbiased* if $H_1 H_2^* = \sqrt{n}H$, where H is a unit Hadamard matrix. A set of unit Hadamard matrices is called *mutually unbiased* if every distinct pair of matrices is unbiased. This are often called “MUHMs”.

The reason we are able to work with mutually unbiased Hadamard matrices instead of mutually unbiased bases is due to the following theorem.

Theorem 4.4. *There exists k mutually unbiased bases in $\mathbb{C}^n$ if and only if there exists $k-1$ mutually unbiased unit Hadamard matrices of order n.*

Proof. First, let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\}$ be orthonormal bases in $\mathbb{C}^n$. We may perform the same change of basis on the bases and arrive at another set of mutually unbiased bases, $\mathcal{B}' = \left\{\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_k\right\}$ where $\mathcal{B}'_1$ is the standard basis in $\mathbb{C}^n$. Since $\mathcal{B}'_1$ is unbiased with each of the other bases, we know that each entry in every other basis vector must have the same absolute value ($\frac{1}{\sqrt{n}}$). We can now create k square matrices, $B_1, B_2, \ldots, B_k$, such that the rows of B_i are the vectors in $\mathcal{B}'_i$. We note that $B_1 = I_n$ and that $\sqrt{n}B_i$ is a unit Hadamard matrix for any $2 \leq i \leq k$. Multiplication gives that the magnitude of each entry in $L = (\sqrt{n}B_i)\left(\sqrt{n}B_j\right)^*$ is $\sqrt{n}$. Moreover, the fact that

$$\left(\frac{1}{\sqrt{n}}L\right)\left(\frac{1}{\sqrt{n}}L\right)^* = \frac{1}{n}\left(\left(\sqrt{n}B_i\right)\left(\sqrt{n}B_j\right)^*\right)\left(\left(\sqrt{n}B_i\right)\left(\sqrt{n}B_j\right)^*\right)^* = \frac{1}{n}\left(n^2 I\right) = nI$$

gives that $\frac{1}{\sqrt{n}}L$ is a Hadamard matrix. Thus, $\{\sqrt{n}B_2, \ldots, \sqrt{n}B_k\}$ is a set of k mutually unbiased unit Hadamard matrices.

Next, we let $\{H_1, \ldots, H_{k-1}\}$ be a set of Hadamard matrices. It is easy to see that

$$\left\{I_n, \frac{1}{\sqrt{n}}H_1, \ldots, \frac{1}{\sqrt{n}}H_{k-1}\right\}$$

is a set of k mutually unbiased bases (where the vectors of the bases are the rows of the matrices). □

For the remainder of the thesis, we will focus on matrices in lieu of bases. When studying mutually unbiased objects, there are two main goals: finding lower and upper bounds on the size of the set of mutually unbiased Hadamard matrices and finding examples that attain these bounds.

Lemma 4.5. *Let $r, s \in \mathbb{T}^n$ such that $\langle r, s\rangle = \alpha$ and define $R = r^*r - I_n$ and $S = s^*s - I_n$. Then*

$$Tr(RS^*) = |\alpha|^2 - n.$$

Proof.

$$\begin{aligned}
\mathrm{Tr}(RS^*) &= \mathrm{Tr}(RS^*) \\
&= \mathrm{Tr}((r^*r)(s^*s)^* - r^*r - (s^*s)^* + I) \\
&= \mathrm{Tr}((r^*r)(s^*s)^*) - \mathrm{Tr}(r^*r) - \mathrm{Tr}(s^*s) + \mathrm{Tr}(I) \\
&= \mathrm{Tr}((r^*r)(s^*s)^*) - n - n + n \\
&= \mathrm{Tr}(r^*(rs^*)s) - n \\
&= \alpha \cdot \mathrm{Tr}(r^*s) - n \\
&= \alpha\overline{\alpha} - n \\
&= |\alpha|^2 - n
\end{aligned}$$

□

Lemma 4.6. *Let $\{H_1, \ldots, H_k\}$ be a set of mutually unbiased Hadamard matrices of order n. Then $\left\{S_{ij} := r_{ij}^* r_{ij} - I | 1 \le i \le k, 2 \le j \le n\right\}$ is linearly independent where r_{ij} is the j^{th} row of H_i.*

Proof. Let $a_{12}, a_{13}, \ldots, a_{k2}, \ldots, a_{kn} \in \mathbb{C}^n$. Select arbitrary x, y such that $1 \le x \le k$ and $2 \le y \le n$. (The fifth implication below utilizes Lemma 4.5 in three separate ways.)

$$
\begin{aligned}
\sum_{i=1}^{k}\sum_{j=2}^{n} a_{ij}S_{ij} = 0 \quad &\implies \quad \left(\sum_{i=1}^{k}\sum_{j=2}^{n} a_{ij}S_{ij}\right) S_{xy}^{*} = 0 \\
&\implies \quad \sum_{i=1}^{k}\sum_{j=2}^{n} a_{ij}S_{ij}S_{xy}^{*} = 0 \\
&\implies \quad \mathrm{Tr}\left(\sum_{i=1}^{k}\sum_{j=2}^{n} a_{ij}S_{ij}S_{xy}^{*}\right) = 0 \\
&\implies \quad \left(\sum_{i\neq x}\sum_{j=2}^{n} a_{ij}\cdot \mathrm{Tr}\left(S_{ij}S_{xy}^{*}\right)\right) + \left(\sum_{j\neq y} a_{xj}\cdot \mathrm{Tr}\left(S_{xj}S_{xy}^{*}\right)\right) \\
&\qquad\quad + \left(a_{xy}\cdot \mathrm{Tr}\left(S_{xy}S_{xy}^{*}\right)\right) = 0 \\
&\implies \quad \left(\sum_{i\neq x}\sum_{j=2}^{n} a_{ij}\cdot (n-n)\right) + \left(\sum_{j\neq y} a_{xj}\cdot (0-n)\right) \\
&\qquad\quad + \left(a_{xy}\cdot \left(n^{2}-n\right)\right) = 0 \\
&\implies \quad n^{2}\cdot a_{xy} - n\sum_{j=2}^{n} a_{xj} = 0
\end{aligned}
$$

Since this must be true for any pair of x and y, it suffices to show

$$
D = I_k \otimes \begin{pmatrix}
n^2-n & -n & -n & & -n \\
-n & n^2-n & -n & \cdots & -n \\
-n & -n & n^2-n & & -n \\
 & \vdots & & \ddots & \vdots \\
-n & -n & -n & \cdots & n^2-n
\end{pmatrix}_{(n-1)\times(n-1)}
$$

is non-singular. For this, we need to show

$$\det\begin{pmatrix} n^2-n & -n & -n & & -n \\ -n & n^2-n & -n & \cdots & -n \\ -n & -n & n^2-n & & -n \\ & \vdots & & \ddots & \vdots \\ -n & -n & -n & \cdots & n^2-n \end{pmatrix} \neq 0.$$

By adding the negative of the first row of the matrix to each of the other rows, followed by subtracting each of the other columns from the first, we see that

$$\det\begin{pmatrix} n & -n & -n & \cdots & -n \\ 0 & n^2 & 0 & \cdots & 0 \\ 0 & 0 & n^2 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n^2 \end{pmatrix} = (n\cdot(n^2)^{n-2}) = n^{2n-3} \neq 0.$$

□

Theorem 4.7. *If $\{H_1,\ldots,H_k\}$ be a set of mutually unbiased unit Hadamard matrices of order n, then $k \leq n$.*

Proof. Let $\{H_1,\ldots,H_k\}$ be a set of mutually unbiased unit Hadamard matrices of order n. Let r_{ij} be the j^{th} row of H_i. We define $S_{ij} = r_{ij}^* r_{ij} - I_n$. Noting that each S_{ij} is Hermitian and has a zero diagonal gives that Span($\{S_{ij}\}$) is a subspace of all Hermitian matrices with a zero diagonal. By Lemma 4.6, we have that S_{ij} are linearly independent. Combining this with the fact that the set of Hermitian matrices with zero diagonal has dimension $2(0+1+2+\cdots+(n-1))$, we have

$$k(n-1) = |S_{ij}| \leq 2(0+1+2+\cdots+(n-1)) = n(n-1),$$

whence the result follows. □

Theorem 4.8. *If $\{H_1, \ldots, H_k\}$ be a set of mutually unbiased real Hadamard matrices of order n, then $k \leq \frac{n}{2}$.*

Proof. We may utilize the same proof as Theorem 4.7 with one small change: since our matrices are real, we have that $\{S_{ij}\}$ is a subset of symmetric, zero diagonal matrices, which has a dimension of $(0+1+2+\cdots+(n-1))$. □

The next theorem will show that this bound is attained in some cases.

Lemma 4.9. *Let p be an odd prime power.*

$$\left|\sum_{k=0}^{p-1} e^{(ak^2+bk)2\pi i/p}\right| = \begin{cases} p & \text{if } a \equiv 0 \pmod{p} \text{ and } b \equiv 0 \pmod{p}, \\ 0 & \text{if } a \equiv 0 \pmod{p} \text{ and } b \not\equiv 0 \pmod{p}, \\ \sqrt{p} & \text{otherwise.} \end{cases}$$

Proof. We will only show the proof for odd prime numbers. The proof for prime powers follows similarly using finite fields. We note that the first case is trivial. Now, if $a \equiv 0 \pmod{p}$ and $b \not\equiv 0 \pmod{p}$, then

$$\sum_{k=0}^{p-1} e^{\frac{2\pi i}{p}(ak^2+bk)} = \sum_{k=0}^{p-1} e^{\frac{2\pi i}{p}(bk)}.$$

Since p is prime, $e^{\frac{2\pi i}{p}b}$ is a primitive p^{th} root of unity. Since we are summing over all powers of $e^{\frac{2\pi i}{p}b}$, the second equality holds.

The third equality is trickier – we will examine the absolute value squared. In the following steps, the fact that p is an odd prime is only used in going from the third last equality to the second last (inside the curly braces, $\{\cdot\}$) to ensure that $2am$ is a primitive p^{th} root of unity for any choice of m, $1 \leq m \leq p-1$. To conserve space, we will use $\mathbf{e}(x) := e^{\frac{2\pi i}{p}x}$.

$$\begin{aligned}
&\left|\sum_{k=0}^{p-1}\mathbf{e}(ak^2+bk)\right|^2\\
=\;&\left(\sum_{k=0}^{p-1}\mathbf{e}(ak^2+bk)\right)\overline{\left(\sum_{\ell=0}^{p-1}\mathbf{e}(a\ell^2+b\ell)\right)}\\
=\;&\left(\sum_{k=0}^{p-1}\mathbf{e}(ak^2+bk)\right)\left(\sum_{\ell=0}^{p-1}\mathbf{e}(-(a\ell^2+b\ell))\right)\\
=\;&\sum_{k=0}^{p-1}\sum_{\ell=0}^{p-1}\left[\mathbf{e}\left(a(k^2-\ell^2)+b(k-\ell)\right)\right]\\
=\;&\sum_{k=0}^{p-1}\sum_{\ell=0}^{p-1}\left[\mathbf{e}\left((k-\ell)(a(k+\ell)+b\right)\right]\\
=\;&\sum_{m=0}^{p-1}\sum_{\ell=0}^{p-1}\left[\mathbf{e}\left(m(a(m+2\ell)+b\right)\right]\\
=\;&\sum_{m=0}^{p-1}\left[\mathbf{e}\left(m(am+b)\right)\sum_{\ell=0}^{p-1}\mathbf{e}\left(2am\ell\right)\right]\\
=\;&\left[\mathbf{e}\left(0\right)\sum_{\ell=0}^{p-1}\mathbf{e}\left(0\right)\right]+\sum_{m=1}^{p-1}\left[\mathbf{e}\left(m(am+b)\right)\left\{\sum_{\ell=0}^{p-1}\mathbf{e}\left(2am\ell\right)\right\}\right]\\
=\;&p+\sum_{m=1}^{p-1}\left[\mathbf{e}\left(m(am+b)\right)\{0\}\right]\\
=\;&p
\end{aligned}$$

□

Theorem 4.10 ([23, 31, 39])**.** *If n is an odd prime power, then there exists n mutually unbiased unit Hadamard matrices of order n. They are*

$$\{H_1, H_2, \ldots, H_n\},$$

where $(H_j)_{k\ell} = e^{(j\ell^2+k\ell)2\pi i/n}$.

Proof. Note that the inner product of the r^{th} row of the H_j and the s^{th} row of H_k takes the following form

$$\sum_{m=0}^{n-1} e^{(jm^2+rm)2\pi i/n}\overline{e^{(km^2+sm)2\pi i/n}} = \sum_{m=0}^{n-1} e^{((j-k)m^2+(r-s)m)2\pi i/n}.$$

We may then utilize the correct case from Lemma 4.9 to give us the desired absolute value of the inner product. □

It is important to note that the size of the sets found in Theorem 4.10 are the same as the upper bound given by Theorem 4.7, so our upper bound is sharp in some situations. However, if we look at any order of unit Hadamard matrices other than a prime power, we run into a problem. Mutually unbiased unit Hadamard matrices have been looked at quite extensively, and even in the first case that is not a prime power, $n = 6$, no example is known that attains the upper bound in Theorem 4.7. In fact, it is generally believed that the maximal set of mutually unbiased unit Hadamard matrices of order 6 is 2 (see [3]).

4.1 Mutually Unbiased Weighing Matrices

We now introduce a natural extension to mutually unbiased Hadamard matrices.

Definition 4.11. Two unit weighing matrices, W_1 and W_2, of order n and weight w are *unbiased* if $W_1 W_2^* = \sqrt{w}W$, where W is a unit weighing matrix of order n and weight w. A set of unit weighing matrices that are pairwise unbiased is called *mutually unbiased*. These are shortened to be called "MUWM".

Except the case where $n = w$, a set of mutually unbiased unit weighing matrices are not equivalent to a set of mutually unbiased bases (as in Theorem 4.4). Instead, we are now dealing with a set of orthonormal bases whose vectors meet at two angles ($\pi/2$ and $\cos^{-1}(\frac{1}{\sqrt{w}})$) instead of just one ($\cos^{-1}(\frac{1}{\sqrt{n}})$). It is for this reason that these sets are termed *biangular*.

When we deal with real weighing matrices, we have the following strong restriction on the weight of the matrices.

Lemma 4.12. *Let W_1 and W_2 be real unbiased weighing matrices of order n and weight w. Then w must be a perfect square.*

Proof. Since both W_1 and W_2 are integer matrices, $W_1 W_2^T = \sqrt{w}L$ must be an integer matrix as well. □

Note that Lemma 4.12 is a special case of a proof for Hadamard matrices found in [8].

However, when we are dealing with unit weighing matrices, we have no such restriction. In fact, we have the following.

4.1.1 Bounds and Assumptions

In this section, we will describe the structure of mutually unbiased unit weighing matrices, as well as examine lower and upper bounds on the size of sets of mutually unbiased unit weighing matrices. We begin with a construction of mutually unbiased unit weighing matrices that is built off of other sets (similar to the Kronecker construction for Hadamard matrices in Lemma 2.13).

Theorem 4.13. *Let $\left\{\mathcal{W}_1,\ldots,\mathcal{W}_k\right\}$ be a collection of sets of mutually unbiased unit weighing matrices of order n_i and weight w. Then there are*

$$\min_{1\le i\le k}\left(\left|\mathcal{W}_i\right|\right)$$

mutually unbiased unit weighing matrices of order $\sum_{i=1}^{k} n_i$ and weight w.

Proof. Let $\mathcal{W}_i=\left\{W_1^{(i)},W_2^{(i)},\ldots,W_{\ell_i}^{(i)}\right\}$ for each $1\le i\le k$ and let

$$m=\min_{1\le i\le k}\left(\left|\mathcal{W}_i\right|\right)=\min_{1\le i\le k}\left(\ell_i\right).$$

Then the set

$$\left\{\left(W_1^{(1)}\oplus\cdots\oplus W_1^{(k)}\right),\left(W_2^{(1)}\oplus\cdots\oplus W_2^{(k)}\right),\ldots,\left(W_m^{(1)}\oplus\cdots\oplus W_m^{(k)}\right)\right\}$$

gives the desired result by noting that $(A\oplus B)(A\oplus B)^*=AA^*\oplus BB^*$. □

Definition 4.14. Let W be a unit weighing matrix of order n and weight w. If $W=W_1\oplus W_2$ for some W_1 and W_2 of orders strictly less than n, then W is said to be *decomposable*[3]. Note

[3]The term *decomposable matrix* is sometimes used to describe a *reducible matrix*. The reader is warned not to confuse the two terms in this thesis.

that since the rows of W must be orthogonal, it follows that W_1 and W_2 are also weighing matrices. We may write W in such a way that $W = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ where each W_i is indecomposable of order n_i. The *block structure* of W is the k-tuple $(n_1, n_2, \ldots, n_k)$.

When two unit weighing matrices have exactly the same block structure, we will be able to utilize the following proposition.

Proposition 4.15. *If two weighing matrices, W_1 and W_2, of the same weight have the same block structure, then W_1 is unbiased with W_2 if and only if each indecomposable block of W_1 is unbiased with the corresponding indecomposable block of W_2.*

Proof. This is easily seen by noting that

$$(W_1^{(1)} \oplus \cdots \oplus W_1^{(m)})(W_2^{(1)} \oplus \cdots \oplus W_2^{(m)})^* = W_1^{(1)} W_2^{(1)*} \oplus \cdots \oplus W_1^{(m)} W_2^{(m)*}.$$

□

The block structures of matrices is repeatedly used in our proofs throughout the thesis by applying the following proposition.

Proposition 4.16. *Let $\{W_1, \ldots, W_k\}$ be a set of mutually unbiased unit weighing matrices of order n and weight w with the same block structure, say $(n_1, \ldots, n_m)$. Then k is bounded above by the maximal size of a set of mutually unbiased weighing matrices of order n_i and weight w, for $1 \leq i \leq k$.*

Proof. This follows from Proposition 4.15. □

When we examine an arbitrary set of mutually unbiased unit weighing matrices, they may not be in a form where Propositions 4.15 and 4.16 may be used. However, we may be able to apply appropriate row and column permutations in such a way that we may utilize those propositions. For example,

$$W_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & - & 0 \\ 0 & 1 & 0 & - \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \end{pmatrix}$$

are two indecomposable weighing matrices which are unbiased with one another. However, with appropriate row and column permutations[4], we may examine

$$W_1' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \end{pmatrix} \text{ and } W_2' = \begin{pmatrix} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix},$$

which are also unbiased with one another, and where Propositions 4.15 and 4.16 may be used. We will call the block structure found in W_1' and W_2' *suitable* and the block structure found in W_1 and W_2 *not suitable*. Throughout the article, we will only concern ourselves with matrices that have a suitable block structure. To this end, we pose an algorithm to determine a matrix's suitable block structure.

Lemma 4.17. *The suitable block structure of a unit weighing matrix of order n can be determined in $O(n^3)$ steps.*

Proof. Let W be a weighing matrix of order n and W' be the equivalent weighing matrix that has a suitable block structure. We define G_W be the graph on n vertices with an edge between vertices i and j if and only if at least one nonzero entry in row i is in the same column as a nonzero entry in row j in W. Two rows of W are in the same indecomposable block of W' if and only if there is a path between the corresponding nodes in G_W. Thus, an

[4]Note that the column permutations must be the same for both matrices to ensure they are still unbiased with one another.

indecomposable block of W' can be found by taking the rows corresponding to all vertices in any connected component of G_W and removing all columns that only have zeroes. The number of indecomposable blocks of W' is the number of connected components of G_W.

In total, this process involves two parts. First, to build the graph, we look at all pairs of rows and examining each column, for a time of $O(n^3)$. Then, we determine the number of connected components, which takes $O(n^2)$ via depth first search for an overall complexity of $O(n^3)$ steps. □

So far, the only upper bounds given are for mutually unbiased Hadamard matrices. In the following theorems, we will show that the number of mutually unbiased weighing matrices also has an upper bound. Each of the following theorems were given in [9], but we provide a more detailed proof here.

For the following four theorems, we will utilize the concept of tensor products, which the reader only needs a vague understanding of to understand fully[5].

Definition 4.18. Let V be a vector space and T be a tensor. T is a *symmetric n-tensor* if

$$T(v_1, v_2, \ldots, v_n) = T(v_{\sigma_1}, v_{\sigma_2}, \ldots, v_{\sigma_n})$$

for all permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Let $S^k(V)$ denote the space of symmetric n-tensors of V.

Lemma 4.19. *Let V be a vector space of dimension n. Then* $\dim\left(S^k(V)\right) = \binom{n+k-1}{k}$.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of V. The basis elements of $S^k(V)$ are $\{v_{a_1} \otimes \cdots \otimes v_{a_k}\}$ where $(a_1, \ldots, a_k)$ is any non-increasing sequence in $\{1, \ldots, n\}$. It is well known that the number of non-increasing sequences is $\binom{n+k-1}{k}$ [15]. □

[5] Kronecker products are a special case of tensor products on matrices, so any reader that is not familiar with tensor products may wish to view them as Kronecker products (see Definition 2.12).

Definition 4.20. A *positive definite matrix*, M, is an $n \times n$ matrix such that for any nonzero vector $v \in \mathbb{C}^n$, $v^*Mv > 0$. Similarly, a *positive semi-definite matrix*, N, is an $n \times n$ matrix such that for any nonzero vector $v \in \mathbb{C}^n$, $v^*Nv \geq 0$.

Definition 4.21. Let $V \subset \mathbb{T}^n$ such that $|V| = k$. Then the *Gramian matrix*, $\mathrm{Gram}(V) = [g_{ij}]$, is a $k \times k$ matrix where $g_{ij} = \langle v_i, v_j \rangle$.

Lemma 4.22. *Let $r \in \mathbb{R}$, M be a positive definite matrix, N be a positive semi-definite matrix and V be a set of unit vectors. Then we have the following.*

(a) $M + N$ is a positive definite matrix.

(b) M has an inverse.

(c) If $r > 0$, then rM is positive definite and rN is positive semi-definite.

(d) Applying simultaneous elementary row and column operations to M gives a positive definite matrix.

(e) $Gram(V)$ is a positive semi-definite matrix.

Proof. (a) Let $v \in \mathbb{C}^n \backslash \{0\}$.

$$v^*(M+N)v = v^*Mv + v^*Nv > 0 + v^*Nv \geq 0 + 0 = 0$$

(b) Let $v \in \mathbb{C}^n \backslash \{0\}$. Since $v^*Mv > 0$, we have that $Mv \neq 0$, so 0 cannot be an eigenvalue of M, and the result follows.

(c) Let $v \in \mathbb{C}^n \backslash \{0\}$.

$$v^*(rM)v = r(v^*Mv) > r \cdot 0 = 0$$

and

$$v^*(rN)v = r(v^*Nv) \geq r \cdot 0 = 0.$$

(d) Let $v \in \mathbb{C}^n \backslash \{0\}$. Let Q represent the elementary row operation you wish to apply. Then $Q^* M Q$ is the matrix after applying the row operations.

$$v^*(Q^*MQ)v = (Qv)^*M(Qv) > 0$$

since $Qv \in \mathbb{C}^n$ and M is positive definite.

(e) Let $V = \{v_1, \ldots, v_m\}$. And let A be the rectangular matrix of m rows where the i^{th} row of A is v_i. Then $\mathrm{Gram}(V) = AA^*$. Let $v \in \mathbb{C}^n \backslash \{0\}$. Then

$$v^* \mathrm{Gram}(V) v = v^*(AA^*)v = (v^*A)(v^*A)^* \geq 0.$$

□

Theorem 4.23 ([9, Equation 3.7]). *Let $V \subset \mathbb{R}^n$ be a set of unit vectors. If $|\langle v, w \rangle| \in \{0, \alpha\}$ for all $v, w \in V$, $v \neq w$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, then*

$$|V| \leq \binom{n+2}{3}. \tag{4.1}$$

Proof. Let $A = \{X_v := v \otimes v \otimes v | v \in V\} \subset S^3(\mathbb{R}^n)$. We claim that A is a set of linearly independent vectors in $S^3(\mathbb{R}^n)$, which would immediately give us our result through the use of Lemma 4.19. To show that A is linearly independent, we will show that the $\mathrm{Gram}(A)$ is non-singular.

To see this, note that $\langle X_v, X_w \rangle = \langle v, w \rangle^3$, which implies that $\mathrm{Gram}(A) = I + \alpha^3 C$ where $\mathrm{Gram}(V) = I + \alpha C$. We have that

$$\mathrm{Gram}(A) = I + \alpha^3 C = (1-\alpha^2)I + \alpha^2(I + \alpha C) = (1-\alpha^2)I + \alpha^2 \mathrm{Gram}(V).$$

From our assumption, we have that $1 - \alpha^2 > 0$ which means that $(1-\alpha^2)I$ is a positive definite matrix (by Lemma 4.22 (c)). And $\mathrm{Gram}(V)$ is the Gramian matrix of a set of vectors, which implies that $\alpha^2\text{Gram}(V)$ is a positive semi-definite matrix (by Lemma 4.22 (d) and (e)). The sum of a positive definite matrix and a positive semi-definite matrix is a positive definite matrix (by Lemma 4.22 (a)). All positive definite matrices have an inverse (by Lemma 4.22 (b)), so $\text{Gram}(A)$ must be non-singular. □

Theorem 4.24 ([9, Equation 5.9]). *Let $V \subset \mathbb{C}^n$ be a set of unit vectors. If $|\langle v,w\rangle| \in \{0,\alpha\}$ for all $v,w \in V$, $v \neq w$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, then*

$$|V| \leq n\binom{n+1}{2}. \tag{4.2}$$

Proof. The proof is nearly identical to Theorem 4.23 on replacing A with $A' = \{X_v := v \otimes v \otimes v^* | v \in V\} \subset S^2(\mathbb{C}^n) \otimes \mathbb{C}^n$. □

If we wish to add a restriction on the value of α, we can obtain a better bound in certain cases.

Theorem 4.25 ([9, Equation 3.9]). *Let $V \subset \mathbb{R}^n$ be a set of unit vectors where $|\langle v,w\rangle| \in \{0,\alpha\}$ for all $v,w \in V$, $v \neq w$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$. If $3-(n+2)\alpha^2 > 0$, then*

$$|V| \leq \frac{n(n+2)(1-\alpha^2)}{3-(n+2)\alpha^2}. \tag{4.3}$$

Proof. Let $h_v = \frac{1}{3}\sum_{i=1}^{n}((v\otimes e_i \otimes e_i) + (e_i \otimes v \otimes e_i) + (e_i \otimes e_i \otimes v))$ and $A = \{X_v := v \otimes v \otimes v | v \in V\} \subset S^3(\mathbb{R}^n)$.

We know that

$$(n+2)(I+\alpha^3 C) - 3(I+\alpha C)$$

is positive semi-definite since it may be obtained through simultaneous row and column permutations of $\text{Gram}(\{h_a\} \cup A)$ (using Lemma 4.22 (d) and (e)). Let v be an eigenvector of $I+\alpha C$ and let $v_0 := v^* v$ for convenience. Since $I+\alpha C$ is positive semi-definite, $(I+\alpha C)v = \lambda v \implies \lambda \geq 0$.

$$v^*([(n+2)(1-\alpha^2)]I-[3-\alpha^2(n+2)](I+\alpha C))v$$

$$=[(n+2)(1-\alpha^2)]v^*v-[3-\alpha^2(n+2)]v^*(I+\alpha C)v$$

$$=[(n+2)(1-\alpha^2)]v^*v-[3-\alpha^2(n+2)]v^*\lambda v$$

$$=[(n+2)(1-\alpha^2)]v_0-[3-\alpha^2(n+2)]\lambda v_0.$$

Since our original matrix was positive semi-definite, we know that this number must be non-negative, which implies

$$[(n+2)(1-\alpha^2)]v_0-[3-\alpha^2(n+2)]\lambda v_0\geq 0 \implies (n+2)(1-\alpha^2)\geq[3-\alpha^2(n+2)]\lambda$$

$$\implies \lambda\leq\frac{(n+2)(1-\alpha^2)}{3-\alpha^2(n+2)}$$

assuming $3-\alpha^2(n+2)>0$.

Since this must be true for all eigenvalues of $I+\alpha C$, we have the following

$$|V|=\mathrm{Tr}(I+\alpha C)=\sum_{i=1}^{n}\lambda\leq\sum_{i=1}^{n}\frac{(n+2)(1-\alpha^2)}{3-\alpha^2(n+2)}=\frac{n(n+2)(1-\alpha^2)}{3-\alpha^2(n+2)}.$$

□

Theorem 4.26 ([9, Equation 5.9])**.** *Let $V\subset\mathbb{C}^n$ be a set of unit vectors. If $|\langle v,w\rangle|\in\{0,\alpha\}$ for all $v,w\in V$, $v\neq w$, where $\alpha\in\mathbb{R}$ and $0<\alpha<1$, then*

$$|V|\leq\frac{n(n+1)(1-\alpha^2)}{2-(n+1)\alpha^2} \tag{4.4}$$

if the denominator is positive.

Proof. Similar to Theorem 4.25. □

It is important to note that in most cases, the bounds involving a specific α are smaller than the ones without, but not always. For example, if we are looking for real vectors with $n = 9$ and $\alpha = \frac{1}{2}$, the first bound, (4.1), gives us $|V| \leq 165$ whereas the second bound, (4.3), gives us $|V| \leq 297$.

The following are immediate corollaries to the previous few theorems.

Corollary 4.27. *Let $\mathcal{W} = \{W_1, \ldots, W_m\}$ be a set of mutually unbiased unit weighing matrices of order n and weight w. Then we have that*

$$m \leq \frac{(n-1)(n+2)}{2}. \tag{4.5}$$

Moreover, if $2w - (n+1) > 0$, then

$$m \leq \frac{w(n-1)}{2w-(n+1)}. \tag{4.6}$$

Proof. Define V to be the set of all rows of $\frac{1}{\sqrt{w}}W_1, \ldots, \frac{1}{\sqrt{w}}W_m$ (note that $|V| = mn$). Since $\mathcal{W}$ is a set of mutually unbiased weighing matrices, we set $\alpha = \frac{1}{\sqrt{w}}$. Moreover, note that since all vectors in V come from a weighing matrix of weight w, we may adjoin the rows of the identity matrix to V without disrupting the bi-angularity (note that now, $|V| = mn + n$). By applying Theorem 4.24 and Theorem 4.26 to V (with the rows of the identity matrix included), we obtain the desired results. □

Corollary 4.28. *Let $\mathcal{W} = \{W_1, \ldots, W_m\}$ be a set of real mutually unbiased weighing matrices of order n and weight w. Then we have that*

$$m \leq \frac{(n-1)(n+4)}{6}. \tag{4.7}$$

Moreover, if $3w - (n+2) > 0$, then

$$m \leq \frac{w(n-1)}{3w-(n+2)}. \tag{4.8}$$

Proof. Similar to Corollary 4.27. □

4.1.2 The Search For Sets

When we study mutually unbiased weighing matrices, our main goal is to find as many matrices in a set as possible. From Corollary 4.27 and Corollary 4.28, we have an upper bound for the number of mutually unbiased weighing matrices. We have also given constructions that will provide us with lower bounds, but before we may utilize any of those constructions, we must find examples of small mutually unbiased weighing matrices. This section demonstrates the searches that were involved with finding such sets.

With unit weighing matrices, an exhaustive computer search is impractical, if not impossible, to perform since each nonzero entry in every matrix has infinitely many choices. To this end, we restricted the entries to small roots of unity in our computer searches. For each type of matrix, we searched for matrices over the m^{th} roots of unity, with $m \leq 24$. The 12^{th} roots of unity seem to be the largest group needed to find some maximal sets. Many of the maximal sets that we found do not match the upper bound given in Corollary 4.27. However, for many of these cases, we will prove smaller upper bounds than those given in Corollary 4.27.

Table 4.1 contains a summary of the various bounds that we have for mutually unbiased weighing matrices.

4.1.3 Mutually Unbiased Weighing Matrices of Weight 2

In Theorem 3.30, we proved that $UW(n,2)$ do not exist for odd orders. For n even, we have the following.

Lemma 4.29. *Let n be even. Then there are at most 2 mutually unbiased weighing matrices of order n and weight 2.*

Proof. Say we have a set of mutually unbiased weighing matrices of the appropriate order and weight. From Theorem 3.30, we know that one of the matrices may be transformed

Table 4.1: A summary of upper bounds and lower bounds on the size of mutually unbiased weighing matrices. For lower bounds, if the upper bound is attained, we will give an explicit example of a set attaining the bound. For upper bounds, we will state the appropriate Theorem, Lemma, etc. Any row that is shaded indicates that there is a gap between the lower and upper bounds.

Type	Lower Bounds			Upper Bounds	
	Largest	Root of Unity	Example	Smallest	Rationale
UW(2,2)	2	4	Theorem 4.10	2	Corollary 4.27
UW(3,2)	0	–	–	0	Theorem 3.30
UW(3,3)	3	3	Theorem 4.10	3	Corollary 4.27
UW(4,2)	2	4	Lemma 4.29	2	Lemma 4.29
UW(4,3)	9	6	Corollary 4.32	9	Corollary 4.27
UW(4,4)	4	4	Theorem 4.10	4	Corollary 4.27
UW(5,2)	0	–	–	0	Theorem 3.30
UW(5,3)	0	–	–	0	Corollary 3.33
UW(5,4)	5	6	Theorem 4.34	5	Theorem 4.34
UW(5,5)	5	5	Theorem 4.10	5	Corollary 4.27
UW(6,2)	2	4	Lemma 4.29	2	Lemma 4.29
UW(6,3)	3	3	Corollary 4.32	3	Theorem 4.31
UW(6,4)	20	6	Theorem 4.35	20	Corollary 4.27
UW(6,5)	2	12	–	8	Corollary 4.27
UW(6,6)	2	12	–	6	Corollary 4.27
UW(7,2)	0	–	–	0	Theorem 3.30
UW(7,3)	3	6	Corollary 4.32	3	Theorem 4.31
UW(7,4)	8	2	Corollary 4.39	8	Theorem 4.38
UW(7,5)	0	–	–	0	Theorem 3.48
UW(7,6)	0	–	–	9	Corollary 4.27
UW(7,7)	7	7	Theorem 4.10	7	Corollary 4.27

into

$$\begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes I_{n/2}.$$

Permute the rows of the second matrix so that there is a nonzero in the top-left entry. The second entry in the top row must be nonzero, otherwise the inner product of the top row of the first and second matrices will be neither 0 nor $\sqrt{2}$. Continue this argument so that the block structure is the same between all matrices in the set of unbiased weighing matrices. By applying Corollary 4.27 (for 2×2 submatrices) and Proposition 4.16, we have our result. □

4.1.4 Mutually Unbiased Weighing Matrices of Weight 3

Lemma 4.30. *A $UW(n,3)$, W_1, is unbiased with W_2 if and only if W_1 has the same block structure as W_2.*

Proof. From Theorem 3.32, we know that W_1 may be transformed into a matrix of the following form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$.

We may assume through row and column permutations and normalization by a unimodular number that the first 3 rows of W_2 have a 1 in the first column.

Assume that the top left block in W_1 is a $UW(3,3)$. In the first row of W_2, if the first three entries are $(1,0,0)$, then the inner product of this row and the first row of W_1 can obviously not be of the desired form. Moreover, if there are two nonzero entries (i.e., either $(1,a,0)$ or $(1,0,a)$), then there must be a third entry in columns 4 through n. The inner product of this row and three different rows in W_1 will simply be a unimodular number (this is true by the structure of W_1), and thus, not in the desired form. This means that the first three entries must all be nonzero. This argument can be made for the second and third row of W_2, and thus, the topleft corner of W_2 is a $UW(3,3)$, as desired.

Now assume that the top left block in W_1 is a $UW(4,3)$. If columns 2, 3 and 4 are all zero in any of the first 3 rows, then the inner product of row 1 in W_1 and that row will give us a unimodular number. If there is exactly 1 nonzero in columns 2, 3 and 4, then the inner product of that row and the fourth row of W_1 will be unimodular. Thus, we know that in the first 3 rows of W_2, all 3 nonzero entries must appear in the first four columns.

We will now show that the first zero in these rows will not be in the same column. Assume that one column has at least 2 zeroes. This means that at least one of columns 2,3 and 4 will be complete (i.e., no more nonzero entries may go into that column). Column 1 is already complete, so in our fourth row, there are either 1 or 2 nonzeroes in the first 3 columns. By taking the inner product of the fourth row of W_2 by the appropriate row in W_1, we will get a unimodular number. Thus, the first zero in the first 4 rows must be in different columns (note that the first zero in row 4 must be in column 1). Furthermore, through appropriate row permutations and negations, the second entry in row 4 must be a 1. The next two entries are clearly nonzero or there is 1-orthogonality within W_2. Thus, in the first 4 rows of W_2, the three nonzero entries must appear in the first 4 rows, with the first zeroes of the rows in different columns (i.e., a $UW(4,3)$).

Once we know that the top left block of W_1 and W_2 are the same, if we examine the bottom right $(n-3)\times(n-3)$ or $(n-4)\times(n-4)$ block, we have a $UW(n-3,3)$ or $UW(n-4,3)$, and we can recursively use the same argument to obtain the desired result. □

Theorem 4.31. *The upper bound on the number of MUWM of the form $UW(n,3)$ is*

$$\begin{cases} 0 & \text{if } n = 5 \\ 3 & \text{if } n \not\equiv 0 \pmod 4 \text{ and } n \neq 5 \\ 9 & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

where $n \geq 3$.

Proof. Using Lemma 4.30 with Proposition 4.16 and the fact that the upper bound for $UW(3,3)$ is 3 and $UW(4,3)$ is 9 via Corollary 4.27, we have that if the matrix contains a $UW(3,3)$ in its block structure, then it acts as a limiting factor, causing the upper bound to be 3. Otherwise, it is 9, which can only occur when n is a multiple of 4. □

Corollary 4.32. *The upper bound given in Theorem 4.31 is tight for all $n \geq 3$ and $n \neq 5$.*

Proof. A computer search has shown the bounds to be tight for $UW(4,3)$ (see Appendix D) and the bound for $UW(3,3)$ is attained through Theorem 4.10. We may construct the $UW(n,3)$ by adjoining the appropriate amount of $UW(4,3)$ and $UW(3,3)$ together along the main diagonals. If n is a multiple of 4, use only $UW(4,3)$s along the main diagonal. Otherwise, it does not matter which blocks are used. A simple induction will show that every integer larger than 5 may be written in the form of $3m+4l$. □

4.1.5 Mutually Unbiased Weighing Matrices of Weight 4

UW(5,4)

Lemma 4.33. *Let W be a unit weighing matrix that is unbiased with*

$$W_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix}$$

where $\omega = e^{i\frac{2\pi}{3}}$. Then every nonzero entry in W is a sixth root of unity.

Proof. The proof of this lemma is given in Appendix B.1. □

Theorem 4.34. *The largest number of mutually unbiased weighing matrices of the form $UW(5,4)$ is 5. Moreover, this bound is tight.*

Proof. By Lemma 3.35, all weighing matrices of order 5 and weight 4 are equivalent to W_5 given in Lemma 4.33. Thus, given a set of mutually unbiased weighing matrices, we may permute and multiply by a unit number the rows and columns of the matrices in such a way that one of them is W_5. By Lemma 4.33, we know that any matrix that is unbiased with W_5 must only contain 0 and the sixth roots of unity. Moreover, the case analysis in Lemma 4.33 shows that there are only 60 possible rows in the other matrices in the set that are not in W_5. An exhaustive computer search was done over these rows, which revealed that the maximal set using only the sixth root of unity contains 5 elements. One collection of these matrices are included in Appendix D. □

Although there are only five matrices, the theoretic upper bound given in (4.4) is attained by vectors that cannot be partitioned into weighing matrices. See Table D.6 in Appendix D.1.

UW(6,4)

This is the first case where the upper bound given in Corollary 4.27 seems too large (20 mutually unbiased weighing matrices). However, relatively quickly, our computer program gave us the following.

Theorem 4.35. *There are 20 mutually unbiased weighing matrices of order 6 and weight 4.*

Proof. A set of matrices attaining this bound can be found in Appendix D, Table D.3. □

Each of the matrices in the set of matrices given are over the sixth root of unity. What is even more special about this set of matrices is that it attains the upper bounds given in both (4.5) and (4.6).

The first four matrices given in Table D.3 are real, which falls just short of the upper bound given in Corollary 4.28. This turns out to be an optimal set of real weighing matrices.

Theorem 4.36. *There are no more than 4 mutually unbiased real weighing matrices of order 6 and weight 4.*

Proof. An exhaustive computer search over real weighing matrices was performed and found that there were no sets of mutually unbiased real weighing matrices of order 6 and weight 4. □

UW(7,4)

Lemma 4.37. *Let W be a unit weighing matrix that is unbiased with*

$$W_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}.$$

Then every nonzero entry in W is either 1 or -1.

Proof. The proof of this lemma is included in Appendix B.2. □

Theorem 4.38. *The maximum number of mutually unbiased weighing matrices of order 7 and weight 4 is 8.*

Proof. Similarly to the proof of Theorem 4.34, one matrix in the set may be transformed into the real weighing matrix W_7 given in Lemma 4.37. Every $UW(7,4)$ is equivalent to this matrix (see Lemma 3.35). By Lemma 4.37, every weighing matrix equivalent to W_7 must also be real, so we may use Corollary 4.28 to provide us with this bound. □

Corollary 4.39. *The bound given in Theorem 4.38 is tight.*

Proof. Using a computer search, we find eight real mutually unbiased weighing matrices $W(7,4)$ given in Appendix D. This achieves the real upper bound given by Corollary 4.28. By Theorem 4.38, this is also the maximal set of $UW(7,4)$, despite not achieving the upper bound of 24 given by Corollary 4.27. □

UW(8,4)

Theorem 4.40. *The maximum number of real mutually unbiased weighing matrices of order 8 and weight 4 is 14.*

Proof. A set of size 14 $W(8,4)$ has been generated in Appendix D. This meets the upper bound given by Corollary 4.28. □

Further investigations into $UW(8,4)$ using large roots of unity have proven fruitless. Odd roots of unity produce maximal sets smaller than that of the real case, and even roots of unity become computationally infeasible after the fourth root of unity, which returns the set of $W(8,4)$ as the maximal set of mutually unbiased unit weighing matrices.

4.2 Unbiased Hadamard Matrices

So far, we have only examined a very special case of unbiasedness. Our selection of the values of n and α in (4.3) and (4.4), as well as imposing a certain structure to our matrices, make it possible to append the identity to the set of weighing matrices. More precisely, considering each row of all weighing matrices in a set of mutually unbiased weighing matrices of order n and the rows of the identity matrix of order n as vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, they form a class of bi-angular vectors. We now make a different selection for the value of α in such a way that it is no longer possible to add the identity matrix and preserve the bi-angularity. Below, in Table 4.2, we give an example of a set of eight Hadamard matrices of order 8 that form a bi-angular set of vectors in $\mathbb{R}^8$, but no rows of the identity matrix can be added to the set and preserve bi-angularity. In the following set, $\alpha = \frac{1}{2}$, but if the identity is added, it would introduce the inner product of $\frac{1}{\sqrt{8}}$ (up to absolute value) and the bi-angularity of the lines would disappear.

Table 4.2: 8 mutually unbiased Hadamard matrices with $\alpha = \frac{1}{2}$

$$\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&-&1&-&-&1&-\\ 1&-&-&1&1&-&-&1\\ 1&-&-&-&-&1&1&1\\ 1&1&-&-&1&1&-&-\\ 1&1&1&-&-&-&-&1\\ 1&-&1&1&-&1&-&-\\ 1&-&1&-&1&-&1&- \end{pmatrix} \quad \begin{pmatrix} 1&1&1&-&1&-&1&1\\ 1&-&1&1&1&1&1&-\\ 1&-&-&1&-&-&1&1\\ 1&1&-&1&1&-&-&-\\ 1&1&-&-&-&1&1&-\\ 1&-&1&-&-&-&-&-\\ 1&-&-&-&1&1&-&1\\ 1&1&1&1&-&1&-&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&-&-&-&1&-&1\\ 1&-&-&-&1&1&1&-\\ 1&-&1&-&-&-&1&1\\ 1&1&1&1&-&1&1&-\\ 1&1&1&-&1&-&-&-\\ 1&-&1&1&1&1&-&1\\ 1&-&-&1&-&-&-&-\\ 1&1&-&1&1&-&1&1 \end{pmatrix} \quad \begin{pmatrix} 1&-&-&-&-&1&-&-\\ 1&1&1&-&-&-&1&-\\ 1&1&-&-&1&1&1&1\\ 1&-&1&1&-&1&1&1\\ 1&-&1&-&1&-&-&1\\ 1&1&1&1&1&1&-&-\\ 1&1&-&1&-&-&-&1\\ 1&-&-&1&1&-&1&- \end{pmatrix}$$

$$\begin{pmatrix} 1&-&1&-&-&1&-&1\\ 1&1&1&-&1&1&1&-\\ 1&1&-&-&-&-&1&1\\ 1&-&-&1&-&1&1&-\\ 1&-&-&-&1&-&-&-\\ 1&1&1&1&-&-&-&-\\ 1&1&-&1&1&1&-&1\\ 1&-&1&1&1&-&1&1 \end{pmatrix} \quad \begin{pmatrix} 1&-&-&1&-&1&-&1\\ 1&-&-&-&1&-&1&1\\ 1&1&-&1&1&1&1&-\\ 1&1&1&1&-&-&1&1\\ 1&-&1&1&1&-&-&-\\ 1&-&1&-&-&1&1&-\\ 1&1&1&-&1&1&-&1\\ 1&1&-&-&-&-&-&- \end{pmatrix}$$

$$\begin{pmatrix} 1&1&1&-&-&1&-&-\\ 1&-&1&-&1&1&1&1\\ 1&-&-&-&-&-&1&-\\ 1&1&-&1&-&1&1&1\\ 1&1&-&-&1&-&-&1\\ 1&-&-&1&1&1&-&-\\ 1&-&1&1&-&-&-&1\\ 1&1&1&1&1&-&1&- \end{pmatrix} \quad \begin{pmatrix} 1&1&-&-&1&-&1&-\\ 1&-&1&1&-&-&1&-\\ 1&-&-&1&1&1&1&1\\ 1&1&1&1&1&-&-&1\\ 1&1&1&-&-&1&1&1\\ 1&-&1&-&1&1&-&-\\ 1&-&-&-&-&-&-&1\\ 1&1&-&1&-&1&-&- \end{pmatrix}$$

The rows of these matrices are generated from the BCH-code [7, 20] of length 7 with weight distribution $\{(0,1),(2,21),(4,35),(6,7)\}$ (see [38] for more information about BCH-codes). Once the codewords are generated, we append a column of zeroes, then perform the following operation onto each entry of the codewords:

$$f(i) = \begin{cases} 1 & \text{if } i = 0, \\ -1 & \text{if } i = 1. \end{cases} \tag{4.9}$$

We were also able to generate 32 Hadamard matrices of order 32 which have inner products in $\{0, \pm 8\}$ through a similar process. The weight distribution of the order 32 matrices is $\{(0,1),(12,310),(16,527),(20,186)\}$. The partition of the vectors into Hadamard matrices is shown in Tables D.7–D.10 in Appendix D.2.

In an attempt to continue this, we have generated the 128^2 codewords from the BCH-code of order 127, but were not able to partition them into the 128 Hadamard matrices needed due to computer memory restrictions. The inner products between the vectors are all in $\{0, \pm 16\}$. We do believe that this set of vectors contains the needed ingredients to make the Hadamard matrices required. Moreover, we pose the following

Conjecture 4.41. Let $n = 2^{2k+1}$. Then there exists a set of n real Hadamard matrices, $\{H_1, H_2, \ldots, H_n\}$, so that the entries of $H_i H_j^t$ $(i \neq j)$ contain exactly two elements, 0 and 2^{k+1} (up to absolute value).[6]

It is important to note that the number of vectors found through Conjecture 4.41 is usually less than the bound given in Theorem 4.25. We believe that the upper bound is too high in this case because the vectors are *flat* (i.e., all contain entries that have the same absolute value). In fact, we think that the upper bounds given in Theorems 4.25 and 4.26 are rarely obtained if V is a set of flat vectors. We feel that there is a different upper bound available for flat vectors that is (generally) smaller than Theorems 4.25 and 4.26.

[6]Since the time that we have published Conjecture 4.41, Nozaki and Suda have released an article that uses coding theory to affirm that this conjecture is true [28, Page 15]. The content of their article, however, is well beyond the scope of this thesis, so we refer the reader to [28] for the full details.

Using the terminology from [4], these matrices form a set of *weakly unbiased Hadamard matrices*. However, it is important to note that the matrices formed here are a very special kind of unbiased Hadamard matrices since the entire set of vectors forms a set of bi-angular lines (whereas the vectors from [4] give possibly tri-angular lines). These matrices seem to form very nice combinatorial objects, which are discussed in further detail in the next section.

Chapter 5

Applications

Life is like a proof,

there is a little box at the end.

– K. B.

5.1 Strongly Regular Graphs

In [9], Calderbank *et al.* determined a way to construct strongly regular graphs from a full line set, namely, a set of vectors that meet the upper bounds in (4.3) or (4.4). They used unit vectors that met the conditions and bounds of Theorem 4.25. Before we can construct our objects, we will need a few definitions.

Definition 5.1. A simple graph $G(V,E)$ on v vertices is called *strongly regular* if for any vertex $w \in V$,

1. The degree of w is k,
2. For each $u \in V$ such that u is adjacent to w, u and w have exactly λ common neighbours.
3. For each $x \in V$ such that x is not adjacent to w, x and w have exactly μ common neighbours.

Strongly regular graphs are denoted $SRG(v,k,\lambda,\mu)$.

The following theorems can be found in most elementary graph theory textbooks.

Theorem 5.2. *Let G be a strongly regular graph of type $SRG(v,k,\lambda,\mu)$. Then*

(a) $(v-k-1)\mu = k(k-\lambda-1)$

(b) The adjacency matrix of G has exactly 3 distinct eigenvalues:

(i) k with multiplicity 1 and

(ii) $\frac{1}{2}\left(\lambda-\mu \pm \sqrt{(\lambda-\mu)^2+4(k-\mu)}\right)$ *with multiplicity* $\frac{1}{2}\left(v-1 \mp \frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$.

(c) The complement of a strongly regular graph is a strongly regular graph with parameters $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$.

Next, we define a special type of strongly regular graph.

Definition 5.3. A finite set of points, P, lines, L, and incidences, $I \subset P \times L$, is a *partial geometry*, denoted $pg(s,t,\alpha)$, if

- Each point is incident with $t+1$ lines.
- Each line is incident with $s+1$ points.
- For each pair of distinct points, there is at most one line incident with both of them.
- If $p \in P$ and $\ell \in L$ are not incident, then there are exactly α pairs $(q,m) \in I$ such that p is incident with m and q is incident with ℓ.

Theorem 5.4. *A $pg(s,t,\alpha)$ generates an*

$$SRG\left((s+1)(st+\alpha)/\alpha\ ,\ s(t+1)\ ,\ s-1+t(\alpha-1)\ ,\ \alpha(t+1)\right).$$

For a good summary of strongly regular graphs and partial geometries, we refer the reader to [11] and the references therein. Strongly regular graphs can be found in [11, Section VII(11)] and partial geometries can be found in [11, Section VI(41)].

A strongly regular graph that satisfies the conditions laid out in Definition 5.3 is called *geometric graph*. If a strongly regular graph's parameters match Theorem 5.4, but the graph does not satisfy the conditions laid out in Definition 5.3, then it is called a *pseudogeometric graph*.

Lemma 5.5. *Let $V \subset \mathbb{R}^n$ be a spanning set whose cardinality matches the upper bound given in (4.3). Moreover, let $G = Gram(V) = I + \alpha C$, where C is a $\{0, \pm 1\}$ matrix and $0 < \alpha < 1$. Then C has two distinct eigenvalues, $-\frac{1}{\alpha}$ and $\frac{|V|-n}{n\alpha}$ with multiplicities $|V| - n$ and n, respectively.*

Proof. Since V spans $\mathbb{R}^n$, we know that the nullity of $\mathrm{Gram}(V)$ is $|V| - n$, so 0 is an eigenvalue with that multiplicity. Therefore, C has $|V| - n$ eigenvalues equal to $-\frac{1}{\alpha}$. For the remaining n eigenvalues, we have that each eigenvalue, λ, of G satisfies $0 \leq \lambda \leq \frac{(n+2)(1-\alpha^2)}{3-\alpha^2(n+2)}$ (through the proof of Theorem 4.25). Since we have attained the upper bound, the last line in the proof of Theorem 4.25 tells us that each $\lambda = \frac{(n+2)(1-\alpha^2)}{3-\alpha^2(n+2)}$. Using the cardinality of our set, and simplifying the expression, we arrive at our result. □

Theorem 5.6 ([9, Proposition 3.12])**.** *If equality holds in (4.3) and V spans $\mathbb{R}^n$, then 'perpendicularity' defines a strongly regular graph.*

Proof. Let $V' = \{v \otimes v | v \in V\} \subset S^2(\mathbb{R}^n)$. Since $|\langle a \otimes a, b \otimes b\rangle| = |\langle a, b\rangle|^2$, $\mathrm{Gram}(V') = I + \alpha^2 D$, where D is a $\{0, 1\}$ matrix. Since V spans $\mathbb{R}^n$, we know that the nullity of $\mathrm{Gram}(V')$ is $|V| - \binom{n+1}{2}$, so 0 is an eigenvalue with that multiplicity. Therefore, D has at least $|V| - \binom{n+1}{2}$ eigenvalues equal to $-\frac{1}{\alpha^2}$. Next, note that the diagonal entries of C^2 are the row sums of D. By the Cayley-Hamilton theorem, we have that $\left(C + \frac{1}{\alpha} I\right)\left(C - \frac{|V|-n}{n\alpha} I\right) = 0$. By expanding this out, and noting that C has a zero diagonal, we have that each diagonal entry of C^2 is exactly $\frac{|V|-n}{n\alpha^2}$. Thus, since all row sums are identical, we have that $\frac{|V|-n}{n\alpha^2}$ is an eigenvalue of C. At this point, we are missing exactly $\binom{n+1}{2} - 1$ eigenvalues.

First, let us examine the trace of A and A^2.

$$0 = \mathrm{Tr}(A) = \frac{|V|-n}{n\alpha^2} + (|V| - \binom{n+1}{2})(-\frac{1}{\alpha^2}) + \sum \lambda.$$

$$\frac{|V|-n}{\alpha^2} = n\frac{|V|-n}{n\alpha^2} = \mathrm{Tr}(A^2) = \frac{|V|-n^2}{n\alpha^2} + (|V| - \binom{n+1}{2})(-\frac{1}{\alpha^4}) + \sum \lambda^2.$$

For simplicity, let $K = \frac{(n+2)(n-1)}{2}$ and $\Delta = 3-(n+2)\alpha^2$. From these, we have

$$\sum_{j=1}^{K} \lambda = -K\frac{1-n\alpha^2}{\alpha^2\Delta}$$

and

$$\sum_{j=1}^{K} \lambda^2 = K\left(\frac{1-n\alpha^2}{\alpha^2\Delta}\right)^2.$$

Thus, by the Cauchy-Schwarz inequality, we have

$$\left(K\left(\frac{1-n\alpha^2}{\alpha^2\Delta}\right)\right)^2 = \left(-K\left(\frac{1-n\alpha^2}{\alpha^2\Delta}\right)\right)^2 = \left(\sum_{j=1}^{K}\lambda\right)^2 = \left(\sum_{j=1}^{K}\lambda\cdot 1\right)^2$$

$$\leq \left(\sum_{j=1}^{K}\lambda^2\right)\cdot\left(\sum_{j=1}^{K}1^2\right) = \left(K\left(\frac{1-n\alpha^2}{\alpha^2\Delta}\right)^2\right)(K) = \left(K\left(\frac{1-n\alpha^2}{\alpha^2\Delta}\right)\right)^2,$$

which implies that each $\lambda = \frac{1-n\alpha^2}{\alpha^2\Delta}$. Since this matrix has exactly three eigenvalues (one of which being the row sums), A is the adjacency matrix of a strongly regular graph.

□

Theorem 5.7 ([9, Section 5])**.** *If equality holds in (4.4) and V spans $\mathbb{C}^n$, then 'perpendicularity' defines a strongly regular graph.*

Proof. Similar to Theorem 5.6. □

When we use the term 'perpendicularity', we refer to the graph where each node represents a row of a weighing matrix, and there is an adjacency between two vertices (say i, j) if v_i is orthogonal with v_j. Of note, in Theorem 5.6, the proof gives the eigenvalues of the *complement* of the graph defined by 'perpendicularity' (i.e., it gives the eigenvalues of the graph defined by 'non-perpendicularity'). For the following theorems, we will be interested in 'perpendicularity'.

Corollary 5.8. *Let $\mathcal{W}$ be a set of m mutually unbiased unit (resp. real) weighing matrices of order n and weight w. If m matches the upper bound given in (4.6) (resp. (4.8)), then the 'perpendicularity' of the rows of the matrices forms a strongly regular graph.*

Proof. Since $\mathcal{W}$ is a set of mutually unbiased weighing matrices, the inner product between any two rows falls in $\{0, \frac{1}{\sqrt{w}}\}$, up to absolute value. So we may apply Theorem 5.6 or Theorem 5.7. □

Theorem 5.9. *Let $\mathcal{W}$ be a set of m mutually unbiased unit (resp. real) weighing matrices of order n and weight w. If m matches the upper bound given in (4.6) (resp. (4.8)), then the strongly regular graph generated in Corollary 5.8 has parameters corresponding to the following partial geometry:*

$$pg\left(n-1, \frac{w(n-w)}{\Delta}, n-w\right)$$

where $\Delta = 2w-(n+1)$ (resp. $\Delta = 3w-(n+2)$).

Proof. Using Theorem 5.6, we can construct a graph (say G) which is strongly regular. Theorem 5.7 gives us the three eigenvalues of our graph. We will be interested in the complement of this graph. We may then use each point in Theorem 5.2 to arrive at the parameters of our strongly regular graph. Then, Theorem 5.4 can be used to give us our result. □

Thus, anytime we have a set of mutually unbiased weighing matrices which meet the bounds given in (4.3) or (4.4), we are able to generate either a pseudogeometric graph or a geometric graph.

Corollary 5.10. *The following SRGs exist:*

(a) $SRG(40, 12, 2, 4)$ *which is geometric.*

(b) $SRG(45,12,3,3)$ *which is pseudogeometric.*

(c) $SRG(63,30,13,15)$ *which is geometric.*

(d) $SRG(120,63,30,36)$ *which is geometric.*

(e) $SRG(126,45,12,18)$ *which is pseudogeometric.*

Proof. As seen in Table 4.1, we have sets of $UW(4,3)$, $W(7,4)$, $W(8,4)$ and $UW(6,4)$ that attain the needed upper bound. By applying Corollary 5.8, we get the desired graphs (a),(c),(d) and (e). Each graph which is geometric was checked via computer computation.

Interestingly, even though Theorem 4.34 limits the number of mutually unbiased weighing matrices, it does not put a restriction on the number of vectors whose pairwise inner products' absolute value are in $\{0,2\}$. In fact, we have found 40 vectors over the sixth root of unity (all having weight 4) such that the inner products' absolute value remain in $\{0,2\}$. These vectors are given in Appendix D.1, and they form the strongly regular graph given in (b).

□

It is important to note that strongly regular graphs with all of these parameters have been previously found, but this is a new method for finding them. The first case where a full set of mutually unbiased weighing matrices will give a strongly regular graph with parameters that are currently unknown is $UW(8,5)$, which will generate an $SRG(288,175,110,100)$.

5.2 Association Schemes

We will now examine sets of vectors that contain more than two angles between them (*multi-angular vectors*). The following definition gives a generalization of strongly regular graphs. General information about association schemes can be found in [11, Section VI(1)].

Definition 5.11. An $m-$association scheme is a set $\mathcal{A} = \{A_0,\ldots,A_m\}$ of $(0,1)$–matrices of order n that satisfy the following conditions:

(a) $A_0 = I$

(b) $\sum_{i=0}^{m} A_i = J$

(c) $A_i = A_i^T$

(d) $A_i A_j = A_j A_i \in Span_{\mathbb{Z}}(\mathcal{A})$

Note that a 2-association scheme is equivalent to a strongly regular graph. We can utilize Hadamard matrices and mutually orthogonal Latin squares to construct association schemes with very large parameters.

Definition 5.12. A *Latin square* is an $n \times n$ matrix defined on the alphabet $\{a_1, \ldots, a_n\}$ if every row and every column contains exactly one a_i for each $1 \le i \le n$.

Example 5.13.

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{pmatrix}$$

is a Latin square of order 4.

Normally, the alphabet $\{1, 2, \ldots, n\}$ is used in Latin squares. However, for our constructions below, we will be using matrices as our alphabet to construct block matrices.

Definition 5.14. Let L_1 and L_2 be two Latin squares of order n defined over the same alphabet. Let r_1 and r_2 be arbitrary rows from L_1 and L_2, respectively. L_1 and L_2 are *suitable Latin squares* if exactly one entry is in common between r_1 and r_2 (for every choice of r_1 and r_2). A set of Latin squares that are pairwise suitable are called *mutually suitable Latin squares* (or *MSLS*).

Mutually suitable Lain squares are very similar to the more common mutually orthogonal Latin squares (more commonly known as “MOLS”).

Example 5.15.

$$\left\{ \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \right\}$$

is a set of mutually suitable Latin squares over the alphabet $\{0,1,2\}$.

Construction 5.16. As input, we need a Hadamard matrix, H, of order m, $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}$, $\mathcal{M}$, a set of mutually suitable Latin squares and $\mathcal{L}$, a Latin square of order K. The association scheme constructed is of order mK^2.

(a) Let $C_i = r_i^t r_i$ where r_i is the i^{th} row of H, $0 \leq i < m$.

(b) Redefine $C_0 = kJ$.

(c) Use $\{C_i\}$ as the alphabet of $\mathcal{M}$ of order n (giving you $n-1$ matrices called M_i, $1 \leq i < n$).

(d) Define M_0 to be ℓJ.

(e) Use $\{M_i\}$ as the alphabet of $\mathcal{L}$ and call this matrix G.

(f) We will examine G^2, which is $mn^2 \times mn^2$. The different classes for our association scheme are the distinct values in this matrix.

Note that part (f) in Construction 5.16 is looking at the Gramian matrix of the vectors that are represented as the rows of G. The association schemes that are constructed have an immense amount of structure associated with them. The various objects used in this construction are included in Appendix E.

Table 5.1: Association schemes created via Construction 5.16

H	$\mathcal{M}$	$\mathcal{L}$	k	ℓ	Order of Association Scheme
H_4	$\mathcal{M}_3$	$\mathcal{L}_3$	2	2	5
H_4	$\mathcal{M}_3$	$\mathcal{L}_3$	3	2	6
H_4	$\mathcal{M}_4$	$\mathcal{L}_4$	1	1	2
H_4	$\mathcal{M}_4$	$\mathcal{L}_4$	1	0	3
H_4	$\mathcal{M}_4$	$\mathcal{L}_4$	0	0	4
H_4	$\mathcal{M}_4$	$\mathcal{L}_4$	2	0	5
H_8	$\mathcal{M}_7$	$\mathcal{L}_7$	2	2	5
H_8	$\mathcal{M}_7$	$\mathcal{L}_7$	2	0	6
H_{12}	$\mathcal{M}_{11}$	$\mathcal{L}_{11}$	2	2	5
H_{12}	$\mathcal{M}_{11}$	$\mathcal{L}_{11}$	2	0	6
H_{20}	$\mathcal{M}_{19}$	$\mathcal{L}_{19}$	2	2	5
H_{20}	$\mathcal{M}_{19}$	$\mathcal{L}_{19}$	2	0	6

When deciding on values for k and ℓ, we do not believe that the value of 2 and 3 impact the fact that the matrices generate association schemes. Instead, we feel that one could use any combination of *sufficiently large* distinct values.

This area of research was inspired by the many applications that can be found in physics, and has grown into a very interesting mathematical area. Knowledge from many areas of mathematics are required to fully explore this field. We have introduced these objects in hopes that we, and others, will utilize them to explore new and interesting areas of mathematics, draw connections to existing areas of mathematics and grasp a deeper understanding of the structure behind such combinatorial objects.

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Appendix A

Detailed Proofs from Chapter 3

A.1 Standardized $UW(n,3)$

(This is a proof of Lemma 3.31.)

Lemma A.1. *Every $UW(n,3)$ is equivalent to a weighing matrix whose top leftmost submatrix is either a $UW(3,3)$ or a $UW(4,3)$.*

Proof. By Theorem 3.23, we alter W so that it is in standard form. This means that the second row has three possibilities, listed below as Case 1, 2 and 3, after further appropriate column permutations (Note that these permutations should leave the shape of the first row intact). When we say that a row is not orthogonal with another row with no further context, it is because it would imply that the set of elements in the two rows would have 1-orthogonality.

1. $\begin{pmatrix} 1 & a & b & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$
2. $\begin{pmatrix} 1 & a & 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 1, so not possible.

For case 1, 3-orthogonality implies $b = \overline{a}$, where $a \in \{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$, and four further subcases arise for the third row:

(a) $\begin{pmatrix} 1 & c & d & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & c & 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 0 & c & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 1, so not possible.

For case (b), we have $c = -1$ by orthogonality with the first row and $c = -a$ by orthogonality with the second row. Similarly, in case (c), we have $c = -1$ and $c = -\overline{a}$. Both of these are not possible. However, case (a) produces a viable option when $c = \overline{d} = \overline{a}$, finishing case 1 and implying that the top 3×3 submatrix is a $UW(3,3)$ of the following form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \overline{a} \\ 1 & \overline{a} & a \end{pmatrix}$$

Note that if $a = e^{-\frac{2\pi i}{3}}$, then swap rows 2 and 3, so we assume $a = e^{2\pi i/3}$.

For case 2, $a = -1$ and we have six subcases for the third row:

(a) $\begin{pmatrix} 1 & b & c & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, with $-1 \prec b$.
(b) $\begin{pmatrix} 1 & b & 0 & c & 0 & 0 & \cdots & 0 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & b & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & 0 & b & c & 0 & 0 & \cdots & 0 \end{pmatrix}$
(e) $\begin{pmatrix} 1 & 0 & b & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 2, so not possible.
(f) $\begin{pmatrix} 1 & 0 & 0 & b & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 1, so not possible.
(g) $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 1, so not possible.

In subcase (a), $b = 1$ by orthogonality with row 2 and $b \in \{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$ by orthogonality with row 1. In case (b), $b = -1$ by orthogonality with row 1 and $-b \in \{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$ by orthogonality with row 2. In case (c), $b = -1$ by orthogonality with row 1, which implies row 2 is not orthogonal with row 3. In case (d), we have a valid configuration by setting $b = c = -1$. We now construct the next row, which gives four more subcases:

(i) $\begin{pmatrix} 0 & 1 & d & f & 0 & 0 & \cdots & 0 \end{pmatrix}$
(ii) $\begin{pmatrix} 0 & 1 & d & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 3, so not possible.
(iii) $\begin{pmatrix} 0 & 1 & 0 & d & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 3, so not possible.
(iv) $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 1-orthogonality with row 3, so not possible.

In case (i), we have a valid row if $d = -f = -1$, finishing all of the cases above, and giving a $UW(4,3)$ in the upper left 4×4 submatrix of the form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{pmatrix}$$

□

A.2 Standardized $UW(n,4)$

(This is a proof of Lemma 3.35.)

Lemma A.2. *All $UW(n,4)$ are equivalent to a $UW(n,4)$ with diagonal blocks consisting of the following matrices: W_5, W_6, W_7, W_8 and $E_{2m}(x)$ where $2 \leq m \leq \frac{n}{2}$ and x is any unimodular number.*

$$W_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix}, \quad W_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 & 0 \\ 1 & \overline{\omega} & \omega & 0 & 0 & 1 \\ 1 & 0 & 0 & - & - & - \\ 0 & 1 & 0 & - & -\overline{\omega} & -\omega \\ 0 & 0 & 1 & - & -\omega & -\overline{\omega} \end{pmatrix} \mathit{for}\ \omega = e^{\frac{2\pi i}{3}},$$

$$W_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}, \quad W_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & - & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & - \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 1 & - \end{pmatrix},$$

$$E_{2m}(x) = \begin{pmatrix} 1 & 1 & 1 & 1 &&&&&&&&&&& \\ 1 & 1 & - & - &&&&&&&&&&& \\ 1 & - & 0 & 0 & 1 & 1 &&&&&&&&& \\ 1 & - & 0 & 0 & - & - &&&&&&&&& \\ && 1 & - & 0 & 0 & 1 & 1 &&&&&&& \\ && 1 & - & 0 & 0 & - & - &&&&&&& \\ &&&& 1 & - & 0 & 0 &&&&&&& \\ &&&& 1 & - & 0 & 0 &&&&&&& \\ &&&&&&&& \ddots &&&&&& \\ &&&&&&&&& 0 & 0 & 1 & 1 && \\ &&&&&&&&& 0 & 0 & - & - && \\ &&&&&&&&& 1 & - & 0 & 0 & 1 & 1 \\ &&&&&&&&& 1 & - & 0 & 0 & - & - \\ &&&&&&&&&&& 1 & - & x & -x \\ &&&&&&&&&&& 1 & - & -x & x \end{pmatrix},$$

where x is any unimodular number.

Proof. To classify all unit weighing matrices of weight four, we apply a brute force depth first search on the rows of our weighing matrices. At each step, we will provide an $m \times n$ matrix which will give m mutually orthogonal rows consisting solely of four unimodular numbers per row (and zeroes otherwise). For convenience, we will only show the columns of the vectors that contain at least one nonzero entry. For example,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 \end{pmatrix}$$

represents two rows of a unit weighing matrix of order n for some $n \geq 6$.

We begin our case analysis by starting with four sequential ones:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

For the second row, we place a 1 in the first column and then only be concerned with the how many nonzero entries there are in the next three columns. Obviously, if there are no nonzero entries, then the first two rows cannot be orthogonal. Thus, we have three cases.

Case 1: Four nonzero entries in the first four columns.
By four orthogonality, we know that one of those entries is -1. We permute the columns to place that negative in the second column and make the other columns negations of one another.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & x & -x \end{pmatrix}$$

For the third row, we will list all candidates that are orthogonal with the first row. These rows can easily be listed by m-orthogonality (see Table A.1). Note that we swap the third and fourth columns and relabel x by $-x$ and arrive at a similar weighing matrix, so those duplicates will be left out of Table A.1. In each case, a is a primitive third root of unity and $b \in \mathbb{T}$. We use μ_3 to denote the set of third roots of unity.

Table A.1: Case analysis part 1 for Lemma 3.35

Row	Inner product with row two implies	Subcase
$\begin{pmatrix} 1 & - & b & -b & 0 & 0 & 0 \end{pmatrix}$	$b = -x = -1$	1A
$\begin{pmatrix} 1 & b & - & -b & 0 & 0 & 0 \end{pmatrix}$	$x = 1$	1B
$\begin{pmatrix} 1 & a & \overline{a} & 0 & 1 & 0 & 0 \end{pmatrix}$	Contradiction since $-a \notin \mu_3$	
$\begin{pmatrix} 1 & 0 & a & \overline{a} & 1 & 0 & 0 \end{pmatrix}$	Contradiction since $a \cdot (-\overline{a}) = -1 \notin \mu_3$	
$\begin{pmatrix} 1 & - & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	Contradiction since $2 \neq 0$	
$\begin{pmatrix} 1 & 0 & - & 0 & 1 & 1 & 0 \end{pmatrix}$	$x = 1$	1C

So we are left with three subcases.

Case 1A:
In the first subcase, we have a 3×4 matrix to which we append one more row. Since columns of a weighing matrix must also be orthogonal, we can fully fill in the final row of the matrix uniquely.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{pmatrix}$$

Case 1B:
We use a similar process as Case 1A in this subcase.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & b & - & -b \\ 1 & -b & - & b \end{pmatrix}$$

By rearranging the second matrix above, we arrive at

$$E_4(x) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & x & x \\ 1 & - & x & -x \end{pmatrix}.$$

Case 1C:
We have the following submatrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 \\ 1 & 0 & - & 0 & 1 & 1 \end{pmatrix}$$

which can be extended in the same was as above:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 \\ 1 & 0 & - & 0 & 1 & 1 \\ 1 & 0 & - & 0 & - & - \end{pmatrix}$$

We will swap the second and third column since the third column is now filled. When we insert the next two rows, we will have a one in the third column. This will force the entries in the fourth column.

$$E_6(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 \\ 1 & - & 0 & 0 & - & - \\ 0 & 0 & 1 & - & x & -x \\ 0 & 0 & 1 & - & -x & x \end{pmatrix}$$

For the block in the bottom right corner, we have two options: we can either have $x = 0$ or a unimodular number. If we take the latter choice, then we have completed our weighing matrix. If we take the second choice, then we are in a similar situation as before.

$$E_8(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 & - & - & 0 & 0 \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - & 0 & 0 & - & - \\ 0 & 0 & 0 & 0 & 1 & - & x & -x \\ 0 & 0 & 0 & 0 & 1 & - & -x & x \end{pmatrix}$$

This process can be continued inductively for any value of $2m$, $m \geq 2$. The matrix that is generated will be called $E_{2m}(x)$.

Case 2: Three nonzero entries in the first four columns.
By three orthogonality, we know that the two nonzero entries are distinct primitive third roots of unity. We consider all rows that can be appended to the following submatrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \end{pmatrix}$$

Note that we ignore any case where there are two rows with the exact same zero placement, since it would have been taken care of in the first case. All cases listed in Table A.2 are from orthogonality with the first row.

Table A.2: Case analysis part 2 for Lemma 3.35

Row	Inner product with row two implies	Subcase
$\begin{pmatrix} 1 & a & \overline{a} & 0 & 0 & 1 & 0 \end{pmatrix}$	$a = \overline{\omega}$	2A
$\begin{pmatrix} 1 & a & 0 & \overline{a} & b & 0 & 0 \end{pmatrix}$	$a = \overline{\omega}$ and $b = \omega$	2B
$\begin{pmatrix} 1 & a & 0 & \overline{a} & 0 & 1 & 0 \end{pmatrix}$	Contradiction since $a\overline{\omega} \neq -1$	
$\begin{pmatrix} 1 & - & 0 & 0 & b & 1 & 0 \end{pmatrix}$	Contradiction since $-\overline{\omega} \notin \mu_3$	
$\begin{pmatrix} 1 & - & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$	Contradiction since $1 - \omega \neq 0$	
$\begin{pmatrix} 1 & 0 & a & \overline{a} & b & 0 & 0 \end{pmatrix}$	$a = \omega$ and $b = \overline{\omega}$	2C
$\begin{pmatrix} 1 & 0 & a & \overline{a} & 0 & 1 & 0 \end{pmatrix}$	Contradiction since $a\overline{\omega} \neq -1$	
$\begin{pmatrix} 1 & 0 & - & 0 & b & 1 & 0 \end{pmatrix}$	Contradiction since $-\omega \notin \mu_3$	
$\begin{pmatrix} 1 & 0 & - & 0 & 0 & 1 & 1 \end{pmatrix}$	Contradiction since $-\omega \neq -1$	
$\begin{pmatrix} 1 & 0 & 0 & - & b & 1 & 0 \end{pmatrix}$	$b = -1$	2D
$\begin{pmatrix} 1 & 0 & 0 & - & 0 & 1 & 1 \end{pmatrix}$	Contradiction since $1 \neq 0$	

We will now work through the four subcases. In each of the subcases to follow, the submatrix on the left is the matrix which we obtained from our analysis above. We then append subsequent rows to each of these by placing a one in the left most column that is not full (i.e., does not already have four nonzeroes). In all of the subcases, placing a one into the appropriate column will make that column full. Since the column is full, we will be able to fill in each entry in the rest of the row by orthogonality with that full column.

Case 2A:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 & 0 \\ 1 & \overline{\omega} & \omega & 0 & 0 & 1 \end{pmatrix} \longrightarrow W_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 & 0 \\ 1 & \overline{\omega} & \omega & 0 & 0 & 1 \\ 1 & 0 & 0 & - & - & - \\ 0 & 1 & 0 & - & -\overline{\omega} & -\omega \\ 0 & 0 & 1 & - & -\omega & -\overline{\omega} \end{pmatrix}$$

Case 2B:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \end{pmatrix} \longrightarrow W_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix}$$

Case 2C:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & 0 & \omega & \overline{\omega} & \omega \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix}$$

By swapping the third and fourth row, we get W_5.
Case 2D:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 & 0 \\ 1 & 0 & 0 & - & - & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 & 0 \\ 1 & 0 & 0 & - & - & 1 \\ 1 & \overline{\omega} & \omega & 0 & 0 & - \\ 0 & 1 & 0 & - & -\overline{\omega} & \omega \\ 0 & 0 & 1 & - & -\omega & \overline{\omega} \end{pmatrix}$$

By swapping rows 3 and 4, followed by negating the sixth column, we arrive back at W_6.

This takes care of all subcases of Case 2.

Case 3: Two nonzero entry in the first four columns of the second row.
By two orthogonality, we know that this entry must be -1. We will look at all rows that may be appended to the following submatrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 \end{pmatrix}$$

Similar to before, we will only look at rows that are orthogonal with the first row, as well as intersect the first two rows in exactly two places (note that 4-intersection was taken care of in Case 1, while 3-intersection was taken care of in Case 2 and 1-intersection implies 1-orthogonality). Moreover, we may swap either columns 3 and 4 or columns 5 and 6 freely.

These rows can be found in Table A.3.

Table A.3: Case analysis part 3 for Lemma 3.35

Row	Inner product with row two implies	Subcase
$\begin{pmatrix} 1 & - & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$	Contradiction since $2 \neq 0$	
$\begin{pmatrix} 1 & 0 & - & 0 & b & 0 & 1 & 0 \end{pmatrix}$	$b = -1$	3A

Case 3A:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \end{pmatrix}$$

From here, we will append one more row and fill out the next row via orthogonality with the first column.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \end{pmatrix}$$

When filling in the next row, there are only a few choices. We are still looking for rows that intersect with each of the first few rows in at most 2 locations (possibly zero) and the first nonzero should be in the second column. A quick search can tell you that there are only four rows that satisfy these conditions (in terms of zero placement). These can be found in Table A.4.

Table A.4: Case analysis part 4 for Lemma 3.35

Row	Row in the partial matrix that implies $a \in \{\pm 1\}$	$b \in \{\pm 1\}$	$c \in \{\pm 1\}$	Subcase
$\begin{pmatrix} 0 & 1 & a & 0 & 0 & b & c & 0 \end{pmatrix}$	1	2	3	3AA
$\begin{pmatrix} 0 & 1 & a & 0 & b & 0 & 0 & c \end{pmatrix}$	1	2	N/A	3AB
$\begin{pmatrix} 0 & 1 & 0 & a & 0 & b & 0 & c \end{pmatrix}$	1	2	N/A	3AC
$\begin{pmatrix} 0 & 1 & 0 & a & b & 0 & c & 0 \end{pmatrix}$	1	2	3	3AD

Note that when we append either of the second or third rows into our matrix, c will be the first nonzero entry in the eighth column, so this implies that $c = 1$ in both cases. We will append this row, and in a manner similar to that of Case 2, we will be able to force the rest of the entries in each matrix by orthogonality with the first few full columns.

Case 3AA:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \end{pmatrix} \longrightarrow W_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}$$

Case 3AB:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & - & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow W_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & - & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & - \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 1 & - \end{pmatrix}$$

Case 3AC:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & 1 \\ 0 & 1 & - & 0 & 1 & 0 & 0 & - \\ 0 & 0 & 1 & - & 0 & 1 & 1 & - \\ 0 & 0 & 0 & 0 & 1 & - & 1 & 1 \end{pmatrix}$$

If we swap rows 5 and 6, and negate the eighth row, we get W_8.
Case 3AD:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}$$

By swapping rows 5 and 6, we arrive at W_7.

$\square$

A.3 Standardized $UW(6,5)$

This section is, by far, the most tedious portion of the thesis. The case analysis that follows will lead to a full classification of $UW(6,5)$. Enough detail will be provided so that, with a pen and paper, the reader can follow along verifying each step.

(This is the proof of Lemma 3.40.)

Lemma A.3. *There are at most 7 inequivalent $UW(6,5)$.*

Proof. By Lemma 3.39, every $UW(6,5)$ is equivalent to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & a & 0 & b & c \\ 1 & -y & 0 & d & f & g \\ 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{pmatrix}$$

We will systematically place constraints on the variables given based on the fact that the rows and columns on the matrix must be orthogonal. Lemma 3.39 has given the structure for the first two rows and columns. The cases will be processed in a depth-first manner by first placing constraints on the third row, simplifying the expressions, then repeating the same process on the fourth row. When we append the fifth row, we will use the orthogonality of the first column with the i^{th} column to give a simplified possibility for each entry (h,j,k and l). Similarly, when adding the final row, we will use orthogonality of the second column and the i^{th} column to determine m,n,p and q.

When appending the third and fourth row, there will be three cases. We know that the first row must be orthogonal to these rows, so we know that by 4-orthogonality that one of the entries in the first five columns must be a -1 and the other two nonzero entries must be the negation of one another.

To make the proof easier to follow, the variables a,b,c,d,f,g,h,j,k,l,m,n,p and q will only be used as placeholders. Once one of these variables has a relationship to another variable, a different variable will be introduced into the matrix. Only x,y and z will be needed to complete the analysis. You may assume that a variable name given in one case is the same as in all children cases (but not sibling cases). A horizontal line will be drawn to signify the current depth of the analysis. r_i will denote the i^{th} row of the current matrix.

Case 1: $y = -1$. This immediately implies that $a = -b$ (we will relabel a to be z).

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & - & z & 0 & -z & c \\ \hline 1 & 1 & 0 & d & f & g \\ 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{pmatrix} \tag{A.1}$$

Since $\langle r_2, r_3 \rangle = 0$, we have that $z = -x$ and $c = -1$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & - & -x & 0 & x & - \\ \hline 1 & 1 & 0 & d & f & g \\ 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.2}$$

At this point, since $\langle r_1, r_4 \rangle = 0$, we have that $d = f = -1$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & - & -x & 0 & x & - \\ 1 & 1 & 0 & - & - & g \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.3}$$

Next, $\langle r_3, r_4 \rangle = 0$ implies that $g = -\bar{x}$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & - & -x & 0 & x & - \\ 1 & 1 & 0 & - & - & -\bar{x} \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.4}$$

We will fill in the fifth row and sixth row uniquely from orthogonality with columns 1 and 2 and temporarily name this matrix $T_1(x)$.

$$T_1(x) := \left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & - & -x & 0 & x & - \\ 1 & 1 & 0 & - & - & -\bar{x} \\ 1 & 0 & - & x & -x & \bar{x} \\ 0 & 1 & - & -x & x & \bar{x} \end{array}\right) \tag{A.5}$$

Case 2: $a = -1$. This immediately implies that $b = -y$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & c \\ \hline 1 & -y & 0 & d & f & g \\ 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.6}$$

We must now branch into three distinct subcases. The three cases represent all of the possibilities for where the negative appears in the fourth row.

Case 2a: $y = 1$. This implies that $d = -f$ (and we will relabel d to z).

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & 1 & - & 0 & - & c \\ 1 & - & 0 & z & -z & g \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.7}$$

At this point, we swap rows 3 and 4 followed by columns 3 and 4 and after appropriate relabelling, arrive at the matrix given in (A.4) so we arrive at $T_1(x)$ from this branch.

Case 2b: $d = -1$. This implies that $f = y$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & c \\ 1 & -y & 0 & - & y & g \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.8}$$

Since $\langle r_2, r_3 \rangle = 0$ and $\langle r_2, r_4 \rangle = 0$, then $\langle r_2, r_3 \rangle$ - $\langle r_2, r_4 \rangle$ = 0. We deduce that $y = -\bar{x}$. From here, $\langle r_2, r_3 \rangle = 0$ implies that $c = -1$ and then $\langle r_3, r_4 \rangle = 0$ gives $c = g$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & -\bar{x} & - & 0 & \bar{x} & - \\ 1 & \bar{x} & 0 & - & -\bar{x} & - \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.9}$$

We fill in the final two rows to arrive at the following matrix which we label as $T_3(x)$.

$$T_3(x) := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & -\bar{x} & - & 0 & \bar{x} & - \\ 1 & \bar{x} & 0 & - & -\bar{x} & - \\ 1 & 0 & -x & x & - & 1 \\ 0 & 1 & - & - & 1 & 1 \end{pmatrix} \tag{A.10}$$

Case 2c: $f = -1$, which implies that $d = y$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & c \\ 1 & -y & 0 & y & - & g \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.11}$$

We will simplify c and g by noting that $\langle r_3, r_4 \rangle = 0$ gives $g = -\bar{y}c$. We will relabel c to be z.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & z \\ 1 & -y & 0 & y & - & -\bar{y}z \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.12}$$

We now append the fifth row to our search, which only resolves the value of h and k,

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & z \\ 1 & -y & 0 & y & - & -\bar{y}z \\ 1 & 0 & -x & j & y & l \\ \hline 0 & 1 & m & n & p & q \end{array}\right) \tag{A.13}$$

Some simplification is possibly from $\langle r_3, r_5\rangle = 0$ giving $l = -xz$ and then $\langle r_2, r_5\rangle = 0$ gives $j = x^2z$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & z \\ 1 & -y & 0 & y & - & -\bar{y}z \\ 1 & 0 & -x & x^2z & y & -xz \\ \hline 0 & 1 & m & n & p & q \end{array}\right) \tag{A.14}$$

Append the final row to give

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & z \\ 1 & -y & 0 & y & - & -\bar{y}z \\ 1 & 0 & -x & x^2z & y & -xz \\ 0 & 1 & m & -x & -\bar{y} & q \\ \hline \end{array}\right) \tag{A.15}$$

We reduce this to two variables by using $\langle c_3, c_4\rangle = 0$, $\langle c_3, c_6\rangle = 0$ and $\langle c_4, c_6\rangle = 0$ (in that order) to give $m = \bar{z}$, $q = -\overline{xz}$ and $z = \pm\bar{x}y$, respectively.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & - & 0 & -y & \pm\bar{x}y \\ 1 & -y & 0 & y & - & \mp\bar{x} \\ 1 & 0 & -x & \mp xy & y & \mp y \\ 0 & 1 & \pm x\bar{y} & -x & -\bar{y} & \mp\bar{y} \\ \hline \end{array}\right) \tag{A.16}$$

Using the fact that $\langle r_4, r_5\rangle = 0 = \langle r_3, r_6\rangle$, we have that $-\bar{y} + \overline{xy} = y - \bar{x}y \implies y + \bar{y} = \bar{x}(y + \bar{y})$. Thus, we have two possibilities: $\bar{x} = 1$ or $y + \bar{y} = 0$. We further branch into subcases (2ca will deal with the $\bar{x} = 1$ and 2cb will deal with $y + \bar{y} = 0$).

Case 2ca: $\bar{x} = 1 \implies x = 1$. Thus, we have the following.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & 1 & - & 0 & 1 \\ 1 & y & - & 0 & -y & \pm y \\ 1 & -y & 0 & y & - & \mp 1 \\ 1 & 0 & - & \mp y & y & \mp y \\ 0 & 1 & \pm\bar{y} & - & -\bar{y} & \mp\bar{y} \end{pmatrix} \tag{A.17}$$

With x out of the picture, we can now see that the lower signs on the $\pm$ and $\mp$ is invalid (see, for example, $\langle r_1, r_5 \rangle$). So we arrive at a $UW(6,5)$ which we will label as $T_4(y)$.

$$T_4(y) := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & 1 & - & 0 & 1 \\ 1 & y & - & 0 & -y & y \\ 1 & -y & 0 & y & - & - \\ 1 & 0 & - & -y & y & -y \\ 0 & 1 & \bar{y} & - & -\bar{y} & -\bar{y} \end{pmatrix} \tag{A.18}$$

Case 2cb: $y + \bar{y} = 0 \implies y = \pm i$. As to not confuse the different $\pm$s, we will split this into two cases again, the first where $y = i$ (case 2cba) and the second where $y = -i$ (case 2cbb).

Case 2cba:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & i & - & 0 & -i & \pm i\bar{x} \\ 1 & -i & 0 & i & - & \mp\bar{x} \\ 1 & 0 & -x & \mp ix & i & \mp i \\ 0 & 1 & \mp ix & -x & i & \pm i \end{pmatrix} \tag{A.19}$$

Since $\langle r_2, r_3 \rangle = 0$, then $\langle r_2, r_3 \rangle - \overline{\langle r_2, r_3 \rangle} = 0$. Thus, we have the following

$$\begin{aligned} \langle r_2, r_3 \rangle - \overline{\langle r_2, r_3 \rangle} = 0 \quad &\implies 2i - (x - \bar{x}) \mp i(x + \bar{x}) = 0 \\ &\implies 2i - 2i\Im(x) \mp 2i\Re(x) = 0 \\ &\implies \pm\Re(x) + \Im(x) = 1 \\ &\implies \Re(x)^2 + \Im(x)^2 \pm 2\Re(x)\Im(x) = 1 \\ &\implies \Re(x)\Im(x) = 0 \\ &\implies x \in \{\pm 1, \pm i\} \end{aligned} \tag{A.20}$$

The fifth implication comes from the fact that x is unimodular. When $x = -1$ or $x = -i$, then $\langle r_2, r_3 \rangle \neq 0$. When $x = 1$, then the lower signs of the $\pm$ does not work ($\langle r_2, r_3 \rangle \neq 0$), and using the upper sign gives $T_4(i)$. When we plug in $x = i$, the upper sign does not work ($\langle r_2, r_3 \rangle \neq 0$), and using the lower sign gives the following matrix, which we will denote T_6.

$$T_6 := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & i & -i & 0 & 1 \\ 1 & i & - & 0 & -i & - \\ 1 & -i & 0 & i & - & -i \\ 1 & 0 & -i & - & i & i \\ 0 & 1 & - & -i & i & -i \end{pmatrix} \tag{A.21}$$

Case 2cbb:

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & -i & - & 0 & i & \mp i\bar{x} \\ 1 & i & 0 & -i & - & \mp\bar{x} \\ 1 & 0 & -x & \pm ix & -i & \pm i \\ 0 & 1 & \pm ix & -x & -i & \mp i \\ \hline \end{array}\right) \tag{A.22}$$

The case analysis for this section is nearly identical to Case 2cba. We use the same pairs of rows' inner products to allow us to find the same contradictions as above. Moreover, $x \in \{\pm 1, \pm i\}$ and when $x = 1$, we get $T_4(-i)$ and when $x = -i$, we get a new matrix which we denote T_7 (note that $T_6^T = T_7$).

$$T_7 := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & -i & i & 0 & 1 \\ 1 & -i & - & 0 & i & - \\ 1 & i & 0 & -i & - & i \\ 1 & 0 & i & - & -i & -i \\ 0 & 1 & - & i & -i & i \end{pmatrix} \tag{A.23}$$

Case 3: $b = -1$. This implies that $a = -y$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & c \\ \hline 1 & -y & 0 & d & f & g \\ 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.24}$$

In the next row, we have three possibilities for the location of the negative.

Case 3a: $y = 1$, which implies that $d = -f$. We then relabel d to be z.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & 1 & - & 0 & - & c \\ 1 & - & 0 & z & -z & g \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.25}$$

The orthogonality of rows 2 and 4 give $g = -1$ and $x = z$. Then, the orthogonality of rows 3 and 4 gives $c = \bar{x}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & 1 & - & 0 & - & \bar{x} \\ 1 & - & 0 & x & -x & - \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{pmatrix} \tag{A.26}$$

The next row can be filled in accordingly.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & 1 & - & 0 & - & \bar{x} \\ 1 & - & 0 & x & -x & - \\ 1 & 0 & -x & - & x & -\bar{x} \\ \hline 0 & 1 & m & n & p & q \end{pmatrix} \tag{A.27}$$

And finally, the last row.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & 1 & - & 0 & - & \bar{x} \\ 1 & - & 0 & x & -x & - \\ 1 & 0 & -x & - & x & -\bar{x} \\ 0 & 1 & x & - & -x & -\bar{x} \\ \hline \end{pmatrix} \tag{A.28}$$

When we swap rows 3 and 4 as well as columns 3 and 4, we get $T_1(-x)$.

Case 3b: $d = -1$. This implies that $f = y$. The fact that $\langle r_3, r_4 \rangle = 0$ gives $g = yc$. We will relabel c to be z.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & z \\ 1 & -y & 0 & - & y & yz \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{pmatrix} \tag{A.29}$$

We now fill in the fifth row to arrive at the following.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & z \\ 1 & -y & 0 & - & y & yz \\ 1 & 0 & h & x & -y & l \\ \hline 0 & 1 & m & n & p & q \end{pmatrix} \tag{A.30}$$

From $\langle r_2, r_5 \rangle = 0$, we have that $h = -xl$. And then since $\langle r_4, r_5 \rangle = 0$, $l = xyz$.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & z \\ 1 & -y & 0 & - & y & yz \\ 1 & 0 & -x^2yz & x & -y & xyz \\ \hline 0 & 1 & m & n & p & q \end{array}\right) \tag{A.31}$$

We can fill in the final row in the following way.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & z \\ 1 & -y & 0 & - & y & yz \\ 1 & 0 & -x^2yz & x & -y & xyz \\ 0 & 1 & x & n & \bar{y} & q \end{array}\right) \tag{A.32}$$

Based on the fact that $\langle c_3, c_4\rangle = 0$, $n = \overline{yz}$. Then $\langle c_5, c_6\rangle = 0$ gives $q = x\bar{y}z$. Finally, $\langle r_2, r_6\rangle = 0$ reveals that $z = \pm\bar{x}$. Putting these three facts together, we arrive at

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & \pm\bar{x} \\ 1 & -y & 0 & - & y & \pm\bar{x}y \\ 1 & 0 & \mp xy & x & -y & \pm y \\ 0 & 1 & x & \pm x\bar{y} & \bar{y} & \pm\bar{y} \end{array}\right) \tag{A.33}$$

Let's look at the upper and lower signs on the $\pm$s separately. First, let us examine the upper signs. $\langle r_3, r_5\rangle + \langle c_2, c_6\rangle = 0 \implies x = \pm 1$ and $\langle c_2, c_6\rangle \implies x \neq 1$, so $x = -1$. We then have the following matrix, which we will denote $T_2(y)$.

$$T_2(y) = \left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & - & 1 & 0 & 1 \\ 1 & y & -y & 0 & - & - \\ 1 & -y & 0 & - & y & -y \\ 1 & 0 & y & - & -y & y \\ 0 & 1 & - & -\bar{y} & \bar{y} & \bar{y} \end{array}\right) \tag{A.34}$$

When we look at the lower signs of the $\pm$ in (A.33), we note that $\langle r_1, r_5\rangle - \langle c_2, c_6\rangle = 0 \implies x = y$, and $\langle r_1, r_6\rangle - \langle c_1, c_6\rangle = 0 \implies x = \pm i$. Thus, we have the following matrix:

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & \pm i & \mp i & 0 & 1 \\ 1 & \pm i & \mp i & 0 & - & \pm i \\ 1 & \mp i & 0 & - & \pm i & - \\ 1 & 0 & - & \pm i & \mp i & \mp i \\ 0 & 1 & \pm i & - & \mp i & \pm i \end{array}\right) \tag{A.35}$$

But if we look carefully, the top of the $\pm$s is equivalent to T_7 and the bottom is equivalent to T_6 (one must simply swap rows 3 and 4 as well as columns 3 and 4).

Case 3c: $f = -1$. This implies that $d = y$. The fact that $\langle r_3, r_4 \rangle = 0$ gives $g = -c$. We will relabel c to be z.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & z \\ 1 & -y & 0 & y & - & -z \\ \hline 1 & 0 & h & j & k & l \\ 0 & 1 & m & n & p & q \end{array}\right) \tag{A.36}$$

We adjoin in the fifth row, which will introduce many simplifications.

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & y & -y & 0 & - & z \\ 1 & -y & 0 & y & - & -z \\ 1 & 0 & h & j & 1 & - \\ \hline 0 & 1 & m & n & p & q \end{array}\right) \tag{A.37}$$

First, $\langle r_1, r_5 \rangle = 0$ gives $h = j = -1$, then $\langle c_1, c_3 \rangle = 0$ gives $x = y$ and finally, $\langle r_2, r_3 \rangle = 0$ gives $z = x$. We will then append the sixth and final row to arrive at the unique matrix, which we will denote $T_5(x)$.

$$T_5(x) := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & x & -x & 0 & 1 \\ 1 & x & -x & 0 & - & x \\ 1 & -x & 0 & x & - & -x \\ 1 & 0 & - & - & 1 & - \\ 0 & 1 & x & -x & - & - \end{pmatrix} \tag{A.38}$$

□

Appendix B

Detailed Proofs from Chapter 4

B.1 Sets of $UW(5,4)$

(This is the proof of Lemma 4.33.)

Lemma B.1. *Let W be a unit weighing matrix that is unbiased with*

$$W_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix}$$

where $\omega = e^{i\frac{2\pi}{3}}$. Then every nonzero entry in W is a sixth root of unity.

Proof. Since $W_5 W^* = 2L$ for some weighing matrix L, we know that each row of W must be orthogonal with exactly one row of W_5 and unbiased with the other four. Moreover, we know that the first nonzero entry in each row of W may be a one. To show the stated lemma, we will show that any viable vector (i.e., a vector in $\mathbb{C}^5$ with exactly four nonzero unimodular entries) that is orthogonal with one row of W_5 and unbiased with the other four only contains entries that are sixth roots of unity.

Using the definition of m-orthogonality and the results given in Proposition 3.25, we can determine that there are at most 11 *different* rows that are orthogonal to each of the rows of W_5, each with exactly one free variable. We will break up the analysis into five distinct cases. Each case will represent the full set of vectors which are orthogonal to a specific row in W_5. For ease, we will use R_i to be the the i^{th} row of W_5. Moreover, the rows of W that we are considering will be labelled r_i for $1 \leq i \leq 55$. The standard brackets around the vector will be dropped for convenience.

Let $b \in \mathbb{T}$ and α a primitive third root of unity (either ω or $\overline{\omega}$). The five main observations that are used throughout the proof are:

(O1) $|1-\alpha+b| = 2 \implies b \in \{\pm\overline{\alpha}\}$,

(O2) $|1+\alpha+b| = 2 \implies b = -\overline{\alpha}$,

(O3) $|3+b| = 2 \implies b = -1$,

(O4) $1+\alpha+\overline{\alpha} = 0$.

(O5) $|1+\alpha+\overline{\alpha}+\alpha b| = 2 \implies |\alpha b| = 2$, which is a contradiction since $|\alpha b| = 1$.

In each of the five cases, we will examine all rows that are orthogonal to the i^{th} row of W_5 (it turns out there are 11 candidates each time). Then, we will show that any free variable (b) is a sixth root of unity or arrive at a contradiction by using one of the five observations above. Note that we will stop each case as soon as a contradiction is found or all free variables are shown to be a sixth root of unity.

Case 1: Consider all rows that are orthogonal with row 1 of W_5:

$$
\begin{array}{cccccc}
(r_1) & 1 & - & b & -b & 0 \\
(r_2) & 1 & b & - & -b & 0 \\
(r_3) & 1 & b & -b & - & 0 \\
(r_4) & 1 & \omega & \overline{\omega} & 0 & b \\
(r_5) & 1 & \overline{\omega} & \omega & 0 & b \\
(r_6) & 1 & \omega & 0 & \overline{\omega} & b \\
(r_7) & 1 & \overline{\omega} & 0 & \omega & b \\
(r_8) & 1 & 0 & \omega & \overline{\omega} & b \\
(r_9) & 1 & 0 & \overline{\omega} & \omega & b \\
(r_{10}) & 0 & 1 & \omega & \overline{\omega} & b \\
(r_{11}) & 0 & 1 & \overline{\omega} & \omega & b
\end{array}
$$

Then,

$$
\begin{array}{llllll}
(a) & |\langle R_2, r_1\rangle| = 2 \implies & |1-\omega+\overline{\omega b}| = 2 & \implies \overline{\omega b} = \pm\overline{\omega} & \implies b \in \{\pm 1\}. \\
(b) & |\langle R_2, r_2\rangle| = 2 \implies & |1+\omega\overline{b}-\overline{\omega}| = 2 & \implies \omega\overline{b} = \pm\omega & \implies b \in \{\pm 1\}. \\
(c) & |\langle R_3, r_3\rangle| = 2 \implies & |1+\overline{\omega b}-\omega| = 2 & \implies \overline{\omega b} = \pm\overline{\omega} & \implies b \in \{\pm 1\}. \\
(d) & |\langle R_2, r_4\rangle| = 2 \implies & |1+1+1+\overline{b}| = 2 & \implies \overline{b} = -1 & \implies b = -1. \\
(e) & |\langle R_2, r_5\rangle| = 2 \implies & |1+\overline{\omega}+\omega+\overline{b}| = 2 & \implies |\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(f) & |\langle R_3, r_6\rangle| = 2 \implies & |1+\omega+\overline{\omega}+\overline{\omega b}| = 2 & \implies |\overline{\omega b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(g) & |\langle R_3, r_7\rangle| = 2 \implies & |1+1+1+\overline{\omega b}| = 2 & \implies \overline{\omega b} = -1 & \implies b = -\overline{\omega}. \\
(h) & |\langle R_4, r_8\rangle| = 2 \implies & |1+1+1+\omega\overline{b}| = 2 & \implies \omega\overline{b} = -1 & \implies b = -\omega. \\
(i) & |\langle R_4, r_9\rangle| = 2 \implies & |1+\overline{\omega}+\omega+\omega\overline{b}| = 2 & \implies |\omega\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(j) & |\langle R_5, r_{10}\rangle| = 2 \implies & |1+\omega+\overline{\omega}+\omega\overline{b}| = 2 & \implies |\omega\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(k) & |\langle R_5, r_{11}\rangle| = 2 \implies & |1+1+1+\omega\overline{b}| = 2 & \implies \omega\overline{b} = -1 & \implies b = -\omega.
\end{array}
$$

Case 2: Consider all rows that are orthogonal with row 2 of W_5:

(r_{12})	1	1	1	b	0
(r_{13})	1	$\overline{\omega}$	ω	b	0
(r_{14})	1	$-\omega$	$-\overline{\omega}b$	0	b
(r_{15})	1	$-\omega\overline{b}$	$-\overline{\omega}$	0	b
(r_{16})	1	b	$-b$	0	$-$
(r_{17})	1	1	0	b	ω
(r_{18})	1	$\overline{\omega}$	0	b	$\overline{\omega}$
(r_{19})	1	0	ω	b	ω
(r_{20})	1	0	1	b	$\overline{\omega}$
(r_{21})	0	1	$\overline{\omega}$	b	ω
(r_{22})	0	1	1	b	1

Then,

$$\begin{array}{llllll}
(a) & |\langle R_1, r_{12}\rangle| = 2 \implies & |1+1+1+\overline{b}| = 2 & \implies \overline{b} = -1 & \implies b = -1. \\
(b) & |\langle R_1, r_{13}\rangle| = 2 \implies & |1+\overline{\omega}+\omega+\overline{b}| = 2 & \implies |\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(c) & |\langle R_1, r_{14}\rangle| = 2 \implies & |1-\overline{\omega}-\omega\overline{b}| = 2 & \implies -\omega\overline{b} = \pm\omega & \implies b \in \{\pm 1\}. \\
(d) & |\langle R_1, r_{15}\rangle| = 2 \implies & |1-\overline{\omega}b-\overline{\omega}| = 2 & \implies -\overline{\omega}b = \pm\omega & \implies b \in \{\pm\overline{\omega}\}. \\
(e) & |\langle R_3, r_{16}\rangle| = 2 \implies & |1+\overline{\omega b}+\overline{\omega}| = 2 & \implies \overline{\omega b} = -\omega & \implies b = -\omega. \\
(f) & |\langle R_3, r_{17}\rangle| = 2 \implies & |1+\overline{\omega}+\omega\overline{b}+\omega| = 2 & \implies |\omega\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(g) & |\langle R_3, r_{18}\rangle| = 2 \implies & |1+1+\omega\overline{b}+1| = 2 & \implies \omega\overline{b} = -1 & \implies b = -\omega. \\
(h) & |\langle R_4, r_{19}\rangle| = 2 \implies & |1+1+\overline{\omega b}+1| = 2 & \implies \overline{\omega b} = -1 & \implies b = -\overline{\omega}. \\
(i) & |\langle R_4, r_{20}\rangle| = 2 \implies & |1+\omega+\overline{\omega b}+\overline{\omega}| = 2 & \implies |\overline{\omega b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(j) & |\langle R_5, r_{21}\rangle| = 2 \implies & |1+1+\omega\overline{b}+1| = 2 & \implies \omega\overline{b} = -1 & \implies b = -\omega. \\
(k) & |\langle R_5, r_{22}\rangle| = 2 \implies & |1+\overline{\omega}+b+\omega| = 2 & \implies |b| = 2. & \rightarrow\leftarrow
\end{array}$$

Case 3: Consider all rows that are orthogonal with row 3 of W_5:

(r_{23})	1	ω	b	$\overline{\omega}$	0
(r_{24})	1	1	b	1	0
(r_{25})	1	ω	b	0	1
(r_{26})	1	1	b	0	ω
(r_{27})	1	$-\overline{\omega}$	0	$-\overline{\omega}b$	b
(r_{28})	1	$-b$	0	$-\omega$	b
(r_{29})	1	$-\omega b$	0	b	$-\overline{\omega}$
(r_{30})	1	0	b	1	1
(r_{31})	1	0	b	$\overline{\omega}$	ω
(r_{32})	0	1	b	ω	ω
(r_{33})	0	1	b	1	$\overline{\omega}$

Then,

$$\begin{array}{llllll}
(a) & |\langle R_1, r_{23}\rangle| = 2 \implies & |1+1+1+\overline{b}| = 2 & \implies \overline{b} = -1 & \implies b = -1. \\
(b) & |\langle R_1, r_{24}\rangle| = 2 \implies & |1+\overline{\omega}+\omega+\overline{b}| = 2 & \implies |\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(c) & |\langle R_2, r_{25}\rangle| = 2 \implies & |1+1+\overline{\omega b}+1| = 2 & \implies \overline{\omega b} = -1 & \implies b = -\overline{\omega}. \\
(d) & |\langle R_2, r_{26}\rangle| = 2 \implies & |1+\omega+\overline{\omega b}+\overline{\omega}| = 2 & \implies |\overline{\omega b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(e) & |\langle R_1, r_{27}\rangle| = 2 \implies & |1-\omega-\omega\overline{b}| = 2 & \implies -\omega\overline{b} = \pm\overline{\omega} & \implies b \in \{\pm\overline{\omega}\}. \\
(f) & |\langle R_1, r_{28}\rangle| = 2 \implies & |1-\overline{b}-\overline{\omega}| = 2 & \implies -\overline{b} = \pm\omega & \implies b \in \{\pm\overline{\omega}\}. \\
(g) & |\langle R_2, r_{29}\rangle| = 2 \implies & |1-\overline{b}-\omega| = 2 & \implies -\overline{b} = \pm\overline{\omega} & \implies b \in \{\pm\omega\}. \\
(h) & |\langle R_4, r_{30}\rangle| = 2 \implies & |1+\omega\overline{b}+\overline{\omega}+\omega| = 2 & \implies |\omega\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(i) & |\langle R_4, r_{31}\rangle| = 2 \implies & |1+\omega\overline{b}+1+1| = 2 & \implies \omega\overline{b} = -1 & \implies b = -\omega. \\
(j) & |\langle R_5, r_{32}\rangle| = 2 \implies & |1+\overline{\omega b}+1+1| = 2 & \implies \overline{\omega b} = -1 & \implies b = -\overline{\omega}. \\
(k) & |\langle R_5, r_{33}\rangle| = 2 \implies & |1+\overline{\omega b}+\omega+\overline{\omega}| = 2 & \implies |\overline{\omega b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow
\end{array}$$

Case 4: Consider all rows that are orthogonal with row 4 of W_5:

$$\begin{array}{cccccc}
(r_{34}) & 1 & b & 1 & 1 & 0 \\
(r_{35}) & 1 & b & \overline{\omega} & \omega & 0 \\
(r_{36}) & 1 & b & 1 & 0 & \overline{\omega} \\
(r_{37}) & 1 & b & \overline{\omega} & 0 & 1 \\
(r_{38}) & 1 & b & 0 & \omega & \overline{\omega} \\
(r_{39}) & 1 & b & 0 & 1 & 1 \\
(r_{40}) & 1 & 0 & -\omega & -\omega b & b \\
(r_{41}) & 1 & 0 & -b & -\overline{\omega} & b \\
(r_{42}) & 1 & 0 & -\overline{\omega}b & b & -\overline{\omega} \\
(r_{43}) & 0 & b & 1 & \overline{\omega} & 1 \\
(r_{44}) & 0 & b & 1 & 1 & \omega
\end{array}$$

Then,

$$\begin{array}{llllll}
(a) & |\langle R_1, r_{34}\rangle| = 2 \implies & |1+\overline{b}+1+1| = 2 & \implies \overline{b} = -1 & \implies b = -1. \\
(b) & |\langle R_1, r_{35}\rangle| = 2 \implies & |1+\overline{b}+\omega+\overline{\omega}| = 2 & \implies |\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(c) & |\langle R_2, r_{36}\rangle| = 2 \implies & |1+\omega\overline{b}+\overline{\omega}+\omega| = 2 & \implies |\omega\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(d) & |\langle R_2, r_{37}\rangle| = 2 \implies & |1+\omega\overline{b}+1+1| = 2 & \implies \omega\overline{b} = -1 & \implies b = -\omega. \\
(e) & |\langle R_3, r_{38}\rangle| = 2 \implies & |1+\overline{\omega b}+1+1| = 2 & \implies \overline{\omega b} = -1 & \implies b = -\overline{\omega}. \\
(f) & |\langle R_3, r_{39}\rangle| = 2 \implies & |1+\overline{\omega b}+\omega+\overline{\omega}| = 2 & \implies |\overline{\omega b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow \\
(g) & |\langle R_1, r_{40}\rangle| = 2 \implies & |1-\overline{\omega}-\overline{\omega b}| = 2 & \implies -\overline{\omega b} = \pm\omega & \implies b \in \{\pm\omega\}. \\
(h) & |\langle R_1, r_{41}\rangle| = 2 \implies & |1-\omega-\overline{b}| = 2 & \implies -\overline{b} = \pm\overline{\omega} & \implies b \in \{\pm\omega\}. \\
(i) & |\langle R_2, r_{42}\rangle| = 2 \implies & |1-\overline{b}-\omega| = 2 & \implies -\overline{b} = \pm\overline{\omega} & \implies b \in \{\pm\omega\}. \\
(j) & |\langle R_5, r_{43}\rangle| = 2 \implies & |\overline{b}+\overline{\omega}+\overline{\omega}+\omega| = 2 & \implies |1-\overline{\omega}-\overline{b}| = 2 & \implies b \in \{\pm\overline{\omega}\}. \\
(k) & |\langle R_5, r_{44}\rangle| = 2 \implies & |\overline{b}+\overline{\omega}+\omega+1| = 2 & \implies |\overline{b}| = 2 & \implies |b| = 2. \rightarrow\leftarrow
\end{array}$$

Case 5: Consider all rows that are orthogonal with row 5 of W_5:

$$\begin{array}{cccccc}
(r_{45}) & b & 1 & \omega & \overline{\omega} & 0 \\
(r_{46}) & b & 1 & 1 & 1 & 0 \\
(r_{47}) & b & 1 & \omega & 0 & \overline{\omega} \\
(r_{48}) & b & 1 & 1 & 0 & 1 \\
(r_{49}) & b & 1 & 0 & 1 & \overline{\omega} \\
(r_{50}) & b & 1 & 0 & \overline{\omega} & 1 \\
(r_{51}) & b & 0 & 1 & \omega & 1 \\
(r_{52}) & b & 0 & 1 & 1 & \omega \\
(r_{53}) & 0 & 1 & -\overline{\omega} & -b & b \\
(r_{54}) & 0 & 1 & -\omega b & -\omega & b \\
(r_{55}) & 0 & 1 & -\omega b & b & -\omega
\end{array}$$

Then,

$$\begin{array}{lllll}
(a) & \lvert\langle R_1, r_{45}\rangle\rvert = 2 \implies & \lvert\overline{b}+1+\overline{\omega}+\omega\rvert = 2 & \implies \lvert\overline{b}\rvert = 2 & \implies \lvert b\rvert = 2. \rightarrow\leftarrow \\
(b) & \lvert\langle R_1, r_{46}\rangle\rvert = 2 \implies & \lvert\overline{b}+1+1+1\rvert = 2 & \implies \overline{b} = -1 & \implies b = -1. \\
(c) & \lvert\langle R_2, r_{47}\rangle\rvert = 2 \implies & \lvert\overline{b}+\omega+\overline{\omega}+\overline{\omega}\rvert = 2 & \implies \lvert 1-\overline{\omega}-\overline{b}\rvert = 2 & \implies b \in \{\pm\overline{\omega}\}. \\
(d) & \lvert\langle R_2, r_{48}\rangle\rvert = 2 \implies & \lvert\overline{b}+\omega+\overline{\omega}+1\rvert = 2 & \implies \lvert\overline{b}\rvert = 2 & \implies \lvert b\rvert = 2. \rightarrow\leftarrow \\
(e) & \lvert\langle R_3, r_{49}\rangle\rvert = 2 \implies & \lvert\overline{b}+\overline{\omega}+\omega+1\rvert = 2 & \implies \lvert\overline{b}\rvert = 2 & \implies \lvert b\rvert = 2. \rightarrow\leftarrow \\
(f) & \lvert\langle R_3, r_{50}\rangle\rvert = 2 \implies & \lvert\overline{b}+1+\omega\rvert = 2 & \implies \overline{b} = -\overline{\omega} & \implies b = -\omega. \\
(g) & \lvert\langle R_1, r_{51}\rangle\rvert = 2 \implies & \lvert\overline{b}+1+\overline{\omega}\rvert = 2 & \implies \overline{b} = -\omega & \implies b = -\overline{\omega}. \\
(h) & \lvert\langle R_4, r_{52}\rangle\rvert = 2 \implies & \lvert\overline{b}+\omega+\overline{\omega}+1\rvert = 2 & \implies \lvert\overline{b}\rvert = 2 & \implies \lvert b\rvert = 2. \rightarrow\leftarrow \\
(i) & \lvert\langle R_1, r_{53}\rangle\rvert = 2 \implies & \lvert 1-\omega-\overline{b}\rvert = 2 & \implies -\overline{b} = \pm\overline{\omega} & \implies b \in \{\pm\omega\}. \\
(j) & \lvert\langle R_5, r_{54}\rangle\rvert = 2 \implies & \lvert 1-\overline{\omega b}-\overline{\omega}\rvert = 2 & \implies -\overline{\omega b} = \pm\omega & \implies b \in \{\pm\omega\}. \\
(k) & \lvert\langle R_5, r_{55}\rangle\rvert = 2 \implies & \lvert a-\omega\overline{b}-\overline{\omega}\rvert = 2 & \implies \lvert 1-\overline{b}-\omega\rvert = 2 & \implies b \in \{\pm\omega\}.
\end{array}$$

□

B.2 Sets of $UW(7,4)$

(This is the proof of Lemma 4.37.)

Lemma B.2. *Let W be a unit weighing matrix that is unbiased with*

$$W_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}.$$

Then every nonzero entry in W is either 1 or -1.

Proof. We can easily see that there are only $\binom{7}{3} = 35$ possible zero placements that are valid in a row of W. We will break up the proof into two different sections. In the first, we will examine all rows that have the same zero placement as one of the rows in W_7. Then, in the second, we will look at the other 28 rows.

For each possible row in the first portion, we will show that any nonzero entry in each row must be real. We will work through the first example in full detail, then put all seven cases in an encoded form into Table B.1. Each case follows very similar to the example shown. In all cases, let a,b and c be arbitrary unimodular numbers that are independent of the other cases.

For example, consider the following row: $\begin{pmatrix} 1 & a & b & c & 0 & 0 & 0 \end{pmatrix}$

- Taking the complex inner product with row 2 of W_7, we have that $|1+a| \in \{0,2\}$ which implies $a \in \{\pm 1\}$.
- Taking the complex inner product with row 3 of W_7, we have that $|1+b| \in \{0,2\}$ which implies $b \in \{\pm 1\}$.
- Taking the complex inner product with row 4 of W_7, we have that $|1+c| \in \{0,2\}$ which implies $c \in \{\pm 1\}$.

Table B.1: Case analysis part 1 for Lemma 4.37

Row	Row in W_7 that implies $a \in \{\pm 1\}$	Row in W_7 that implies $b \in \{\pm 1\}$	Row in W_7 that implies $c \in \{\pm 1\}$
$\begin{pmatrix} 1 & a & b & c & 0 & 0 & 0 \end{pmatrix}$	2	3	4
$\begin{pmatrix} 1 & a & 0 & 0 & b & c & 0 \end{pmatrix}$	1	3	4
$\begin{pmatrix} 1 & 0 & a & 0 & b & 0 & c \end{pmatrix}$	1	2	4
$\begin{pmatrix} 1 & 0 & 0 & a & 0 & b & c \end{pmatrix}$	1	2	3
$\begin{pmatrix} 0 & 1 & a & 0 & 0 & b & c \end{pmatrix}$	1	2	6
$\begin{pmatrix} 0 & 1 & 0 & a & b & 0 & c \end{pmatrix}$	1	2	5
$\begin{pmatrix} 0 & 0 & 1 & a & b & c & 0 \end{pmatrix}$	1	3	5

Then, for the second portion of the case analysis, we will create a table of the remaining 28 zero placements. In each case, the inner product of the row and a specific row in W_7 gives us a single unimodular value, which cannot equal two.

For example, consider the following row: $\begin{pmatrix} 1 & a & b & 0 & c & 0 & 0 \end{pmatrix}$. Taking the complex inner product with row 4 of W_7, we have that $|1| \in \{0,2\}$ which is clearly a contradiction. Table B.2 shows which row in W_7 does not work with the corresponding case.

□

Table B.2: Case analysis part 2 for Lemma 4.37

Row	Row in W_7 that gives a contradiction
$\begin{pmatrix} 1 & a & b & 0 & c & 0 & 0 \end{pmatrix}$	4
$\begin{pmatrix} 1 & a & b & 0 & 0 & c & 0 \end{pmatrix}$	6
$\begin{pmatrix} 1 & a & b & 0 & 0 & 0 & c \end{pmatrix}$	7
$\begin{pmatrix} 1 & a & 0 & b & c & 0 & 0 \end{pmatrix}$	5
$\begin{pmatrix} 1 & a & 0 & b & 0 & c & 0 \end{pmatrix}$	3
$\begin{pmatrix} 1 & a & 0 & b & 0 & 0 & c \end{pmatrix}$	7
$\begin{pmatrix} 1 & a & 0 & 0 & b & 0 & c \end{pmatrix}$	7
$\begin{pmatrix} 1 & a & 0 & 0 & 0 & b & c \end{pmatrix}$	7
$\begin{pmatrix} 1 & 0 & a & b & c & 0 & 0 \end{pmatrix}$	5
$\begin{pmatrix} 1 & 0 & a & b & 0 & c & 0 \end{pmatrix}$	6
$\begin{pmatrix} 1 & 0 & a & b & 0 & 0 & c \end{pmatrix}$	2
$\begin{pmatrix} 1 & 0 & a & 0 & b & c & 0 \end{pmatrix}$	6
$\begin{pmatrix} 1 & 0 & a & 0 & 0 & b & c \end{pmatrix}$	6
$\begin{pmatrix} 1 & 0 & 0 & a & b & c & 0 \end{pmatrix}$	5
$\begin{pmatrix} 1 & 0 & 0 & a & b & 0 & c \end{pmatrix}$	5
$\begin{pmatrix} 1 & 0 & 0 & 0 & a & b & c \end{pmatrix}$	1
$\begin{pmatrix} 0 & 1 & a & b & c & 0 & 0 \end{pmatrix}$	4
$\begin{pmatrix} 0 & 1 & a & b & 0 & c & 0 \end{pmatrix}$	3
$\begin{pmatrix} 0 & 1 & a & b & 0 & 0 & c \end{pmatrix}$	2
$\begin{pmatrix} 0 & 1 & a & 0 & b & c & 0 \end{pmatrix}$	4
$\begin{pmatrix} 0 & 1 & a & 0 & b & 0 & c \end{pmatrix}$	4
$\begin{pmatrix} 0 & 1 & 0 & a & b & c & 0 \end{pmatrix}$	3
$\begin{pmatrix} 0 & 1 & 0 & a & 0 & b & c \end{pmatrix}$	3
$\begin{pmatrix} 0 & 1 & 0 & 0 & a & b & c \end{pmatrix}$	1
$\begin{pmatrix} 0 & 0 & 1 & a & b & c & 0 \end{pmatrix}$	2
$\begin{pmatrix} 0 & 0 & 1 & a & b & 0 & c \end{pmatrix}$	2
$\begin{pmatrix} 0 & 0 & 1 & 0 & a & b & c \end{pmatrix}$	1
$\begin{pmatrix} 0 & 0 & 0 & 1 & a & b & c \end{pmatrix}$	1

Appendix C

List of Real and Unit Weighing Matrices

C.1 Real Weighing Matrices of Weight 5

Every real weighing matrix of weight 5 is equivalent to one that is the direct sum of the following 7 families of matrices, $W_6, W_8, W_{12}, W_{14}, W_{16}, W_{4t+4}$ and W_{4t+2}.

$$W_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & - & - & 0 & 1 \\ 1 & - & - & 0 & 1 & - \\ 1 & - & 0 & 1 & - & 1 \\ 1 & 0 & 1 & - & - & - \\ 0 & 1 & - & 1 & - & - \end{pmatrix}$$

$$W_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 & 1 & 0 \\ 1 & - & 1 & - & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & 1 & - \\ 0 & 0 & 0 & 1 & - & 1 & - & 1 \end{pmatrix}$$

$$W_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & - & 0 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & - & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 1 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 0 \\ 0 & 1 & 0 & 0 & - & 0 & 1 & 0 & 0 & 0 & 1 & - \\ 0 & 0 & 1 & - & 0 & - & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & - & 0 & - & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$W_{14} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & - & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & - & 0 & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & - & 0 & 0 & - & 0 & - & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & - & 0 & 0 & - & 0 & - & - & 0 & 0 & 0 \\
0 & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & - & 0 & 0 & 0 & 1 & 0 & 0 & - & 0 & - & 0 \\
0 & 1 & 0 & 0 & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 0 & 0 \\
0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & - \\
0 & 0 & 1 & 0 & - & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & - & 0 & - & 1 & 0 & 0 & 0 & 0 & 0 & - \\
0 & 0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 0 & 0 & - & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 1 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & 1 & - & 0
\end{pmatrix}$$

$$W_{16} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & - & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & - & 0 & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & - & 0 & 0 & - & 0 & - & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & - & 0 & 0 & - & 0 & - & - & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & - & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & - & - & 0 & 0 \\
0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & - & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & - & 0 \\
0 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 0 & 0 & - & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 1 & 0 & 0 & - & 0 & 0 & - \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 0 & 0 & 1 & 0 & 0 & - & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & - & 1
\end{pmatrix}$$

$$
W_{4t+2} = \left(\begin{array}{ccccccccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & - & - & 0 & 1 \\
1 & - & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & - & 0 & 0 & 0 & 0 & - & - & 0 & 1 \\
1 & 0 & 0 & 0 & - & - & 0 & 0 & - & - \\
0 & 1 & 0 & 0 & - & - & 0 & 0 & 1 & 1 \\
 & & 1 & - & 0 & 0 & & & & & 1 & 1 & 1 & 0 \\
 & & 1 & - & 0 & 0 & & & & & - & - & 0 & 1 \\
 & & 1 & 0 & - & 1 & & & & & 0 & 0 & - & - \\
 & & 0 & 1 & - & 1 & & & & & 0 & 0 & 1 & 1 \\
 & & & & & & 1 & - & 0 & 0 \\
 & & & & & & 1 & - & 0 & 0 \\
 & & & & & & 1 & 0 & - & 1 \\
 & & & & & & 0 & 1 & - & 1 \\
 & & & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 0 \\
 & & & & & & & & & & & & & & & & & & & - & - & 0 & 1 \\
 & & & & & & & & & & & & & & & & & & & 0 & 0 & - & - \\
 & & & & & & & & & & & & & & & & & & & 0 & 0 & 1 & 1 \\
 & & & & & & & & & & & & & & & 1 & - & 0 & 0 & 1 & 0 & - & 1 \\
 & & & & & & & & & & & & & & & 1 & - & 0 & 0 & 0 & - & 1 & - \\
 & & & & & & & & & & & & & & & 1 & 0 & - & 1 & - & 1 & 0 & 0 \\
 & & & & & & & & & & & & & & & 0 & 1 & - & 1 & 1 & - & 0 & 0
\end{array}\right)
$$

$$
W_{4t+4} = \left(\begin{array}{ccccccccccccccccccccccccc}
1&1&1&1&1&0 \\
1&1&-&-&0&1 \\
1&-&0&0&0&0&1&1&1&0 \\
1&-&0&0&0&0&-&-&0&1 \\
1&0&0&0&-&-&0&0&-&- \\
0&1&0&0&-&-&0&0&1&1 \\
&&1&-&0&0&&&&&1&1&1&0 \\
&&1&-&0&0&&&&&-&-&0&1 \\
&&1&0&-&1&&&&&0&0&-&- \\
&&0&1&-&1&&&&&0&0&1&1 \\
&&&&&&1&-&0&0 \\
&&&&&&1&-&0&0 \\
&&&&&&1&0&-&1 \\
&&&&&&0&1&-&1 \\
&&&&&&&&&&&&&&\ddots \\
&&&&&&&&&&&&&&&&&&&1&1&1&0 \\
&&&&&&&&&&&&&&&&&&&-&-&0&1 \\
&&&&&&&&&&&&&&&&&&&0&0&-&- \\
&&&&&&&&&&&&&&&&&&&0&0&1&1 \\
&&&&&&&&&&&&&&&1&-&0&0&1&-&0&0&1&0 \\
&&&&&&&&&&&&&&&1&-&0&0&-&1&0&0&0&1 \\
&&&&&&&&&&&&&&&1&0&-&1&0&0&0&0&-&- \\
&&&&&&&&&&&&&&&0&1&-&1&0&0&0&0&1&1 \\
&&&&&&&&&&&&&&&&&&&1&0&-&1&-&1 \\
&&&&&&&&&&&&&&&&&&&0&1&-&1&1&-
\end{array}\right)
$$

C.2 List of Unit Weighing Matrices

Given here is a list of unit matrices of weight 4. Recall that all unit weighing matrices of weight 4 are equivalent to a weighing matrix that is made up of W_5, W_6, W_7, W_8 and E_{2m}. We now give examples of $UW(n,4)$ with n small. Note that we make no claim about equivalence of the matrices, only that this list is an upper bound on the number of inequivalent matrices.

The following table gives the number of decompositions of n without showing the decompositions.

Table C.1: Decompositions of unit weighing matrices of type $UW(n,4)$

$UW(4,4)$	E_4
$UW(5,4)$	W_5
$UW(6,4)$	E_6
	W_6
$UW(7,4)$	W_7
$UW(8,4)$	$E_4 \oplus E_4$
	E_8
	W_8
$UW(9,4)$	$E_4 \oplus W_5$
$UW(10,4)$	$E_4 \oplus E_6$
	$E_4 \oplus W_6$
	$W_5 \oplus W_5$
	E_{10}
$UW(11,4)$	$E_4 \oplus W_7$
	$W_5 \oplus E_6$
	$W_5 \oplus W_6$
$UW(12,4)$	$E_4 \oplus E_4 \oplus E_4$
	$E_4 \oplus E_8$
	$E_4 \oplus W_8$
	$W_5 \oplus W_7$
	$E_6 \oplus E_6$
	$E_6 \oplus W_6$
	$W_6 \oplus W_6$
	E_{12}
$UW(13,4)$	$E_4 \oplus E_4 \oplus W_5$
	$W_5 \oplus W_8$
	$W_5 \oplus E_8$
	$E_6 \oplus W_7$
	$W_6 \oplus W_7$
$UW(14,4)$	$E_4 \oplus E_4 \oplus E_6$
	$E_4 \oplus E_4 \oplus W_6$
	$E_4 \oplus W_5 \oplus W_5$
	$E_4 \oplus E_{10}$
	$E_6 \oplus E_8$
	$E_6 \oplus W_8$
	$W_6 \oplus E_8$
	$W_6 \oplus W_8$
	$W_7 \oplus W_7$
	E_{14}

Table C.2: Number of decompositions of unit weighing matrices of type $UW(n,4)$

n	#	n	#	n	#	n	#
1	0	26	91	51	2401	76	49960
2	0	27	73	52	3445	77	46836
3	0	28	128	53	3089	78	61251
4	1	29	103	54	4379	79	57587
5	1	30	173	55	3952	80	74976
6	2	31	142	56	5563	81	70630
7	1	32	236	57	5034	82	91488
8	3	33	194	58	7015	83	86422
9	1	34	313	59	6391	84	111485
10	4	35	265	60	8852	85	105496
11	3	36	424	61	8082	86	135445
12	8	37	357	62	11087	87	128477
13	5	38	555	63	10177	88	164323
14	10	39	476	64	13884	89	156137
15	7	40	737	65	12778	90	198849
16	16	41	634	66	17296	91	189343
17	11	42	961	67	15987	92	240258
18	23	43	837	68	21517	93	229138
19	17	44	1256	69	19937	94	289613
20	34	45	1098	70	26647	95	276750
21	25	46	1621	71	24789	96	348615
22	46	47	1433	72	32967	97	333611
23	36	48	2102	73	30731	98	418702
24	68	49	1860	74	40607	99	401394
25	52	50	2687	75	37987	100	502179

Appendix D

Sets of Mutually Unbiased Weighing Matrices

This section includes a library of sets of weighing matrices whose size equal the smallest upper bound that is known. To save space, we define $\omega := e^{2\pi i/3}$ and $\underline{\omega} := -\omega$.

Table D.1: 9 mutually unbiased weighing matrices of order 4 and weight 3, $UW(4,3)$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \omega & 0 \\ 1 & - & 0 & \omega \\ 1 & 0 & \underline{\overline{\omega}} & \underline{\overline{\omega}} \\ 0 & 1 & \underline{\overline{\omega}} & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \overline{\omega} & 0 \\ 1 & - & 0 & \overline{\omega} \\ 1 & 0 & \underline{\omega} & \underline{\omega} \\ 0 & 1 & \underline{\omega} & \overline{\omega} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \underline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & \omega & 0 \\ 1 & 0 & \underline{\overline{\omega}} & - \\ 0 & 1 & \underline{\omega} & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & 0 & \omega \\ 1 & \overline{\omega} & \overline{\omega} & 0 \\ 1 & 0 & \underline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & - & \overline{\omega} \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & 0 & \overline{\omega} \\ 1 & \overline{\omega} & 1 & 0 \\ 1 & 0 & - & \underline{\omega} \\ 0 & 1 & \underline{\overline{\omega}} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \omega & 1 & 0 \\ 1 & \underline{\overline{\omega}} & 0 & \omega \\ 1 & 0 & - & \underline{\overline{\omega}} \\ 0 & 1 & \underline{\omega} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & \omega & 0 \\ 1 & \underline{\overline{\omega}} & 0 & \overline{\omega} \\ 1 & 0 & \underline{\overline{\omega}} & \underline{\omega} \\ 0 & 1 & - & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & \overline{\omega} & 0 \\ 1 & \underline{\overline{\omega}} & 0 & 1 \\ 1 & 0 & \underline{\omega} & - \\ 0 & 1 & \underline{\overline{\omega}} & \overline{\omega} \end{pmatrix}$$

Table D.2: 5 mutually unbiased weighing matrices of order 5 and weight 4, $UW(5,4)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \overline{\omega} & 0 & 1 \\ 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\ 1 & 0 & \omega & \overline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & - & 0 \\ 1 & \omega & \overline{\omega} & 0 & - \\ 1 & \overline{\omega} & 0 & \underline{\overline{\omega}} & \underline{\omega} \\ 1 & 0 & \omega & \underline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & \overline{\omega} & \underline{\overline{\omega}} & \underline{\overline{\omega}} \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & - & 1 & 0 \\ 1 & \omega & \underline{\omega} & 0 & - \\ 1 & \overline{\omega} & 0 & \omega & \underline{\omega} \\ 1 & 0 & \underline{\overline{\omega}} & \overline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & \underline{\omega} & \omega & \underline{\overline{\omega}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \underline{\omega} & 0 & \omega & \underline{\omega} \\ 1 & - & 1 & 1 & 0 \\ 1 & \underline{\overline{\omega}} & \overline{\omega} & 0 & - \\ 1 & 0 & \omega & \overline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & \underline{\omega} & \underline{\overline{\omega}} & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & 0 & \underline{\overline{\omega}} & \overline{\omega} \\ 1 & - & - & - & 0 \\ 1 & \underline{\overline{\omega}} & \underline{\omega} & 0 & 1 \\ 1 & 0 & \underline{\overline{\omega}} & \underline{\omega} & \omega \\ 0 & 1 & \overline{\omega} & \omega & \underline{\overline{\omega}} \end{pmatrix}$$

Table D.3: 20 mutually unbiased weighing matrices of order 6 and weight 4, $UW(6,4)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 \\ 1 & - & 0 & 0 & - & - \\ 0 & 0 & 1 & - & 1 & - \\ 0 & 0 & 1 & - & - & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & - & 0 & 0 \\ 1 & 1 & - & 1 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & - \\ 1 & - & 0 & 0 & - & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & - & - \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & - & - \\ 1 & - & 1 & - & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & - \\ 0 & 0 & 1 & 1 & - & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & - \\ 1 & 1 & 0 & 0 & - & 1 \\ 1 & - & 1 & 1 & 0 & 0 \\ 1 & - & - & - & 0 & 0 \\ 0 & 0 & 1 & - & 1 & 1 \\ 0 & 0 & 1 & - & - & - \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & \omega & 0 & \omega & 0 \\ 1 & \overline{\omega} & \omega & 0 & \underline{\overline{\omega}} & 0 \\ 1 & 0 & \underline{\overline{\omega}} & \underline{\omega} & 0 & 1 \\ 1 & 0 & \underline{\overline{\omega}} & \overline{\omega} & 0 & - \\ 0 & 1 & 0 & 1 & \overline{\omega} & \omega \\ 0 & 1 & 0 & - & \overline{\omega} & \underline{\overline{\omega}} \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & \omega & 0 & \underline{\overline{\omega}} & 0 \\ 1 & \overline{\omega} & \omega & 0 & \omega & 0 \\ 1 & 0 & \underline{\overline{\omega}} & \underline{\omega} & 0 & - \\ 1 & 0 & \underline{\overline{\omega}} & \overline{\omega} & 0 & 1 \\ 0 & 1 & 0 & 1 & \underline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & 0 & - & \underline{\omega} & \omega \end{pmatrix}$$

$$\begin{pmatrix} 1 & \underline{\omega} & \underline{\overline{\omega}} & 0 & \omega & 0 \\ 1 & \overline{\omega} & \underline{\overline{\omega}} & 0 & \underline{\overline{\omega}} & 0 \\ 1 & 0 & \omega & \underline{\omega} & 0 & - \\ 1 & 0 & \omega & \overline{\omega} & 0 & 1 \\ 0 & 1 & 0 & 1 & \overline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & 0 & - & \overline{\omega} & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & \underline{\overline{\omega}} & 0 & \underline{\overline{\omega}} & 0 \\ 1 & \overline{\omega} & \underline{\overline{\omega}} & 0 & \omega & 0 \\ 1 & 0 & \omega & \underline{\omega} & 0 & 1 \\ 1 & 0 & \omega & \overline{\omega} & 0 & - \\ 0 & 1 & 0 & 1 & \underline{\omega} & \omega \\ 0 & 1 & 0 & - & \underline{\omega} & \underline{\overline{\omega}} \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & 0 & \omega & 0 & \omega \\ 1 & \overline{\omega} & 0 & \omega & 0 & \underline{\overline{\omega}} \\ 1 & 0 & \underline{\omega} & \underline{\overline{\omega}} & 1 & 0 \\ 1 & 0 & \overline{\omega} & \underline{\overline{\omega}} & - & 0 \\ 0 & 1 & 1 & 0 & \omega & \overline{\omega} \\ 0 & 1 & - & 0 & \underline{\overline{\omega}} & \overline{\omega} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \underline{\omega} & 0 & \omega & 0 & \underline{\overline{\omega}} \\ 1 & \overline{\omega} & 0 & \omega & 0 & \omega \\ 1 & 0 & \underline{\omega} & \underline{\overline{\omega}} & - & 0 \\ 1 & 0 & \overline{\omega} & \underline{\overline{\omega}} & 1 & 0 \\ 0 & 1 & 1 & 0 & \underline{\overline{\omega}} & \underline{\omega} \\ 0 & 1 & - & 0 & \omega & \underline{\omega} \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & 0 & \underline{\overline{\omega}} & 0 & \omega \\ 1 & \overline{\omega} & 0 & \underline{\overline{\omega}} & 0 & \underline{\overline{\omega}} \\ 1 & 0 & \underline{\omega} & \omega & - & 0 \\ 1 & 0 & \overline{\omega} & \omega & 1 & 0 \\ 0 & 1 & 1 & 0 & \underline{\overline{\omega}} & \overline{\omega} \\ 0 & 1 & - & 0 & \omega & \overline{\omega} \end{pmatrix} \quad \begin{pmatrix} 1 & \underline{\omega} & 0 & \underline{\overline{\omega}} & 0 & \underline{\overline{\omega}} \\ 1 & \overline{\omega} & 0 & \underline{\overline{\omega}} & 0 & \omega \\ 1 & 0 & \underline{\omega} & \omega & 1 & 0 \\ 1 & 0 & \overline{\omega} & \omega & - & 0 \\ 0 & 1 & 1 & 0 & \omega & \underline{\omega} \\ 0 & 1 & - & 0 & \underline{\overline{\omega}} & \underline{\omega} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \omega & \underline{\omega} & 0 & 0 & \underline{\omega} \\ 1 & \underline{\overline{\omega}} & \overline{\omega} & 0 & 0 & \underline{\omega} \\ 1 & 0 & 0 & 1 & \omega & \overline{\omega} \\ 1 & 0 & 0 & - & \underline{\overline{\omega}} & \overline{\omega} \\ 0 & 1 & \omega & \underline{\omega} & 1 & 0 \\ 0 & 1 & \omega & \overline{\omega} & - & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & \underline{\omega} & 0 & 0 & \overline{\omega} \\ 1 & \underline{\overline{\omega}} & \overline{\omega} & 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 & 1 & \underline{\overline{\omega}} & \underline{\omega} \\ 1 & 0 & 0 & - & \omega & \underline{\omega} \\ 0 & 1 & \omega & \underline{\omega} & - & 0 \\ 0 & 1 & \omega & \overline{\omega} & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & \overline{\omega} & 0 & 0 & \underline{\omega} \\ 1 & \underline{\overline{\omega}} & \underline{\omega} & 0 & 0 & \underline{\omega} \\ 1 & 0 & 0 & 1 & \underline{\overline{\omega}} & \overline{\omega} \\ 1 & 0 & 0 & - & \omega & \overline{\omega} \\ 0 & 1 & \underline{\overline{\omega}} & \underline{\omega} & - & 0 \\ 0 & 1 & \underline{\overline{\omega}} & \overline{\omega} & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \omega & \overline{\omega} & 0 & 0 & \overline{\omega} \\ 1 & \underline{\overline{\omega}} & \underline{\omega} & 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 & 1 & \omega & \underline{\omega} \\ 1 & 0 & 0 & - & \underline{\overline{\omega}} & \underline{\omega} \\ 0 & 1 & \underline{\overline{\omega}} & \underline{\omega} & 1 & 0 \\ 0 & 1 & \underline{\overline{\omega}} & \overline{\omega} & - & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & 0 & \underline{\omega} & \underline{\omega} & 0 \\ 1 & \underline{\overline{\omega}} & 0 & \overline{\omega} & \underline{\omega} & 0 \\ 1 & 0 & 1 & 0 & \overline{\omega} & \omega \\ 1 & 0 & - & 0 & \overline{\omega} & \underline{\overline{\omega}} \\ 0 & 1 & \underline{\omega} & \omega & 0 & 1 \\ 0 & 1 & \overline{\omega} & \omega & 0 & - \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & 0 & \underline{\omega} & \overline{\omega} & 0 \\ 1 & \underline{\overline{\omega}} & 0 & \overline{\omega} & \overline{\omega} & 0 \\ 1 & 0 & 1 & 0 & \underline{\omega} & \underline{\overline{\omega}} \\ 1 & 0 & - & 0 & \underline{\omega} & \omega \\ 0 & 1 & \underline{\omega} & \omega & 0 & - \\ 0 & 1 & \overline{\omega} & \omega & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \omega & 0 & \overline{\omega} & \underline{\omega} & 0 \\ 1 & \underline{\overline{\omega}} & 0 & \underline{\omega} & \underline{\omega} & 0 \\ 1 & 0 & 1 & 0 & \overline{\omega} & \underline{\overline{\omega}} \\ 1 & 0 & - & 0 & \overline{\omega} & \omega \\ 0 & 1 & \underline{\omega} & \overline{\omega} & 0 & - \\ 0 & 1 & \overline{\omega} & \underline{\overline{\omega}} & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \omega & 0 & \overline{\omega} & \overline{\omega} & 0 \\ 1 & \underline{\overline{\omega}} & 0 & \underline{\omega} & \overline{\omega} & 0 \\ 1 & 0 & 1 & 0 & \underline{\omega} & \omega \\ 1 & 0 & - & 0 & \underline{\omega} & \overline{\omega} \\ 0 & 1 & \underline{\omega} & \underline{\overline{\omega}} & 0 & 1 \\ 0 & 1 & \overline{\omega} & \underline{\overline{\omega}} & 0 & - \end{pmatrix}$$

Table D.4: 8 mutually unbiased real weighing matrices of order 7 and weight 4, $W(7,4)$.

$$\begin{pmatrix} 1&1&1&1&0&0&0\\ 1&-&0&0&1&1&0\\ 1&0&-&0&-&0&1\\ 1&0&0&-&0&-&-\\ 0&1&-&0&0&1&-\\ 0&1&0&-&1&0&1\\ 0&0&1&-&-&1&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&-&-&0&0&0\\ 1&-&0&0&-&1&0\\ 1&0&1&0&1&0&1\\ 1&0&0&1&0&-&-\\ 0&1&1&0&0&1&-\\ 0&1&0&1&-&0&1\\ 0&0&1&-&-&-&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&0&-&-&0\\ 1&-&1&-&0&0&0\\ 1&0&-&0&1&0&1\\ 1&0&0&1&0&1&-\\ 0&1&1&0&0&1&1\\ 0&1&0&-&1&0&-\\ 0&0&1&1&1&-&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&0&0&1&1&0\\ 1&-&-&-&0&0&0\\ 1&0&1&0&-&0&-\\ 1&0&0&1&0&-&1\\ 0&1&-&0&0&-&-\\ 0&1&0&-&-&0&1\\ 0&0&1&-&1&-&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&1&-&0&0&0\\ 1&-&0&0&1&-&0\\ 1&0&-&0&-&0&-\\ 1&0&0&1&0&1&1\\ 0&1&-&0&0&-&1\\ 0&1&0&1&1&0&-\\ 0&0&1&1&-&-&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&0&1&-&0\\ 1&-&-&1&0&0&0\\ 1&0&1&0&-&0&1\\ 1&0&0&-&0&1&-\\ 0&1&-&0&0&1&1\\ 0&1&0&1&-&0&-\\ 0&0&1&1&1&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&-&1&0&0&0\\ 1&-&0&0&-&-&0\\ 1&0&1&0&1&0&-\\ 1&0&0&-&0&1&1\\ 0&1&1&0&0&-&1\\ 0&1&0&-&-&0&-\\ 0&0&1&1&-&1&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&0&-&1&0\\ 1&-&1&1&0&0&0\\ 1&0&-&0&1&0&-\\ 1&0&0&-&0&-&1\\ 0&1&1&0&0&-&-\\ 0&1&0&1&1&0&1\\ 0&0&1&-&1&1&0 \end{pmatrix}$$

Table D.5: 14 mutually unbiased real weighing matrices of order 8 and weight 4, $W(8,4)$.

$$\begin{pmatrix} 1&1&1&1&0&0&0&0\\ 1&1&-&-&0&0&0&0\\ 1&-&1&-&0&0&0&0\\ 1&-&-&1&0&0&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&-&-\\ 0&0&0&0&1&-&1&-\\ 0&0&0&0&1&-&-&1 \end{pmatrix} \begin{pmatrix} 1&1&1&-&0&0&0&0\\ 1&1&-&1&0&0&0&0\\ 1&-&0&0&1&-&0&0\\ 1&-&0&0&-&1&0&0\\ 0&0&1&1&0&0&1&1\\ 0&0&1&1&0&0&-&-\\ 0&0&0&0&1&1&1&-\\ 0&0&0&0&1&1&-&1 \end{pmatrix} \begin{pmatrix} 1&1&0&0&1&1&0&0\\ 1&1&0&0&-&-&0&0\\ 1&-&1&1&0&0&0&0\\ 1&-&-&-&0&0&0&0\\ 0&0&1&-&0&0&1&-\\ 0&0&1&-&0&0&-&1\\ 0&0&0&0&1&-&1&1\\ 0&0&0&0&1&-&-&- \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&1&0&1&-&0\\ 1&0&0&1&0&-&1&0\\ 1&0&0&-&1&0&0&-\\ 1&0&0&-&-&0&0&1\\ 0&1&1&0&1&0&0&1\\ 0&1&1&0&-&0&0&-\\ 0&1&-&0&0&1&1&0\\ 0&1&-&0&0&-&-&0 \end{pmatrix} \begin{pmatrix} 1&1&0&0&1&-&0&0\\ 1&1&0&0&-&1&0&0\\ 1&-&0&0&0&0&1&1\\ 1&-&0&0&0&0&-&-\\ 0&0&1&1&0&0&1&-\\ 0&0&1&1&0&0&-&1\\ 0&0&1&-&1&1&0&0\\ 0&0&1&-&-&-&0&0 \end{pmatrix} \begin{pmatrix} 1&0&0&1&1&0&0&1\\ 1&0&0&1&-&0&0&-\\ 1&0&0&-&0&1&1&0\\ 1&0&0&-&0&-&-&0\\ 0&1&1&0&0&1&-&0\\ 0&1&1&0&0&-&1&0\\ 0&1&-&0&1&0&0&-\\ 0&1&-&0&-&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&0&0&0&0&1&-\\ 1&1&0&0&0&0&-&1\\ 1&-&0&0&1&1&0&0\\ 1&-&0&0&-&-&0&0\\ 0&0&1&1&1&-&0&0\\ 0&0&1&1&-&1&0&0\\ 0&0&1&-&0&0&1&1\\ 0&0&1&-&0&0&-&- \end{pmatrix} \begin{pmatrix} 1&0&0&1&1&0&0&-\\ 1&0&0&1&-&0&0&1\\ 1&0&0&-&1&0&0&1\\ 1&0&0&-&-&0&0&-\\ 0&1&1&0&0&1&1&0\\ 0&1&1&0&0&-&-&0\\ 0&1&-&0&0&1&-&0\\ 0&1&-&0&0&-&1&0 \end{pmatrix} \begin{pmatrix} 1&0&0&1&0&1&1&0\\ 1&0&0&1&0&-&-&0\\ 1&0&0&-&0&1&-&0\\ 1&0&0&-&0&-&1&0\\ 0&1&1&0&1&0&0&-\\ 0&1&1&0&-&0&0&1\\ 0&1&-&0&1&0&0&1\\ 0&1&-&0&-&0&0&- \end{pmatrix}$$

$$\begin{pmatrix} 1&0&1&0&0&1&0&-\\ 1&0&1&0&0&-&0&1\\ 1&0&-&0&1&0&1&0\\ 1&0&-&0&-&0&-&0\\ 0&1&0&1&1&0&-&0\\ 0&1&0&1&-&0&1&0\\ 0&1&0&-&0&1&0&1\\ 0&1&0&-&0&-&0&- \end{pmatrix} \begin{pmatrix} 1&0&1&0&1&0&1&0\\ 1&0&1&0&-&0&-&0\\ 1&0&-&0&1&0&-&0\\ 1&0&-&0&-&0&1&0\\ 0&1&0&1&0&1&0&1\\ 0&1&0&1&0&-&0&-\\ 0&1&0&-&0&1&0&-\\ 0&1&0&-&0&-&0&1 \end{pmatrix} \begin{pmatrix} 1&1&0&0&0&0&1&1\\ 1&-&0&0&0&0&-&1\\ 1&0&1&0&0&-&0&-\\ 1&0&-&0&0&1&0&-\\ 0&1&0&1&-&0&-&0\\ 0&1&0&-&1&0&-&0\\ 0&0&1&1&1&1&0&0\\ 0&0&1&-&-&1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&0&0&0&0&-&-\\ 1&-&0&0&0&0&1&-\\ 1&0&1&0&0&1&0&1\\ 1&0&-&0&0&-&0&1\\ 0&1&0&1&1&0&1&0\\ 0&1&0&-&-&0&1&0\\ 0&0&1&1&-&-&0&0\\ 0&0&1&-&1&-&0&0 \end{pmatrix} \begin{pmatrix} 1&0&1&0&1&0&-&0\\ 1&0&1&0&-&0&1&0\\ 1&0&-&0&0&1&0&1\\ 1&0&-&0&0&-&0&-\\ 0&1&0&1&0&1&0&-\\ 0&1&0&1&0&-&0&1\\ 0&1&0&-&1&0&1&0\\ 0&1&0&-&-&0&-&0 \end{pmatrix}$$

D.1 Vectors of Dimension 5 and Weight 4

A set of 40 vectors in $\mathbb{C}^5$ such that there are exactly 4 unimodular entries (and one zero) whose pairwise inner product falls within $\{0,2\}$ can be found in Table D.6. Based on the structure, we may add the rows of the identity, normalize all of the vectors and attain the upper bound given in (4.4) for $n = 5$ and $\alpha = \frac{1}{2}$.

D.2 Hadamard Matrices of Order 32

In Tables D.7,D.8,D.9 and D.10, we show the partition of the 32^2 vectors into 32 Hadamard matrices of order 32 (denoted by $H_1, H_2, \ldots, H_{32}$). Each section represents one Hadamard matrix, and each hexadecimal number represents one row of the matrix (where each digit represents four entries). The most significant binary digit represents the left-most entry of the 4-tuple and the least significant binary digit represents the right-most digit. For example, `4259F1BA` represents

$$\underbrace{0100}_{4}\underbrace{0010}_{2}\underbrace{0101}_{5}\underbrace{1001}_{9}\underbrace{1111}_{F}\underbrace{0001}_{1}\underbrace{1011}_{B}\underbrace{1010}_{A}.$$

Then, we apply (4.9) to give us

$$\underbrace{1-11}_{4}\underbrace{11-1}_{2}\underbrace{1-1-}_{5}\underbrace{-11-}_{9}\underbrace{----}_{F}\underbrace{111-}_{1}\underbrace{-1--}_{B}\underbrace{-1-1}_{A}.$$

Table D.6: 40 vectors in $\mathbb{C}^5$ that meet the upper bound in Theorem 4.26. $\omega = e^{\frac{2\pi i}{6}}$.

(1	1	1	1	0)
(1	1	1	–	0)
(1	1	–	1	0)
(1	1	–	–	0)
(1	ω	ω^2	0	1)
(1	ω	ω^2	0	–)
(1	ω	ω^5	0	1)
(1	ω	ω^5	0	–)
(1	ω^2	0	ω	ω^2)
(1	ω^2	0	ω	ω^5)
(1	ω^2	0	ω^4	ω^2)
(1	ω^2	0	ω^4	ω^5)
(1	–	1	1	0)
(1	–	1	–	0)
(1	–	–	1	0)
(1	–	–	–	0)
(1	ω^4	ω^2	0	1)
(1	ω^4	ω^2	0	–)
(1	ω^4	ω^5	0	1)
(1	ω^4	ω^5	0	–)
(1	ω^5	0	ω	ω^2)
(1	ω^5	0	ω	ω^5)
(1	ω^5	0	ω^4	ω^2)
(1	ω^5	0	ω^4	ω^5)
(1	0	ω	ω^2	ω)
(1	0	ω	ω^2	ω^4)
(1	0	ω	ω^5	ω)
(1	0	ω	ω^5	ω^4)
(1	0	ω^4	ω^2	ω)
(1	0	ω^4	ω^2	ω^4)
(1	0	ω^4	ω^5	ω)
(1	0	ω^4	ω^5	ω^4)
(0	1	ω^2	ω	ω)
(0	1	ω^2	ω	ω^4)
(0	1	ω^2	ω^4	ω)
(0	1	ω^2	ω^4	ω^4)
(0	1	ω^5	ω	ω)
(0	1	ω^5	ω	ω^4)
(0	1	ω^5	ω^4	ω)
(0	1	ω^5	ω^4	ω^4)

Table D.7: H_1 through H_8

H_1	00000000	4259F1BA	203AEEB5	50967C6E	59F1BA84	47FC04A7
	4E9BC24D	4B3E3750	62631F0F	7C6EA12C	1E0DBE23	259F1BA8
	32F56361	750967C6	176A78C9	67C6EA12	6EA12CF8	70AC92DB
	55338973	0CC233F7	12CF8DD4	05A5F51D	3B92A58B	79CB5431
	1BA84B3E	5C544F99	6B04D9E5	3E375096	2CF8DD42	0967C6EA
	3750967C	295D285F				
H_2	6EF49ECD	2D2FA8E1	755CD5F3	1BFDF90B	31A55E78	7C3B1319
	3FE02535	5826CF27	727E6854	4DCBFF54	5663B46A	7B19AEBE
	129A3FE1	0055B235	24486E0B	44AC39BE	0E10C978	38C29892
	4AE942F3	15B88246	438E8419	514109CD	67935827	2A0D1546
	69D6236A	3687E3DF	093274DF	1CDF44AC	60B1E580	5F047280
	236AD3AC	07770F92				
H_3	477DB9D5	1F0EC4C7	6A07A301	182C7960	0384325E	5CD5F2EB
	5BF74F4C	529089A6	636065EB	114BBF8A	7FEA9372	2EFE288A
	405F0472	71AFE83F	4E1A7F3F	2799EE60	0AE3F4B4	0DC14913
	355663B4	20BB53C7	78C82ED5	29DC952D	768D5598	3274DE13
	1669022D	3C31A55E	4938C298	04A68FF9	6D251EA6	6442D84C
	55B23401	3B1318F9				
H_4	050E9174	10E3A107	19D1D5D8	475760CE	7935826D	20C438E9
	6BAFBD8C	629DC953	2E8143A4	3529089A	4E651411	7770F920
	52BA50BD	02799EE6	3C1B7C45	40206F5C	5CFF2BF0	7E428DFF
	27B3377B	29F64C36	65EAC6C1	1EA6DA4A	0B4BEA39	325E0708
	3B6C73D7	5B882462	7007F6B2	49121B83	6CD8B21E	1794AE95
	0C3CE5AB	55CD5F2F				
H_5	7357CBAB	66BAFBD8	7475760C	017C11CA	5FD07DC7	2AA6712F
	3F342A72	5FAF16E9	14EE4A97	3F4B415C	61E72D51	4D1FF013
	0621C743	381697D5	4A3D4DB4	149121B9	7328A085	740A1D22
	01037AE4	66C590F6	2D84CC88	58F2C060	065EAC6D	4A42269A
	13CCF730	2DFBA7A6	3869FCFB	4D609B3D	2AD91A01	588DAB4E
	13B39C1E	6198467F				
H_6	76595ADF	03503D19	53C64177	1CF59DB7	0D154654	0E3A1063
	4C63E1D9	1FDACB80	288A5DFC	5EAC6C0D	372FFD52	4109CCA3
	12B0E6FA	6496D70B	119FB0CD	4F4CB7EE	396A861F	007F6B2E
	50E91740	69FCFA71	781C2192	5D833A3A	3400AB65	42269A94
	25E07086	75760CE8	3A45D028	2BA50BCB	26CF26B1	7B3377A5
	67B9813C	6AD3AC46				
H_7	7A4F666F	0F4601A9	5195068A	7A300D41	56B7BB2D	23BEDCEB
	44075DD7	3171513F	24E30A62	56C8D003	0864BC0E	435A8B5E
	1DF6E753	1AAB31DA	23C1B7C5	6880EBBB	0F396A87	7D6DDBC8
	7D12B0E6	447836F9	4325E070	310E3A11	68FF8095	362C87B6
	1AD45AF4	3653EC98	6FDD3D32	249C614C	1D898C7D	51EA6DA4
	081BD720	6FA2561C				
H_8	0BCA574B	621C7421	70864BC0	4993A6F1	27328A09	2E00FED6
	523BEDCF	32DFBA7A	5C7E9682	7EC3308D	77F14452	3C9AC137
	171513E7	10621C75	4EE4A963	1E276738	058F2C06	195068AA
	02F82394	3BEDCEA5	0CBD58D9	2045859B	5B099910	79B43F1F
	47D6DDBC	656B7BB3	554CE25D	6C590F6C	35A8B5E8	6B2E00FE
	40A1D22E	2977F144				

Table D.8: H_9 through H_{16}

H_9	09B3C9AD	7B9813CC	7DC6BFA1	2C2CD205	2A727E68	513E62E3
	74A1794B	13679359	00D40F47	254B14EF	2315B882	3F9F4E1B
	4DB4947A	44D35290	0FED65C0	1A0055B3	614C4938	5859A409
	72FFD526	6712E555	068AA32A	6E7523BF	36F888F1	1C5EF9DE
	30A6249C	5760CE8E	682B8FD2	428DFEFD	5E070864	4BEA3817
	39C1E276	15393F34				
H_{10}	1853124E	16166903	146FF7E5	4680156D	78B745FB	0FC7BCDB
	241DDC3E	1A2A8CA8	266442D8	2A58A773	6F23EB6E	631F0EC5
	3DCC09E6	3FB59700	48C56E20	4ABCF0C6	61669023	6D5A7588
	0182C796	28213995	0DBE223D	536D251E	5F51C0B5	5114BBF8
	76F23EB6	03FB5970	5D285E53	338972AB	31F0EC4D	7ACEDB1D
	748BA050	44F98B8B				
H_{11}	1FA5A0AE	2FD78B75	111E0DBF	347FC04B	5347FC05	789D9CE0
	46541A2A	040DEB90	48116167	216C2664	6D70AC93	5D028748
	3AC46D5A	1F5B76F2	6D8E7ACF	46AACC76	0AB64681	0A4890DD
	53B92A59	48EFB73B	2F295D29	76D8E7AD	3A3ABB06	6335D7DE
	2192F038	762631F1	04F33DCC	78634ABC	34811617	63CB0182
	5DFC5114	11E0DBE3				
H_{12}	6C0CBD59	3BB87C90	1740A1D2	2E554CE3	47836F89	05DA9E33
	35FD07DD	40F4601B	526E5FFA	201037AE	77A4F667	1E72D50D
	2767383C	29224371	49C614C4	6249C614	4EB11B56	3CCF7302
	5C2B24B7	1037AE40	7E9682B8	5B5C2B25	6B7BB2CB	02AD91A1
	0B9FE57E	0CE8EAEC	653EC986	79E18D2A	70D3F9F5	1905DA9F
	328A084F	55195068				
H_{13}	5017C11C	3D32DFBA	11CA02F8	7CEF1C5E	415C7E96	784993A7
	0427328B	37D12B0E	69022C2D	4F1905DB	5E52BA51	63E1D899
	760CE8EA	5AF435A8	0081BD72	39945043	28DFEFC9	223C1B7D
	1B29F64C	269A9484	45FAF16F	0EC4C63F	2C796030	54B14EE5
	72AA6713	4BBF8A22	67475760	3377A4F7	156C8D01	6DA4A3D4
	0A6249C6	1F8F79B5				
H_{14}	702D2FA9	3DB362C8	0524486F	33F61985	6B856497	6CA7D930
	41DDC3E4	34D4A422	261B29F6	0206F5C8	0B613322	0C438E85
	1E8C0351	794AE943	53124E30	62E2A27D	4F98B8A9	770F920E
	2F7CEF1C	5430F397	17EBC5BB	3A91DF6F	46FF7E43	65C01FDA
	7E6854E4	48BA050E	21399451	285E52BB	19AEBEF6	10C9781C
	5D57357D	5A7588DA				
H_{15}	50C3CE5B	01D775A3	1A7F3E9D	37052449	08B0B349	3E62E2A3
	39405F04	7DB9D48F	66119FB1	4C4938C2	2B8FD2D0	25CAA99D
	6F76595B	74DE1265	06F5C804	61332216	302799EE	6854E4FC
	4B6B8565	1318F877	73FCAFC2	7A9B6928	0F920EEE	1D5D833A
	452EFE28	59A408B1	22E8143A	5E86B516	143A45D0	57E173FC
	420C438F	2CAD6F77				
H_{16}	2DAE1593	24C9D379	1DDC3E48	669022C3	61B29F64	442D84CC
	2A8CA834	58A77255	430F396B	745FAF17	1AFE83EF	36065EAD
	08310E3B	737D12B0	0156C8D1	0F13B39C	3124E30A	06747576
	384325E0	56E20918	23EB6EDE	68D5598E	7A1AD45A	13994505
	7D3869FD	14BBF8A2	51C0B4BF	4D4A4226	6FF7E429	4A68FF81
	3F619847	5F85CFF2				

Table D.9: H_{17} through H_{24}

H_{17}	22977F14	1CA02F82	7AB1B033	3FCAFC2E	2CD20459	580C163C
	01FDACB8	3E1D898D	74F4CB7E	1332216C	4B14EE4B	0E6FA256
	59DB639F	1D775A21	3058F2C0	579E18D2	0FB8D7F5	4486E0A5
	67EC3309	23400AB7	002AD91B	687E3DE7	12E554CF	2D0571FA
	45519506	4AC39BE8	663B46AA	7523BEDD	69A94844	7B66C590
	56496D71	318F8763				
H_{18}	25B5C2B3	60CE8EAE	7201037A	2A27CC5D	06DF111F	2462B710
	377A4F67	427328A1	5F2EAB9B	2BF0B9FE	6E8BF5E3	094D1FF1
	73D676D9	6F5C8040	1A55E786	36AD3AC4	089A6A52	1B829225
	6119FB0D	50BCA575	516BD0D6	43A45D02	7C447837	070864BC
	14109CCB	15C7E968	4DE1264F	7D930D94	38E84189	4C3653EC
	5EF9DE38	393F342A				
H_{19}	13E62E2B	39BE8958	1DA35566	503D1807	0129A3FF	7A65BF74
	2216C266	571FA5A0	4CB7EE9E	25347FC1	7D4702D3	37FBF215
	45D02874	14C4938C	6F888F07	61CDF44A	2C53B92B	7302799E
	0F6CD8B2	595ADEED	5E78634A	3E9C34FF	4B955339	060B1E58
	2B71048C	66EF49ED	1A81E8C1	084E6515	68AA32A0	30D94FB2
	7420C439	42F295D3				
H_{20}	028748BA	1048C56E	0E457B4D	49B97FEA	722BDA61	3CB0182C
	0722BDA7	2E7F95F8	2BDA60E5	35D7DEC6	7EE9E996	40DEB900
	778E2F7C	6983915F	30722BDB	3915ED31	7B4C1C8B	5ED30723
	60E457B5	6541A2A8	6C266442	15ED3073	22BDA60F	27185312
	4C1C8AF7	192F0384	0BE08E50	1C8AF699	457B4C1D	57B4C1C9
	521134D4	5B76F23E				
H_{21}	4844D352	11B569D6	71513E63	0D3F9F4F	4F666EF5	64BC0E10
	4601A81F	3D4DB494	18D2AF3C	21C7420D	2F823940	5DA9E321
	639EB3B7	1697D471	7836F889	037AE402	5A8B5E86	045859A5
	6DDBC8FA	6AF9755D	767383C4	412315B8	54CE25CB	1FF0129B
	7F14452E	26E5FFAA	3308CFD9	3A6F0933	342A727E	53EC986C
	28A084E7	0A1D22E8				
H_{22}	2F563607	4CE25CAB	39EB3B6D	308CFD87	4BC0E10C	3EC986CA
	75F7B19A	11616691	64680157	0EBBAD11	42A727E6	1806A07B
	099910B6	541A2A8C	07DC6BFB	7C907770	28748BA0	2631F0ED
	5D7DEC66	1F241DDC	1643DB36	21134D4A	72D50C3D	6D0FC7BD
	6A2D7A1A	00FED65C	7BB2CAD7	634ABCF0	5A5F51C1	45859A41
	37AE4020	5338972B				
H_{23}	78E2F7CE	1B569D62	4FB261B2	41F71AFF	22437053	76A78C83
	2561CDF4	5E2DD17F	67383C4E	1513E62F	2B24B6B9	697D4703
	590F6CD8	03AEEB45	4890DC15	574A1795	46D5A758	34FE7D39
	0AC92DAF	12315B88	71853124	2C060B1E	6E5FFAA4	0DEB9008
	601A81E9	048C56E2	3D99BBD3	5068AA32	7FC04A69	1C7420C5
	3ABB0674	33DCC09E				
H_{24}	075DD689	598ED1AA	332216C2	7F3E9C35	0A37FBF3	0918ADC4
	7254B14F	5AA1879D	609B3C9B	3D676D8F	4B415C7E	717BE778
	15925B5D	21ED9B16	462B7104	7C11CA02	6EDE47D6	3E483BB8
	22C2CD21	54E4FCD0	2C87B66C	486E0A49	18F87627	16BD0D6A
	1BD72010	300D40F5	63B46AAC	57CBAAE7	6DF111E1	45042733
	047280BE	2FA8E05B				

Table D.10: H_{25} through H_{32}

H_{25}	16E8BF5F	597007F6	483BB87C	712E554D	2FFD526E	5393F342
	3A10621D	42D84CC8	251EA6DA	0D40F461	4C9D3785	3EB6EDE4
	467EC331	7BCDA1F9	5DD6880F	6E20918A	6A861E73	57357CBB
	124E30A6	64C3653E	07A300D5	7F6B2E00	7588DAB4	03058F2C
	30F396A9	21B82923	1C0B4BEB	34551950	18ADC412	09E67B98
	6065EAC7	2B5BDD97				
H_{26}	47A9B692	27CC5C55	20EEE1F2	62B71048	3BC717BE	70789D9C
	173FCAFC	02D2FA8F	775A203B	799EE604	32216C26	4E4FCD0A
	29089A6A	6C73D677	524486E1	55663B46	0C163CB0	05F04728
	101D775B	19FB0CC3	408B0B35	0B348117	6595ADEF	496D70AD
	3503D181	5BA2FD79	7EBC5BA3	3CE5AA19	6B516BD0	2E2A27CD
	1ED9B164	5C8040DE				
H_{27}	58737D12	680156C9	4FE7D387	7F95F85C	38972AA7	7DEC66BA
	1F71AFE9	6A78C82F	4D9E4D61	2F038432	544F98B9	71D08311
	3AEEB441	41A2A8CA	06A07A31	0A9C9F9A	134D4A42	1134D4A4
	34ABCF0C	04D9E4D7	5636065F	2146FF7F	1D08310F	66442D84
	73A91DF7	36D251EA	43DB362C	08E5017C	233F6199	5A0AE3F4
	643DB362	2D7A1AD4				
H_{28}	2EAB9ABF	109CCA29	274DE127	791F5B76	4ECE7078	5C01FDAC
	70F920EE	32A0D154	55E78634	6CF26B05	0571FA5A	77DB9D49
	49ECCDDF	47280BE0	1E580C16	6BD0D6A2	206F5C80	3B46AACC
	5B23400B	6236AD3A	7E3DE6D1	0C9781C2	6514109D	025347FD
	29892718	52C53B93	197AB1B1	17BE778E	0BB53C65	400AB647
	3C64176B	35826CF3				
H_{29}	4947A9B6	29A3FE03	10B61332	6BFA0FB9	27E6854E	5BDD9657
	320BB53D	1EF3687F	20918ADC	022C2CD3	4075DD69	62C87B66
	5CAA99C5	3C4ECE70	65BF74F4	5598ED1A	70524487	79603058
	7E173FCA	52EFE288	198467ED	3B39C1E2	0B1E580C	6C8D002B
	357CBAAF	77254B15	055B2341	2ED4F191	17C11CA0	4702D2FB
	0C69579E	4E30A624				
H_{30}	0EEE1F24	239405F0	1C2192F0	6E0A4891	1546541A	4A1794AF
	7BE778E2	75A203AF	51BFDF91	0789D9CE	3F1EF369	2AF3C31A
	315B8824	4452EFE2	6928F536	676D8E7B	7CC5C545	09CCA283
	2DD17EBD	7280BE08	58D8197B	43705245	4D352908	24B6B857
	604F33DC	1B032F57	1264E9BD	36793583	00AB6469	5FFAA4DC
	383C4ECE	569D6236				
H_{31}	5A203AEF	163CB018	64E9BC25	1879CB55	33A3ABB0	6AACC768
	71048C56	0D6A2D7A	5ADEECB3	7FBF2147	0D94FB26	7F41F71B
	4FCD0A9C	280BE08E	71FA5A0A	4F33DCC0	18871D09	26B04D9F
	28F536D2	546541A2	03D1806B	549B97FE	418871D1	032F5637
	335D7DEC	3D1806A1	264E9BC3	16C26644	64176A79	4176A78D
	6A521134	3DE6D0FD				
H_{32}	24370525	4A9629DD	08CFD867	3EE35FD1	50427329	561CDF44
	3784993B	01A81E8D	2D50C3CF	69579E18	75DD6881	7CBAAE6B
	121B8293	7383C4EC	0E91740A	1B7C4479	07F6B2E0	31DA3556
	1D22E814	4CC885B0	6F093275	666EF49F	5925B5C3	38BDF3BC
	45AF435A	14452EFE	603058F2	43F1EF37	2269A948	5F7B19AE
	7AE40206	2B0E6FA2				

Appendix E

Combinatorial Objects Used in Construction 5.16

These are the objects that were used in Construction 5.16 in Table 5.1.

E.1 Hadamard Matrices

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{pmatrix}$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \\ 1 & 1 & - & - & - & - & 1 & 1 \end{pmatrix}$$

$$H_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - \\ 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 \\ 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - \\ 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - \\ 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\ 1 & 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 \\ 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & - \\ 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 \\ 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & - \end{pmatrix}$$

$$H_{20} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - \\
1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - \\
1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - \\
1 & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - \\
1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 \\
1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - \\
1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - \\
1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & -
\end{pmatrix}$$

E.2 Latin Squares

Each of the Latin squares of order n are defined on the alphabet $\{0, 1, \ldots, n-1\}$.

$$\mathcal{L}_3 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\mathcal{L}_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$\mathcal{L}_7 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 0 & 6 & 4 \\ 2 & 5 & 4 & 6 & 3 & 0 & 1 \\ 3 & 2 & 6 & 5 & 1 & 4 & 0 \\ 4 & 0 & 3 & 1 & 6 & 2 & 5 \\ 5 & 6 & 0 & 4 & 2 & 1 & 3 \\ 6 & 4 & 1 & 0 & 5 & 3 & 2 \end{pmatrix}$$

$$\mathcal{L}_{11} = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 9 & 5 & 7 & 10 & 0 & 6 & 4 & 3 & 8 \\
2 & 9 & 3 & 10 & 6 & 8 & 1 & 0 & 7 & 5 & 4 \\
3 & 5 & 10 & 4 & 1 & 7 & 9 & 2 & 0 & 8 & 6 \\
4 & 7 & 6 & 1 & 5 & 2 & 8 & 10 & 3 & 0 & 9 \\
5 & 10 & 8 & 7 & 2 & 6 & 3 & 9 & 1 & 4 & 0 \\
6 & 0 & 1 & 9 & 8 & 3 & 7 & 4 & 10 & 2 & 5 \\
7 & 6 & 0 & 2 & 10 & 9 & 4 & 8 & 5 & 1 & 3 \\
8 & 4 & 7 & 0 & 3 & 1 & 10 & 5 & 9 & 6 & 2 \\
9 & 3 & 5 & 8 & 0 & 4 & 2 & 1 & 6 & 10 & 7 \\
10 & 8 & 4 & 6 & 9 & 0 & 5 & 3 & 2 & 7 & 1
\end{pmatrix}$$

$$\mathcal{L}_{19} = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 14 & 17 & 9 & 11 & 8 & 4 & 12 & 18 & 0 & 10 & 5 & 16 & 3 & 7 & 6 & 15 & 13 \\
2 & 14 & 3 & 15 & 18 & 10 & 12 & 9 & 5 & 13 & 1 & 0 & 11 & 6 & 17 & 4 & 8 & 7 & 16 \\
3 & 17 & 15 & 4 & 16 & 1 & 11 & 13 & 10 & 6 & 14 & 2 & 0 & 12 & 7 & 18 & 5 & 9 & 8 \\
4 & 9 & 18 & 16 & 5 & 17 & 2 & 12 & 14 & 11 & 7 & 15 & 3 & 0 & 13 & 8 & 1 & 6 & 10 \\
5 & 11 & 10 & 1 & 17 & 6 & 18 & 3 & 13 & 15 & 12 & 8 & 16 & 4 & 0 & 14 & 9 & 2 & 7 \\
6 & 8 & 12 & 11 & 2 & 18 & 7 & 1 & 4 & 14 & 16 & 13 & 9 & 17 & 5 & 0 & 15 & 10 & 3 \\
7 & 4 & 9 & 13 & 12 & 3 & 1 & 8 & 2 & 5 & 15 & 17 & 14 & 10 & 18 & 6 & 0 & 16 & 11 \\
8 & 12 & 5 & 10 & 14 & 13 & 4 & 2 & 9 & 3 & 6 & 16 & 18 & 15 & 11 & 1 & 7 & 0 & 17 \\
9 & 18 & 13 & 6 & 11 & 15 & 14 & 5 & 3 & 10 & 4 & 7 & 17 & 1 & 16 & 12 & 2 & 8 & 0 \\
10 & 0 & 1 & 14 & 7 & 12 & 16 & 15 & 6 & 4 & 11 & 5 & 8 & 18 & 2 & 17 & 13 & 3 & 9 \\
11 & 10 & 0 & 2 & 15 & 8 & 13 & 17 & 16 & 7 & 5 & 12 & 6 & 9 & 1 & 3 & 18 & 14 & 4 \\
12 & 5 & 11 & 0 & 3 & 16 & 9 & 14 & 18 & 17 & 8 & 6 & 13 & 7 & 10 & 2 & 4 & 1 & 15 \\
13 & 16 & 6 & 12 & 0 & 4 & 17 & 10 & 15 & 1 & 18 & 9 & 7 & 14 & 8 & 11 & 3 & 5 & 2 \\
14 & 3 & 17 & 7 & 13 & 0 & 5 & 18 & 11 & 16 & 2 & 1 & 10 & 8 & 15 & 9 & 12 & 4 & 6 \\
15 & 7 & 4 & 18 & 8 & 14 & 0 & 6 & 1 & 12 & 17 & 3 & 2 & 11 & 9 & 16 & 10 & 13 & 5 \\
16 & 6 & 8 & 5 & 1 & 9 & 15 & 0 & 7 & 2 & 13 & 18 & 4 & 3 & 12 & 10 & 17 & 11 & 14 \\
17 & 15 & 7 & 9 & 6 & 2 & 10 & 16 & 0 & 8 & 3 & 14 & 1 & 5 & 4 & 13 & 11 & 18 & 12 \\
18 & 13 & 16 & 8 & 10 & 7 & 3 & 11 & 17 & 0 & 9 & 4 & 15 & 2 & 6 & 5 & 14 & 12 & 1
\end{pmatrix}$$

E.3 Mutually Suitable Latin Squares

The mutually suitable Latin squares that were used in the thesis were generated from mutually orthogonal Latin squares. These sets of mutually orthogonal Latin squares can be obtained through the use of finite fields (see [11, Construction 3.29]). Once we have obtained the full set, we may make the transformation $((i,j),k) \longrightarrow ((k,j),i)$, where $((i,j),k)$ that the (i,j) entry is a k. The set obtained is a set of mutually suitable Latin squares (see [21, Lemma 9] for a proof of this).