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# Phenomenological Implications of the Generalized Uncertainty Principle

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Various theories of Quantum Gravity argue that near the Planck scale, the Heisenberg Uncertainty Principle should be replaced by the so called Generalized Uncertainty Principle (GUP). We show that the GUP gives rise to two additional terms in any quantum mechanical Hamiltonian, proportional to $\beta p^4$ and $\beta^2 p^6$ respectively, where $\beta \sim 1/(M_{Pl}c)^2$ is the GUP parameter. These terms become important at or above the Planck energy. Considering only the first of these, and treating it as a perturbation, we show that the GUP affects the Lamb shift, Landau levels, reflection and transmission coefficients of a potential step and potential barrier, and the current in a Scanning Tunnel Microscope (STM). Although these are too small to be measurable at present, we speculate on the possibility of extracting measurable predictions in the future.

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## I. INTRODUCTION

Although, there are various approaches to Quantum Gravity, e.g. String Theory and Canonical Quantum Gravity, to our knowledge, none of them has made a single prediction which can be experimentally tested at present (or in the near future). Even if Supersymmetry is observed in the Large Hadron Collider (LHC), it would at best confirm the existence of an essential ingredient of the String Theory, and would hardly be an evidence in favor of the theory itself. Given this situation, it is important to try to extract testable predictions. There has been recent attempts in this direction, and although some of them do compute Quantum Gravity effects, the smallness of the Planck length (and largeness of the Planck energy) too often renders these effects minuscule [1]. In this paper, we explore a few well understood low energy systems and show that Quantum Gravity does predict corrections for them. These corrections are once again, generically quite small to be measurable. However, we argue that (i) they could signal a new intermediate length scale between the electroweak and the Planck scale, and (ii) study of other related systems could give rise to predictions which can perhaps be tested [1].

Our main ingredient is the so-called Generalized Uncertainty Principle (GUP), which has been argued from various approaches to Quantum Gravity and Black Hole Physics, using a combination of thought experiments and series of arguments [3]. These indicate that there exists a minimum measurable length [4], the Planck length, $\ell_{Pl} \approx 10^{-33}$ cm. The prediction is largely model independent, and can be understood as follows: the Heisenberg Uncertainty Principle (HUP), whereby uncertainty in position decreases with increasing energies ($\Delta x \sim \hbar/\Delta p$),

breaks down for energies close to the Planck scale, at which point the corresponding Schwarzschild radius becomes comparable to the Compton wavelength (both being approximately equal to the Planck length). Higher energies result in a further increase of the Schwarzschild radius, resulting in $\Delta x \approx \ell_{Pl}^2 \Delta p/\hbar$. Consistent with the above, the following form of GUP has been proposed, postulated to hold at all scales [3]

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}[1 + \beta\left((\Delta p)^2 + <p>^2\right) + 2\beta\left(\Delta p_i^2 + <p_i>^2\right)] , \ i = 1, 2, 3 \quad (1)$$

where $[\beta] = (momentum)^{-2}$ and we will assume that $\beta = \beta_0/(M_{Pl}c)^2 = \ell_{Pl}^2/2\hbar^2$ while $M_{Pl}$ is the Planck mass, $M_{Pl}c^2 = $ (Planck energy) $\approx 10^{19}$ GeV. It is evident that the parameter $\beta_0$ is dimensionless. But, what determines its value? It is normally assumed that $\beta_0$ is not far from unity. We will see in this article, that on the one hand, $\beta_0 \approx 1$ renders the effects of Quantum Gravity on everyday quantum phenomena too small to be measurable. On the other hand, if one does not impose the above condition *a priori*, current experiments predict large upper bounds on it, which are consistent with current observations, and may indeed signal the existence of a new length scale. Note that any new such intermediate length scale, $\ell_{inter} \equiv \sqrt{\beta_0}\ell_{Pl}$ cannot exceed the electroweak length scale $\sim 10^{17} \ \ell_{Pl}$ (as otherwise it would have been observed), this tells us that $\beta_0$ cannot exceed about $10^{34}$. (The factor of 2 in the last term in Eq.(1) follows from Eq.(2) below).

It was shown in [5], using standard methods, that the above inequality follows from the modified Heisenberg algebra

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j) . \quad (2)$$

This form ensures, via the Jacobi identity, that $[x_i, x_j] = 0 = [p_i, p_j]$ [6]. Note that the above algebra does not admit of a simple representation in position space. How-

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[1] Some preliminary results in this direction were published in [2].

ever, defining

$$x_i = x_{0i} \tag{3}$$
$$p_i = p_{0i}\left(1 + \beta p_0^2\right) \tag{4}$$

where $p_0^2 = \sum_{j=1}^{3} p_{0j}p_{0j}$ and with $x_{0i}, p_{0j}$ satisfying the canonical commutation relations

$$[x_{0i}, p_{0j}] = i\hbar\,\delta_{ij}, \tag{5}$$

it is easy to show that Eq.(2) is satisfied, to order $\beta$. Henceforth, we neglect terms of order $\beta^2$ and higher.

Here, $p_{0i}$ can be interpreted as the momentum at low energies (having the usual representation in position space, i.e. $p_{0i} = -i\hbar d/dx_{0i}$), and $p_i$ as that at higher energies.

Using (4), any Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r})\ , \quad \vec{r} = (x_1, x_3, x_3) \tag{6}$$

can be written as [7]

$$H = \frac{p_0^2}{2m} + V(\vec{r}) + \frac{\beta}{m}p_0^4 + \mathcal{O}(\beta^2) \tag{7}$$
$$\equiv H_0 + H_1 + \mathcal{O}(\beta^2)\ , \tag{8}$$

where

$$H_0 = \frac{p_0^2}{2m} + V(\vec{r})\ \text{ and }\ H_1 = \frac{\beta}{m}p_0^4 = \frac{\beta\hbar^4}{m}\nabla^4\ , \tag{9}$$

where in the last step, we have specialized to the position representation. Thus, we see that *any* system with a well defined quantum (or even classical) Hamiltonian $H_0$, is perturbed by $H_1$, defined above, near the Planck scale. In other words, Quantum Gravity effects are in some sense universal! It remains to compute the corrections to various phenomena due to the Hamiltonian $H_1$. Before we do that, we note that using (7) to write the time-dependent Schrödinger equation

$$H\psi(\vec{r}, t) = i\hbar\frac{\partial\psi}{\partial t}\ , \tag{10}$$

and going through the usual set of steps (multiplying (10) by $\psi^\star$, and subtracting it from the complex conjugate of (10) multiplied by $\psi$), and making a few further manipulations, one arrives at the following charge and current densities and the conservation equation

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{11}$$
$$\rho = |\psi|^2 \tag{12}$$
$$\vec{J} = \frac{\hbar}{2mi}\left[\psi^\star\vec{\nabla}\psi - \psi\vec{\nabla}\psi^\star\right]$$
$$- \frac{\beta\hbar^3}{mi}\left[\left(\psi^\star\vec{\nabla}\nabla^2\psi - \psi\vec{\nabla}\nabla^2\psi^\star\right)\right.$$
$$\left. + \left(\nabla^2\psi^\star\vec{\nabla}\psi - \nabla^2\psi\vec{\nabla}\psi^\star\right)\right] \tag{13}$$
$$\equiv \vec{J_0} + \vec{J_1}\ , \tag{14}$$

where $\vec{J_0}$ is the usual quantum mechanical expression and $\vec{J_1}$ is the additional $\beta$-dependent term due to GUP. It is satisfying that the modified Hamiltonian (7) does admit of a (new) conserved current. Next, we study its effect on a number of quantum mechanical systems, with various $V(\vec{r})$.

## II. GUP AND THE LAMB SHIFT

For the Hydrogen atom, $V(\vec{r}) = -k/r$ ($k = e^2/4\pi\epsilon_0$, $e = $ electronic charge, $r = \sqrt{|\vec{r}|}$), for which, to first order, the perturbing Hamiltonian $H_1$ shifts the wave-functions to [8]

$$|\psi_{nlm}\rangle_1 = |\psi_{nlm}\rangle + \sum_{\{n'l'm'\}\neq\{nlm\}}\frac{e_{n'l'm'|nlm}}{E_n - E_{n'}}|n'l'm'\rangle \tag{15}$$

where $n, l, m$ have their usual significance, and

$$e_{n'l'm'|nlm} \equiv \langle n'l'm'|H_1|nlm\rangle\ . \tag{16}$$

Using $p_0^2 = 2m[H_0 + k/r]$, we get [7]

$$H_1 = (4\beta m)\left[H_0^2 + k\left(\frac{1}{r}H_0 + H_0\frac{1}{r}\right) + \left(\frac{k}{r}\right)^2\right]\ . \tag{17}$$

Thus,

$$\frac{e_{n'l'm'|nlm}}{4\beta m} = (E_n)^2\,\delta_{nn'} + k\,(E_n + E_{n'})\,\langle n'l'm'|\frac{1}{r}|nlm\rangle$$
$$+ k^2\langle n'l'm'|\frac{1}{r^2}|nlm\rangle\ . \tag{18}$$

From the orthogonality of spherical harmonics, it follows that the above are non-vanishing if and only if $l' = l$ and $m' = m$. Thus, the first order shift in the ground state wave-function is given by (in the position representation)

$$\Delta\psi_{100}(\vec{r}) \equiv \psi_{100(1)}(\vec{r}) - \psi_{100}(\vec{r}) = \frac{e_{200|100}}{E_1 - E_2}\psi_{200}(\vec{r})$$
$$= \frac{928\sqrt{2}\beta m E_0}{81}\,\psi_{200}(\vec{r})\ , \tag{19}$$

where we have used the following:
(i) the first term in the sum in Eq.(15) ($n' = 2$) dominates, since $E_n = -E_0/n^2$ ($E_0 = e^2/8\pi\epsilon_0 a_0 = k/2a_0 = 13.6\ eV$, $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.3\times 10^{-11}$ metre, $m = $ electron mass $= 0.5\ MeV/c^2$),
(ii) $\psi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\theta, \phi)$,
(iii) $R_{10} = 2a_0^{-3/2}e^{-r/a_0}$ and
    $R_{20} = (2a_0)^{-3/2}\left(2 - r/a_0\right)e^{-r/2a_0}$,
(iv) $Y_{00}(\theta, \phi) = 1/(\sqrt{4\pi})$,
where $E_0$ is the lowest (ground state) energy level of the Hydrogen atom and $a_0$ is the Bohr radius.
Next, consider the expression for the Lamb shift for the $n^{th}$ level of the Hydrogen atom ($\alpha \equiv e^2/4\pi\epsilon_0\hbar c \approx 1/137$) [9]

$$\Delta E_n = \frac{4\alpha^2}{3m^2}\left(\ln\frac{1}{\alpha}\right)|\psi_{nlm}(0)|^2\ . \tag{20}$$

Varying $\psi_{nlm}(0)$, the additional contribution to the Lamb shift due to GUP in proportion to its original value is given by

$$\frac{\Delta E_{n(GUP)}}{\Delta E_n} = 2\frac{\Delta|\psi_{nlm}(0)|}{\psi_{nlm}(0)} \ . \qquad (21)$$

Thus, for the Ground State, using $\psi_{100}(0) = a_0^{-3/2}\pi^{-1/2}$ and $\psi_{200}(0) = a_0^{-3/2}(8\pi)^{-1/2}$, we get

$$
\begin{aligned}
\frac{\Delta E_{0(GUP)}}{\Delta E_0} &= 2\frac{\Delta|\psi_{100}(0)|}{\psi_{100}(0)} = \frac{928\beta m E_0}{81} \\
&\approx 10\beta_0 \frac{m}{M_{Pl}}\frac{E_0}{M_{Pl}c^2} \\
&\approx 10 \times \left(0.42 \times 10^{-22}\right) \times \left(1.13 \times 10^{-27}\right)\beta_0 \\
&\approx 0.47 \times 10^{-48}\ \beta_0 \ .
\end{aligned}
\qquad (22)
$$

The above result may be interpreted in two ways. First, if one assumes $\beta_0 \sim 1$, then it predicts a non-zero, but virtually *unmeasurable* effect of Quantum Gravity/GUP. On the other hand, if such an assumption is not made, the current accuracy of precision measurement of Lamb shift of about 1 part in $10^{12}$ [7, 10], sets the following upper bound on $\beta_0$

$$\beta_0 < 10^{36} \ . \qquad (23)$$

This bound is weaker than that set by the electroweak scale, but not incompatible with it. Moreover, with more accurate measurements in the future, this bound is expected to get reduced by several orders of magnitude, in which case, it could signal a new and intermediate length scale between the electroweak and the Planck scale.

### III.   THE LANDAU LEVELS

Next consider a particle of mass $m$ and charge $e$ in a constant magnetic field $\vec{B} = B\hat{z}$, described by the vector potential $\vec{A} = Bx\hat{y}$ in the Landau gauge. The corresponding Hamiltonian is

$$
\begin{aligned}
H_0 &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 \qquad (24)\\
&= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{eB}{m}\,xp_y + \frac{e^2B^2}{2m}\,x^2 \ . \qquad (25)
\end{aligned}
$$

Since $p_y$ commutes with $H$, replacing it with its eigenvalue $\hbar k$, we get

$$H_0 = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k}{m\omega_c}\right)^2 \ , \qquad (26)$$

where $\omega_c = eB/m$ is the cyclotron frequency. This is nothing but the Hamiltonian of a harmonic oscillator in the $x$ direction, with its equilibrium position given

by $x_0 \equiv \hbar k/m\omega_c$. Consequently, the eigenfunctions and eigenvalues are given by

$$\psi_{k,n}(x,y) = e^{iky}\phi_n(x - x_0) \qquad (27)$$

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) \ , \ n \in N \ , \qquad (28)$$

where $\phi_n$ are the harmonic oscillator wave-functions.

Following the procedure outlined in Appendix A, the GUP corrected Lagrangian, coupled minimally to a $U(1)$ gauge potential yields the GUP corrected Hamiltonian after a Legendre transformation. The final result is [Eq.(119)]

$$
\begin{aligned}
H &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + \frac{\beta}{m}\left(\vec{p} - e\vec{A}\right)^4 \\
&= H_0 + 4\beta m H_0^2
\end{aligned}
\qquad (29)
$$

where in the last step we have inverted Eq.(24) to write $(\vec{p} - e\vec{A})$ in terms of $H_0$. Evidently, the eigenfunctions remain unchanged. This alone guarantees, for example, that the GUP will have no effect at all on phenomena such as the Quantum Hall Effect [11], the Bohm-Aharonov effect [12], and Dirac Quantization [13]. However, the eigenvalues shift by

$$
\begin{aligned}
\Delta E_{n(GUP)} &= 4\beta m\langle\phi_n|H_0^2|\phi_n\rangle \\
&= 4\beta m(\hbar\omega_c)^2\left(n + \frac{1}{2}\right)^2 , \quad (30)
\end{aligned}
$$

$$\text{or}\ \ \frac{\Delta E_{n(GUP)}}{E_n} = 4\beta m(\hbar\omega_c)\left(n + \frac{1}{2}\right) \qquad (31)$$

$$\approx \beta_0\frac{m}{M_{Pl}}\frac{\hbar\omega_c}{M_{Pl}c^2} \ . \qquad (32)$$

For an electron in a magnetic field of 10 $T$, $\omega_c \approx 10^3\ GHz$ and we get

$$
\begin{aligned}
\frac{\Delta E_{n(GUP)}}{E_n} &\approx \left(0.42 \times 10^{-22}\right) \times \left(5.48 \times 10^{-32}\right)\beta_0 \\
&= 2.30 \times 10^{-54}\beta_0 \ .
\end{aligned}
\qquad (33)
$$

Thus, Quantum Gravity/GUP does affect the Landau levels. However, once again, assuming $\beta_0 \sim 1$ renders the correction too small to be measured. Without this assumption, an accuracy of 1 part in $10^3$ in direct measurements of Landau levels using a Scanning Tunnel Microscope (STM) (which is somewhat optimistic) [14], the upper bound on $\beta_0$ follows

$$\beta_0 < 10^{50} \ . \qquad (34)$$

This bound is far weaker than that set by electroweak measurements, but compatible with the latter (as was the case for the Lamb shift). Once again, it is expected that the above accuracy will increase significantly with time, predicting a tighter bound on $\beta_0$, as well as perhaps an intermediate length scale.

## IV. POTENTIAL STEP

Next, we study the one dimensional potential step given by

$$V(x) = V_0 \ \theta(x) \ , \tag{35}$$

where $\theta(x)$ is the usual step function. Assuming $E > V_0$, the Schrödinger equation to the left and right of the barrier are given respectively by

$$d^2\psi_< + k^2\psi_< - \ell_{Pl}^2 d^4\psi_< = 0 \ , \ x \leq 0 \tag{36}$$
$$d^2\psi_> + k_1^2\psi_> - \ell_{Pl}^2 d^4\psi_> = 0 \ , \ x \geq 0 \tag{37}$$
$$k = \sqrt{2mE/\hbar^2} \ , k_1 = \sqrt{2m(E - V_0)/\hbar^2} \ , \tag{38}$$

where $d^n \equiv d^n/dx^n$. Assuming solutions of the form $\psi_{<,>} = e^{mx}$, we get

$$m^2 + k^2 - \ell_{Pl}^2 m^4 = 0 \ , \ x \leq 0 \tag{39}$$
$$m^2 + k_1^2 - \ell_{Pl}^2 m^4 = 0 \ , \ x \geq 0 \tag{40}$$

with the following solution sets to leading order in $\beta$, each consisting of 4 values of $m$

$$x \leq 0 : m = \{\pm ik', \pm 1/\ell_{Pl}\} \ , k' \equiv k(1 - \beta\hbar^2 k^2) \tag{41}$$
$$x \geq 0 : m = \{\pm ik_1', \pm 1/\ell_{Pl}\} \ , k_1' \equiv k_1(1 - \beta\hbar^2 k_1^2) \tag{42}$$

and the wavefunctions

$$\psi_< = Ae^{ik'x} + Be^{-ik'x} + A_1 e^{x/\ell_{Pl}} \ , \ x \leq 0 \tag{43}$$
$$\psi_> = Ce^{ik_1'x} + D_1 e^{-x/\ell_{Pl}} \ , \ x \geq 0 \tag{44}$$

where we have omitted the left-mover from $\psi_>$ and the exponentially growing terms from both $\psi_<$ and $\psi_>$. Note that the $\ell_{Pl}$-dependent decaying terms are a result of the GUP induced fourth order Schrödinger equation. They are independent of both $E$ and $V_0$, and appear to be *non-perturbative* in nature. Now the boundary conditions at $x = 0$ consist of 4 equations (instead of the usual 2)

$$d^n\psi_<|_0 = d^n\psi_>|_0 \ , \ n = 0, 1, 2, 3 \ , \tag{45}$$

giving rise to the following

$$A + B + A_1 = C + D_1 \tag{46}$$
$$ik'(A - B) + \frac{A_1}{\ell_{Pl}} = ik_1'C - \frac{D_1}{\ell_{Pl}} \tag{47}$$
$$-k'^2(A + B) + \frac{A_1}{\ell_{Pl}^2} = -k_1'^2 C + \frac{D_1}{\ell_{Pl}^2} \tag{48}$$
$$-ik'^3(A - B) + \frac{A_1}{\ell_{Pl}^3} = -ik_1'^3 C - \frac{D_1}{\ell_{Pl}^3} \ . \tag{49}$$

The above equations have the following solutions to leading order in $\beta$

$$\frac{B}{A} = \frac{k' - k_1'}{k' + k_1'} \tag{50}$$
$$\frac{C}{A} = \frac{2k'}{k' + k_1'} \tag{51}$$
$$\frac{A_1}{A} = k'(k' - k_1')\ell_{Pl}^2 \tag{52}$$
$$\frac{D_1}{A} = -k'(k' - k_1')\ell_{Pl}^2 \ . \tag{53}$$

Note that $A_1$ and $D_1$ are of the order $\ell_{Pl}^2 \ (\sim \beta)$, and that they vanish for $V_0 = 0$ (when $k' = k_1'$). In other words, the decaying terms are absent for the free particle. Computing the conserved current using (13), we get

$$J_< = k' \left[|A|^2 - |B|^2\right] \tag{54}$$
$$J_> = k_1'|C|^2 \ . \tag{55}$$

Naturally, the reflection and transmission coefficients are defined as

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k' - k_1'}{k' + k_1'}\right)^2 \tag{56}$$
$$= \left(\frac{k - k_1}{k + k_1}\right)^2 \left(1 - 4\beta\hbar^2 kk_1\right) \tag{57}$$
$$T = \frac{k_1'}{k'}\left|\frac{C}{A}\right|^2 = \left(\frac{2k'}{k' + k_1'}\right)^2 \tag{58}$$
$$= \frac{4kk_1}{(k + k_1)^2}\left(1 + \beta\hbar^2(k - k_1)^2\right) \tag{59}$$
$$R + T = 1 \ , \tag{60}$$

Note that the GUP affects both $R$ and $T$. In deriving Eqs.(57) and (59), we have used Eqs.(41) and (42) to leading order in $\beta$. Also, the conservation equation (60) would not hold if we had not included the exponential solutions in Eqs.(43-44).

## V. POTENTIAL BARRIER

A potential barrier of height $V_0$ from $x = 0$ and $x = a$ in Eq.(7) is given by

$$V(x) = V_0 \ [\theta(x) - \theta(x - a)] \ , \tag{61}$$

where $\theta(x)$ is the usual step function. In this case, we assume $E < V_0$. The Schrödinger equation in the three regions (which, henceforth, are denoted for brevity $R_1$, $R_2$, and $R_3$ for $x \leq 0$, $0 \leq x \leq a$, and $x \geq a$, respectively,) are given respectively by

$$d^2\psi_< + k^2\psi_< - \ell_{Pl}^2 d^4\psi_< = 0 \ , \ \text{in } R_1 \tag{62}$$
$$d^2\psi_> + k_1^2\psi_> - \ell_{Pl}^2 d^4\psi_> = 0 \ , \ \text{in } R_2 \tag{63}$$
$$d^2\psi_{>>} + k^2\psi_{>>} - \ell_{Pl}^2 d^4\psi_{>>} = 0 \ , \ \text{in } R_3 \tag{64}$$
$$k = \sqrt{2mE/\hbar^2} \ , \ k_1 = \sqrt{2m(V_0 - E)/\hbar^2} \ , \tag{65}$$

where $d^n \equiv d^n/dx^n$. Assuming solutions of the form $\psi_{<,>,>>} = e^{mx}$, we get

$$m^2 + k^2 - \ell_{Pl}^2 m^4 = 0 \ , \ \text{in } R_1 \tag{66}$$
$$m^2 + k_1^2 - \ell_{Pl}^2 m^4 = 0 \ , \ \text{in } R_2 \tag{67}$$
$$m^2 + k^2 - \ell_{Pl}^2 m^4 = 0 \ , \ \text{in } R_3 \tag{68}$$

with the following solution sets to leading order in $\beta$ and $\ell_{Pl}$, each consisting of 4 values of $m$

$$R_1 : m = \{\pm ik', \pm 1/\ell_{Pl}\} \ , k' \equiv k(1 - \beta\hbar^2 k^2) \ , \tag{69}$$
$$R_2 : m = \{\pm ik_1', \pm 1/\ell_{Pl}\} \ , k_1' \equiv k_1(1 - \beta\hbar^2 k_1^2) \ , \tag{70}$$
$$R_3 : m = \{\pm ik', \pm 1/\ell_{Pl}\} \ , k' \equiv k(1 - \beta\hbar^2 k^2) \ , \tag{71}$$

and the wavefunctions in $R_1$, $R_2$, and $R_3$, respectively, are

$$\psi_< = Ae^{ik'x} + Be^{-ik'x} + A_1 e^{x/\ell_{Pl}} , \tag{72}$$

$$\psi_> = Fe^{k_1'x} + Ge^{-k_1'x} + H_1 e^{x/\ell_{Pl}} + L_1 e^{-x/\ell_{Pl}}, \tag{73}$$

$$\psi_{>>} = Ce^{ik_1'x} + D_1 e^{-x/\ell_{Pl}} , \tag{74}$$

where we have omitted the left-mover from $\psi_{>>}$ and the exponentially growing terms from both $\psi_<$ and $\psi_{>>}$. Note once again the $\ell_{Pl}$-dependent decaying terms. Now the boundary conditions consist of 8 equations, 4 each from $x = 0$ and $x = a$

$$d^n \psi_<|_0 = d^n \psi_>|_0 \ , \ n = 0, 1, 2, 3 , \tag{75}$$

$$d^n \psi_>|_a = d^n \psi_{>>}|_a \ , \ n = 0, 1, 2, 3 , \tag{76}$$

giving rise to the following

$$A + B + A_1 = F + G + H + L_1 \tag{77}$$

$$ik'(A - B) + \frac{A_1}{\ell_{Pl}} = k_1'(F - G) + \frac{H_1 - L_1}{\ell_{Pl}} \tag{78}$$

$$-k'^2(A + B) + \frac{A_1}{\ell_{Pl}^2} = k_1'^2(F + G) + \frac{H_1 + L_1}{\ell_{Pl}^2} \tag{79}$$

$$-ik'^3(A - B) + \frac{A_1}{\ell_{Pl}^3} = k_1'^3(F - G) + \frac{H_1 - L_1}{\ell_{Pl}^3} \tag{80}$$

$$Fe^{k_1'a} + Ge^{-k_1'a} + H_1 e^{a/\ell_{Pl}} + L_1 e^{-a/\ell_{Pl}}$$
$$= Ce^{ik'a} + D_1 e^{-a/\ell_{Pl}} \tag{81}$$

$$k_1'\left(Fe^{k_1'a} - Ge^{-k_1'a}\right) + \frac{H_1}{\ell_{Pl}} e^{a/\ell_{Pl}} - \frac{L_1}{\ell_{Pl}} e^{-a/\ell_{Pl}}$$
$$= ik'Ce^{ik'a} - \frac{D_1}{\ell_{Pl}} e^{-a/\ell_{Pl}} \tag{82}$$

$$k_1'^2\left(Fe^{k_1'a} + Ge^{-k_1'a}\right) + \frac{H_1}{\ell_{Pl}^2} e^{a/\ell_{Pl}} + \frac{L_1}{\ell_{Pl}^2} e^{-a/\ell_{Pl}}$$
$$= -k'^2 Ce^{ik'a} + \frac{D_1}{\ell_{Pl}^2} e^{-a/\ell_{Pl}} \tag{83}$$

$$k_1'^3\left(Fe^{k_1'a} - Ge^{-k_1'a}\right) + \frac{H_1}{\ell_{Pl}^3} e^{a/\ell_{Pl}} - \frac{L_1}{\ell_{Pl}^3} e^{-a/\ell_{Pl}}$$
$$= -ik'^3 Ce^{ik'a} - \frac{D_1}{\ell_{Pl}^3} e^{-a/\ell_{Pl}}. \tag{84}$$

These have the solutions to leading order in $\beta$

$$\frac{B}{A} = \frac{(k'^2 + k_1'^2)(e^{2k_1'a} - 1)}{e^{2k_1'a}(k' + ik_1')^2 - (k' - ik_1')^2} \tag{85}$$

$$\frac{C}{A} = \frac{4ik'k_1' e^{-ik'a} e^{k_1'a}}{e^{2k_1'a}(k' + ik_1')^2 - (k' - ik_1')^2} \tag{86}$$

$$\frac{F}{A} = \frac{-2k'(k' - ik_1')}{e^{2k_1'a}(k' + ik_1')^2 - (k' - ik_1')^2} \tag{87}$$

$$\frac{G}{A} = \frac{2e^{2k_1'a} k'(k' + ik_1')}{e^{2k_1'a}(k' + ik_1')^2 - (k' - ik_1')^2} . \tag{88}$$

Computing the conserved current as given in (13), we get

$$J_< = k' \left[|A|^2 - |B|^2\right] \tag{89}$$

$$J_{>>} = k'|C|^2 . \tag{90}$$

Thus, the reflection and transmission coefficients are given by

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{(k'^2 + k_1'^2)(e^{2k_1'a} - 1)}{e^{2k_1'a}(k' + ik_1')^2 - (k' - ik_1')^2}\right|^2 \tag{91}$$

$$= \left[1 + \frac{(2k'k_1')^2}{(k'^2 + k_1'^2)^2 \sinh^2(k_1'a)}\right]^{-1} \tag{92}$$

$$T = \left|\frac{C}{A}\right|^2 = \left|\frac{4ik'k_1' e^{-ik'a} e^{k_1'a}}{e^{2k_1'a}(k' + ik_1')^2 - (k' - ik_1')^2}\right|^2 \tag{93}$$

$$= \left[1 + \frac{(k'^2 + k_1'^2)^2 \sinh^2(k_1'a)}{(2k'k_1')^2}\right]^{-1} \tag{94}$$

$$R + T = 1 . \tag{95}$$

Note that the GUP affects both $R$ and $T$. Once again, the conservation equation (95) would not hold if we had not included the exponential solutions in Eqs(72-74).

From Eq.(94) above, and using the definitions of $k, k_1, k', k_1'$, it can be shown that when $k_1 a \gg 1$, the transmission coefficient is approximately

$$T = T_0 \left[1 + \frac{4m\beta(2E - V_0)^2}{V_0} + \frac{2\beta a}{\hbar} [2m(V_0 - E)]^{3/2}\right] , \tag{96}$$

$$T_0 = \frac{16E(V_0 - E)}{V_0^2} e^{-2k_1 a} , \tag{97}$$

$T_0$ being the 'usual' tunnelling amplitude. Now $T$ is proportional to the current $I$ flowing between the tip and a sample in a Scanning Tunnel Microscope (STM). The current is usually amplified using an amplifier of gain $\mathcal{G}$. Thus, the enhancement in current due to GUP is given by

$$\frac{\delta I}{I_0} = \frac{\delta T}{T_0} = 4\beta_0 \frac{m(2E - V_0)^2}{V_0} + \frac{2\beta a}{\hbar} [2m(V_0 - E)]^{3/2}$$
$$= \frac{4\beta_0 m}{M_{Pl}} \frac{(2E - V_0)^2/V_0}{E_{Pl}}$$
$$+ 4\sqrt{2}\beta_0 \frac{a}{\ell_{Pl}} \left(\frac{m}{M_{Pl}}\right)^{3/2} \left(\frac{V_0 - E}{E_{Pl}}\right)^{3/2} . \tag{98}$$

Then, assuming the following approximate (but realistic) values [15]

$$m = m_e = 0.5 \ MeV/c^2 \tag{99}$$

$$E, V_0 = 10 \ eV \tag{100}$$

$$a = 10^{-10} \ m \tag{101}$$

$$I = 10^{-9} \ A \tag{102}$$

$$\mathcal{G} = 10^9 \tag{103}$$

we get

$$k_1 = 10^{10} \ m^{-1} \tag{104}$$

$$T_0 = e^{-1} \tag{105}$$

$$\frac{\delta I}{I_0} = \frac{\delta T}{T_0} = 10^{-48}\beta_0 \tag{106}$$

$$\delta\mathcal{I} \equiv \mathcal{G}\delta I = 10^{-48}\beta_0 \ A \ . \tag{107}$$

Thus, due to the excess current $\delta\mathcal{I}$ added up to the charge of just one electron, $e = 10^{-19} \ C$, one would have to wait for a time

$$\tau = \frac{e}{\delta\mathcal{I}} = 10^{29}\beta_0^{-1} \ s \ . \tag{108}$$

If $\beta_0 \approx 1$, this is much bigger than the age of our universe ($10^{18} \ s$)! However, if the quantity $\delta\mathcal{I}$ can be increased by a factor of about $10^{21}$, say by a combination of increase in $I$ and $\mathcal{G}$, and by a larger value of $\beta_0$, the above time will be reduced to about a year ($\approx 10^8 \ s$), and one can hope that the effect of GUP can be measured.

Conversely, if such a GUP induced current cannot be measured in such a time-scale, it will put an upper bound

$$\beta_0 < 10^{21} \ . \tag{109}$$

Note that this is more stringent than the two previous examples, and is in fact consistent with that set by the electroweak scale! In practice however, it may be easier to experimentally determine the *apparent barrier height* $\Phi_A \equiv V_0 - E$, and the (logarithmic) rate of increase of current with the gap. From Eq.(96) they are related by [15]

$$\sqrt{\Phi_A} = \frac{\hbar}{\sqrt{8m}} \left| \frac{d\ln I}{da} \right| \left( 1 - \frac{\beta\hbar^2}{4} \left| \frac{d\ln I}{da} \right|^2 \right) \ . \tag{110}$$

The cubic deviation from the linear $\sqrt{\Phi_A}$ vs $\left| \frac{d\ln I}{da} \right|$ curve predicted by GUP may be easier to spot and the value of $\beta$ estimated with improved accuracies.

## VI.  DISCUSSION

The above analysis, especially Eqs.(23), (34) and (109) indicate that a much larger coefficient of the additional term in the GUP (than previously thought) is not ruled out by current observations. These translate to intermediate length scales $\ell_{inter} \sim 10^{18}\ell_{Pl}, 10^{25}\ell_{Pl}$ and $10^{10}\ell_{Pl}$ respectively, of which the first two are far bigger than the electroweak scale, and the last, although smaller, may get further constrained with increased accuracies. In any case, more accurate measurements of the quantum phenomena studied here, or others, are required to tighten the above bounds. Then one might be able to see whether a true intermediate length scale emerges. It is not inconceivable that such a new length scale may show up in future experiments in the LHC. On the other hand, it is quite possible that $\beta_0 \sim 1$, the effects of GUP on low energy phenomena are negligible, and there is no intermediate length scale, supporting a recent argument [16].

Perhaps more importantly, our study reveals the universality of GUP effects, meaning that the latter can potentially be tested in a wide class of quantum mechanical systems, in which they maybe more pronounced. Possibilities include statistical mechanical systems (where a large number of particles may help in the enhancement), study of normally forbidden quantum processes to see if the GUP allows them, systems which may be affected by a fractional power of $\beta$, and GUP effects in cosmology. Any signature of testable predictions in one or more of the above (or perhaps others) could open a much needed low-energy 'window' to Quantum Gravity Phenomenology.

## Appendix A: Minimal Coupling to Electromagnetism

For the Hamiltonian (7) the equations of motion are

$$\dot{x}_0 = \frac{\partial H}{\partial p} = \frac{p_0}{m} + \frac{4\beta}{m}p_0{}^3 \tag{111}$$

its inverse

$$p_0 = m\dot{x}_0 - 4\beta(m\dot{x}_0)^3 \tag{112}$$

$$\text{and} \ \ \dot{p}_0 = -\frac{\partial H}{\partial x_0} = -\frac{\partial V}{\partial x_0} \ . \tag{113}$$

From Eqs.(111) and (112), we get the modified Newton's law

$$m\ddot{x}_0 \left( 1 - 12\beta m^2\dot{x}_0^2 \right) = -\frac{\partial V}{\partial x_0} \ , \tag{114}$$

which is also the Euler-Lagrangian equation for the following GUP modified Lagrangian

$$L = p_0\dot{x}_0 - H = \frac{1}{2}m\dot{x}_0^2 - V(x_0) - \frac{\beta}{m}(m\dot{x}_0)^4 \ . \tag{115}$$

Following the procedure outlined in [17], Section 16, we couple the above non-relativistic one-dimensional Lagrangian to a $U(1)$ gauge potential $A^\mu = (\phi, A)$ (relativistic and higher dimensional generalizations are straightforward)

$$L = \frac{1}{2}m\dot{x}_0^2 - V(x) - \frac{\beta}{m}(m\dot{x}_0)^4 + e\left( A\dot{x}_0 - \phi \right) \ . \tag{116}$$

The generalizations of (112) and (7) are

$$p_0 = m\dot{x}_0 + eA - 4\beta m^3\dot{x}_0^3 \tag{117}$$

$$H = \frac{1}{2}m\dot{x}_0^2 + e\phi - 3\beta m^3\dot{x}_0^4 \ . \tag{118}$$

Eliminating $\dot{x}_0$ in favor of $p$, it follows that

$$H = e\phi + \frac{1}{2m}\left(p_0 - eA\right)^2 + \frac{\beta}{m}\left(p_0 - eA\right)^4 \quad (119)$$

$$\equiv H_0 + H_1 . \quad (120)$$

Note that for $\phi = 0$, the eigenfunctions of $H_0$ and $H$ are identical, albeit with different eigenvalues. We have used this fact in the section on Landau levels. Also, the above form ensures that the classical gauge invariance of the action translates into the multiplication by a phase of the quantum wavefunction, under a gauge transformation.

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