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Erdős-Ko-Rado theorem for $\{0, \pm 1\}$-vectors

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Abstract

The main object of this paper is to determine the maximum number of $\{0, \pm 1\}$-vectors subject to the following condition. All vectors have length n, exactly k of the coordinates are $+1$ and one is -1, $n \geq 2k$. Moreover, there are no two vectors whose scalar product equals the possible minimum, -2. Thus, this problem may be seen as an extension of the classical Erdős-Ko-Rado theorem. Rather surprisingly there is a phase transition in the behaviour of the maximum at $n = k^2$. Nevertheless, our solution is complete. The main tools are from extremal set theory and some of them might be of independent interest.

1 Introduction

Let $[n] = \{1, \ldots, n\}$. Then any subset of the power set $2^{[n]}$ is called a family of subsets, or family for short. Another way of looking at families is by associating with a set $R \subset [n]$ its characteristic vector $v(F) = (x_1, \ldots, x_n)$ with $x_i = 1$ for $i \in F$ and $x_i = 0$ for $i \notin F$.

This association of a family $\mathcal{F}$ with a family of vectors $\mathcal{V} = \{v(F) : F \in \mathcal{F}\}$ provides a fruitful connection between some geometric problems concerning $\mathbb{R}^n$ and families of subsets with restrictions on sizes of pairwise intersections. The by now classic result of Frankl and Wilson [4] is a good example. Kahn and Kalai [7] gave it a twist to deduce counterexamples to the famous Borsuk conjecture.

Raigorodskii [12] succeeded in improving the bounds in geometric applications by enlarging the scope of vectors from $\{0, 1\}$-vectors to $\{0, \pm 1\}$-vectors.

Extremal problems for $\{0, \pm 1\}$-vectors were considered before (cf. [1]), but no systematic investigation happened so far. The aim of the present paper is to consider extending the classical Erdős-Ko-Rado Theorem [2] to this setting. Before stating the main results, let us introduce some definitions.

Definition 1. *For $0 \leq l, k < n$ define $\mathcal{V}(n, k, l) \subset \mathbb{R}^n$ to be the set of all $\{0, \pm 1\}$-vectors having exactly k coordinates equal $+1$ and l coordinates equal -1.*

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We tacitly assume that $n \geq k + l$ and also $k \geq l$. Indeed, for $l > k$ one can replace a family $\mathcal{V}$ of vectors by $-\mathcal{V} = \{-v : v \in \mathcal{V}\}$ and have the number of $+1$'s prevail. Note that

$$|\mathcal{V}(n, k, l)| = \binom{n}{k}\binom{n-k}{l}. \tag{1}$$

With this notation families of k-sets are just subsets of $\mathcal{V}(n, k, 0)$.

For vectors v, w their *scalar product* is denoted by $\langle v, w\rangle$. If both v and w are $\{0, 1\}$-vectors, then their scalar product is non-negative with $\langle v(F), v(G)\rangle = 0$ iff $F \cap G = \emptyset$.

Definition 2. *A family $\mathcal{F} \subset 2^{[n]}$ is called intersecting, if $F \cap G \neq \emptyset$ holds for all $F, G \in \mathcal{F}$.*

For completeness let us state the Erdős-Ko-Rado Theorem.

Theorem 1 ([2])**.** *Suppose that $n \geq 2k > 0$ and $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{2}$$

For $n \geq 2k$ the minimum possible scalar product in $\mathcal{V}(n, k, l)$ is $-2l$. Two vectors achieve this iff the -1 coordinates in each of them match $+1$ coordinates in the other.

Definition 3. *A family $\mathcal{V}$ of vectors is called intersecting, if the scalar product of any two vectors in $\mathcal{V}$ exceeds the minimum of the scalar product in $\mathcal{V}(n, k, l)$.*

By analogy with the Erdős-Ko-Rado Theorem define

$$m(n, k, l) = \{\max |\mathcal{V}| : \mathcal{V} \subset \mathcal{V}(n, k, l), \mathcal{V} \text{ is intersecting}\}.$$

With this terminology the Erdős-Ko-Rado Theorem can be stated as

$$m(n, k, 0) = \binom{n-1}{k-1} \quad \text{for} \quad n \geq 2k.$$

The present paper is mostly devoted to the complete determination of $m(n, k, 1)$. The surprising fact is that the situation is very different from the case $l = 0$, the Erdős-Ko-Rado Theorem. Namely, for $n > k^2$ the Erdős-Ko-Rado-type construction is no longer optimal.

Definition 4. *For $n \geq 2k$ define $\mathcal{E}(n, k, 1) = \{(x_1, \ldots, x_n) \in \mathcal{V}(n, k, 1) : x_1 = 1\}$.*

Clearly, $\mathcal{E}(n, k, 1)$ is intersecting with

$$|\mathcal{E}(n, k, 1) = k\binom{n-1}{k}. \tag{3}$$

Theorem 2. *For $2k \leq n \leq k^2$*

$$m(n, k, 1) = k\binom{n-1}{k} \qquad \textit{holds.} \tag{4}$$

The proof of (4) is much simpler in the range $2k \le n \le 3k-1$. It can be done using Katona's Circle Method (cf. [9]). We are going to present this case in Section 5. However, for $n \ge 3k$ the proof of (4) is much harder and much more technical. It is postponed till Section 7.

What happens for $n > k^2$?

Suppose that $\mathcal{V} \subset \mathcal{V}(n,k,1)$ is intersecting. Adding an extra 0 in the $n+1$'st coordinate makes it possible to consider $\mathcal{V}$ as a subset of $\mathcal{V}(n+1,k,1)$. Adding the $\binom{n}{k}$ vectors that have -1 in the $n+1$'st coordinate produce an intersecting family $\mathcal{P}(\mathcal{V}) \subset \mathcal{V}(n+1,k,1)$. This implies the following inequality:

$$m(n+1,k,1) \ge m(n,k,1) + \binom{n}{k}. \tag{5}$$

As a counterpart for (5) we prove

Theorem 3. *For $n \ge k^2$ one has*

$$m(n+1,k,1) = m(n,k,1) + \binom{n}{k}. \tag{6}$$

As an immediate consequence we have the following.

Corollary 4. *For $n > k^2$ one has*

$$m(n,k,1) = m(k^2,k,1) + \binom{k^2}{k} + \binom{k^2+1}{k} + \ldots + \binom{n-1}{k}.$$

The paper is organized as follows. In the next section we present a brief summary of the paper, containing the ideas and the logic of the proofs. In Section 3 we discuss the constructions of intersecting families. In Section 4 we summarize all the necessary material from extremal set theory, in particular, concerning shadows, shifting and cross-intersecting families, and prove several auxiliary statements. In Section 5 we prove (4) in the case $2k \le n \le 3k-1$. In Section 6 we prove Theorem 3. In Section 7 we prove (4) in the case $3k \le n \le k^2$.

Throughout the paper we assume that $n \ge 2k$ and that $k > 1$.

2 Summary

First, for the whole proof we assume that the families are shifted, i.e., 1's normally appear before 0's and -1's, and 0's appear before -1's among the coordinate positions of vectors from our family. The precise definition of shifting and the proof of the fact that it preserves the property that the family is intersecting is discussed in Section 4.1.

We start with the sketch of the proof of Theorem 3. The goal is to show that the number of the vectors containing $+1$ or -1 on the last coordinate is at most $\binom{n}{k}$. These vectors fall into two groups: $\mathcal{A}$, with -1 at the end, and $\mathcal{B}$, with 1 at the end. We fix another coordinate $i < n+1$ and take the vectors from $\mathcal{A}$, denoted $\mathcal{A}(i)$, that have 1 on i-th position, and vectors from $\mathcal{B}$, denoted $\mathcal{B}(-i)$, that have -1 on i-th position.

For each such i we may treat the resulting families of vectors as two set families of $(k-1)$-sets, which are cross-intersecting. We remark that, due to shifting, $\mathcal{B}(-i) \subset \mathcal{A}(i)$ and, in particular, $\mathcal{B}(-i)$ is intersecting. Then we bound the expression $|\mathcal{A}(i)| + k|\mathcal{B}(-i)|$ from above using Theorem 9 from Section 4.4. Finally we average the result over all possible i, obtaining a desired bound on $|\mathcal{A}| + |\mathcal{B}|$. Note that the coefficient k is needed so that we get the expression $|\mathcal{A}| + |\mathcal{B}|$ after the averaging.

Next, we sketch the proof of Theorem 2. For $2k \le n \le 3k-1$ the theorem is obtained via a direct application of Katona's circle method. The general form of this method is discussed in Section 4.3. The only trick is to choose a good subset of vertices to which we can apply the method.

The case of $3k \le n \le k^2$ is the most difficult part of the proof. The idea is to apply again the argument with the averaging used for the proof of Theorem 3. However, in this case there is a major complication. If one takes a look at Theorem 9 with given parameters (the size of the ground set is $n-1$, since i and $n+1$ are not present, and the size of each set in $\mathcal{A}(i), \mathcal{B}(-i)$ is $k-1$), the maximum value of the expression $|\mathcal{A}(i)| + k|\mathcal{B}(-i)|$ is $(k+1)\binom{n-2}{k-2}$ and is attained in the case when both are the trivial intersecting families. However, if we look at the Erdős-Ko-Rado-type family, which is expected to be maximal in this range, and the corresponding sets $\mathcal{A}(i), \mathcal{B}(i)$, then they are indeed the trivial intersecting families for all i except for $i = 1$. In that case we have the other extreme: $\mathcal{B}(1)$ is empty and $\mathcal{A}(1) = \binom{[n-1]}{k-1}$. Therefore, if we apply the bound from Theorem 9 for all i blindly, then after the averaging we get a worse bound, with the difference of $(k+1)\binom{n-2}{k-2} - \binom{n-1}{k-1}$ from the size of the Erdős-Ko-Rado-type family.

The idea to circumvent it is as follows. If $\mathcal{B}(-1)$ is empty, then we are home. If not, then, due to the fact that the whole family is shifted, we may conclude that there is a relatively big set I, $I \subset [n]$, such that for all $i \in I$ there are many sets in $\mathcal{B}(-i)$ that do not contain element $\{1\}$. The precise statement is Proposition 24. The next step is to use Corollary 21 of a theorem due to the first author from Section 4.7, which roughly states that we can bound non-trivially the size of the intersecting family from above, provided that we know that it is far from trivial intersecting family, that is, if there are many sets that do not contain the element with the biggest degree. By non-trivial we mean a bound that is smaller than the size of the trivial intersecting family.

The result of the manipulations presented in the previous paragraph is a non-trivial bound on the size of each $\mathcal{B}(-i)$, $i \in I$, provided that $\mathcal{B}(-1)$ is non-empty. Finally, we bound the size of each $|\mathcal{A}(i)| + k|\mathcal{B}(-i)|$ using Theorem 18 from Section 4.6, which is a refined version of Theorem 9. Using this bound, we apply the same averaging as in Theorem 3 and show that in all cases the size of $|\mathcal{A}| + |\mathcal{B}|$ is at most the size of $|\mathcal{A}| + |\mathcal{B}|$ when $\mathcal{B}(-1)$ is empty. Speaking very roughly, the sets in $\mathcal{B}(-1)$ force the sets $\mathcal{B}(-i)$ to be small, and thus force the whole sum $|\mathcal{A}| + |\mathcal{B}|$ to be small. The large part of Section 7 is devoted to the calculations that ensure that it is indeed the case.

Theorem 18, which gives very fine-grained bounds on $|\mathcal{A}(i)| + k|\mathcal{B}(-i)|$ depending on the size of $\mathcal{B}(-i)$, is itself one of the complicated parts of the proof. First, we do a detailed analysis of the maximal cross-intersecting families using and refining Kruskal-Katona Theorem in Section 4.5. The results of this section may be interesting in their own right, as they provide a better understanding of the structure of cross-intersecting families. The language of truncated characteristic vectors, introduced in Section 4.5 seems to be very convenient. Lemmas 16, 17 allow us to reduce the wide array of different cross-intersecting families to a few, one of which is guaranteed to be maximum w.r.t. the expression we maximize. The proof of Theorem 18 itself is a more technical counterpart.

3 Comparing the constructions

To get some intuition for the problem, we start with the comparison of the constructions of intersecting families briefly discussed in the introduction.

The first intersecting family $\mathcal{E}(n,k,1)$ is the Erdős-Ko-Rado-type family, mentioned in the introduction, in which all the vectors have 1 on the first position. We have $e(n,k,1) := |\mathcal{E}(n,k,1)| = k\binom{n-1}{k}$. Note that we have $v(n,k,1) = |\mathcal{V}(n,k,1)| = (k+1)\binom{n}{k+1}$. Therefore, we have $e(n,k,1)/v(n,k,1) = k/n$.

The second family $\mathcal{P}(n,k,1)$ consists of all the vectors for which the last non-zero coordinate is -1. It is easy to see that this is indeed an intersecting family. We have $p(n,k,1) := |\mathcal{P}(n,k,1)| = \binom{n}{k+1}$. Therefore, we have $p(n,k,1)/v(n,k,1) = 1/(k+1)$.

Proposition 5. *The inequality* $e(n+1,k,1) - e(n,k,1) \geq p(n+1,k,1) - p(n,k,1)$ *holds iff* $n \leq k^2$. *We have equality iff* $n = k^2$.

Proof. The proof is a matter of simple calculations:

$$e(n+1,k,1) - e(n,k,1) = k\binom{n}{k} - k\binom{n-1}{k} = k\binom{n-1}{k-1},$$

$$p(n+1,k,1) - p(n,k,1) = \binom{n+1}{k+1} - \binom{n}{k+1} = \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$ □

The second construction of intersecting families allows for the following generalization, described in the introduction. Assume we are given an intersecting family $\mathcal{F} \subset \mathcal{V}(n,k,1)$. We can construct an intersecting family $\mathcal{P}(\mathcal{F}) \subset \mathcal{V}(n+1,k,1)$ in the following way:

$$\mathcal{P}(\mathcal{F}) = \{(\mathbf{v},0) : \mathbf{v} \in \mathcal{F}\} \bigcup \{\mathbf{w} = (w_1,\ldots,w_{n+1}) : \mathbf{w} \in V(n+1,k,1), w_{n+1} = -1\}.$$

Since $\mathcal{F}$ is intersecting, $\mathcal{P}(\mathcal{F})$ is intersecting as well. We have

$$|\mathcal{P}(\mathcal{F})| - |\mathcal{F}| = \binom{n}{k} = p(n+1,k,1) - p(n,k,1). \tag{7}$$

We denote by $\mathcal{P}^s(\mathcal{F})$ the result of s consecutive applications of operation $\mathcal{P}(*)$ to the family $\mathcal{F}$. This gives us the following composite construction of an intersecting family $\mathcal{C}(n,k,1) \subset \mathcal{V}(n,k,1)$.

$$\mathcal{C}(n,k,1) = \begin{cases} \mathcal{E}(n,k,1), & \text{if } n \le k^2; \\ \mathcal{P}^{n-k^2}\big(\mathcal{E}(n,k,1)\big), & \text{if } n > k^2. \end{cases} \tag{8}$$

We denote the cardinality of $\mathcal{C}(n,k,1)$ by $c(n,k,1)$. We have $c(n,k,1) = e(n,k,1)$ for $n \le k^2$ and, due to to (7), $c(n,k,1) = e(k^2,k) - p(k^2,k) + p(n,k)$ for $n > k^2$. By Proposition 5, $\mathcal{C}(n,k,1)$ is the biggest intersecting family among the ones discussed in this subsection. In what follows we prove that $\mathcal{C}(n,k,1)$ has maximum cardinality among intersecting families in $\mathcal{V}(n,k,1)$.

Remark. Due to the fact that equality is possible in Proposition 5, there is a slightly different intersecting family that has exactly the same cardinality as $\mathcal{C}(n,k,1)$. Its definition is almost the same, we only have to replace k^2 by k^2+1 in (8).

4 Auxiliaries from extremal set theory

In this section we present several auxiliary results and techniques that we'll use in the latter sections. Some of the results presented here are well-known, while the others appear to be new and may be of independent interest.

4.1 Shifting

We start with *shifting* (left compression). For a given pair of indices $i < j \in [n]$ and a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we define an (i,j)-shift $\mathbf{v}_{i,j}$ of $\mathbf{v}$ in the following way. If $v_i \ge v_j$, then $\mathbf{v}_{i,j} = \mathbf{v}$. If $v_i < v_j$, then $\mathbf{v}_{i,j} = (v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n)$, that is, it is obtained from $\mathbf{v}$ by interchanging its i-th and j-th coordinate.

Next, we define an (i,j)-shift $\mathcal{Q}_{i,j}$ of $\mathcal{Q}$ for a finite system of vectors $\mathcal{Q} \subset \mathbb{R}^n$. We take a vector $\mathbf{v} \in \mathcal{Q}$ and replace it with $\mathbf{v}_{i,j}$, if $\mathbf{v}_{i,j}$ is not already in $\mathcal{Q}$. If it is, then we leave $\mathbf{v}$ in the system. Formally,

$$\mathcal{Q}_{i,j} = \{\mathbf{v}_{i,j} : \mathbf{v} \in \mathcal{Q}\} \cup \{\mathbf{v} : \mathbf{v}, \mathbf{v}_{i,j} \in \mathcal{Q}\}.$$

We call a system $\mathcal{Q}$ *shifted*, if $\mathcal{Q} = \mathcal{Q}_{i,j}$ for all $i < j \in [n]$. Any system of vectors may be made shifted by means of a finite number of (i,j)-shifts. Here is the crucial lemma concerning shifting:

Lemma 6. *For any $\mathcal{Q} \subset \mathbb{R}^n$ and any $i < j \in [n]$ we have*

$$\min\{\langle \mathbf{v}, \mathbf{w}\rangle : \mathbf{v}, \mathbf{w} \in \mathcal{Q}\} \le \min\{\langle \mathbf{v}', \mathbf{w}'\rangle : \mathbf{v}', \mathbf{w}' \in \mathcal{Q}_{i,j}\}.$$

Proof. Take any two vectors $\mathbf{v} = (v_1, \dots, v_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathcal{Q}$. We denote by $\mathbf{v}', \mathbf{w}'$ the result of the (i, j)-shift in $\mathcal{Q}$ applied to $\mathbf{v}, \mathbf{w}$ (that is, for $\mathbf{v}$ we have $\mathbf{v}' = \mathbf{v}$ or $\mathbf{v}_{i,j}$, depending on whether $\mathbf{v}_{i,j}$ is in $\mathcal{Q}$ or not). If $\mathbf{v}' = \mathbf{v}$ and $\mathbf{w}' = \mathbf{w}$, then, obviously, $\langle \mathbf{v}', \mathbf{w}'\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$. Moreover, we have the same if both $\mathbf{v}' \ne \mathbf{v}$ and $\mathbf{w}' \ne \mathbf{w}$. Therefore, the only nontrivial case we need to consider is when $\mathbf{v}' \ne \mathbf{v}$ and $\mathbf{w}' = \mathbf{w}$.

The reasons for $\mathbf{v}'$ being different from $\mathbf{v}$ are unambiguous: $v_i < v_j$ and $\mathbf{v}_{i,j} \notin \mathcal{Q}$. For $\mathbf{w}'$, however, there are two possible reasons not to be shifted. The first one is that $w_i \ge w_j$ and, thus, $\mathbf{w} = \mathbf{w}_{i,j}$. Then

$$\langle \mathbf{v}', \mathbf{w}'\rangle - \langle \mathbf{v}, \mathbf{w}\rangle = v_i w_j + v_j w_i - v_i w_i - v_j w_j = (v_j - v_i)(w_i - w_j) \ge 0.$$

The second possible reason is that $w_i < w_j$, but $\mathbf{w}_{i,j} \in \mathcal{Q}$. Then

$$\langle \mathbf{v}', \mathbf{w}'\rangle = \langle \mathbf{v}_{i,j}, \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}_{i,j}\rangle.$$

The last scalar product is, in fact, between two vectors from $\mathcal{Q}$. Therefore, in all cases we have exhibited a pair of vectors from $\mathcal{Q}$, that have a scalar product smaller than or equal to $\langle \mathbf{v}', \mathbf{w}'\rangle$. □

Applied to our case, the lemma states that, given an intersecting family of vectors, we may replace it with a shifted family of vectors, and the shifted family is intersecting as well.

4.2 Shadows

Given a family $\mathcal{F} \subset \binom{[n]}{k}$, we define its shadow $\sigma(\mathcal{F}) \subset \binom{[n]}{k-1}$ as a family of all $(k-1)$-element sets that are contained in one of the sets from $\mathcal{F}$. More generally, if $l < k$, then the l-th shadow $\sigma^l(\mathcal{F})$ is the set of all $(k-l)$-element sets that are contained in of of $F \in \mathcal{F}$. The famous Kruskal-Katona theorem [8], [11] gives a sharp lower bound on the size of the shadow of $\mathcal{F}$ in terms of k and $|\mathcal{F}|$. We are going to discuss it in the forthcoming paragraphs.

However, we need an analogous relation for the set system and its shadow, but for sets of specific type. Fix a cyclic permutation π of $[n]$. Consider the set $\mathcal{U}(\pi, k) \subset \binom{[n]}{k}$ of n k-sets, each of which forms an interval in the permutation π. That is, they are composed of cyclically consecutive elements in the permutation.

Lemma 7. *For any set system $\mathcal{F} \subset \mathcal{U}(\pi, k)$ we have $|\sigma^l(\mathcal{F}) \cap \mathcal{U}(\pi, k-l)| \ge \min\{|\mathcal{F}| + l, n\}$.*

Proof. It is clearly sufficient to prove that $|\sigma(\mathcal{F}) \cap \mathcal{U}(\pi, k-1)| \ge \min\{|\mathcal{F}| + 1, n\}$. If $|\mathcal{F}| = |\mathcal{U}(\pi, k)| = n$, then $\sigma(\mathcal{F}) = \mathcal{U}(\pi, k-1)$ and the statement is obvious. Therefore, we may assume that $\mathcal{U}(\pi, k) \setminus \mathcal{F} \ne \emptyset$.

Split the family $\mathcal{F}$ into subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_s$, each of which form an "interval". That is, each $\mathcal{F}_i$ is a maximal sequence of different sets $F_i^1, \dots, F_i^d \in \mathcal{F}$, in which each pair of consecutive sets intersect in a $(k-1)$-element set. Clearly, this is a partition of $\mathcal{F}$ into equivalence classes. Moreover, sets from different subfamilies intersect in less than $k-1$ elements. Therefore, $\sigma(\mathcal{F}) = \bigsqcup_i \sigma(\mathcal{F}_i)$.

For each subfamily we have $|\sigma(\mathcal{F}_i) \cap \mathcal{U}(\pi, k-1)| = |\mathcal{F}_i| + 1$. This is due to the fact that each $F \in \mathcal{F}_i$ contains two $(k-1)$-element sets from $\mathcal{U}(\pi, k-1)$, while each $F' \in \sigma(\mathcal{F}_i) \cap \mathcal{U}(\pi, k-1)$ is contained in either one or two sets from $\mathcal{F}_i$, and there are exactly two sets that are contained in one set from $\mathcal{F}_i$. Informally speaking, these are the "left shadow" of F_i^1 and the "right shadow" of F_i^d. These two shadow sets are different, since $\mathcal{U}(\pi, k) \setminus \mathcal{F}_i \neq \emptyset$. Knowing the "degrees" of the sets, we get the desired equality by simple double counting.

Finally, putting the statements for different i together, we get that $|\sigma(\mathcal{F}) \cap \mathcal{U}(\pi, k-1)| \geq |\mathcal{F}| + s \geq |\mathcal{F}| + 1$. Repeating the argument l times yields the result. □

4.3 General form of Katona's circle method

For this subsection only we adopt the language of graph theory. Consider a graph $G = (V, E)$, which is vertex-transitive. That is, the group $Aut(G)$ of automorphisms acts transitively on V. For a given vertex $v \in V$ we denote by S_v the stabilizer of v in $Aut(G)$, which is a subgroup of all automorphisms of G that map v to itself. A basic observation in group theory states that the size of the stabilizer is the same for all the vertices of G. Indeed, if v, w are two vertices of G and $\sigma \in Aut(G)$ maps v to w, then $S_v = \sigma^{-1} S_w \sigma$ and, therefore, $|S_w| = |S_v|$. Moreover, $|S_v| = |S_{vw}|$, where S_{vw} is the set of elements of $Aut(G)$ that maps v into w. We have as well $|Aut(G)| = |G||S_v|$, where $|G|$ is the number of vertices in G.

We remind the reader that $\alpha(G)$ is the independence number of G, that is, the maximum number of vertices that are pairwise non-adjacent. The following lemma is a special case of Lemma 1 from [10].

Lemma 8 (Katona, [10]). *Let G be a vertex-transitive graph. Let $H \subset G$ be a subgraph of G. Then $\alpha(G) \leq \frac{\alpha(H)}{|H|}|G|$.*

Remark. We formulated the lemma for the independence number, since it meets our demands. However, an analogue of it may be formulated for some other graph characteristics.

Proof. For any $\sigma \in Aut(G)$ we denote an induced subgraph of G on the set of vertices $\sigma(V(H)) = \{\sigma(v) : v \in V(H)\}$ by $\sigma(H)$. The proof of the lemma goes by simple double counting. Before doing the crucial double counting step, we remark that the union over all $\sigma \in Aut(G)$ of $\sigma(V(H))$ covers each vertex exactly $|H||S_v|$ times. Take any independent set I in G.

$$|I| = \frac{1}{|H||S_v|} \sum_{\sigma \in Aut(G)} |I \cap \sigma(H)| \leq \frac{1}{|H||S_v|} \sum_{\sigma \in Aut(G)} \alpha(H) = \frac{|Aut(G)|}{|S_v|} \frac{\alpha(H)}{|H|} = \frac{\alpha(H)}{|H|}|G|. \quad (9)$$

□

There is a natural connection between this lemma and intersecting families, which goes via Kneser graphs. A *Kneser graph* $KG_{n,k}$ is graph which set of vertices is $\binom{[n]}{k}$, and two vertices are adjacent iff the corresponding sets are disjoint. By definition the value of $\alpha(KG_{n,k})$ is the size of a maximum intersecting family in $\binom{[n]}{k}$.

4.4 An inequality for cross-intersecting families of sets

In this subsection we prove a theorem about two families that we need for the proof of Theorem 3.

Theorem 9. *Let $n \geq 2k$, $c \geq 1$. Consider two families $\mathcal{A} \subset \binom{[n]}{k}$, $\mathcal{B} \subset \mathcal{A}$. Assume further that for any $B \in \mathcal{B}, A \in \mathcal{A}$ we have $B \cap A \neq \emptyset$. Then*

$$|\mathcal{A}| + c|\mathcal{B}| \leq \max\Big\{\binom{n}{k}, (c+1)\binom{n-1}{k-1}\Big\}. \tag{10}$$

Remark. Informally speaking, the theorem states that the sum is maximized in one of the two cases: either $\mathcal{B}$ is empty and we may take $\mathcal{A}$ to be $\binom{[n]}{k}$, or when $\mathcal{A} = \mathcal{B}$, and each of them is a trivial intersecting family, that is, in which all sets contain a fixed element.

Proof. The proof is an application of Katona's cyclic permutation method. The following proposition is the key step.

Proposition 10. *Fix a cyclic permutation π of $[n]$. Consider the set $\mathcal{U}(\pi, k) \subset \binom{[n]}{k}$ from Section 4.2. Consider two subfamilies $\mathcal{A}(\pi) = \mathcal{A} \cap \mathcal{U}(\pi, k)$ and $\mathcal{B}(\pi) = \mathcal{B} \cap \mathcal{U}(\pi, k)$. Denote $a = |\mathcal{A}(\pi)|, b = |\mathcal{B}(\pi)|$. Then*

$$a + cb \leq \max\{n, (c+1)k\}. \tag{11}$$

Proof. If the set $\mathcal{B}(\pi)$ is empty, then the statement is trivial, since $a \leq |\mathcal{U}(\pi, k)| = n$. Henceforth we assume that $|\mathcal{B}(\pi)| = s > 0$. We pass to the complements of the sets from $\mathcal{B}(\pi)$, considering the set $\overline{\mathcal{B}}(\pi) = \{\bar{B} : B \in \mathcal{B}(\pi)\}$. On the one hand, we know that for each $A \in \mathcal{A}(\pi)$ and $\bar{B} \in \overline{\mathcal{B}}(\pi)$ we have $A \not\subseteq \bar{B}$. In other words, $A \notin \sigma^{n-2k}\big(\overline{\mathcal{B}}(\pi)\big)$. On the other hand, by Lemma 7 we have $\big|\sigma^{n-2k}\big(\overline{\mathcal{B}}(\pi)\big)\big| \geq \min\{|\mathcal{B}(\pi)| + n - 2k, n\} = n - 2k + s$, for if $\big|\sigma^{n-2k}\big(\overline{\mathcal{B}}(\pi)\big)\big| = n$, then $\mathcal{A}(\pi)$ and, consequently, $\mathcal{B}(\pi)$ is forced to be empty.

Combining these two facts, we get $|\mathcal{A}(\pi)| \leq n - (n - 2k + s) = 2k - s$. From the following chain $s = |\mathcal{B}(\pi)| \leq |\mathcal{A}(\pi)| \leq 2k - s$ we conclude that $s \leq k$. Finally,

$$|\mathcal{A}(\pi)| + c|\mathcal{B}(\pi)| \leq 2k - s + cs = 2k + (c-1)s \leq (c+1)k.$$

□

Knowing Proposition 10, the rest of the proof of the lemma is a standard double-counting argument, which was, in fact, carried out in the proof of Lemma 8. We take the circle $U(\pi, k)$ as a subgraph H from Lemma 8. In parallel to (9), we get

$$|\mathcal{A}| + c|\mathcal{B}| \leq \max\{n, (c+1)k\}\frac{1}{n}\binom{n}{k} = \max\Big\{\binom{n}{k}, (c+1)\binom{n-1}{k-1}\Big\}.$$

□

4.5 Analysis of the Kruskal-Katona's Theorem

For $i \le j$ denote $[i,j] = \{i, i+1, \dots, j\}$. We introduce a lexicographical order $<$ on the sets from $\binom{[n]}{k}$ by setting $A < B$ iff either $B \subset A$ or the minimal element of $A \setminus B$ is less than the minimal element of $B \setminus A$. We note that, in particular, $A < A$.

For $0 \le m \le \binom{n}{k}$ let $\mathcal{L}(m,k)$ be the collection of m largest sets with respect to this order. In the proof we are going to use the famous Kruskal-Katona Theorem [8], [11] in the form due to Hilton [5]:

Theorem 11. *Suppose that $\mathcal{A} \subset \binom{[n]}{a}, \mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. Then the same holds for the families $\mathcal{L}(|\mathcal{A}|, a), \mathcal{L}(|\mathcal{B}|, b)$.*

We may demonstrate the power of Theorem 11 immediately, proving the following corollary of it and Theorem 9:

Corollary 12. *The statement of Theorem 9 holds even if we replace the condition $\mathcal{B} \subset \mathcal{A}$ by $\mathcal{B} \subset \binom{[n]}{k}, |\mathcal{B}| \le |\mathcal{A}|$.*

Proof. The proof is straightforward. One has to pass to the families $\mathcal{L}(|\mathcal{A}|, k), \mathcal{L}(|\mathcal{B}|, k)$. Then the condition $|\mathcal{B}| \le |\mathcal{A}|$ is equivalent to $\mathcal{L}(|\mathcal{B}|, k) \subset \mathcal{L}(|\mathcal{A}|, k)$. After we just have to apply Theorem 9 to the families $\mathcal{L}(|\mathcal{A}|, k), \mathcal{L}(|\mathcal{B}|, k)$. □

To avoid trivialities, for the whole section we assume that $a + b \le n$.

We say, that two sets S and T *intersect strongly*, if there exists a positive integer j satisfying: (1) $S \cap T \cap [j] = \{j\}$. (2) $S \cup T \supset [j]$. Let S be a finite s-element set and $t \ge s$ an integer. Define the set family $\mathcal{L}(S,t)$ by

$$\mathcal{L}(S,t) = \{T \in \binom{[n]}{t} : T < S\}$$

Proposition 13. *Let A and B be an a-element and a b-element set, respectively. Assume that $n \ge a+b$. Then $\mathcal{L}(A,a)$ and $\mathcal{L}(B,b)$ are cross-intersecting iff A and B intersect strongly.*

Proof. First suppose that $\mathcal{L}(A,a)$ and $\mathcal{L}(B,b)$ are cross-intersecting. Then A and B intersect. Let j be the smallest integer contained in both. If $A \cup B$ contains $[j]$, then both properties (1) and (2) from the definition of strong intersecting sets are satisfied. Otherwise, there exists $i < j$ such that $i \notin A \cap B$. since $n \ge a + b$, there exists an a-element set C satisfying $C \cap [i] = A \cap [i] \cup \{i\}$, which is *disjoint* with B. At the same time, $C < A$, contradicting the assumption that $\mathcal{L}(A,a)$ and $\mathcal{L}(B,b)$ are cross-intersecting.

Now suppose that A and B intersect strongly. Let $C < A$. We claim that C and B intersect strongly. If $C \cap [j] = A \cap [j]$, then it follows directly from (1) and (2). Otherwise, let i be the first element where they differ. Since $C < A$, $i \in C \backslash A, i < j$. Now both conditions (1) and (2) hold for C and B with $j = h$. Repeating the same argument for any D with $D < B$ gives that C and D intersect strongly. In particular, they intersect non-trivially. Consequently, $\mathcal{L}(A,a)$ and $\mathcal{L}(B,b)$ are cross-intersecting. □

Definition 1. *We say that* $\mathcal{A} \subset \binom{[n]}{a}$ *and* $\mathcal{B} \subset \binom{[n]}{b}$ *form a maximal cross-intersecting pair, if whenever* $\mathcal{A}' \subset \binom{[n]}{a}$ *and* $\mathcal{B}' \subset \binom{[n]}{b}$ *are cross-intersecting with* $\mathcal{A}' \supset \mathcal{A}$ *and* $\mathcal{B}' \supset \mathcal{B}$*, then necessarily* $\mathcal{A} = \mathcal{A}'$ *and* $\mathcal{B} = \mathcal{B}'$ *holds.*

Now we are in a position to prove the following strengthening of Theorem 11. We believe that this proposition is of independent interest. It was definitely of great use in proving Theorem 2.

Proposition 14. *Let* a *and* b *be positive integers,* $a + b \le n$*. Let* P *and* Q *be non-empty subsets of* $[n]$ *with* $|P| \le a$*,* $|Q| \le b$*. Suppose that* P *and* Q *intersect strongly in their last element. That is, there exists* j*, such that* $P \cap Q = \{j\}$ *and* $P \cup Q = [j]$*. Then* $\mathcal{L}(P, a)$ *and* $\mathcal{L}(Q, b)$ *form a maximal pair of cross-intersecting families.*

Inversely, if $\mathcal{L}(m, a)$ *and* $\mathcal{L}(r, b)$ *form a maximal pair of cross-intersecting families, then it is possible to find sets* P *and* Q *such that* $\mathcal{L}(m, a) = \mathcal{L}(P, a)$*,* $\mathcal{L}(r, b) = \mathcal{L}(Q, b)$ *and* P, Q *satisfy the above condition.*

Proof. If P and Q satisfy the condition then $\mathcal{L}(P, a)$ and $\mathcal{L}(Q, b)$ form a pair of cross-intersecting families by Proposition 13. We have to show that it is a maximal pair. Let A be the a-set, following $\mathcal{L}(P, a)$ in the lexicographical order. Then A is the set obtained from P by omitting the element $\{j\}$ and adjoining the interval $[j+1, p]$, where $p = a - |P| + 1$. Note that $A \cap Q = \emptyset$, therefore there are members of $\mathcal{L}(Q, b)$ containing Q, which are disjoint with A. The same argument applies to the set B, following the initial segment $\mathcal{L}(Q, b)$. Thus, the maximality of the pair $\mathcal{L}(P, a)$ and $\mathcal{L}(Q, b)$ is proved.

Next, let A, B be the last member of $\mathcal{L}(m, a)$ and $\mathcal{L}(r, b)$, respectively. Let j be the smallest element of $A \cap B$. Note that the cross-intersecting property of the two families implies that j is well-defined. Define $P = A \cap [j], Q = B \cap [j]$.

First we prove that $P \cup Q = [j]$. Suppose the contrary and let $i < j$ be an element that is not contained in $P \cup Q$. Then $P' = (P - \{j\}) \cup i$ precedes P in the lexicographical order and hence also $P' < A$. Consequently, all a-element supersets of P' precede A as well. Thus they are all members of $\mathcal{L}(m, a)$. However, $P' \cap B = \emptyset$ and, since $a + b \le n$ we have that some superset P' is disjoint to B, a contradiction.

The proof is almost complete. We have just proved that P and Q are strongly intersecting. By Proposition 13 the families $\mathcal{L}(P, a)$ and $\mathcal{L}(Q, b)$ form a pair of cross-intersecting families. These families contain $\mathcal{L}(m, a)$ and $\mathcal{L}(r, b)$, respectively. By the maximality of the pair $\mathcal{L}(m, a)$ and $\mathcal{L}(r, b)$, $\mathcal{L}(m, a) = \mathcal{L}(P, a), \mathcal{L}(r, b) = \mathcal{L}(Q, b)$ must hold. □

In what follows we will use the following simple statement:

Proposition 15. *Let* $\mathcal{A} \subset \binom{n}{a}$ *and* $\mathcal{B} \subset \binom{n}{b}$ *be cross-intersecting. Then we have* $|\mathcal{A}| + c|\mathcal{B}| \le \max\left\{\binom{n}{a}, c\binom{n}{b}\right\}$.

Proof. Consider the bipartite graph with one part being $\binom{n}{a}$ and the other part being $\binom{n}{b}$, with two sets connected by an edge if one is disjoint from the other. Assign weight 1 to each vertex in $\binom{n}{a}$ and weight c to any vertex in $\binom{n}{b}$. Then the proposition essentially states that the independent set of the biggest weight in this graph coincides with one of its two parts. It is an easy consequence of the fact that the graph is regular in each of its parts. □

Since we are interested in bounding $|\mathcal{L}(m,a)| + c|\mathcal{L}(r,b)|$ from above, we may w.l.o.g. restrict our attention only to maximal intersecting pairs of families, which, by Proposition 14, have the type $\mathcal{L}(P,a), \mathcal{L}(Q,b)$ for some $P, Q \subset [n]$. Assume that $P \cap Q = \{i\}$. Then we call $\mathbf{w} = (w_1, \ldots, w_i)$ a *truncated characteristic vector* of P and $\mathcal{L}(P,a)$, if $w_j = 1 \Leftrightarrow j \in P$. We define $\mathbf{v} = (v_1, \ldots, v_i)$ analogously w.r.t. Q. We note that $w_i = v_i = 1$. We would also say that the family $\mathcal{L}(P,a)$ is defined by $\mathbf{w}$. For a given set $R \subset [n]$ its *characteristic vector* $\mathbf{r} = (r_1, \ldots, r_n)$ is defined as follows: $r_j = 1$ iff $j \in R$.

This easy proposition gives a very useful consequence concerning the size of a maximal cross-intersecting pair of families $\mathcal{L}(m,a), \mathcal{L}(r,b)$.

Lemma 16. *Consider two maximal cross-intersecting families $\mathcal{A} = \mathcal{L}(P,a)$, $\mathcal{B} = \mathcal{L}(Q,b)$. Let $\mathbf{w} = (w_1, \ldots, w_i), \mathbf{v} = (v_1, \ldots, v_i)$ be the truncated characteristic vectors of P, Q, respectively. Take any $j < i$ such that $(v_j, v_{j+1}) = (1,0)$.*

Consider the following two pairs of cross-intersecting families. The first pair is $\mathcal{A}', \mathcal{B}'$, where $\mathcal{B}'$ is defined by the vector $\mathbf{v}' = (v_1, \ldots, v_j)$ and $\mathcal{A}'$ is defined by $\mathbf{w}' = (w_1, \ldots, w_{j-1}, 1)$. The second pair is $\mathcal{A}'', \mathcal{B}''$, where $\mathcal{B}''$ is defined by the vector $\mathbf{v}'' = (v_1, \ldots, v_j, 1)$ and $\mathcal{A}''$ is defined by $\mathbf{w}'' = (w_1, \ldots, w_j, 1)$.

Then for any $c > o$ we have

$$|\mathcal{A}| + c|\mathcal{B}| \leq \max\{|\mathcal{A}'| + c|\mathcal{B}'|, |\mathcal{A}''| + c|\mathcal{B}''|\}. \tag{12}$$

Before going into the proof of the lemma, we introduce the following notation. For disjoint $X, Y \subset [n]$ and a family $\mathcal{F} \subset \binom{n}{k}$ we define $\mathcal{F}(X\bar{Y}) = \{F - X : X \subset F \in \mathcal{F}, Y \cap F = \emptyset\}$. For a vector $\mathbf{v}$ we define $\mathcal{F}(\mathbf{v}) = \mathcal{F}(X\bar{Y})$, where X is the set of 1-coordinates of $\mathbf{v}$, and Y is the set of 0-coordinates.

Proof. We remark that, by definition, we have $\mathcal{B}'' \subset \mathcal{B} \subset \mathcal{B}'$ and $\mathcal{A}'' \supset \mathcal{A} \supset \mathcal{A}'$. It it easier for us to argue in terms of characteristic vectors rather than sets.

All the vectors $\mathbf{u}$ from $\mathcal{B}' - \mathcal{B}''$ have the first $j+1$ coordinates the same as $\mathbf{u} = (v_1, \ldots, v_j, 0)$. Analogously, all the vectors from $\mathcal{A}'' - \mathcal{A}'$ have the first $j+1$ coordinates the same as $\mathbf{p} = (w_1, \ldots, , w_j, 1)$. Therefore, to prove the lemma one has to show that

$$|\mathcal{A}(\mathbf{p})| + c|\mathcal{B}(\mathbf{u})| \leq \max\{|\mathcal{A}'' - \mathcal{A}'|, c|\mathcal{B}' - \mathcal{B}''|\}. \tag{13}$$

Since the scalar product of $\mathbf{u}$ and $\mathbf{p}$ is zero, the families $\mathcal{A}(\mathbf{p})$ and $\mathcal{B}(\mathbf{u})$ are cross-intersecting. Therefore, (13) follows immediately from Proposition 15 and we are done. □

Using almost identical proof, one may get the following twin of Lemma 16:

Lemma 17. *Consider two maximal cross-intersecting families* $\mathcal{A} = \mathcal{L}(P, a),\ \mathcal{B} = \mathcal{L}(Q, b)$. *Let* $\mathbf{w} = (w_1, \ldots, w_i), \mathbf{v} = (v_1, \ldots, v_i)$ *be the truncated characteristic vectors of* P, Q, *respectively. Take any* $j < i$ *such that* $(v_j, v_{j+1}) = (1, 1)$.

Consider the following cross-intersecting family. It is a pair $\mathcal{A}', \mathcal{B}'$, *where* $\mathcal{B}'$ *is defined by the vector* $\mathbf{v}' = (v_1, \ldots, v_j)$ *and* $\mathcal{A}'$ *is defined by* $\mathbf{w}' = (w_1, \ldots, w_{j-1}1)$.

Then for any $c > o$ *we have*

$$|\mathcal{A}| + c|\mathcal{B}| \le \max\left\{|\mathcal{A}'| + c|\mathcal{B}'|, \binom{n}{a}\right\}. \tag{14}$$

We remark that in the proof of Lemma 17 instead of $\mathcal{A}'', \mathcal{B}''$ from Lemma 16 we use $\mathcal{A}'' = \binom{n}{a}, \mathcal{B}'' = \emptyset$.

4.6 A sharpening of Theorem 9

In the proof of Theorem 23 we need a sharpened version of Theorem 9, which proof relies on the material from the previous subsection.

Theorem 18. *Let* $k, n \in \mathbb{N}$,*where* $k \ge 2$ *and* $n \ge 2k + 1$. *Consider two families* $\mathcal{A} \subset \binom{[n]}{k}$, $\mathcal{B} \subset \mathcal{A}$. *Assume further that for any* $B \in \mathcal{B}, A \in \mathcal{A}$ *we have* $B \cap A \ne \emptyset$. *Let* $2 \le i \le k$ *and* $|\mathcal{B}| \le \binom{n-1}{k-1} - \binom{n-i}{k-1}$. *Then for any* $c \ge 1$

$$|\mathcal{A}| + c|\mathcal{B}| \le \max\Big\{(c+1)\binom{n-1}{k-1} - c\binom{n-i}{k-1} + \binom{n-i}{k-i+1}, \\ (c+1)\binom{n-1}{k-1} - (c-1)\binom{n-2}{k-1}, \binom{n}{k}\Big\}. \tag{15}$$

Proof. If $\mathcal{B}$ is empty, then the statement is obvious, therefore, we assume the opposite for the rest of the proof. By Theorem 11, $\mathcal{L}_0 = \mathcal{L}(|\mathcal{A}|, k)$ and $\mathcal{L}_1 = \mathcal{L}(|\mathcal{B}|, k)$ are cross-intersecting. W.l.o.g. we may assume that this is a maximal cross-intersecting pair, and, thus, by Proposition 14, $\mathcal{L}_0 = \mathcal{L}(P, k)$ and $\mathcal{L}_1 = \mathcal{L}(Q, k)$ for some $P, Q, |P| \le k, |Q| \le k$. Consider the truncated characteristic vector $\mathbf{v} = (v_1, \ldots, v_l)$ of $\mathcal{L}_1$. We know for sure that $v_1 = 1$, since $\mathcal{B}$ is intersecting. We consider several cases depending on the form of $\mathbf{v}$.

(i) Assume that $v_1 = v_2 = v_3 = 1$. Then, applying Lemma 17 with $j = 2$, we get that $|\mathcal{L}_0| + c|\mathcal{L}_1| \le \max\Big\{|\mathcal{L}_0'| + c|\mathcal{L}_1'|, \binom{n}{k}\Big\}$, where $\mathcal{L}_1'$ is defined by the characteristic vector $(1, 1)$ and $\mathcal{L}_0'$ is defined by the vector $(0, 1)$. It is easy to see that

$$|\mathcal{L}_0'| + c|\mathcal{L}_1'| = \binom{n-1}{k-1} + \binom{n-2}{k-1} + c\binom{n-2}{k-2} = (c+1)\binom{n-1}{k-1} - (c-1)\binom{n-2}{k-1}.$$

(ii) Assume that $v_1 = v_2 = 1, v_3 = 0$. We first apply Lemma 16 with $j = 2$, obtaining two possible candidates for maximum cross-intersecting families. One of them is the pair $\mathcal{L}_0', \mathcal{L}_1'$. The other one is $\mathcal{L}_0'', \mathcal{L}_1''$, where $\mathcal{L}_1''$ is defined by the characteristic vector $(1, 1, 1)$. Therefore, for these two families we may apply **(i)** and we are done.

(iii) Assume that $v_2 = v_3 = \ldots = v_{j-1} = 0, v_j = 1$. We either apply Lemma 17 with j directly, or first apply Lemma 16, and then Lemma 17. In either case, we conclude that $|\mathcal{L}_0| + c|\mathcal{L}_1| \le \max\Big\{|\hat{\mathcal{L}}_0| + c|\hat{\mathcal{L}}_1|, \binom{n}{k}\Big\}$, where $\hat{\mathcal{L}}_0$ is defined by the characteristic vector $\mathbf{w} = (w_1, \ldots, w_j)$, where $w_1 = 0$ and $w_2 = \ldots = w_j = 1$ and $\hat{\mathcal{L}}_1$ is defined by the characteristic vector $\mathbf{u} = (u_1, \ldots, u_j)$, where $u_1 = u_j = 1, u_2 = \ldots = u_{j-1} = 0$.

Due to the inequality $|\mathcal{B}| \le \binom{n-1}{k-1} - \binom{n-i}{k-1}$ it is easy to see that $j \le i$. Calculating the cardinalities of $\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_0$, we get

$$|\mathcal{L}'_0| + c|\mathcal{L}'_1| = (c+1)\binom{n-1}{k-1} - c\binom{n-j}{k-1} + \binom{n-j}{k-j+1} =: f(j).$$

Indeed, $\hat{\mathcal{L}}_0$ consists of all the sets containing $\{1\}$, or $[2, j]$. The family $\hat{\mathcal{L}}_1$ contains all the sets containing $\{1\}$ and at least one of $[2, j]$. Therefore, the last thing we have to show is that for $j \ge 3$ $f(j)$ increases as j increases. We note that for $j = 2$ the value of $f(j)$ is exactly the second expression from the maximum in (15). We have

$$f(j+1) - f(j) = c\Big(\binom{n-j}{k-1} - \binom{n-j-1}{k-1}\Big) - \Big(\binom{n-j}{k-j+1} - \binom{n-j-1}{k-j}\Big) = \\ = c\binom{n-j-1}{k-2} - \binom{n-j-1}{k-j+1} \ge 0.$$

This is due to the fact that $c \ge 1$ and

$$\frac{\binom{n-j-1}{k-2}}{\binom{n-j-1}{k-j+1}} = \prod_{l=2}^{j-2} \frac{n-k-l}{k-l} \ge 1,$$

since $n > 2k$. This completes the proof of the theorem. □

For our purposes it would be more convenient to apply the following slightly modified version of the theorem:

Corollary 19. *In the conditions of Theorem 18 let* $2 \le i \le k$ *and* $|\mathcal{B}| \le \binom{n-1}{k-1} - \binom{n-i}{k-1} + x$ *for some natural* x*. Then*

$$|\mathcal{A}| + c|\mathcal{B}| \le \max\Big\{(c+1)\binom{n-1}{k-1} - c\binom{n-i}{k-1} + \binom{n-i}{k-i+1} + kx, \\ (c+1)\binom{n-1}{k-1} - (c-1)\binom{n-2}{k-1}, \binom{n}{k}\Big\}. \quad (16)$$

Proof. The proof is basically the same, the only thing one has to notice that, if $|\mathcal{B}|$ gets bigger, then $|\mathcal{A}|$ can only get smaller. Therefore, if $|\mathcal{B}| = \binom{n-1}{k-1} - \binom{n-i-1}{k-1} + y$, $1 \le y \le x$, then one just remove y elements from $|\mathcal{B}|$, applies the same proof, and puts the elements back, adding cy to the right hand side. Note that we only add cx to the first expression, since the other two appear in the proof only when $|\mathcal{B}|$ is relatively small. □

4.7 Intersecting families with conditions on maximum degree

For a given family $\mathcal{F}$ let $d(\mathcal{F}) = \max_{l\in[n]} |\{F \in \mathcal{F} : l \in F\}|$ be the maximal degree of an element of $[n]$ in $\mathcal{F}$. The other important ingredient of the proof of Theorem 2 is the following theorem due to Frankl [3]:

Theorem 20. *Suppose that $n > 2k$, $3 \le i \le k+1$, $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ is intersecting and*

$$d(\mathcal{F}) \le \binom{n-1}{k-1} - \binom{n-i}{k-1} =: h(i).$$

Then

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-i}{k-1} + \binom{n-i}{k-i+1} =: g(i).$$

It is not difficult to see that $g(3) = g(4)$ and $g(i) < g(j)$ if $4 \le i < j$. We are interested in the following corollary of this theorem:

Corollary 21. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. If for some $4 \le i \le k+1$*

$$|\mathcal{F}| - d(\mathcal{F}) \ge \binom{n-i}{k-i+1}, \tag{17}$$

Then $|\mathcal{F}| \le g(i)$ holds. In particular, if $|\mathcal{F}| - d(\mathcal{F}) > 0$, then $|\mathcal{F}| \le g(k+1)$ holds.

Proof. W.l.o.g. assume that $d(\mathcal{F}) = |\mathcal{F}(1)|$. Consider the families $\mathcal{F}(\bar{1})$, $\mathcal{F}(1)$. They are cross-intersecting and $|\mathcal{F}(\bar{1})| \ge \binom{n-i}{k-i+1}$. By Theorem 11 the families $\mathcal{L}(|\mathcal{F}(\bar{1})|, k)$ and $\mathcal{L}(|\mathcal{F}(1)|, k-1)$ are cross-intersecting. By (17) $\mathcal{L}(|\mathcal{F}(\bar{1})|, k)$ contains all the sets that contain $[2,k]$, therefore, each set from $\mathcal{L}(|\mathcal{F}(1)|, k-1)$ contains one of $[2,k]$ and, consequently, $d(\mathcal{F}) = |\mathcal{F}(1)| \le h(i)$. We apply Theorem 20 and get the desired conclusion. □

5 Proof of Theorem 2 in the case $2k \le n \le 3k-1$

In this section we show that $\mathcal{E}(n,k,1)$ has the maximum cardinality among intersecting families for $2k \le n \le 3k-1$.

The proof is based on the application of the general Katona's circular method directly for $\mathcal{V}(n,k,1)$. Consider the following subfamily $\mathcal{H}$ of $\mathcal{V}(n,k,1)$:

$$\mathcal{H} = \{\mathbf{v} = (v_1, \ldots, v_n) : \text{ for some } i \in [n] \ v_i = \ldots = v_{i+k-1} = 1, \ v_{i-k} = -1\}.$$

We remark that all indices are modulo n. That is, it is a usual Katona's circle for k-sets, but in which each k-set gets an extra -1-coordinate, which is at distance k from the 1-part along the circle.

Take an intersecting family $\mathcal{F} \subset \mathcal{V}(n,k,1)$. We claim that $|\mathcal{F} \cap \mathcal{H}| \le k$. Denote by $\mathcal{F}' \subset \binom{[n]}{k}$ the family of sets of 1's from $\mathcal{F}$, and similarly for $\mathcal{H}'$. We claim that $\mathcal{H}' \cap \mathcal{F}'$ is an intersecting family. Assume that there are two sets $F_1', F_2' \in \mathcal{H}' \cap \mathcal{F}'$, that are disjoint. Assume for simplicity that $F_2' = [k+1, 2k]$. Then F_1' is obliged to contain $\{1\}$, since any cyclic interval of length k in $[n] \setminus [k+1, 2k]$ contains $\{1\}$, provided that $n \le 3k-1$. Therefore, the corresponding vector $F_1 \in \mathcal{F} \cap \mathcal{H}$ has 1 on the first coordinate position. At the same time, by definition of $\mathcal{H}$, the vector F_2 has -1 on the first coordinate position. Interchanging the roles of F_1, F_2, we get that both of them have -1 in front of 1 of the other vector. Moreover, their sets of 1's do not intersect. Therefore, they form a minimal scalar product. Therefore, $\mathcal{H}' \cap \mathcal{F}'$ is indeed intersecting, and $|\mathcal{H} \cap \mathcal{F}| = |\mathcal{H}' \cap \mathcal{F}'| \le k$.

The rest of the argument is an application of Lemma 8 to the following graph. Let G have the set of vertices $\mathcal{V}(n,k,1)$, with two vertices connected if the corresponding vectors have scalar product -2. Then $|\mathcal{F}| \le \alpha(G)$ and the considerations from the previous paragraph give that $\alpha(G|_{\mathcal{H}}) = k)$. On the other hand, by Lemma 8 we have

$$\alpha(G) \le \frac{\alpha(G|_{\mathcal{H}})}{|\mathcal{H}|}|\mathcal{V}(n,k,1)| = \frac{k}{n}|\mathcal{V}(n,k,1)| = e(n,k,1).$$

The last equality was obtained in Section 3.

6 Proof of Theorem 3

For any intersecting family $\mathcal{G} \subset \mathcal{V}(n+1,k,1)$ we introduce the following notations. By $\mathcal{G}(i), \mathcal{G}(-i), \mathcal{G}(\bar{i})$ we denote the subfamilies of $\mathcal{G}$, that have $1, -1, 0$ as an i-th coordinate, respectively. It is important to mention that we consider them as families of vectors on the set of coordinates with the i-th coordinate excluded. The definition extends in an obvious way on the family $\mathcal{G}(I_1, \bar{I}_2, -I_3)$, where $I_1, I_2, I_3 \subset [n]$ are non-intersecting sets of indices.

Consider a maximum intersecting family $\mathcal{F} \subset \mathcal{V}(n+1,k,1)$. Based on the conclusion of Section 4.1, we may and will assume that $\mathcal{F}$ is shifted. Denote $\mathcal{A} = \mathcal{F}(-(n+1))$ and $\mathcal{B} = \mathcal{F}(n+1)$.

Proposition 22. *We have* $m(n+1,k,1) - m(n,k,1) \le |\mathcal{A}| + |\mathcal{B}|$.

Proof. First, by definition we have $|\mathcal{F}| = m(n+1,k,1)$. Second, consider a family $\mathcal{F}(\overline{n+1})$. It is a subfamily in $\mathcal{V}(n,k,1)$, moreover, it is intersecting. Therefore, $|\mathcal{F}'| \le m(n,k,1)$. Finally, $m(n+1,k,1) = |\mathcal{F}| = |\mathcal{F}(\overline{n+1})| + |\mathcal{A}| + |\mathcal{B}|$. ☐

For a given $i \in [n]$ consider two families of sets $\mathcal{B}(-i), \mathcal{A}(i)$. We have $\mathcal{B}(-i), \mathcal{A}(i) \subset \binom{[n]-\{i\}}{k-1}$. Moreover, $\mathcal{B}(-i) \subset \mathcal{A}(i)$ due to shifting, and we may apply Theorem 9 to these two families with $c = k$ and obtain

$$|\mathcal{A}(i)| + k|\mathcal{B}(-i)| \le \max\left\{\binom{n-1}{k-1}, (k+1)\binom{n-2}{k-2}\right\}.$$

Summing this inequality over all $i \in [n]$, we get

$$\sum_{i=1}^{n} |\mathcal{A}(i)| + k|\mathcal{B}(-i)| = k(|\mathcal{A}| + |\mathcal{B}|) \le n\max\left\{\binom{n-1}{k-1}, (k+1)\binom{n-2}{k-2}\right\}$$

$$|\mathcal{A}| + |\mathcal{B}| \le \max\Big\{ \binom{n}{k}, \frac{n(k+1)}{k} \binom{n-2}{k-2} \Big\}. \quad (18)$$

Maximum in the right hand side of (18) is attained on the first expression if

$$\binom{n}{k} = \frac{n(n-1)}{k(k-1)} \binom{n-2}{k-2} \ge \frac{n(k+1)}{k} \binom{n-2}{k-2},$$

which is equivalent to $n \ge k^2$.

Proof of Theorem 3. The bound $m(n+1,k,1) \le m(n,k,1) + \binom{n}{k}$ was, in fact, already proven in this section. In the notations above, consider $\mathcal{F}, \mathcal{A}, \mathcal{B}$. On the one hand, by (18) and the discussion after this inequality, we have $|\mathcal{A}| + |\mathcal{B}| \le \binom{n}{k}$ for $n \ge k^2$. Applying Proposition 22, we get the bound.

The bound $m(n+1,k,1) \ge m(n,k,1) + \binom{n}{k}$ was already obtained in Section 3 (and mentioned in the introduction). □

7 Proof of Theorem 2 in the case $3k \le n \le k^2$

Looking at equation (18) in the case $n < k^2$, we see that $m(n+1,k,1) - m(n,k,1) \le \frac{n(k+1)}{k}\binom{n-2}{k-2}$. On the other hand, from Section 3 we know, that $e(n+1,k) - e(n,k) = k\binom{n-1}{k-1} = \frac{(n-1)k}{k-1}\binom{n-2}{k-2}$. We have $\frac{n(k+1)}{k} - \frac{(n-1)k}{k-1} = \frac{k^2-n}{k(k-1)}$. Using Theorem 18 and Corollary 21, we are going to improve the inequality (18), so that it matches the bound given by the construction $\mathcal{E}(n,k,1)$. To complete the proof of Theorem 2, it is enough to prove the following theorem.

Theorem 23. *Let $k \ge 3$, $3k-1 \le n < k^2$. We have $m(n+1,k,1) - m(n,k,1) = \frac{(n-1)k}{k-1}\binom{n-2}{k-2} = e(n+1,k) - e(n,k)$.*

Before proving the theorem, we state and prove the following proposition:

Proposition 24. *There is a subset $I \subset [n]$, $|I| \ge n - \lceil 3k/2 \rceil$, such that for every $l \in I$ we have $|\mathcal{B}(\bar{1}, -l)| \ge \frac{1}{3}|\mathcal{B}(-1)|$.*

Proof. Take any $B \in \mathcal{B}(-1)$. Then, due to the fact that $\mathcal{F}$ is shifted, if we swap -1, which is on the first coordinate position, and some of the 0's in B, we obtain a set $B' \in \mathcal{B}(\bar{1})$. Since there are $n-k-1$ 0's in each $B \in \mathcal{B}(-1)$, we may obtain $n-k-1$ sets $B' \in \mathcal{B}(\bar{1})$ out of each B. Moreover, any two sets obtained are different. Indeed, for B any two vectors obtained out of it have different positions of -1, while any two vectors obtained from different $B_1, B_2 \in \mathcal{B}(-1)$ have different sets of 1's. Thus, $|\mathcal{B}(\bar{1}, -2)| + \ldots + |\mathcal{B}(\bar{1}, -n)| \ge (n-k-1)|\mathcal{B}(-1)|$. By pigeonhole principle we get that one of the summands must be at least $\frac{n-k-1}{n-1}|\mathcal{B}(-1)|$. This is the first element from I.

Once we have found the $(i-1)$-th element, which satisfies the inequality from the proposition, we add it to I, and delete this element from all the vectors from $\mathcal{B}(-1)$ and from the ground set. The set of already found elements we denote I_{i-1}. After $(i-1)$ steps each of the sets from the modified $\mathcal{B}(-1)$ has at least $n-k-i$ 0's, while the total number of coordinates is $n-i$. Therefore, the inequality from the previous paragraph, modified for this case, looks like $\sum_{j\in[2,n]\setminus I_{i-1}} |\mathcal{B}(\bar{1},-j)| \ge (n-k-i)|\mathcal{B}(-1)|$, and by pigeon-hole principle we can find an element l_i such that $|\mathcal{B}(\bar{1},-l_i)| \le \frac{n-k-i}{n-i}|\mathcal{B}(-1)|$, which is bigger than $\frac{1}{3}|\mathcal{B}(-1)|$ for $i \le n-\lceil 3k/2\rceil$. $\square$

Proof of Theorem 23. We argue in terms used in Section 6. We consider several cases depending on the size of $\mathcal{B}(-1)$.

(i) First, assume that $\mathcal{B}(-1)$ is empty. Then, applying Theorem 9 with $c=k$, we get $|\mathcal{A}(i)|+k|\mathcal{B}(-i)| \le (k+1)\binom{n-2}{k-2}$ for $i=2,\dots,n$ and $|\mathcal{A}(1)|+k|\mathcal{B}(-1) \le \binom{n-1}{k-1}$. Therefore,

$$\sum_{i=1}^{n} |\mathcal{A}(i)|+k|\mathcal{B}(-i)| = k(|\mathcal{A}|+|\mathcal{B}|) \le \binom{n-1}{k-1} + (n-1)(k+1)\binom{n-2}{k-2}$$

$$|\mathcal{A}|+|\mathcal{B}| \le \Big(\frac{n-1}{(k-1)k}+\frac{(n-1)(k+1)}{k}\Big)\binom{n-2}{k-2} = \frac{(n-1)k}{k-1}\binom{n-2}{k-2} = e(n+1,k)-e(n,k).$$

It is apparent from the calculations above that the theorem follows if we succeed to show that for some $I\subset[2,n]$ we have $\sum_{l\in I}\big(|\mathcal{A}(l)|+k|\mathcal{B}(-l)|\big)+|k\mathcal{B}(-1)| \le |I|(k+1)\binom{n-2}{k-2}$. This is exactly what are we going to do in a range of cases.

(ii) In this case assume that $k\ge 4$. Assume that $|\mathcal{B}(-1)| \ge 3\binom{n-5}{k-4}$. Then, by Proposition 24, we can find $I\subset[2,n]$, $|I| \ge n-\lceil 3k/2\rceil$, such that we have $|\mathcal{B}(\bar{1},-l)| \ge \binom{n-5}{k-4}$. For any $l\in I$ consider the collection $\mathcal{B}(-l)$. Due to shifting, we know that the maximum degree $d(\mathcal{B}(-l))$ is equal to the number of sets from $\mathcal{B}(-l)$ that have 1 on the first coordinate position. Therefore, we know that $|\mathcal{B}(-l)|-d(\mathcal{B}(-l))| = |\mathcal{B}(\bar{1},-l)| \ge \binom{n-5}{k-4}$ and, thus, we may apply Corollary 21 to $\mathcal{B}(-l)$ with $i=4$ and n,k replaced by $n-1,k-1$. We obtain that

$$|\mathcal{B}(-l)| \le \binom{n-2}{k-2} - \binom{n-5}{k-2} + \binom{n-5}{k-4}.$$

For each $l\in I$ we apply Corollary 19 to $\mathcal{B}(-l),\mathcal{A}(-l)$ with $c=k$ and $i=4$ and obtain, that

$$|\mathcal{A}(l)|+k|\mathcal{B}(-l)| \le \max\Big\{(k+1)\binom{n-2}{k-2} - k\binom{n-5}{k-2} + (k+1)\binom{n-5}{k-4},$$
$$(k+1)\binom{n-2}{k-2} - (k-1)\binom{n-3}{k-2}, \binom{n-1}{k-1}\Big\}.$$

If the maximum of the right hand side is attained on the third summand, then for a single $l\in I$

$$|\mathcal{A}(l)|+k|\mathcal{B}(-l)|+|\mathcal{A}(1)|+k|\mathcal{B}(-1)| \le \binom{n-1}{k-1} + (k+1)\binom{n-2}{k-2}$$

and we are done. If the maximum is attained on the second expression, then $|I| \geq 3$ for $n \geq 3k-1, k \geq 4$, and for three different l_1, l_2, l_3 we have

$$\sum_{j=1}^{3}(|\mathcal{A}(l_j)| + k|\mathcal{B}(-l_j)|) + k|\mathcal{B}(-1)| - (3k+3)\binom{n-2}{k-2} = k\binom{n-2}{k-2} - 3(k-1)\binom{n-3}{k-2} \leq$$

$$\leq (2k - 3(k-1))\binom{n-3}{k-2} \leq 0,$$

since $k \geq 3$, and we are done. Finally, if the maximum is attained on the first summand, then

$$\frac{1}{|I|}\Big(\sum_{l\in I}(|\mathcal{A}(l)| + k|\mathcal{B}(-l)|) + k|\mathcal{B}(-1)|\Big) - (k+1)\binom{n-2}{k-2} \leq$$

$$\leq -k\binom{n-5}{k-2} + (k+1)\binom{n-5}{k-4} + \frac{k}{|I|}\binom{n-2}{k-2}. \tag{19}$$

Before continuing, we need the following two bounds:

$$\frac{\binom{n-5}{k-4}}{\binom{n-5}{k-2}} = \frac{(k-2)(k-3)}{(n-k-2)(n-k-1)} \leq \frac{1}{4}.$$

$$\binom{n-2}{k-2} = \frac{(n-4)(n-3)(n-2)}{(n-k)(n-k-1)(n-k-2)}\binom{n-5}{k-2} \leq \frac{27}{8}\binom{n-5}{k-2}.$$

Using these three bounds, we get

$$(19) \leq -\Big(-\frac{3}{4}k + \frac{1}{4}\Big)\binom{n-5}{k-2} + \frac{27}{8}\frac{k}{n - \lceil 3k/2 \rceil}\binom{n-5}{k-2} \leq$$

$$\leq \Big(-\frac{3}{4}k + \frac{1}{4} + \frac{27k}{8(\lfloor 3k/2 \rfloor - 1)}\Big)\binom{n-5}{k-2} \leq 0,$$

where the last inequality holds for $k = 4, 5$ and, thus, for all $k \geq 4$. This case is settled.

(iii) Assume that for some $5 \leq i \leq k+1$ we have

$$3\binom{n-i-1}{k-i} \leq |\mathcal{B}(-1)| \leq 3\binom{n-i}{k-i+1}.$$

Similarly to the previous case, for $I \subset [n], |I| = n - \lceil \frac{3}{2}k \rceil$, we obtain

$$|\mathcal{B}(-l)| \leq \binom{n-2}{k-2} - \binom{n-i-1}{k-2} + \binom{n-i-1}{k-i}.$$

We again apply Corollary 19 for each $\mathcal{B}(-l), \mathcal{A}(-l)$. Again, if the maximum of the expression from the inequality (16) is attained on one of the last two expressions, we are done. If it is attained on the first one, then

$$\begin{aligned}
&\frac{1}{|I|}\Big(\sum_{l\in I}(|\mathcal{A}(l)|+k|\mathcal{B}(-l)|)+k|\mathcal{B}(-1)|\Big)-(k+1)\binom{n-2}{k-2}\le\\
&\le -k\binom{n-i-1}{k-2}+(k+1)\binom{n-i-1}{k-i}+\frac{k}{n-\lceil 3k/2\rceil}|\mathcal{B}(-1)|\le\\
&\le -k\binom{n-i-1}{k-2}+(k+1)\binom{n-i-1}{k-i}+\frac{3k}{\lfloor 3k/2\rfloor-1}\binom{n-i}{k-i+1}. \qquad (20)
\end{aligned}$$

To proceed further, we need the following bounds:

$$\frac{\binom{n-i-1}{k-i}}{\binom{n-i-1}{k-2}}=\prod_{j=2}^{i-1}\frac{k-j}{n-k-j+1}\le\Big(\frac{1}{2}\Big)^{i-2}\le\frac{1}{8}.$$

$$\frac{\binom{n-i}{k-i+1}}{\binom{n-i-1}{k-2}}\le\frac{(n-i)\prod_{j=2}^{i-2}(k-j)}{\prod_{j=1}^{i-2}(n-k-j)}\le\frac{3}{4}.$$

Now we may continue.

$$(20)\le\Big(-\frac{7}{8}k+\frac{1}{8}+\frac{9k}{4(\lfloor 3k/2\rfloor-1)}\Big)\binom{n-i-1}{k-2}\le 0,$$

where the last inequality holds for $k=3$ and, thus, for all $k\ge 3$.

(iv) The proof so far have covered all the cases for $k\ge 4$. Therefore, it remains to consider the case $k=3$. Then the only feasible value of n is $n=8$.

Proposition 25. *The family $\mathcal{B}(-1)$ is 2-intersecting, that is, every pair of sets have at least two elements in common.*

Proof. Assume the contrary: there are two vectors $\mathbf{v},\mathbf{w}\in\mathcal{F}$, $\mathbf{v}=(-1,v_2,\dots,v_n,1)$ and $\mathbf{w}=(-1,w_2,\dots,w_n,1)$, such there is at most 1 coordinate position in the interval $[2,n]$ in which both vectors have 1. If there is no such coordinate, then, applying $(1,n+1)$-shifting operation to $\mathbf{v}$, we get two vectors that have scalar product -2. On the other hand, $\mathcal{F}$ is shifted and, thus, $\mathbf{v}_{1,n+1}\in\mathcal{F}$. This is a contradiction.

Assume further, that there is one such coordinate, say, i. First we do a $(1,i)$-shift of $\mathbf{v}$, obtaining the vector $\mathbf{v}_{1,i}$, which has a common 1 with $\mathbf{w}$ in the $(n+1)$st coordinate. Since $n+1>2k+1$, there is at least one coordinate position, say, j, on which both $\mathbf{v}_{1,i}$ and $\mathbf{w}$ have 0. We apply a $(j,n+1)$-shift to $\mathbf{w}$. The vectors $\mathbf{v}_{1,i}$ and $\mathbf{w}_{j,n+1}$ have scalar product -2 and both lie in $\mathcal{F}$, a contradiction. □

In case $k=3$ each set from $\mathcal{B}(-1)$ is a 2-element set and, therefore, Proposition 25 gives that $|\mathcal{B}(-1)|\le 1$. Arguing as in the part (i) of the proof, it is sufficient to show that

$|\mathcal{A}(1)| + k|\mathcal{B}(-1)| \le \binom{n-1}{k-1} = \binom{7}{2}$. If $\mathcal{B}(-1) = \emptyset$, then the above inequality is obvious and we are done. If $\mathcal{B}(-1) = 1$, then $|\mathcal{A}(1)| \le \binom{n-1}{k-1} - \binom{n-k}{k-1} = \binom{7}{2} - \binom{5}{2}$. Therefore,

$$|\mathcal{A}(1)| + k|\mathcal{B}(-1)| = \binom{7}{2} - \binom{5}{2} + 3 < \binom{7}{2}.$$

This concludes the proof of the theorem. □

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References

[1] M. Deza, P. Frankl, *Every large set of equidistant* $(0, +1, -1)$*-vectors forms a sunflower* Combinatorica 1 (1981), 225–231.

[2] P. Erdős, C. Ko, R. Rado, *Intersection theorems for systems of finite sets*, The Quarterly Journal of Mathematics, 12 (1961) N1, 313–320.

[3] P. Frankl, *Erdos-Ko-Rado theorem with conditions on the maximal degree*, Journal of Combinatorial Theory, Series A 46 (1987), N2, 252–263.

[4] P. Frankl, R. Wilson, *Intersection theorems with geometric consequences*, Combinatorica 1 (1981), 357–368.

[5] A.J.W. Hilton, *The Erdos-Ko-Rado theorem with valency conditions*, (1976), unpublished manuscript

[6] A.J.W. Hilton, E.C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford 18 (1967), 369-384.

[7] J. Kahn, G. Kalai, *A counterexample to Borsuk's conjecture*, Bulletin of the American Mathematical Society 29 (1993), 60-62.

[8] G. Katona, *A theorem of finite sets*, "Theory of Graphs, Proc. Coll. Tihany, 1966", Akad, Kiado, Budapest, 1968; Classic Papers in Combinatorics (1987), 381-401.

[9] G. Katona, *A simple proof of the Erdős–Ko–Rado Theorem*, J. Combin. Theory, Ser. B 13 (1972), 183–184.

[10] G.O.H. Katona, *Extremal problems for hypergraphs*, Combinatorics, Mathematical Centre Tracts 56 (1974), Part 2, 13–42.

[11] J.B. Kruskal, *The Number of Simplices in a Complex*, Mathematical optimization techniques 251 (1963), 251-278.

[12] A.M. Raigorodskii, *Borsuk's problem and the chromatic numbers of some metric spaces*, Russian Math. Surveys, 56 (2001), N1, 103 - 139.