A DEGREE VERSION OF THE HILTON–MILNER THEOREM

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ABSTRACT. An intersecting family of sets is trivial if all of its members share a common element. Hilton and Milner proved a strong stability result for the celebrated Erdős–Ko–Rado theorem: when $n > 2k$, every non-trivial intersecting family of k-subsets of $[n]$ has at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ members. One extremal family $\mathcal{HM}_{n,k}$ consists of a k-set S and all k-subsets of $[n]$ containing a fixed element $x \notin S$ and at least one element of S. We prove a degree version of the Hilton–Milner theorem: if $n = \Omega(k^2)$ and $\mathcal{F}$ is a non-trivial intersecting family of k-subsets of $[n]$, then $\delta(F) \le \delta(\mathcal{HM}_{n.k})$, where $\delta(\mathcal{F})$ denotes the minimum (vertex) degree of $\mathcal{F}$. Our proof uses several fundamental results in extremal set theory, the concept of kernels, and a new variant of the Erdős–Ko–Rado theorem.

1. Introduction

A family $\mathcal{F}$ of sets is called *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. A fundamental problem in extremal set theory is to study the properties of intersecting families. For positive integers k, n, let $[n] = \{1, 2, \ldots, n\}$ and $\binom{V}{k}$ denote the family of all k-element subsets (k-subsets) of V. We call a family on V *k-uniform* if it is a subfamily of $\binom{V}{k}$. A *full star* is a family that consists of all the k-subsets of $[n]$ that contains a fixed element. We call an intersecting family $\mathcal{F}$ *trivial* if it is a subfamily of a full star. The celebrated Erdős–Ko–Rado (EKR) theorem [3] states that, when $n \ge 2k$, every k-uniform intersecting family on $[n]$ has at most $\binom{n-1}{k-1}$ members, and the full star shows that the bound $\binom{n-1}{k-1}$ is best possible. Hilton and Milner [14] proved the uniqueness of the extremal family in a stronger sense: if $n > 2k$, every non-trivial intersecting family of k-subsets of $[n]$ has at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ members. It is easy to see that the equality holds for the following family, denoted by $\mathcal{HM}_{n,k}$, which consists of a k-set S and all k-subsets of $[n]$ containing a fixed element $x \notin S$ and at least one vertex of S. For more results on intersecting families, see a recent survey by Frankl and Tokushige [10].

Given a family $\mathcal{F}$ and $x \in V(\mathcal{F})$, we denote by $\mathcal{F}(x)$ the subfamily of $\mathcal{F}$ consisting of all the members of $\mathcal{F}$ that contain x, i.e., $\mathcal{F}(x) := \{F \in \mathcal{F} : x \in F\}$. Let $d_{\mathcal{F}}(x) := |\mathcal{F}(x)|$ be the *degree* of x. Let $\Delta(\mathcal{F}) := \max_x d_{\mathcal{F}}(x)$ and $\delta(\mathcal{F}) := \min_x d_{\mathcal{F}}(x)$ denote the maximum and minimum degree of $\mathcal{F}$, respectively. There were extremal problems in set theory that considered the maximum or minimum degree of families satisfying certain properties. For example, Frankl [7] extended the Hilton–Milner theorem by giving sharp upper bounds on the size of intersecting families with certain maximum degree. Bollobás, Daykin, and Erdős [1] studied the minimum degree version of a well-known conjecture of Erdős [2] on matchings.

Huang and Zhao [15] recently proved a minimum degree version of the EKR theorem, which states that, if $n > 2k$ and $\mathcal{F}$ is a k-uniform intersecting family on $[n]$, then $\delta(\mathcal{F}) \le \binom{n-2}{k-2}$, and the equality holds only if $\mathcal{F}$ is a full star. This result implies the EKR theorem immediately: given a k-uniform intersecting family $\mathcal{F}$, by recursively deleting elements with the smallest degree until $2k$ elements are left, we derive that

$$|\mathcal{F}| \le \binom{n-2}{k-2} + \binom{n-3}{k-2} + \cdots + \binom{2k-1}{k-2} + \binom{2k-1}{k-1} = \binom{n-1}{k-1}.$$

Frankl and Tokushige [11] gave a different proof of the result of [15] for $n \ge 3k$. Generally speaking, a minimum degree condition forces the sets of a family to be distributed somewhat evenly and thus the size of a family that is required to satisfy a property might be smaller than the one without degree condition.

Date: October 8, 2018.
2010 *Mathematics Subject Classification.* 05D05.
Key words and phrases. Intersecting families, Hilton–Milner theorem, Erdős–Ko–Rado theorem.
JH is supported by FAPESP (Proc. 2013/03447-6, 2014/18641-5). HH is partially supported by a Simons Collaboration Grant. YZ is partially supported by NSF grant DMS-1400073.

Unless the extremal family is very regular, an extremal problem under the minimum degree condition seems harder than the original extremal problem because one cannot directly apply the *shifting method* (a powerful tool in extremal set theory).

In this paper we study the minimum degree version of the Hilton–Milner theorem.

Theorem 1. *Suppose $k \ge 4$ and $n \ge ck^2$, where $c = 30$ for $k = 4, 5$ and $c = 4$ for $k \ge 6$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family, then $\delta(\mathcal{F}) \le \delta(\mathcal{HM}_{n,k}) = \binom{n-2}{k-2} - \binom{n-k-2}{k-2}$.*

Han and Kohayakawa [12] recently determined the maximum size of a non-trivial intersecting family that is not a subfamily of $\mathcal{HM}_{n,k}$, which is $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2$. Later Kostochka and Mubayi [17] determined the maximum size of a non-trivial intersecting family that is not a subfamily of $\mathcal{HM}_{n,k}$ or the extremal families given in [12] for sufficiently large n. Furthermore, Kostochka and Mubayi [17, Theorem 8] characterized *all* maximal intersecting 3-uniform families $\mathcal{F}$ on $[n]$ for $n \ge 7$ and $|\mathcal{F}| \ge 11$. Using a different approach, Polcyn and Ruciński [18, Theorem 4] characterized *all* maximal intersecting 3-uniform families $\mathcal{F}$ on $[n]$ for $n \ge 7$, in particular, there are fifteen such families, including the full star and $\mathcal{HM}_{n,3}$. It is straightforward to check that all these families have minimum degree at most 3 – this gives the following proposition.

Proposition 2. *If $n \ge 7$ and $\mathcal{F} \subseteq \binom{[n]}{3}$ is a non-trivial intersecting family, then $\delta(\mathcal{F}) \le \delta(\mathcal{HM}_{n,3}) = 3$.*

In order to prove Theorem 1, we prove a new variant of the EKR theorem, which is closely related to the EKR theorem for direct products given by Frankl (see Theorem 7).

Theorem 3. *Given integers $k \ge 3$, $\ell \ge 4$, and $m \ge k\ell$, let T_1, T_2, T_3 be three disjoint ℓ-subsets of $[m]$. If $\mathcal{F}$ is a k-uniform intersecting family on $[m]$ such that every member intersects all of T_1, T_2, T_3, then $|\mathcal{F}| \le \ell^2 \binom{m-3}{k-3}$.*

Theorem 3 becomes trivial when $\ell = 1$ because every family $\mathcal{F}$ of k-sets that intersect T_1, T_2, T_3 satisfies $|\mathcal{F}| \le \binom{m-3}{k-3}$. Our bound in Theorem 3 is asymptotically tight because a star with a center in $T_1 \cup T_2 \cup T_3$ contains about $\ell^2 \binom{m-3}{k-3}$ k-sets that intersect T_1, T_2, T_3.

It was shown in [15] that one can derive the minimum degree version of the EKR theorem for $n = \Omega(k^2)$ by using the Hilton–Milner Theorem and simple averaging arguments (thus the difficulty of the result in [15] lies in deriving the tight bound $n \ge 2k+1$). However, we can not use this naive approach to prove Theorem 1 for sufficiently large n. Indeed, let $\mathcal{F}$ be a non-trivial intersecting family that is not a subfamily of $\mathcal{HM}_{n,k}$. The result of Han and Kohayakawa [12] says that $|\mathcal{F}|$ is asymptotically at most $(k-1)\binom{n-2}{k-2}$, and in turn, the average degree of $\mathcal{F}$ is asymptotically at most $\frac{k(k-1)}{k-2}\binom{n-3}{k-3}$. Unfortunately, this is much larger than $\delta(\mathcal{HM}_{n,k}) \approx k\binom{n-3}{k-3}$ as k is fixed and n is sufficiently large.

Our proof of Theorem 1 applies several fundamental results in extremal set theory as well as Theorem 3. The following is an outline of our proof. Let $\mathcal{F}$ be a non-trivial intersecting family such that $\delta(\mathcal{F}) > \delta(\mathcal{HM}_{n,k})$. For every $u \in [n]$, we obtain a lower bound for $|\mathcal{F} \setminus \mathcal{F}(u)|$ by applying the assumption on $\delta(\mathcal{F})$ and the Frankl–Wilson theorem [5, 19] on the maximum size of t-intersecting families. If $k = 4, 5$, then we derive a contradiction by considering the *kernel* of $\mathcal{F}$ (a concept introduced by Frankl [6]). When $k \ge 6$, we separate two cases based on $\Delta(\mathcal{F})$. When $\Delta(\mathcal{F})$ is large, assume that $|\mathcal{F}(u)| = \Delta(\mathcal{F})$ and let $\mathcal{F}_2 := \mathcal{F} \setminus \mathcal{F}(u)$. A result of Frankl [9] implies that $\mathcal{F}(u)$ contains three edges $E_i := \{u\} \cup T_i$, $i \in [3]$, where T_1, T_2, T_3 are pairwise disjoint. Since $\mathcal{F}_2$ is intersecting and every member of $\mathcal{F}_2$ meets each of T_1, T_2, T_3, Theorem 3 gives an upper bound on $|\mathcal{F}_2|$, which contradicts the lower bound that we derived earlier. When $\Delta(\mathcal{F})$ is small, we apply the aforementioned result of Frankl [7] to obtain an upper bound on $|\mathcal{F}|$, which contradicts the assumption on $\delta(\mathcal{F})$.

2. Tools

2.1. **Results that we need.** Given a positive integer t, a family $\mathcal{F}$ of sets is called *t-intersecting* if $|A \cap B| \ge t$ for all $A, B \in \mathcal{F}$. A t-intersecting EKR theorem was proved in [3] for sufficiently large n. Later Frankl [5] (for $t \ge 15$) and Wilson [19] (for all t) determined the exact threshold for n.

Theorem 4. [5, 19] *Let $n \geq (t+1)(k-t+1)$ and let $\mathcal{F}$ be a k-uniform t-intersecting families on $[n]$. Then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.*

As mentioned in Section 1, Frankl [7] determined the maximum possible size of an intersecting family under a maximum degree condition.

Theorem 5. [7] *Suppose $n > 2k$, $3 \leq i \leq k+1$, $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting. If $\Delta(\mathcal{F}) \leq \binom{n-1}{k-1} - \binom{n-i}{k-1}$, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-i}{k-1} + \binom{n-i}{k-i+1}$.*

Given a k-uniform family $\mathcal{F}$, a *matching of size s* is a collection of s vertex-disjoint sets of $\mathcal{F}$. A well-known conjecture of Erdős [2] states that if $n \geq (s+1)k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ satisfies $|\mathcal{F}| > \max\{\binom{n}{k} - \binom{n-s}{k}, \binom{k(s+1)-1}{k}\}$, then $\mathcal{F}$ contains a matching of size $s+1$. Frankl [9] verified this conjecture for $n \geq (2s+1)k - s$.

Theorem 6. [9] *Let $n \geq (2s+1)k - s$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$. If $|\mathcal{F}| > \binom{n}{k} - \binom{n-s}{k}$, then $\mathcal{F}$ contains a matching of size $s+1$.*

Frankl [8] proved an EKR theorem for direct products.

Theorem 7. [8] *Suppose $n = n_1 + \cdots + n_d$ and $k = k_1 + \cdots + k_d$, where $n_i \geq k_i$ are positive integers. Let $X_1 \cup \cdots \cup X_d$ be a partition of $[n]$ with $|X_i| = n_i$, and*

$$\mathcal{H} = \left\{F \in \binom{[n]}{k} : |F \cap X_i| = k_i \text{ for } i = 1, \ldots, d\right\}.$$

If $n_i \geq 2k_i$ for all i and $\mathcal{F} \subseteq \mathcal{H}$ is intersecting, then

$$\frac{|\mathcal{F}|}{|\mathcal{H}|} \leq \max_i \frac{k_i}{n_i}.$$

Note that the $d = 1$ case of Theorem 7 is the EKR theorem.

2.2. **Kernels of intersecting families.** Frankl introduced the concept of *kernels* (and called them *bases*) for intersecting families in [6]. Given $\mathcal{F} \subseteq \binom{V}{k}$, a set $S \subseteq V$ is called a *cover* of $\mathcal{F}$ if $S \cap A \neq \emptyset$ for all $A \in \mathcal{F}$. For example, if $\mathcal{F}$ is intersecting, then every member of $\mathcal{F}$ is a cover. Given an intersecting family $\mathcal{F}$, we define its *kernel* $\mathcal{K}$ as

$$\mathcal{K} := \{S : S \text{ is a cover of } \mathcal{F} \text{ and any } S' \subsetneq S \text{ is not a cover of } \mathcal{F}\}.$$

An intersecting family $\mathcal{F}$ is called *maximal* if $\mathcal{F} \cup \{G\}$ is not intersecting for any k-set $G \notin F$. Note that, when proving Theorem 1, we may assume that $\mathcal{F}$ is maximal because otherwise we can add more k-sets to $\mathcal{F}$ such that the resulting intersecting family is still non-trivial and satisfies the minimum degree condition. We observe the following fact on the kernels.

Fact 8. *If $n \geq 2k$ and $\mathcal{F} \in \binom{[n]}{k}$ is a maximal intersecting family, then $\mathcal{K}$ is also intersecting.*

Proof. Suppose there are $K_1, K_2 \in \mathcal{K}$ such that $K_1 \cap K_2 = \emptyset$. Since $n \geq 2k$, we can find two disjoint k-sets F_1, F_2 on $[n]$ such that $K_i \subseteq F_i$ for $i = 1, 2$. For $i = 1, 2$, since K_i is a cover of $\mathcal{F}$, F_i intersects all members of $\mathcal{F}$. Since $\mathcal{F}$ is maximal, we derive that $F_1, F_2 \in \mathcal{F}$. This contradicts the assumption that F_1, F_2 are disjoint. □

For $i \in [k]$, let $\mathcal{K}_i := \mathcal{K} \cap \binom{[n]}{i}$. If an intersecting family $\mathcal{F}$ is non-trivial, then $\mathcal{K}_1 = \emptyset$. Below we prove an upper bound for $|\mathcal{K}_i|$, $3 \leq i \leq k$, where the $i = k$ case was given by Erdős and Lovász [4].

Lemma 9. *For $3 \leq i \leq k$, we have $|\mathcal{K}_i| \leq k^i$.*

In order to prove Lemma 9, We use a result of Håstad, Jukna, and Pudlák [13, Lemma 3.4]. Given a family $\mathcal{F}$, the *cover number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the size of the smallest cover of $\mathcal{F}$.

Lemma 10. [13] *If $\mathcal{F}$ is an i-uniform family with $|\mathcal{F}| > k^i$, then there exists a set Y such that $\tau(\mathcal{F}_Y) \geq k+1$, where $\mathcal{F}_Y := \{F \setminus Y : F \in \mathcal{F}, F \supseteq Y\}$.*

Proof of Lemma 9. Suppose $|\mathcal{K}_i| > k^i$ for some $3 \le i \le k$. Then by Lemma 10, there exists a set Y such that $\tau((\mathcal{K}_i)_Y) \ge k+1$. In particular, $(\mathcal{K}_i)_Y$ is nonempty, namely, there exists $K \in \mathcal{K}_i$ such that $Y \subsetneq K$. By the definition of $\mathcal{K}$, this implies that Y is not a cover of $\mathcal{F}$, so there exists $F \in \mathcal{F}$ such that $F \cap Y = \emptyset$. Since each member of $\mathcal{K}_i$ is a cover of $\mathcal{F}$, each of them intersects F. This implies that $\tau((\mathcal{K}_i)_Y) \le |F| = k$, a contradiction. □

3. Proof of Theorem 3

In this section we derive Theorem 3 from Theorem 7.

Proof of Theorem 3. Let $\mathcal{F}_r$ consist of all the subsets of $\mathcal{F}$ that intersect with $T_1 \cup T_2 \cup T_3$ in exactly r elements. Then $\mathcal{F} = \mathcal{F}_3 \cup \mathcal{F}_4 \cup \cdots \cup \mathcal{F}_k$. Let $X_1 = T_1$, $X_2 = T_2$, $X_3 = T_3$, $X_4 = [m] \setminus (T_1 \cup T_2 \cup T_3)$, and $k_1 = k_2 = k_3 = 1$, $k_4 = k-3$. Since $m \ge k\ell$, we have $1/\ell \ge (k-3)/(m-3\ell)$. Since $\ell \ge 2$, we can apply Theorem 7 to conclude that

$$|\mathcal{F}_3| \le \ell^3 \binom{m-3\ell}{k-3} \cdot \frac{1}{\ell} = \ell^2 \binom{m-3\ell}{k-3}.$$

Note that a set $S \in \mathcal{F}_4$ intersects T_1, T_2, T_3 with either $1,1,2$ or $1,2,1$ or $2,1,1$ elements. We partition $\mathcal{F}_4$ into three subfamilies accordingly. Our assumption implies

$$\frac{k-4}{m-3\ell} \le \frac{2}{\ell} \le \frac{1}{2}.$$

We can apply Theorem 7 to each subfamily of $\mathcal{F}_4$ and obtain that

$$|\mathcal{F}_4| \le 3\binom{\ell}{2}\ell^2\binom{m-3\ell}{k-4} \cdot \frac{2}{\ell} = 3(\ell-1)\ell^2\binom{m-3\ell}{k-4}.$$

Finally, for $5 \le r \le k$, we claim that $|\mathcal{F}_r| \le \ell^2\binom{3\ell-3}{r-3}\binom{m-3\ell}{k-r}$. Indeed, let $X_1 = T_1 \cup T_2 \cup T_3$, $X_2 = [m] \setminus X_1$, $k_1 = r$ and $k_2 = k-r$. Note that $|X_2| = m - 3\ell \ge 2(k-r)$ and $r/(3\ell) \ge (k-r)/(m-3\ell)$. If $|X_1| = 3\ell \ge 2r$, then Theorem 7 gives that

$$|\mathcal{F}_r| \le \binom{3\ell-1}{r-1}\binom{m-3\ell}{k-r} < \ell^2\binom{3\ell-3}{r-3}\binom{m-3\ell}{k-r}.$$

When $3\ell \le 2r$, we have $r \ge 6$ because $\ell \ge 4$. Hence,

$$\binom{3\ell}{r} < \frac{(3\ell)^3}{r(r-1)(r-2)}\binom{3\ell-3}{r-3} \le \frac{18\ell^2}{(r-1)(r-2)}\binom{3\ell-3}{r-3} < \ell^2\binom{3\ell-3}{r-3},$$

and the trivial bound on $|\mathcal{F}_r|$ gives that

$$|\mathcal{F}_r| \le \binom{3\ell}{r}\binom{m-3\ell}{k-r} < \ell^2\binom{3\ell-3}{r-3}\binom{m-3\ell}{k-r}$$

as claimed. Summing up the bounds for $|\mathcal{F}_3|, |\mathcal{F}_4|$ and $|\mathcal{F}_r|$ for $r \ge 5$, we have

$$\begin{aligned}|\mathcal{F}| &= |\mathcal{F}_3| + |\mathcal{F}_4| + \sum_{r=5}^{k}|\mathcal{F}_r| \\ &\le \ell^2\binom{m-3\ell}{k-3} + 3(\ell-1)\ell^2\binom{m-3\ell}{k-4} + \ell^2\sum_{r=5}^{k}\binom{3\ell-3}{r-3}\binom{m-3\ell}{k-r} = \ell^2\binom{m-3}{k-3},\end{aligned}$$

because $\binom{m-3}{k-3} = \sum_{i=0}^{k-3}\binom{m-3l}{k-3-i}\binom{3l-3}{i}$. □

4. Proof of Theorem 1

We start with some simple estimates. First, for $n \geq ck^2$, $c \geq 1$ and $1 \leq t \leq k-1$, we have

$$\frac{\binom{n-2k+t-1}{k-2}}{\binom{n-t-1}{k-2}} = \frac{(n-2k+t-1)\cdots(n-3k+t+2)}{(n-t-1)\cdots(n-t-k+2)} \geq \left(1 - \frac{2k-2t}{n-t-k+2}\right)^{k-2}$$
$$\geq 1 - \frac{2(k-t)(k-2)}{n-t-k+2} \geq \frac{c-2}{c}. \tag{4.1}$$

Similarly, one can show that $\binom{n-k-2}{k-3} \geq \frac{c-1}{c}\binom{n-3}{k-3}$. Second, if $\delta(\mathcal{F}) > \binom{n-2}{k-2} - \binom{n-k-2}{k-2}$, then we have

$$|\mathcal{F}| > \frac{n}{k}\left(\binom{n-2}{k-2} - \binom{n-k-2}{k-2}\right) > n\binom{n-k-2}{k-3}$$
$$\geq \frac{(c-1)n}{c}\binom{n-3}{k-3} > \frac{(c-1)}{c}(k-2)\binom{n-2}{k-2}. \tag{4.2}$$

Lemma 11. *Suppose $k \geq 4$ and $n \geq 4k^2$, $\mathcal{F} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family such that $\delta(\mathcal{F}) > \delta(\mathcal{HM}_{n,k}) = \binom{n-2}{k-2} - \binom{n-k-2}{k-2}$. Then for any $u \in [n]$,*

(i) there exists $E, E' \in \mathcal{F}$ such that $u \notin E \cup E'$ and $|E \cap E'| = 1$;
(ii) $|\mathcal{F} \setminus \mathcal{F}(u)| > \frac{k-2}{2}\binom{n-2}{k-2}$.

Proof. Given $u \in [n]$, write $\mathcal{F}_1 = \mathcal{F}(u)$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. If $|\mathcal{F}_2| = 1$, then $\mathcal{F} \subseteq \mathcal{HM}_{n,k}$, and thus $\delta(\mathcal{F}) \leq \delta(\mathcal{HM}_{n,k})$, a contradiction. So assume that $|\mathcal{F}_2| \geq 2$.

Let $t = \min|E \cap E'|$ among all distinct $E, E' \in \mathcal{F}_2$. Obviously $1 \leq t \leq k-1$, and $\mathcal{F}_2$ is a t-intersecting family on $[2,n]$. Then since $n > 4k^2 \geq (k-t+1)(t+1)+1$, we get $|\mathcal{F}_2| \leq \binom{n-t-1}{k-t}$ by Theorem 4. Note that there exist $E, E' \in \mathcal{F}_2$ such that $|E \cap E'| = t$. Since every set in $\mathcal{F}_1$ must intersect both E and E', for every $x \notin E \cup E' \cup \{u\}$, by the inclusion-exclusion principle, we have

$$|\mathcal{F}_1(x)| \leq \binom{n-2}{k-2} - 2\binom{n-k-2}{k-2} + \binom{n-2k+t-2}{k-2}. \tag{4.3}$$

Let $X = [n] \setminus (E \cup E' \cup \{u\})$ and thus $|X| = n-1-(2k-t)$. Suppose $x \in X$ attains the minimum degree in $\mathcal{F}_2$ among all elements of X. Since $|\mathcal{F}(x)| = |\mathcal{F}_1(x)| + |\mathcal{F}_2(x)| > \delta(\mathcal{HM}_{n,k})$, by (4.3) we have

$$|\mathcal{F}_2(x)| > \binom{n-k-2}{k-2} - \binom{n-2k+t-2}{k-2}.$$

By the definition of x we get

$$|\mathcal{F}_2| > \frac{|X|}{k-t}\left(\binom{n-k-2}{k-2} - \binom{n-2k+t-2}{k-2}\right) \geq \frac{|X|(k-t)}{k-t}\binom{n-2k+t-2}{k-3}$$
$$= (k-2)\binom{n-2k+t-1}{k-2},$$

where the factor $k-t$ comes from the fact that every member $F \in \mathcal{F}_2$ is counted at most $k-t$ times – because $|F \cap E_1| \geq t$. By (4.1) with $c = 4$ and $k \geq 4$, we get

$$|\mathcal{F}_2| > \frac{k-2}{2}\binom{n-t-1}{k-2} \geq \binom{n-t-1}{k-2},$$

which, together with $|\mathcal{F}_2| \leq \binom{n-t-1}{k-t}$, implies that $t = 1$, so (i) holds. Since $t = 1$, the first inequality above gives (ii). □

Proof of Theorem 1. First assume that $k \geq 6$ and $n \geq 4k^2$. Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family such that $\delta(\mathcal{F}) > \delta(\mathcal{HM}_{n,k}) = \binom{n-2}{k-2} - \binom{n-k-2}{k-2}$. Suppose $u \in [n]$ attains the maximum degree of $\mathcal{F}$ and write $\mathcal{F}' := \mathcal{F} \setminus \mathcal{F}(u)$. If $|\mathcal{F}(u)| > \binom{n-1}{k-1} - \binom{n-3}{k-1}$, then by Theorem 6, the $(k-1)$-uniform family $\{E \setminus \{u\} : E \in \mathcal{F}(u)\}$ contains a matching $\mathcal{M} = \{T_1, T_2, T_3\}$ of size 3. Every member of $\mathcal{F}'$ must intersect each of T_1, T_2, T_3. By Theorem 3, we have $|\mathcal{F}'| \leq (k-1)^2\binom{n-4}{k-3}$. On the other hand, Lemma 11 Part (ii)

implies that $|\mathcal{F}'| > \frac{k-2}{2}\binom{n-2}{k-2} = \frac{n-2}{2}\binom{n-3}{k-3} > 2(k-1)^2\binom{n-3}{k-3}$ because $n \geq 4k^2 \geq 4(k-1)^2+2$. This gives a contradiction.

We thus assume that $|\Delta(\mathcal{F})| \leq \binom{n-1}{k-1} - \binom{n-3}{k-1}$. By Theorem 5,

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-3}{k-1} + \binom{n-3}{k-2} = \frac{3n-2k-2}{n-2}\binom{n-2}{k-2} \leq 3\binom{n-2}{k-2}.$$

Since $\delta(\mathcal{F}) > \binom{n-2}{k-2} - \binom{n-k-2}{k-2}$, by (4.2), we have $|\mathcal{F}| > \frac{3}{4}(k-2)\binom{n-2}{k-2}$. The upper and lower bounds for $|\mathcal{F}|$ together imply $k < 6$, a contradiction.

Now assume that $k = 4, 5$ and $n \geq 30k^2$. Since $\mathcal{F}$ is intersecting, each member of $\mathcal{F}$ is a cover of $\mathcal{F}$ and thus contains as a subset a minimal cover, which is a member of the kernel $\mathcal{K}$. Thus $|\mathcal{F}| \leq \sum_{i=1}^k |\mathcal{K}_i|\binom{n-i}{k-i}$. We know $\mathcal{K}_1 = \emptyset$ because $\mathcal{F}$ is non-trivial. We observe that $|\mathcal{K}_2| \leq 1$ – otherwise assume $uv, uv' \in \mathcal{K}_2$ (recall that $\mathcal{K}_2$ is intersecting). By the definition of $\mathcal{K}_2$, every $E \in \mathcal{F} \setminus \mathcal{F}(u)$ contains both v and v' so every $E, E' \in \mathcal{F} \setminus \mathcal{F}(u)$ satisfy that $|E \cap E'| \geq 2$, contradicting Lemma 11 Part (i). By Lemma 9,

$$|\mathcal{F}| \leq \binom{n-2}{k-2} + \sum_{i=3}^{k} k^i \binom{n-i}{k-i}.$$

Since $n \geq 30k^2$, for any $3 \leq i \leq k$, we have

$$k^{i-2}\binom{n-i}{k-i} = \binom{n-2}{k-2} \cdot k^{i-2} \cdot \frac{k-2}{n-2} \cdot \frac{k-3}{n-3} \cdots \frac{k-i+1}{n-i+1} \leq \binom{n-2}{k-2}\frac{1}{30^{i-2}}.$$

Thus

$$|\mathcal{F}| \leq \binom{n-2}{k-2} + k^2\binom{n-2}{k-2}\sum_{i=3}^{k}\frac{1}{30^{i-2}} \leq \binom{n-2}{k-2}\left(1 + \frac{k^2}{29}\right).$$

On the other hand, by (4.2), we have $|\mathcal{F}| > \frac{29}{30}(k-2)\binom{n-2}{k-2} > \frac{28}{29}(k-2)\binom{n-2}{k-2}$. Hence, $28(k-2) < 29 + k^2$, contradicting $4 \leq k \leq 5$. This completes the proof of Theorem 1. ☐

5. Concluding Remarks

The main question arising from our work is whether Theorem 1 holds for all $n \geq 2k+1$. Proposition 2 confirms this for $k = 3$. Another question is whether the following generalization of Theorems 3 and 7 is true. We say a family $\mathcal{H}$ of sets has the *EKR property* if the largest intersecting subfamily of $\mathcal{H}$ is trivial.

Conjecture 12. *Suppose $n = n_1 + \cdots + n_d$ and $k \geq k_1 + \cdots + k_d$, where $n_i > k_i \geq 0$ are integers. Let $X_1 \cup \cdots \cup X_d$ be a partition of $[n]$ with $|X_i| = n_i$, and*

$$\mathcal{H} := \left\{ F \subseteq \binom{[n]}{k} : |F \cap X_i| \geq k_i \text{ for } i = 1, \ldots, d \right\}.$$

If $n_i \geq 2k_i$ for all i and $n_i > k - \sum_{j=1}^d k_j + k_i$ for all but at most one $i \in [d]$ such that $k_i > 0$, then $\mathcal{H}$ has the EKR property.

The assumptions on n_i cannot be relaxed for the following reasons. If $n_i < 2k_i$ for some i, then $\mathcal{H}$ itself is intersecting and $|\mathcal{H}(x)| < |\mathcal{H}|$ for any $x \in [n]$. If $n_i \leq k - \sum_{j=1}^d k_j + k_i$ for distinct i_1, i_2 such that $k_{i_1}, k_{i_2} > 0$, then for any $x \in [n]$, the union of $\mathcal{H}(x)$ and $\{F \in \mathcal{H} : X_{i_1} \subseteq F \text{ or } X_{i_2} \subseteq F\}$ is a larger intersecting family than $\mathcal{H}(x)$.

When $k = k_1 + \cdots + k_d$, Conjecture 12 follows from Theorem 7, in particular, the $d = 1$ case is the EKR theorem. A recent result of Katona [16] confirms Conjecture 12 for the case $d = 2$ and $n_1, n_2 \geq 9(k - \min\{k_1, k_2\})^2$. We can prove Conjecture 12 in the following case.

Theorem 13. *Given positive integers $d \leq k$, $2 \leq t_1 \leq t_2 \leq \cdots \leq t_d$ with $t_2 \geq k - d + 2$, there exists n_0 such that the followings holds for all $n \geq n_0$. If $T_1, \ldots, T_d$ are disjoint subsets of $[n]$ such that $|T_i| = t_i$ for all i, then*

$$\mathcal{H} := \left\{ F \subseteq \binom{[n]}{k} : |F \cap T_i| \geq 1 \text{ for } i = 1, \ldots, d \right\}$$

has the EKR property.

We omit the proof of Theorem 13 here because the purpose of this paper is to prove Theorem 1. Moreover, when $d = 3$ and $t_1 = t_2 = t_3 = k - 1$, our n_0 is $\Omega(k^4)$ so we cannot replace Theorem 3 by Theorem 13 in our main proof. Nevertheless, it would be interesting to know the smallest n_0 such that Theorem 13 holds.

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