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Journal of Sound and Vibration 263 (2003) 377–397
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Forced response statistics of mistuned bladed disks: a stochastic reduced basis approach

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Received 22 February 2002; accepted 23 May 2002

Abstract

This paper presents a stochastic reduced basis approach for predicting the forced response statistics of mistuned bladed-disk assemblies. In this approach, the system response in the frequency domain is represented using a linear combination of complex stochastic basis vectors with undermined coefficients. The terms of the preconditioned stochastic Krylov subspace are used here as basis vectors. Two variants of the stochastic Bubnov–Galerkin scheme are employed for computing the undetermined terms in the reduced basis representation, which arise from how the condition for orthogonality between two random vectors is interpreted. Explicit expressions for the response quantities can then be derived in terms of the random system parameters, which allow for the possibility of efficiently computing the response statistics in the post-processing stage. Numerical studies are presented for mistuned cyclic assemblies of mono-coupled single-mode components. It is demonstrated that the accuracy of the response statistical moments computed using stochastic reduced basis methods can be orders of magnitude better than classical perturbation methods.

1. Introduction

Bladed-disk assemblies represent one of the most commonly encountered and practical examples of rotationally periodic structures in engineering. There exist a number of computationally efficient schemes to analyze such perfectly periodic structures since the dynamics of the entire system can be computed from the analysis of only one typical sector (one blade and the corresponding portion of the disk) using the theory of cyclic symmetry. Unfortunately, in practice, small differences in the blades' structural properties can destroy the perfect cyclic

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doi:10.1016/S0022-460X(02)01058-1

symmetry. This problem known as mistuning [1] arises due to the stochastic nature of manufacturing processes, deviations in material properties, and in-service degradation.

Mistuning is known to have a potentially dramatic effect on the free vibration behavior of coupled blades, since it can lead to the spatial localization of energy around one or a few blades. One well-established and important result is that weakly coupled systems are highly sensitive to mistuning and this sensitivity depends primarily on the ratio of mistuning to interblade coupling strengths [2,3]. In the case of the forced vibration, mistuning can lead to significant increases in the amplitudes and stresses of blades compared to their perfectly tuned counterparts [4]. Another characteristic is the increase in amplitude of the maximum-responding blade at any frequency, which could result in a significant reduction in fatigue life. Also, moderately weakly coupled systems are found to be more sensitive than strongly coupled ones through a greater increase in component response amplitudes [5,6]. A detailed exposition of factors that influence the sensitivity of disordered periodic systems to mistuning has been presented in the literature, see, for example, Refs. [7–9].

A driving factor for research on the mistuning problem has been the ever increasing need for efficient and accurate computational models to predict the existence of rogue blades that exhibit failure due to excessive stress levels. In order to take into account the influence of mistuning in the design of bladed-disks assemblies, a large population of blades must be analyzed. This suggests the application or development of probabilistic analysis procedures, where the blades' properties are considered as random variables, see Ref. [1] for a review of recent developments in dynamic analysis of uncertain systems.

A direct but computationally expensive way to accurately generate the response statistics remains the expensive Monte Carlo simulation (MCS) method. In this approach, samples of the uncertain system parameters are generated in accordance with their probability density function (pdf), and the equations of motion are solved for each realization of these parameters. Subsequently, the statistics of the response amplitudes, stress and life fatigue can be estimated. This statistical information about the system response can then be employed to plan and interpret test results, and also to design systems that are more insensitive to mistuning.

A major disadvantage of simulation techniques is that the computational cost may become prohibitive, particularly for systems that are required to be analyzed using high-fidelity finite element models. This has motivated the development of reduced order modelling techniques [8–13] to make simulation schemes more efficient. However, in this line of approach, a trade-off must be made between the accuracy of the reduced order model and computational cost.

Another popular approach to mistuning analysis involves the application of perturbation techniques to analytically approximate the response statistics [5,7,14–17]. Since perturbation schemes are computationally very efficient, they can be readily applied to large-scale finite element models. Further, the resulting explicit expression for the response allows for the possibility of gaining physical insights into the dynamics of mistuned systems. Hence, these schemes, if accurate, can allow reliable statistical assessments during the turbomachinery design process. However, the accuracy of perturbation methods tends to deteriorate significantly for large coefficients of variation of the random system parameters and increasing frequency of excitation. In the particular case of mistuned bladed disks, the accuracy depends on the relative magnitudes of coupling strength, mistuning strength, and the material damping properties.

The inherent limitations of perturbation methods were illustrated in Ref. [7], where two perturbation approaches were presented for mistuning analysis. In the first approach, the forced response amplitude of each component system is obtained directly as a perturbation of the tuned system. It was found that for strongly coupled systems (i.e., when the ratio of mistuning strength to coupling is less than or equal to unity), the accuracy deteriorates significantly if the ratio of mistuning strength to damping ratio is of order greater than one. In the second approach, the modal properties of the free undamped mistuned system are first approximated using a perturbation method. Subsequently, a modal analysis is carried out to compute the forced response of each component system. This approach can be applied to strongly coupled systems with any damping, but its accuracy depends on the ratio of mistuning strength to coupling strength. Also, it was found that for weakly coupled systems (i.e., when the ratio of mistuning strength to coupling is greater than unity), only direct simulation techniques could provide accurate results for the response statistics.

More recently, Nair and Keane [18,19] introduced a class of stochastic reduced basis methods (SRBMs) to solve random algebraic equations arising from discretization of linear stochastic partial differential equations in space, time, and the random dimension of the problem. This approach essentially involves approximating the random solution process using the terms of the preconditioned stochastic Krylov subspace as basis vectors. It was shown for a class of problems that SRBMs can be orders of magnitude more accurate than traditional perturbation methods. A more detailed exposition of the theoretical underpinnings of SRBMs can be found in Refs. [20,21].

The objective of the present paper is to use SRBMs to develop an efficient numerical scheme for statistical analysis of the forced response of mistuned bladed disks. In particular, the focus is on examining the application of SRBMs to statistical analysis of disordered periodic systems. The main challenge in analyzing disordered periodic systems arises from the fact that small perturbations in the structural properties can lead to significant changes in the dynamic response. Note that this class of problems was not investigated in earlier work on SRBMs [18–21].

In the context of the forced response problem, the application of SRBMs leads to a representation of the frequency response using a linear combination of complex stochastic basis vectors with undetermined coefficients. Motivated by the theoretical analysis in Refs. [20,21], the terms of the preconditioned stochastic Krylov subspace are employed as basis vectors. Note that for the choice of preconditioner used in the present investigation and the random parameterization of the system, the basis vectors become equivalent to the terms of the perturbation series. Subsequently, two variants of the stochastic Bubnov–Galerkin (BG) scheme are presented for computing the undetermined terms in the reduced basis representation—an exact and a zero order BG scheme are presented. It is shown that the present approach leads to an explicit expression for the response as a function of the random system parameters, which enables a complete statistical characterization of the system response in a computationally efficient fashion.

Extensive numerical studies on a model problem are presented to demonstrate that highly accurate results can be obtained for the first two statistical moments of the response and the mean of the maximum blade amplitude. The results obtained using SRBMs are compared with the classical second order perturbation method (PM2) and benchmark results computed using MCS. The results clearly demonstrate that SRBMs can be up to orders of magnitude more accurate than the perturbation method. This paper concludes with an outline of some directions for further research on SRBMs for dynamic analysis.

2. Simplified model and equations of motion

The system studied here is a simplified discrete model of continuously shrouded bladed-disk assemblies [2,5,7]. It consists of a cyclic chain of N masses (each has one grounded spring and damper) interconnected by identical linear springs representing the interblade coupling. An advantage of this simple model is that it allows a straightforward modelling and can help in gaining a preliminary understanding of the computational properties of SRBMs applied to mistuning problems.

The mass and damping of each blade is considered to be identical and represented by m and c, respectively. Mistuning is assumed to originate only from the variations among the component systems' stiffnesses. The modal stiffness of the ith blade is modelled by $k_i = k_0(1+\theta_i)$, where k_0 is the nominal blade's stiffness and θ_i is the random variation in the ith blade stiffness. θ_i are assumed to be uncorrelated zero-mean Gaussian random variables with standard deviation σ. Note that for the model problem under consideration, the total number of random variables p equals the total number of degrees of freedom (d.o.f.) N. Let $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ denote the system mass, stiffness and damping matrices, respectively. Assuming that mistuning affects only the stiffness matrix $\mathbf{K}$, the equations of motion in the frequency domain can be written in the form

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{q}(\boldsymbol{\theta}) = \mathbf{F}, \tag{1}$$

where $\mathbf{A}(\boldsymbol{\theta}) = -\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} + \mathbf{K}(\boldsymbol{\theta})$ is the random dynamic stiffness matrix, ω is the external excitation frequency, $\mathrm{j} = \sqrt{-1}$, $\boldsymbol{\theta} = \{\theta_i\}$, $i = 1, \ldots, p$ is the vector of p random system parameters, and $\mathbf{q}(\boldsymbol{\theta})$ is the random displacement response. $\mathbf{F}$ is the external excitation vector chosen to be the engine order excitation force (see, for example, Ref. [7]), i.e., $\mathbf{F} = F_m\{\mathrm{e}^{\mathrm{j}\phi_i}\}^{\mathrm{T}}$, $\quad \phi_i = 2\pi n(i-1)/N$, $\quad i = 1, \ldots, N$, where F_m is the amplitude of the excitation force, ϕ_i is the phase angle of force for the ith blade component and n is the engine order. Note that for simplicity of notation, the dependence of the dynamic stiffness matrix and the response on the excitation frequency in the equations that follow are not explicitly indicated.

If $\mathbf{K}_0$ is the stiffness matrix of the perfectly tuned system, then $\mathbf{K}(\boldsymbol{\theta})$ can be expanded as

$$\mathbf{K}(\boldsymbol{\theta}) = \mathbf{K}_0 + \delta\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{p} \mathbf{K}_i\, \theta_i, \tag{2}$$

where $\delta\mathbf{K}$ is the deviation of the stiffness matrix due to mistuning and $\mathbf{K}_i$ is a deterministic matrix related to the baseline tuned structure. Note that this representation is chosen here for the sake of convenience. When the random system parameters appear non-linearly in the stiffness matrix, a similar expression can be derived by expanding $\mathbf{K}(\boldsymbol{\theta})$ in terms of orthogonal random polynomials; see, for example, Ref. [22]. Using Eq. (2), the random dynamic stiffness matrix $\mathbf{A}(\boldsymbol{\theta})$ can be written as

$$\mathbf{A}(\boldsymbol{\theta}) = \mathbf{A}_0 + \delta\mathbf{A}, \tag{3}$$

where $\mathbf{A}_0 = -\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} + \mathbf{K}_0$ and $\delta\mathbf{A} = \delta\mathbf{K}$.

Note that for parametric studies of the effects of mistuning on the forced response of the model problem under consideration, Eq. (1) can be rewritten as follows:

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{q}(\boldsymbol{\theta}) = \mathbf{F}/m, \tag{4}$$

where the random dynamic stiffness matrix in this case is given by

$$\mathbf{A}(\boldsymbol{\theta}) = -\omega^2\mathbf{I} + \mathrm{j}\omega(2\zeta\omega_0)\mathbf{I} + \omega_0^2\mathbf{A}^{und.}(\boldsymbol{\theta}), \tag{5}$$

where $\mathbf{I}$ is identity matrix, $\zeta = c/2\sqrt{k_0 m}$ is the viscous damping ratio, c is the viscous damping coefficient, $\omega_0^2 = k_0/m$ is the nominal blade natural frequency and

$$\mathbf{A}^{und.}(\boldsymbol{\theta}) = \begin{bmatrix} 1+2R^2+\theta_1 & -R^2 & 0 & \cdots & -R^2 \\ -R^2 & 1+2R^2+\theta_2 & -R^2 & 0 & 0 \\ 0 & -R^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -R^2 \\ -R^2 & 0 & \cdots & -R^2 & 1+2R^2+\theta_N \end{bmatrix}$$

is the dynamic matrix of the undamped system. $R^2 = k_c/k_0$ is a non-dimensional interblade coupling parameter, where k_c is the coupling stiffness. Note that θ_i is a non-dimensional mistuning parameter. Hence, the same representation of the random dynamic stiffness given in Eq. (3) can be used, where

$$\mathbf{A}_0 = -\omega^2\mathbf{I} + \mathrm{j}\omega(2\zeta\omega_0)\mathbf{I} + \omega_0^2 \begin{bmatrix} 1+2R^2 & -R^2 & 0 & \cdots & -R^2 \\ -R^2 & 1+2R^2 & -R^2 & 0 & 0 \\ 0 & -R^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -R^2 \\ -R^2 & 0 & \cdots & -R^2 & 1+2R^2 \end{bmatrix}$$

and

$$\delta\mathbf{A} = \omega_0^2 \begin{bmatrix} \theta_1 & & & & \\ & \theta_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \theta_N \end{bmatrix}.$$

In the next section, for the sake of generality, the stochastic reduced basis representation of the random displacement response $\mathbf{q}(\boldsymbol{\theta})$ will be derived using Eqs. (1)–(3).

3. Stochastic reduced basis representation

The fundamental idea of SRBMs is to approximate the solution of Eq. (1) using a subspace spanned by a set of stochastic basis vectors [18,19]. A theoretical justification was recently presented for employing the terms of the preconditioned stochastic Krylov subspace as basis vectors, see, for example, Refs. [20,21]. It was shown that the solution of a linear random algebraic system of equations can be approximated to an arbitrary degree of accuracy using this set of basis vectors. For the representation of the random stiffness matrix in Eq. (2), and further by employing the matrix $\mathbf{A}_0^{-1}$ as a preconditioner, it can be shown that the terms of the preconditioned stochastic Krylov subspace coincides with the perturbation series. This implies that the same results can be obtained by using the terms of the perturbation series as stochastic basis vectors.

In the present study, three basis vectors are used to represent the solution of Eq. (1) as

$$\hat{\mathbf{q}}(\boldsymbol{\theta}) = \sum_{i=0}^{2} \xi_i \psi_i(\boldsymbol{\theta}) = \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}, \tag{6}$$

where $\boldsymbol{\Psi} = [\psi_0(\boldsymbol{\theta})\psi_1(\boldsymbol{\theta})\psi_2(\boldsymbol{\theta})] \in \mathbb{C}^{N\times 3}$ and $\boldsymbol{\xi} = \{\xi_0, \xi_1, \xi_2\}^{\mathrm{T}} \in \mathbb{C}^3$ denote the matrix of complex stochastic basic vectors and the vector of undetermined coefficients, respectively.

The three terms of the PM2 are chosen as basis vectors. As mentioned earlier, this is equivalent to employing the first three basis vectors spanning the preconditioned stochastic Krylov subspace. It is assumed that $\mathbf{q}(\boldsymbol{\theta})$ can be well approximated in the subspace spanned by $\psi_0, \psi_1(\boldsymbol{\theta})$ and $\psi_2(\boldsymbol{\theta})$. The first basis vector ψ_0 is obtained by solving for the frequency response of the tuned system, i.e.,

$$\psi_0 = \mathbf{A}_0^{-1}\mathbf{F}. \tag{7}$$

The other two basis vectors are given by

$$\psi_1(\boldsymbol{\theta}) = \sum_{i=1}^{p} \frac{\partial \mathbf{q}}{\partial \theta_i}\theta_i \tag{8}$$

and

$$\psi_2(\boldsymbol{\theta}) = \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial \theta_i\,\partial \theta_j}\theta_i\theta_j. \tag{9}$$

The response sensitivities appearing in Eqs. (8) and (9) can be computed as

$$\frac{\partial \mathbf{q}}{\partial \theta_i} = -\mathbf{A}_0^{-1}\frac{\partial \mathbf{K}}{\partial \theta_i}\psi_0 \tag{10}$$

and

$$\frac{\partial^2 \mathbf{q}}{\partial \theta_i\,\partial \theta_j} = \mathbf{A}_0^{-1}\left(\frac{\partial \mathbf{K}}{\partial \theta_i}\mathbf{A}_0^{-1}\frac{\partial \mathbf{K}}{\partial \theta_j}\mathbf{q}_0 + \frac{\partial \mathbf{K}}{\partial \theta_j}\mathbf{A}_0^{-1}\frac{\partial \mathbf{K}}{\partial \theta_i}\mathbf{q}_0\right). \tag{11}$$

It can be clearly seen from the preceding equations that sensitivity analysis of large-scale systems across a broad range of excitation frequencies will be computationally expensive. This is because, at each frequency point of interest, one needs to compute an independent set of stochastic basis vectors for obtaining the statistics of the response. In other words, the matrix $\mathbf{A}_0$ appearing in Eqs. (7), (10) and (11) needs to be repeatedly inverted at each frequency of interest. This results in a significant increase in computational cost particularly when the size of the system is large and/or the response at a large number of frequency points is to be computed. The efficiency of the basis vector computation procedure can be readily improved by employing the eigenvectors of the tuned system to approximately compute the sensitivities of $\mathbf{q}$ in the modal basis [21].

In the particular case of cyclic structures, $\mathbf{A}_0$ is a circulant matrix. Since its eigenvectors coincide with the eigenvectors of the Fourier matrix $\mathbf{E}$ [23] that diagonalizes $\mathbf{A}_0$, the basis vectors can be computed very efficiently. The expression for $\mathbf{E}$ is given in Appendix A. When the system components have multiple d.o.f., $\mathbf{A}_0$ is a block-circulant matrix that can be block-diagonalized using the transformation $(\mathbf{E}^* \otimes \mathbf{I})\mathbf{A}_0(\mathbf{E}\otimes\mathbf{I})$, where $*$, $\otimes$ and $\mathbf{I}$ denote the complex conjugate transpose of a matrix, the Kronecker product and an identity matrix of size equal to that of a block in $\mathbf{A}_0$ (i.e., of a blade-disk sector), respectively. Note that the computation of the sensitivities of $\mathbf{q}$ in the modal domain can be even more efficient if a smaller set of nominal modes is used. This means that a specific family of modes can be used to arrive at a reduced dynamic stiffness matrix of the tuned system. Therefore, in the sensitivity computation, for a large number of frequency points, this reduced matrix of size of the number of selected modes is inverted at a low cost.

4. Computation of undetermined coefficients

To compute the undetermined coefficients in the stochastic reduced basis representation, stochastic variants of the BG scheme [20] are used. This involves defining a stochastic residual error vector by substituting Eq. (6) into Eq. (1) gives

$$\mathbf{r}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi} - \mathbf{F}. \tag{12}$$

For simplicity of notation, the dependence of ψ_1 and ψ_2 on the random vector $\boldsymbol{\theta}$ will not be explicitly shown in the equations that follow. In the BG scheme, the undetermined coefficients are evaluated by enforcing the condition that $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}(\boldsymbol{\theta})$. Two variants of the BG scheme are presented next for the computation of $\boldsymbol{\xi}$, which arises from the way the orthogonality condition for two random vectors is interpreted.

4.1. Zero order BG scheme

Here the undetermined coefficient vector $\boldsymbol{\xi}$ is determined by enforcing that the stochastic residual $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}(\boldsymbol{\theta})$ in an approximate sense. By considering the inner product of two random vector functions in the Hilbert space of random variables, the following condition results:

$$\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{r}(\boldsymbol{\theta}) \rangle = 0, \tag{13}$$

where superscript $*$ denotes the complex conjugate transpose and $\langle \cdot \rangle$ denotes the ensemble average. Since Eq. (13) can be interpreted as a zero order condition [20], this formulation is henceforth referred to as SRBM-BG$_0$. Eq. (13) leads to the following (3×3) reduced deterministic system of equations for the coefficients ξ_0, ξ_1 and ξ_2

$$\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi} - \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{F} \rangle = 0. \tag{14}$$

The deterministic system of equations to be solved for the vector of undetermined coefficients $\boldsymbol{\xi}$ can be written in a compact form as

$$\mathbf{A}_{SRBM.BG_0}\boldsymbol{\xi} = \mathbf{F}_{SRBM.BG_0}, \tag{15}$$

where $\mathbf{A}_{SRBM.BG_0} = \langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta}) \rangle$ and $\mathbf{F}_{SRBM.BG_0} = \langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{F} \rangle$ denote the reduced dynamic stiffness matrix and the force vector, respectively. Explicit expressions for their elements are given in Appendix B.

Once the coefficients ξ_0, ξ_1 and ξ_2 are computed by solving the reduced order problem in Eq. (15), the mean $(\mu_{\hat{\mathbf{q}}})$ and covariance matrix $\left(\sum_{\hat{\mathbf{q}}}\right)$ of the system response at each excitation

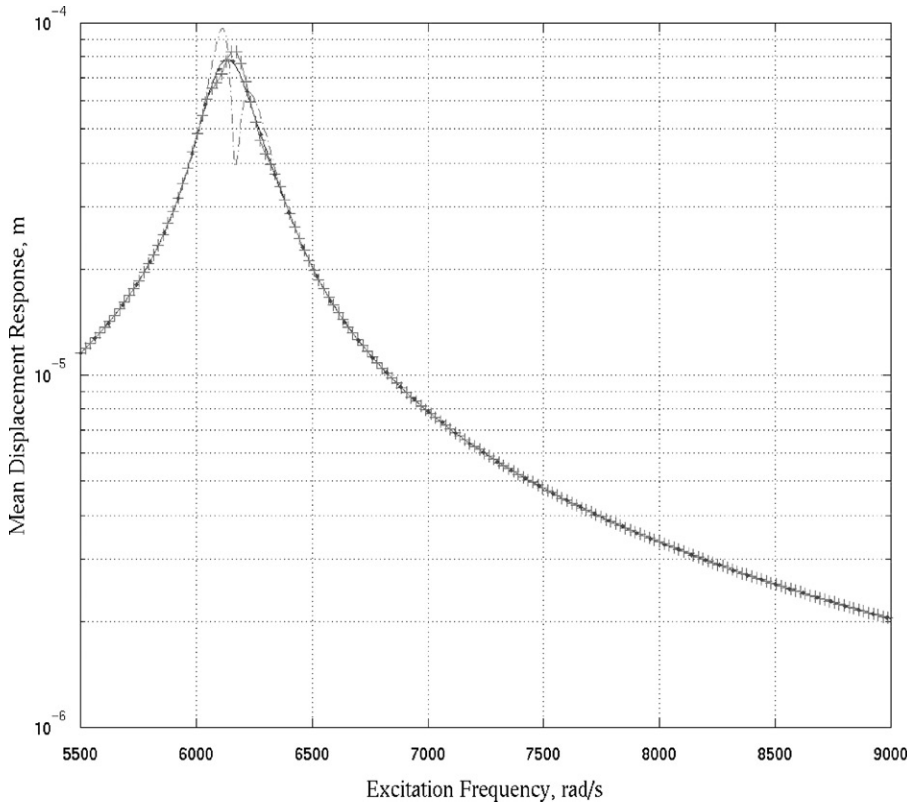

Fig. 1. Mean of the first component amplitude as a function of excitation frequency, for $R = 0.1$, $\zeta = 0.01$ and $\sigma = 2\%$. The solid line (—) represents exact results obtained by MCS, the dots represent SRBM-BG, the plusses (+) represent SRBM-BG_0 and the dash–dotted line (–.) represents PM2 results.

frequency can be computed as

$$(\mu_{\hat{\mathbf{q}}}) = \langle \hat{\mathbf{q}} \rangle = \xi_0 \langle \psi_0(\boldsymbol{\theta}) \rangle + \xi_1 \langle \psi_1(\boldsymbol{\theta}) \rangle + \xi_2 \langle \psi_2(\boldsymbol{\theta}) \rangle \tag{16}$$

and

$$\left(\sum_{\hat{\mathbf{q}}} \right) = \langle \hat{\mathbf{q}}(\boldsymbol{\theta}) \hat{\mathbf{q}}^*(\boldsymbol{\theta}) \rangle = \langle \boldsymbol{\Psi}(\boldsymbol{\theta}) \boldsymbol{\xi} \boldsymbol{\xi}^* \boldsymbol{\Psi}(\boldsymbol{\theta})^* \rangle = \sum_{i=0}^{p} \sum_{j=0}^{p} \xi_i \xi_j^* \langle \psi_i(\boldsymbol{\theta}) \psi_j^*(\boldsymbol{\theta}) \rangle. \tag{17}$$

Compact expressions for the mean and covariance matrix for the case when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables are given in Appendix C.

4.2. Exact BG scheme

By specifying that the stochastic residual error is orthogonal to the approximating space of basis vectors with probability one, an alternative formulation can be derived, henceforth referred

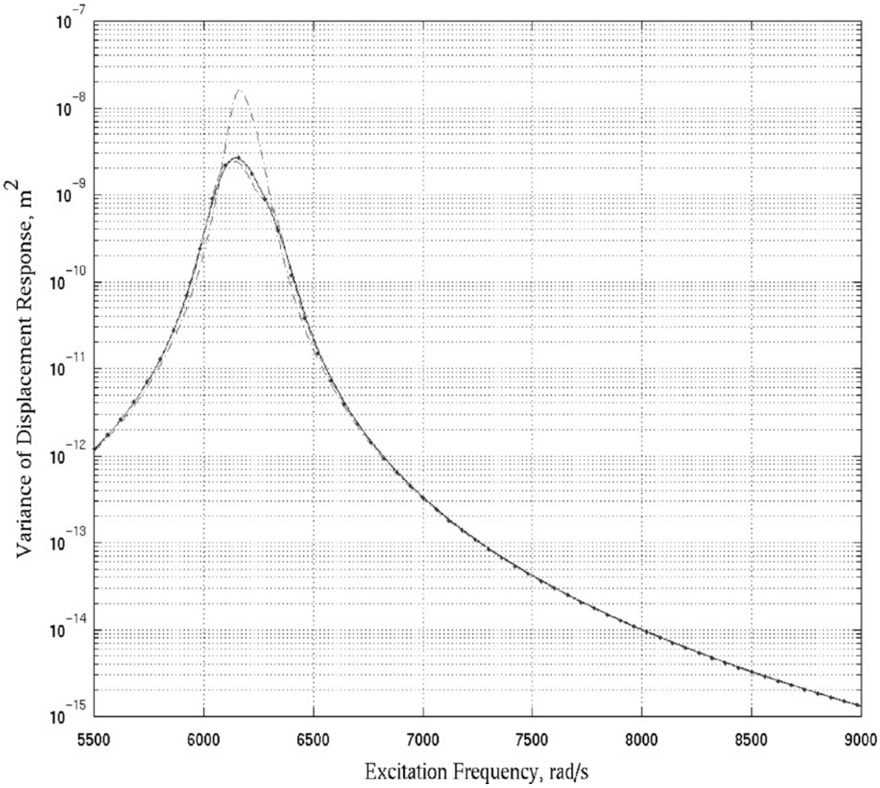

Fig. 2. Variance of the first component amplitude as a function of excitation frequency, for $R = 0.1$, $\zeta = 0.01$ and $\sigma = 2\%$. The solid line (—) represents exact results obtained by MCS, the dots represent SRBM-BG, the dashed line (– –) line represents SRBM-BG$_0$ and the dash–dotted line (–.) represents PM2 results.

to as SRBM-BG. In contrast to the SRBM-BG$_0$ formulation, this projection scheme leads to random function models for the undetermined coefficients since the following reduced order *random* system of equations has to be solved:

$$\mathbf{A}_{SRBM.BG}\boldsymbol{\xi} = \mathbf{F}_{SRBM.BG}, \tag{18}$$

where $\mathbf{A}_{SRBM.BG} = \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})$ and $\mathbf{F}_{SRBM.BG} = \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{F}$ are the reduced order random dynamic stiffness matrix and the reduced order force vector, respectively.

It can be seen that explicit computation of a random function description of the undetermined coefficients will involve the symbolic inversion of $\mathbf{A}_{SRBM.BG}$. Note that this is possible here since only three vectors are used in the reduced basis representation of the random displacement response $\mathbf{q}(\boldsymbol{\theta})$. However, since the resulting approximation is a highly non-linear function of the random system parameters, analytical characterization of the response statistics is no longer readily possible. Fortunately, MCS schemes can be applied to compute efficiently the response statistics by sampling the stochastic reduced basis representation with random function models for the undetermined coefficients.

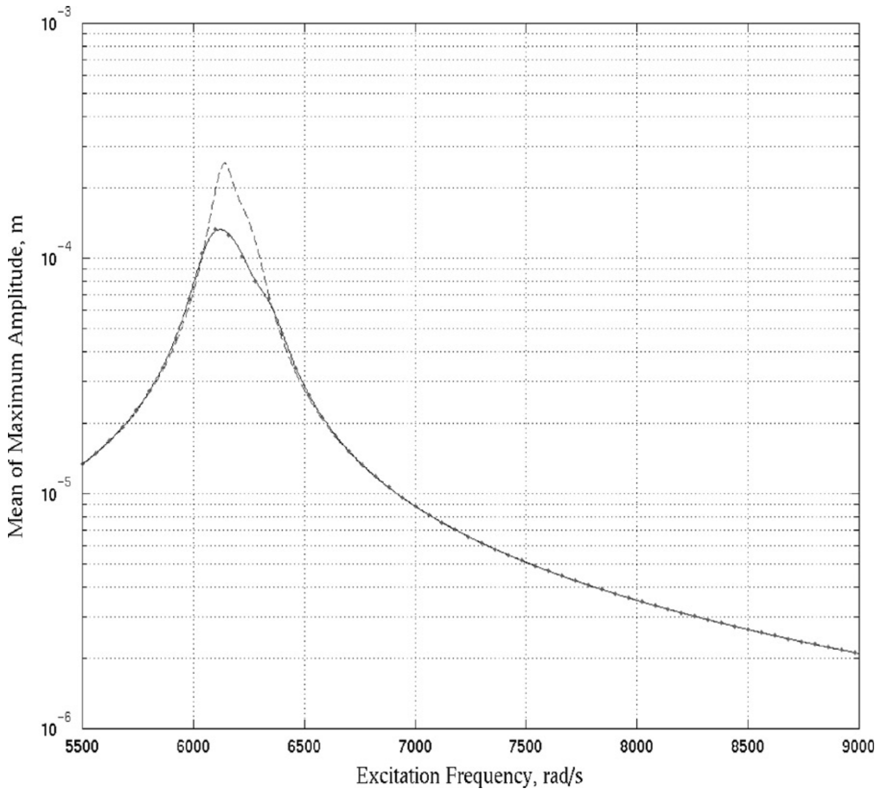

Fig. 3. Mean of the maximum amplitude among the blades across the frequency region of interest, for $R = 0.1$, $\zeta = 0.01$ and $\sigma = 2\%$. The solid line (—) represents exact results obtained by MCS, the dots represent SRBM-BG and the dashed line (– –) represents PM2 results.

5. A note on theoretical aspects

For the sake of completeness, some important theoretical properties of SRBMs which were derived in Refs. [20,21] are cited. The first result mentioned in the previous section states that the solution of a linear random algebraic system of equations with a non-singular coefficient matrix always lie in the stochastic Krylov subspace. This guarantees that nearly exact results can be computed provided a sufficient number of stochastic basis vectors are deployed in the response representation. However, the computational cost and memory requirements increase significantly when the higher order basis vectors are used. Fortunately, for many problems of practical interest, three basis vectors are sufficient to achieve highly accurate results; see also Section 6.

A desirable feature of any stochastic subspace projection scheme is that some measure of the error in the approximated solution must converge when the number of basis vectors is increased. In Ref. [21], it was proved for SRBM-BG$_0$ that the **A**-norm of the error is mean square convergent. For the exact BG scheme, it was conjectured that the **A**-norm of the error converges in probability. However, these results hold only for Hermitian positive-definite matrices. In the context of frequency response analysis of linear stochastic structural systems, the coefficient matrix $\mathbf{A}(\boldsymbol{\theta})$ turns out to be complex symmetric. For such non-Hermitian matrices, convergence

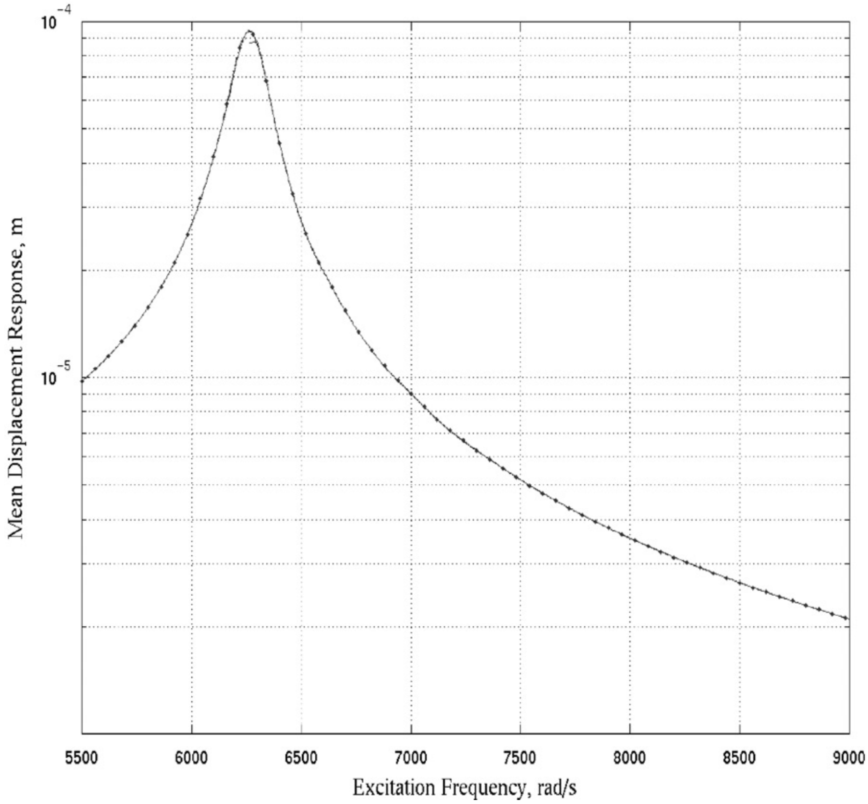

Fig. 4. Mean of the first component amplitude as a function of excitation frequency, for $R = 0.325$, $\zeta = 0.01$ and $\sigma = 2\%$. The solid line (—) represents exact results obtained by MCS, the dots represent SRBM-BG, the dashed line (– –) line represents SRBM-BG$_0$ and the dash–dotted line (–.) represents PM2 results.

results can be established for the $\mathbf{L}_2$ norm of the residual only if an oblique stochastic subspace projection scheme is used. This involves incorporating the Petrov–Galerkin condition that the residual error is orthogonal to the stochastic subspace $\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})$. Equations similar to those presented in Appendix B can be readily derived for the reduced order terms when this oblique stochastic subspace projection scheme is employed, see Ref. [20] for details.

However, in the present study, results are presented only for the orthogonal BG projection scheme. It is shown that, even though this scheme is not provably optimal for non-Hermitian matrices, highly accurate results can be obtained for the response statistics.

6. Results and discussions

The methods developed here are applied to a coupled 10-blade assembly. The values used for the model parameters [24] are given by $m = 0.0114$ kg and $k_0 = 430\,000$ N/m (the nominal blade natural frequency $\omega_0 = 6141.6$ rad/s). The viscous damping ratio is fixed as $\zeta = 0.01$. Results are obtained using both the zero-order and exact BG schemes. The mean and variance of the

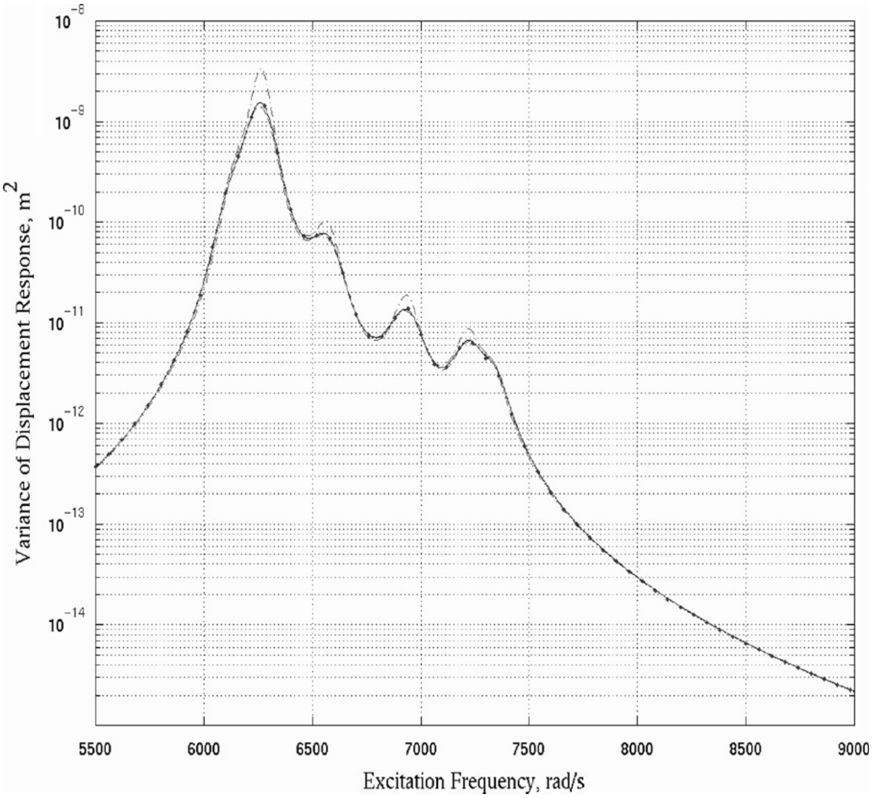

Fig. 5. Variance of the first component amplitude as a function of excitation frequency, for $R = 0.325$, $\zeta = 0.01$ and $\sigma = 2\%$, same legend as in Fig. 4.

frequency response of a typical d.o.f., i.e., the first blade component are shown. These quantities are compared to the statistics computed using the PM2 and benchmark results generated by applying MCS.

Two different values are considered for the mistuning standard deviation: $\sigma = 2$ % and 5%. For each case, three different values of the non-dimensional coupling strength parameter are considered: weak interblade coupling $R = 0.1$, moderately weak interblade coupling $R = 0.325$ and strong interblade coupling $R = 0.5$. The mean maximum amplitude experienced by the mistuned assemblies across the frequency region of interest are computed for the case of weak coupling, i.e., $R = 0.1$. It will therefore be possible to assess the accuracy of results provided by the proposed methods and their limitations. Note that in all the figures presented, the first two statistical moments are plotted as functions of the frequency of the first engine order excitation.

Figs. 1 and 2 shows the mean and standard deviation of the first component, obtained by various methods for $\sigma = 2$ % and $R = 0.1$. In contrast to PM2, SRBM-BG and SRBM-BG$_0$ yield very accurate results for this set of parameters. However, the mean maximum amplitude in the frequency domain was generated only by SRBM-BG. The results obtained are presented in Fig. 3 and compared with PM2 and MCS. It can be readily observed from the results that PM2 fails to predict the maximum response statistics, especially at the frequencies clustered around the nominal natural frequency.

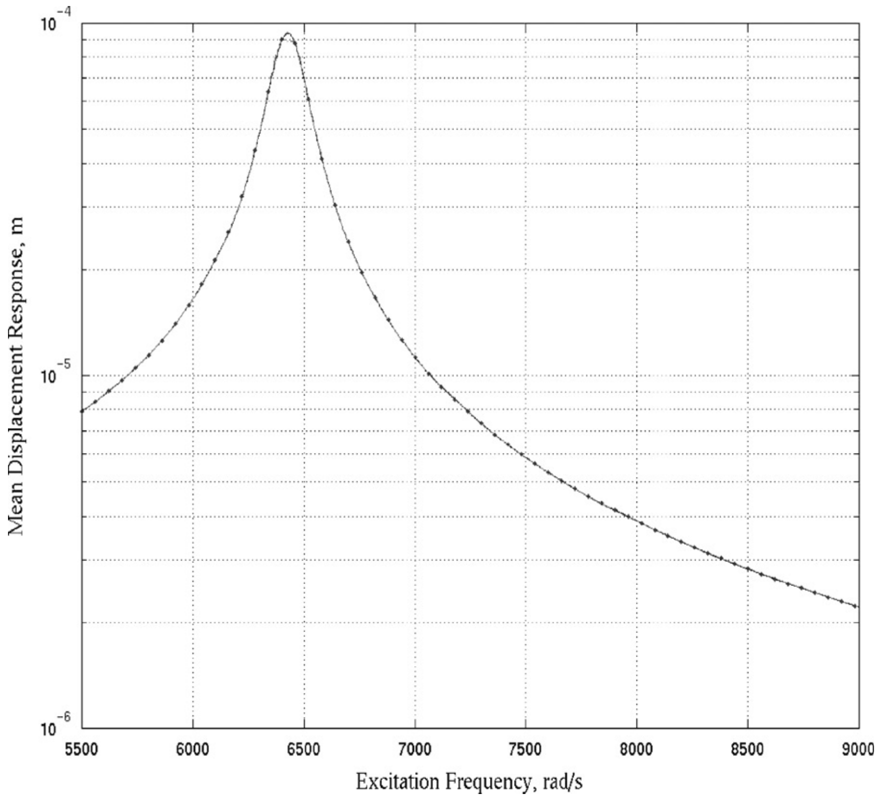

Fig. 6. Mean of the first component amplitude as a function of excitation frequency, for $R = 0.5$, $\zeta = 0.01$ and $\sigma = 2\%$, same legend as in Fig. 4.

As found in Ref. [7], in the case of weak coupling configuration ($R = 0.1$), the perturbation method gives highly erroneous results when the mistuning parameters are perturbed. To circumvent this problem, Wei and Pierre proposed a modified perturbation method. In their approach, the coupling parameter is used as a perturbation quantity rather than the mistuning parameters. Furthermore, for systems where the ratio of mistuning to coupling $\sigma/R^2 > O(1)$, it was concluded that only the MCS technique could provide accurate estimates of the response statistics. Fig. 3 shows that under the same conditions of coupling and mistuning strengths, SRBM-BG gives highly accurate results.

Figs. 4 and 5 display the mean and standard deviation of the first component amplitudes when $\sigma = 2\%$ and the interblade coupling is moderately weak, i.e., $R = 0.325$. It can be seen that the results obtained using SRBM-BG and SRBM-BG$_0$ agree very well with MCS results. In contrast, PM2 misses the peak mean amplitude and also fails to predict the second moment at different excitation frequencies; see Fig. 5. When the interblade coupling is strong, i.e., $R = 0.5$, SRBM-BG and SRBM-BG$_0$ can still be applied even when the damping ratio is low ($\zeta = 1\%$). This is illustrated in Figs. 6 and 7. In contrast to the stochastic reduced basis approach, the traditional PM2 is valid only for ($\sigma/R^2 \leqslant O(1)$ and $\sigma/\zeta \leqslant O(1)$.

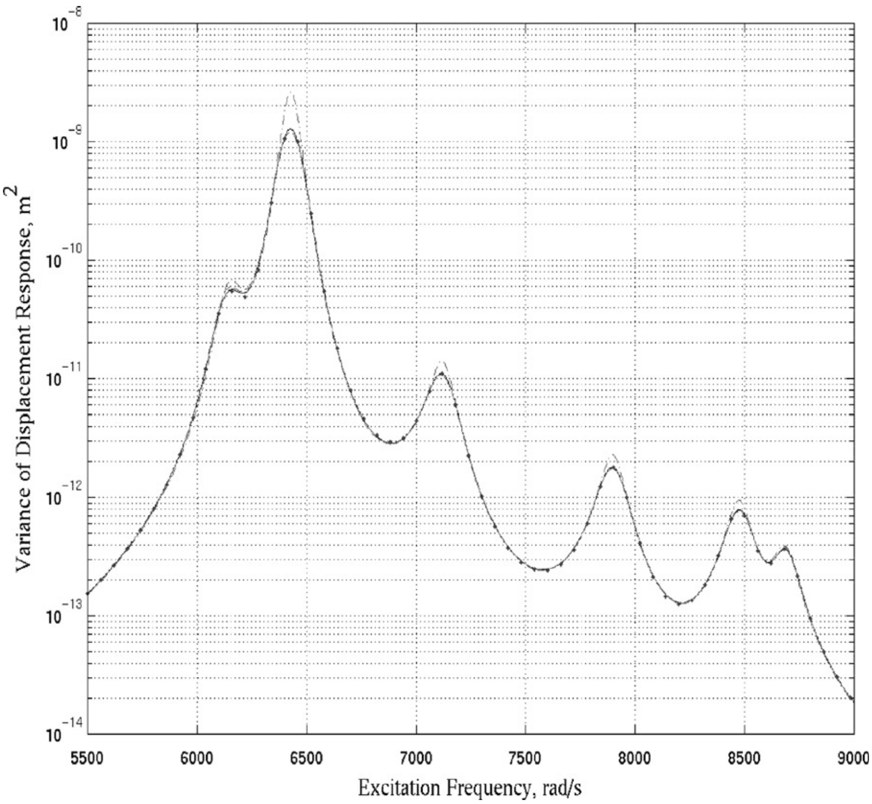

Fig. 7. Variance of the first component amplitude as a function of excitation frequency, for $R = 0.5$, $\zeta = 0.01$ and $\sigma = 2\%$, same legend as in Fig. 4.

When the mistuning strength σ is increased to 5%, the results for the mean and variance of the component amplitudes are expected to deteriorate. For $R = 0.1$, SRBM-BG provides highly accurate results despite the fact that the coupling is weak and the damping is low. This is displayed in Figs. 8 and 9. The results also show that SRBM-BG$_0$ misses the higher amplitudes within the region of clustered resonant frequencies. It appears that this formulation tends to break down for this level of disorder. It can also be observed that PM2 fails for this case, since the perturbation series tends to diverge for this parameter setting.

Fig. 10 shows the mean of the maximum amplitude among the blades. It can be seen that while PM2 clearly over predicts, SRBM-BG gives highly accurate results. Finally, the interblade coupling is increased to moderately weak ($R = 0.325$) and strong ($R = 0.5$) values. The results for these cases are shown in Figs. 11–14. It is found that, at resonance frequencies, PM2 fails to predict the statistical properties of the component amplitudes. As observed for the earlier cases, SRBM-BG yields highly accurate results.

In summary, the numerical results obtained for this example problem clearly demonstrates the accuracy of SRBMs. In particular, the response statistical moments computed using SRBM-BG can be orders of magnitude more accurate than the classical perturbation method.

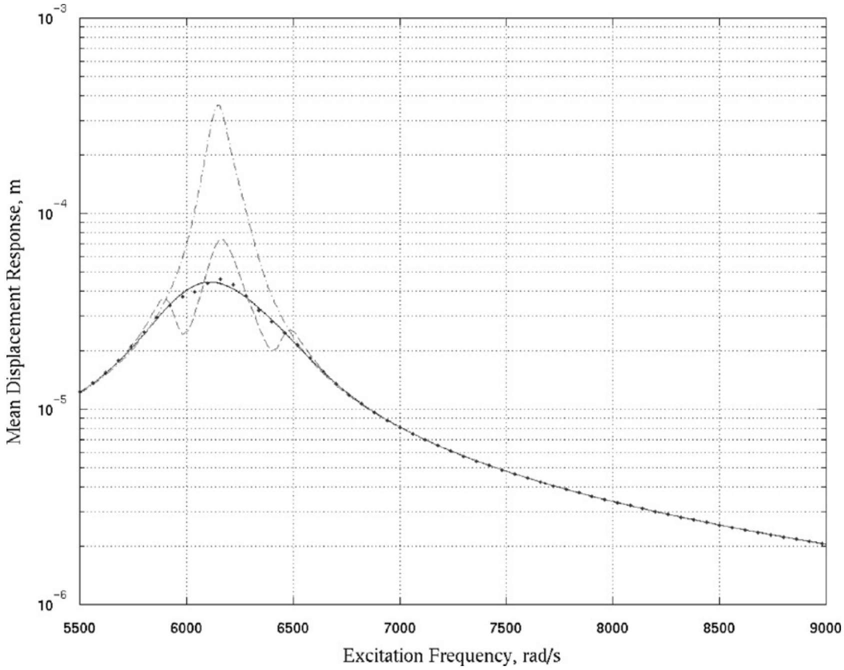

Fig. 8. Mean of the first component amplitude as a function of excitation frequency, for $R = 0.1$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 4.

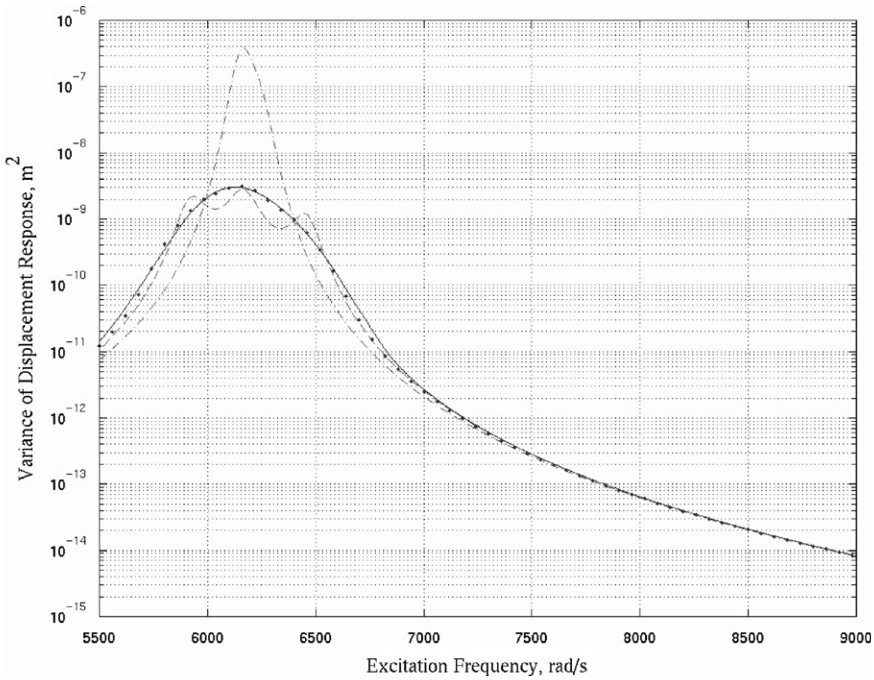

Fig. 9. Variance of the first component amplitude as a function of excitation frequency, for $R = 0.1$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 4.

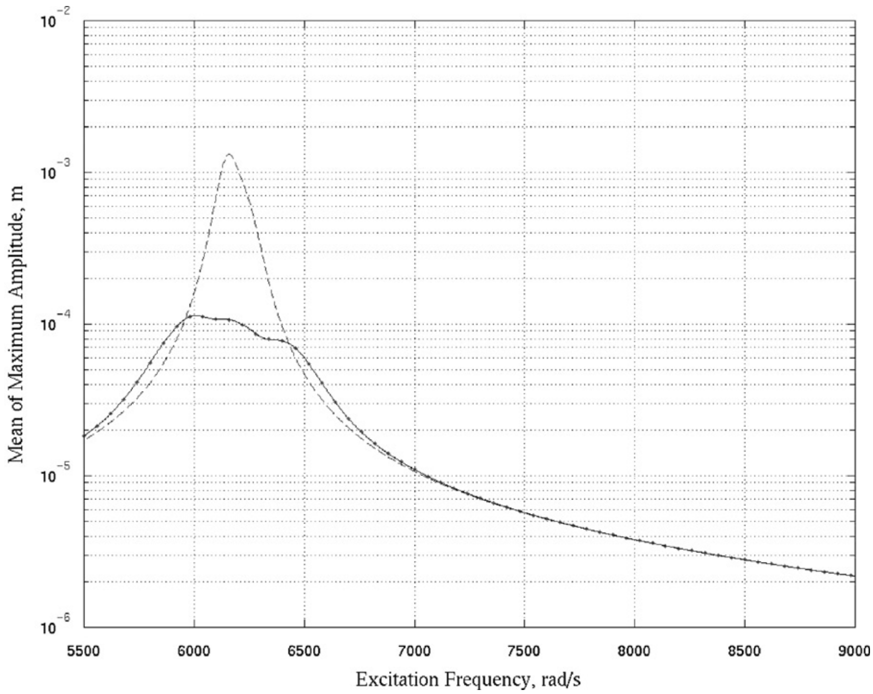

Fig. 10. Mean of the maximum amplitude among the blades across the frequency region of interest, for $R = 0.1$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 3.

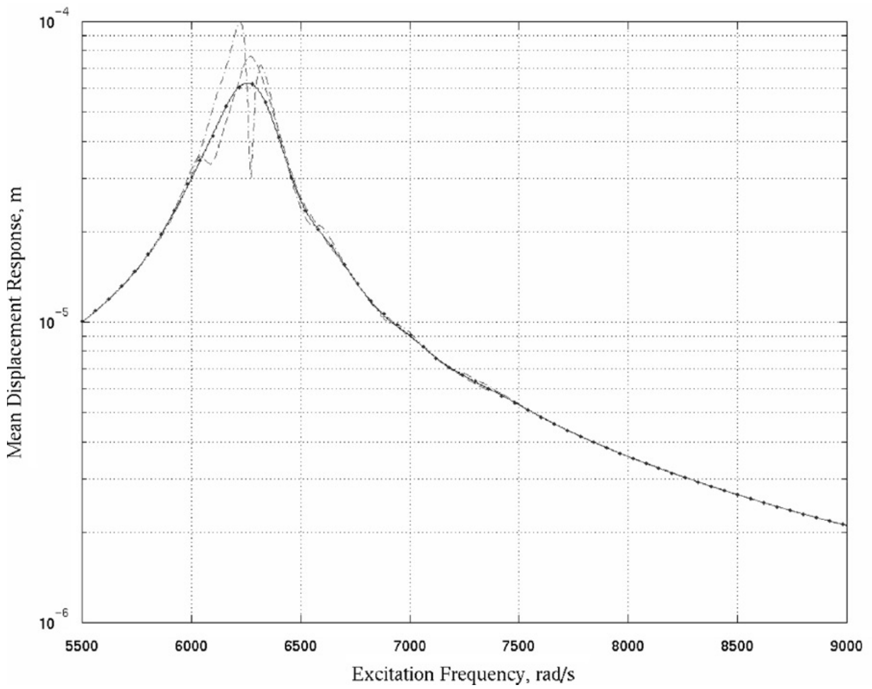

Fig. 11. Mean of the first component amplitude as a function of excitation frequency, for $R = 0.325$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 4.

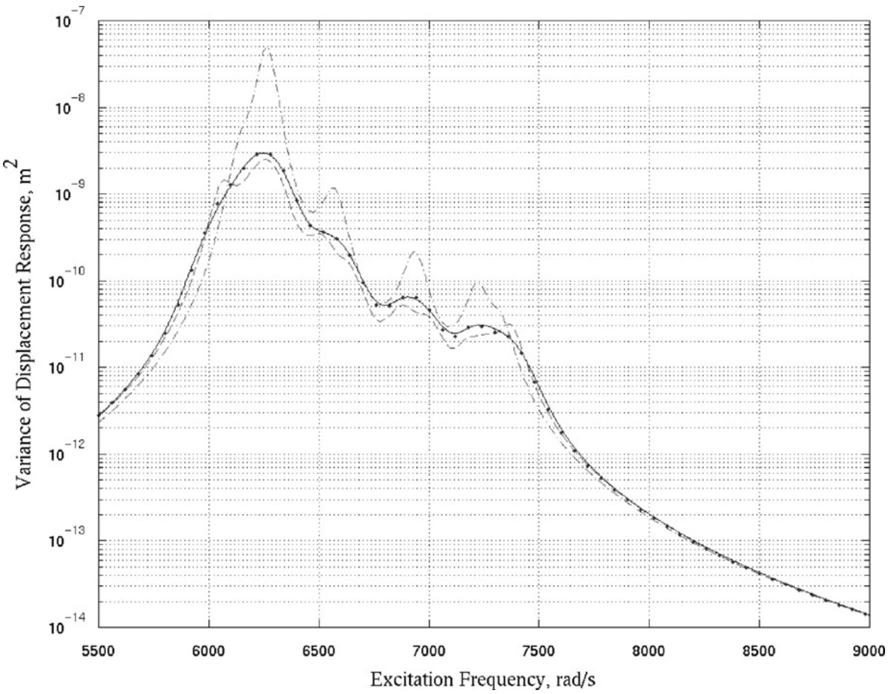

Fig. 12. Variance of the first component amplitude as a function of excitation frequency, for $R = 0.325$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 4.

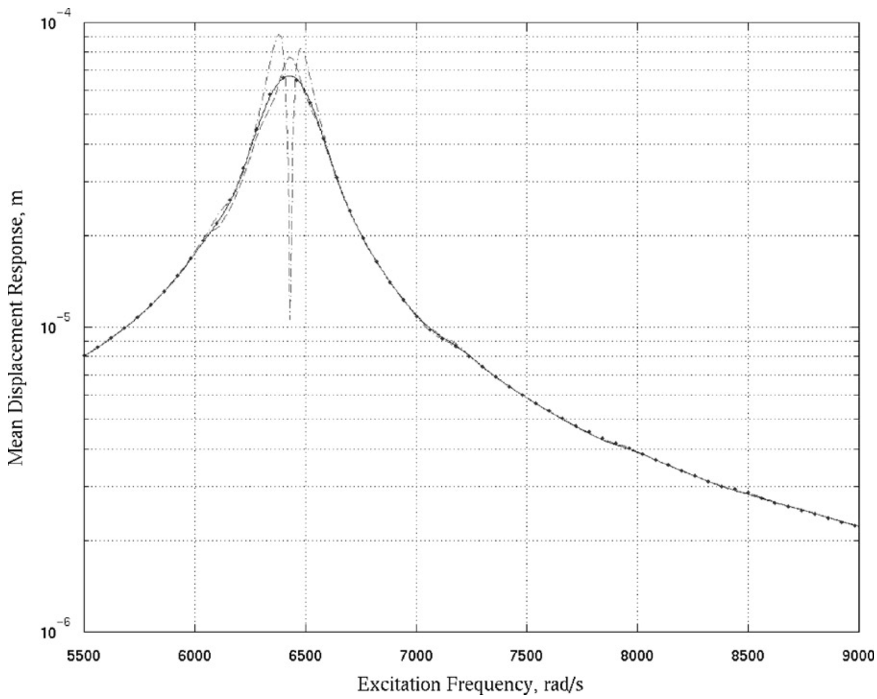

Fig. 13. Mean of the first component amplitude as a function of excitation frequency, for $R = 0.5$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 4.

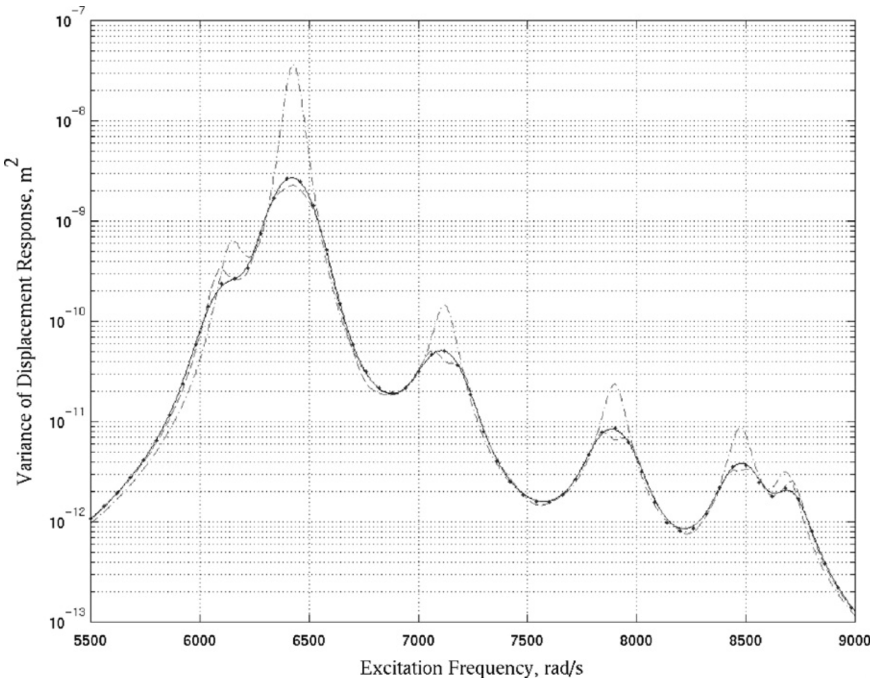

Fig. 14. Variance of the first component amplitude as a function of excitation frequency, for $R = 0.5$, $\zeta = 0.01$ and $\sigma = 5\%$, same legend as in Fig. 4.

7. Concluding remarks

In this paper, a stochastic reduced basis approach is presented for computing the forced response statistics of mistuned bladed-disk assemblies. The fundamental idea is to approximate the frequency domain response using the subspace spanned by the first three terms of the preconditioned stochastic Krylov subspace. For the model problem considered, this is shown to be equivalent to employing the terms of the perturbation series as stochastic basis vectors. Subsequently, two stochastic variants of the Bubnov–Galerkin (BG) scheme were presented for computing the undetermined coefficients in the reduced basis representation. It is shown that this allows one to arrive at explicit expressions for the system response as a function of the random system parameters. This, in turn, enables an efficient statistical characterization of the system response. Some theoretical properties of the BG scheme applied to stochastic processes are also outlined.

Extensive numerical studies on a model problem are presented to test the range of applicability of the present approach. The results computed using the stochastic reduced basis methods (SRBMs) have been compared with the classical second order perturbation method and benchmark results generated using Monte Carlo simulation. It is shown that SRBMs give accurate results for the response statistics across a wide range of coupling strength, mistuning strength, and damping properties. In particular, the results clearly demonstrate that SRBMs can be orders of magnitude more accurate than the perturbation method, particularly for the mean of the maximum blade displacement.

Even though, the results presented here are for a simple model problem, SRBMs can be applied to mistuning analysis of bladed disks analyzed using large-scale finite element models. It also remains to be seen whether employing the oblique stochastic subspace projection outlined in this paper can further improve the accuracy of the response statistics.

Acknowledgements

This work was supported by the University Technology Partnership for Design—a partnership between Rolls Royce Plc., BAE Systems and the Universities of Southampton, Cambridge and Sheffield.

Appendix A

Expression of the Fourier matrix of size N:

$$\mathbf{E} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)(N-1)} \end{bmatrix}, \quad w = \mathrm{e}^{2\mathrm{j}\pi/N}, \quad \mathrm{j} = \sqrt{-1}. \tag{A.1}$$

Appendix B

Here, the expressions for the elements of the reduced order matrix $\mathbf{A}_{SRBM.BG_0}$ and reduced force vector $\mathbf{F}_{SRBM.BG_0}$ are presented when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables with standard deviation σ.

$$\mathbf{A}_{SRBM.BG_0}(1,1) = \psi_0^* \mathbf{A}_0 \psi_0, \tag{B.1}$$

$$\mathbf{A}_{SRBM.BG_0}(1,2) = \sigma^2 \psi_0^* \sum_{i=1}^{p} \left\{ \frac{\partial \mathbf{K}}{\partial \theta_i} \frac{\partial \mathbf{q}}{\partial \theta_i} \right\}, \tag{B.2}$$

$$\mathbf{A}_{SRBM.BG_0}(1,3) = \sigma^2 \psi_0^* \mathbf{A}_0 \sum_{i=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial \theta_i^2}, \tag{B.3}$$

$$\mathbf{A}_{SRBM.BG_0}(2,1) = \sigma^2 \sum_{i=1}^{p} \left\{ \frac{\partial \mathbf{q}^*}{\partial \theta_i} \frac{\partial \mathbf{K}}{\partial \theta_i} \right\} \psi_0, \tag{B.4}$$

$$\mathbf{A}_{SRBM.BG_0}(2,2) = \sigma^2 \sum_{i=1}^{p} \left(\frac{\partial \mathbf{q}^*}{\partial \theta_i} \mathbf{A}_0 \frac{\partial \mathbf{q}}{\partial \theta_i} \right), \tag{B.5}$$

$$\mathbf{A}_{SRBM.BG_0}(2,3) = \sum_{i,j,k,l=1}^{p} \langle \theta_i\theta_j\theta_k\theta_l \rangle \left(\frac{\partial \mathbf{q}^*}{\partial \theta_i} \frac{\partial \mathbf{K}}{\partial \theta_j} \frac{\partial^2 \mathbf{q}}{\partial \theta_k \partial \theta_l} \right), \tag{B.6}$$

$$\mathbf{A}_{SRBM.BG_0}(3,1) = \sigma^2 \left\{ \sum_{i=1}^{p} \frac{\partial^2 \mathbf{q}^*}{\partial \theta_i^2} \right\} \mathbf{A}_0 \boldsymbol{\psi}_0, \tag{B.7}$$

$$\mathbf{A}_{SRBM.BG_0}(3,2) = \sum_{i,j,k,l=1}^{p} \langle \theta_i\theta_j\theta_k\theta_l \rangle \left(\frac{\partial^2 \mathbf{q}^*}{\partial \theta_i \partial \theta_j} \frac{\partial \mathbf{K}}{\partial \theta_k} \frac{\partial \mathbf{q}}{\partial \theta_l} \right), \tag{B.8}$$

$$\mathbf{A}_{SRBM.BG_0}(3,3) = \sum_{i,j,k,l=1}^{p} \langle \theta_i\theta_j\theta_k\theta_l \rangle \left(\frac{\partial^2 \mathbf{q}^*}{\partial \theta_i \partial \theta_j} \mathbf{A}_0 \frac{\partial^2 \mathbf{q}}{\partial \theta_k \, \partial \theta_l} \right). \tag{B.9}$$

Note that $\langle \theta_i\theta_j\theta_k\theta_l \rangle = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, where δ_{ij} is the Kronecker delta function.

$$\mathbf{F}_{SRBM.BG_0} = \left\{ \boldsymbol{\psi}_0 \quad 0 \quad \sigma^2 \textstyle\sum_{i=1}^{p} \dfrac{\partial^2 \mathbf{q}}{\partial \theta_i^2} \right\}^* \mathbf{F}. \tag{B.10}$$

Appendix C

Here, we present expressions for the mean ($\mu_{\hat{\mathbf{q}}}$) and covariance $\left(\sum_{\hat{\mathbf{q}}}\right)$ of the system response for the SRBM-BG$_0$ scheme. We consider the case when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables with standard deviation σ.

Using Eq. (16), the mean frequency response can be computed as

$$(\mu_{\hat{\mathbf{q}}}) = \left\langle \xi_0\psi_0 + \xi_1 \sum_{i=1}^{p} \frac{\partial \mathbf{q}}{\partial \theta_i} \theta_i + \xi_2 \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial \theta_i \partial \theta_j} \theta_i \theta_j \right\rangle \tag{C.1}$$

or

$$(\mu_{\hat{\mathbf{q}}}) = \xi_0\psi_0 + \xi_2\sigma^2 \sum_{i=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial \theta_i^2}. \tag{C.2}$$

Similarly, the covariance of the frequency response can be computed as

$$\begin{aligned} \left(\sum_{\hat{\mathbf{q}}} \right) &= \xi_0\xi_0^*\psi_0\psi_0^* + \sigma^2\xi_0\xi_2^*\psi_0 \sum_{i=1}^{p} \frac{\partial^2 \mathbf{q}^*}{\partial^2 \theta_i} + \sigma^2 \xi_1\xi_1^* \sum_{i=1}^{p} \frac{\partial \mathbf{q}}{\partial \theta_i} \frac{\partial \mathbf{q}^*}{\partial \theta_i} \\ &\quad + \sigma^2\xi_2\xi_0^* \left\{ \sum_{i=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial^2 \theta_i} \right\} \psi_0^* + \xi_2\xi_2^* \sum_{i,j,k,l=1}^{p} \langle \theta_i\theta_j\theta_k\theta_l \rangle \frac{\partial^2 \mathbf{q}}{\partial \theta_i \, \partial \theta_j} \frac{\partial^2 \mathbf{q}^*}{\partial \theta_k \, \partial \theta_l}. \end{aligned} \tag{C.3}$$

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