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Temporal Aggregation of Seasonally Near-Integrated Processes*

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Abstract

We investigate the implications that temporally aggregating, either by average sampling or systematic (skip) sampling, a seasonal process has on the integration properties of the resulting series at both the zero and seasonal frequencies. Our results extend the existing literature in three ways. First, they demonstrate the implications of temporal aggregation for a general seasonally integrated process with S seasons. Second, rather than only considering the aggregation of seasonal processes with exact unit roots at some or all of the zero and seasonal frequencies, we consider the case where these roots are local-to-unity such that the original series is near-integrated at some or all of the zero and seasonal frequencies. These results show, among other things, that systematic sampling, although not average sampling, can impact on the non-seasonal unit root properties of the data; for example, even where an exact zero frequency unit root holds in the original data it need not necessarily hold in the systematically sampled data. Moreover, the systematically sampled data could be near-integrated at the zero frequency even where the original data is not. Third, the implications of aggregation on the deterministic kernel of the series are explored.

Keywords: Aggregation, systematic sampling, average sampling, seasonal (near-) unit roots, demodulation.

JEL classification: C12, C22.

1 Introduction

The use of temporally aggregated data is relatively common in empirical applications using macroeconomic and financial data. In practice temporal aggregation amounts to converting higher frequencies (e.g. monthly) observed in the data to lower frequency components (e.g. quarterly or annual). As such, temporal aggregation therefore has important implications for the time series

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properties of the resulting series (see, among others, Wei, 2006, Casals, Jerez and Satoca, 2009, and Silvestrini and Veredas, 2008), not least because the underlying data transformation has the potential to confuse unobservable cycles with observable cycles in the data. As discussed in Pons (2006), the sampling effect known as aliasing (see, in particular, Koopmans, 1974) implies, for example, that when a unit root in seasonally unadjusted quarterly data is detected, it is not possible to state whether that unit root is also present at a monthly frequency with the same period or occurs at another monthly frequency with a period which is not observable when using quarterly observations. Interestingly, this also has an impact on the long-run component because, for example, a zero-frequency unit root appearing in a quarterly data set could be the result of sampling a monthly series with a root of modulus one (with a slight abuse of language, we will refer to roots of modulus one simply as unit roots in what follows) present at a frequency other than zero; see, for example, Granger and Siklos, (1995).

In this paper we extend the results of Pons (2006) in a number of empirically important directions. First, we generalise his results for the case of quarterly data to the effects of temporally aggregating a seasonal time series observed with a general number of seasons, S. We will focus primarily on systematic sampling (also sometimes referred to as skip sampling) and average sampling but other forms of temporal aggregation are also covered by our general framework. Second, we use a more general local-to-unity framework which allows us to show the effects of aggregation on seasonal cycles for both exact unit root and local to unity (or near-unit root) processes. Third, our analysis also covers the effects of aggregation on the deterministic part of the seasonal process. We will consider the impact of data aggregation by both systematic sampling and average sampling. In obtaining our analytical results we use properties of circulant matrices associated with near-integrated processes at the zero, Nyquist and seasonal harmonic frequencies and discuss their connection with the demodulation operator introduced by Granger and Hatanaka (1964).[1]

We show how temporal aggregation modifies the demodulation operator associated with near-integrated processes at the zero, Nyquist and harmonic seasonal frequencies. In particular we show that a (near-) unit root at the zero frequency component of the data is always preserved under temporal aggregation, regardless of whether systematic or average sampling is used. However, where systematic sampling is used, the proximity of this zero frequency root to unity can depend on the proximity to unity of roots at the seasonal frequencies in the original data. In particular, even where an exact zero frequency unit root holds in the original data it is not guaranteed to hold in the systematically sampled data if the relevant seasonal frequencies in the original data do not admit exact unit roots. Moreover, under systematic sampling near-integrated behaviour at the zero frequency can obtain in the temporally aggregated data even where the original data does not contain a (near-) unit root at the zero frequency. In contrast, (near-) unit roots at the Nyquist frequency are preserved only when $Q := S/S_A$, S_A denoting the number of seasons per year after temporal aggregation has been applied, is odd. When Q is even these roots either vanish from the series under average sampling or are shifted to the zero frequency under systematic sampling.

[1] Further details on complex demodulation can be found in, *inter alia*, Granger and Hatanaka (1964, Chapter 10) and Bloomfield (2000, Chapter 7). In the context considered in this paper, the complex demodulation operator, $e^{\pm i\omega t}$, will be used to shift a complex-valued autoregressive process, such as $\left(1 - \phi e^{\pm i\omega} L\right) y_t = \varepsilon_t$, from its original frequency, ω, to the zero frequency.

Under systematic sampling, (near-) unit roots at the seasonal harmonic frequencies, say ω_k, are shifted under temporal aggregation to frequency $Q\omega_k$. This also occurs under average sampling if $\sin\left(\frac{Q\omega_k}{2}\right) \neq 0$, whereas these roots vanish from the temporally aggregated series otherwise. We also show that the demodulation operator impacts on the terms in the usual trigonometric form of deterministic seasonality in the same way so that qualitatively similar implications are seen for the deterministic component in the temporally aggregated data.

The outline of the remainder of the paper is as follows. In section 2, we detail the general seasonally near-integrated process which we will consider and detail some relevant large sample properties of the series. In section 3, we analytically explore the effects of temporal aggregation on both the stochastic and deterministic components of the data using a framework which contains both systematic and average sampling as special cases. Section 4 concludes. Mathematical proofs and additional supporting material are provided in an on-line supplementary appendix.

2 Seasonally Near-Integrated Processes

Consider the univariate seasonal time series $\{y_{Sn+s}\}$, observed with seasonal periodicity S (e.g. for monthly data, $S = 12$) over $T := SN$ seasonal cycles (e.g. years) which satisfies the data generating process [DGP],

$$y_{Sn+s} = x_{Sn+s} + \mu_{Sn+s} \tag{2.1}$$

$$\alpha(L)x_{Sn+s} = u_{Sn+s}, \; s = 1-S, ..., 0, \;\; n = 1, 2, \dots, N. \tag{2.2}$$

For simplicity we will assume in what follows that the initial conditions $x_{1-S}, \dots, x_0$ are all zero.[2] In (2.1)–(2.2), $\mu_{Sn+s} := \delta' z_{Sn+s}$ is a purely deterministic component, further details on which will be given in section 3.2. The shocks u_{Sn+s} are assumed to follow a mean zero weakly stationary process, precise assumptions on which will be given below. Finally, $\alpha(L) := 1 - \sum_{j=1}^{P} \alpha_j^* L^j$ is a P-th order, $P \leq S$, autoregressive polynomial, L denoting the conventional lag operator such that $L^{Sj+k} y_{Sn+s} = y_{S(n-j)+s-k}$. We assume that this polynomial can be factorised as $\alpha(L) = \prod_{k=0}^{\lfloor S/2 \rfloor} \omega_k(L)^{h_k}$, $\lfloor \cdot \rfloor$ denoting the integer part of its argument, and where $h_k \in \{1, 0\}$, such that $P := \sum_{k=0}^{\lfloor S/2 \rfloor} f_k h_k$, where $\omega_k(L)$ is a polynomial of order f_k whose coefficients are all real. Hence, if $h_k = 1$ ($h_k = 0$) then so $\omega_k(L)$ will be (not be) a factor of $\alpha(L)$. In this factorisation, $\omega_0(L) := (1 - \alpha_0 L)$ associates the parameter α_0 with the zero frequency $\omega_0 := 0$, $\omega_k(L) := [1 - 2(\alpha_k \cos\omega_k - \beta_k \sin\omega_k)L + (\alpha_k^2 + \beta_k^2)L^2]$ corresponds to the conjugate (harmonic) seasonal frequencies $(\omega_k, 2\pi - \omega_k)$, $\omega_k = 2\pi k/S$, with the associated parameters α_k and β_k, $k = 1, \dots, S^*$, $S^* := \lfloor (S-1)/2 \rfloor$, and, for S even, $\omega_{S/2}(L) := (1 + \alpha_{S/2} L)$ associates the parameter $\alpha_{S/2}$ with the Nyquist frequency $\omega_{S/2} := \pi$.[3]

[2] Relaxing this condition induces the presence of an additional term in $X_0 := (x_{1-S}, ..., x_0)'$ in the right member of the general solution (and approximations thereof) to the stochastic difference equation in (2.2) which appear in, for example, Equations (2.6) and (2.7). Here the coefficient on X_0 would depend on n if any of the roots are local-to-unity but would be fixed otherwise. However, for the purposes of establishing the unit root properties of the aggregated series this additional term is irrelevant and so to simplify our presentation we set the initial conditions to zero. Moreover, it is also worth noting that allowing the initial conditions to be of $o_p(N^{1/2})$ would not alter any of the large sample results which follow.

[3] In what follows, it is understood that terms relating to frequency π are to be omitted when S is odd and that where reference is made to the Nyquist frequency this is understood only to apply where S is even.

In order to allow for the presence of (near-) unit root behaviour at some or all of the zero and seasonal frequency components of the data, we will adopt a local-to-unity framework for the factors of $\alpha(L)$. Specifically, a local-to-unit root at the zero frequency is obtained by setting $\alpha_0 = \alpha_{0,T} = \exp\left(\frac{c_0}{T}\right) \cong \left(1 + \frac{c_0}{T}\right)$, where c_0 is a finite constant. For the harmonic seasonal frequency components, a complex pair of local-to-unit roots at frequency ω_k is obtained setting $\alpha_k = \alpha_{k,T} = \exp\left(\frac{c_k}{T}\right) \cong \left(1 + \frac{c_k}{T}\right)$ and $\beta_k = 0$, where c_k is a finite constant. Notice that setting $\beta_k = 0$ imposes that $\omega_k(L) = \left(1 - \alpha_k e^{-\mathrm{i}\omega_k} L\right)\left(1 - \alpha_k e^{\mathrm{i}\omega_k} L\right)$, where $\mathrm{i} := \sqrt{-1}$, and, hence, that these roots occur at frequency ω_k, as seems natural given the seasonal identification of the harmonic frequencies. Finally, for the Nyquist frequency component, a local-to-unit root is obtained by setting $\alpha_{S/2} = \alpha_{S/2,T} = \exp(\frac{c_{S/2}}{T}) \cong (1 + \frac{c_{S/2}}{T})$, where $c_{S/2}$ is again a finite constant. Under this structure, $\{y_{Sn+s}\}$ admits both single ($k = 0, S/2$) and pairs of complex conjugate ($k = 1, \ldots, S^*$) roots with modulus in the neighborhood of unity at frequencies ω_k, $k = 0, ..., \lfloor S/2 \rfloor$. These roots are stable for $c_k < 0$, explosive for $c_k > 0$, and are exact unit roots when $c_k = 0$, $k = 0, \ldots, \lfloor S/2 \rfloor$. Notice, therefore, that the structure considered here allows for different local-to-unity parameters, some of which could be zero such that a pure unit root obtains, to hold at each of the zero and seasonal frequencies. Crucially, (near-) unit roots will only occur in y_{Sn+s} at those frequencies, ω_k, for which $h_k = 1$, $k = 0, \ldots, \lfloor S/2 \rfloor$.

The shocks u_{Sn+s} in (2.2) are taken to satisfy the conditions of Assumption 1 below.

Assumption 1. *The error process* $\{u_{Sn+s}\}$ *is i.i.d. with* $E(u_{Sn+s}) = 0$ *and* $E(u^2_{Sn+s}) = \sigma^2 < \infty$.

Remark 1: In order to simplify our presentation, Assumption 1 does not allow for stationary autocorrelation or conditional heteroskedasticity in u_{Sn+s}. However, our assumptions could be weakened to allow for these effects without qualitatively altering the results presented in this paper. In particular, we could permit u_{Sn+s} to follow a stationary and invertible ARMA process thereby allowing non-zero stable roots to occur and these could be identified with any observable spectral frequency. Doing so would introduce additional nuisance parameters, deriving from the stationary autocorrelation, into the large sample results given in Lemma 1 below (see del Barrio Castro, Rodrigues and Taylor, 2018, for details on how our key results given below in Equation (2.7) and Lemma 1 would need to be modified when u_{Sn+s} is serially correlated) but would not alter the conclusions drawn from them regarding the integrational properties of the aggregated data. Moreover, the implied ARMA dynamics in the time aggregated data arising from stationary and invertible ARMA shocks in the original data is already well documented in the literature; see, for example, Chapter 20 of Wei (2006) for an excellent summary. □

Under the local-to-unity framework outlined above, we can write (2.2) as

$$\left(\Delta_0^{c_0}\right)^{h_0} \left(\Delta_{S/2}^{c_{S/2}}\right)^{h_{S/2}} \prod_{k=1}^{S^*} \left(\Delta_k^{c_k}\right)^{h_k} x_{Sn+s} = u_{Sn+s} \tag{2.3}$$

where $\Delta_0^{c_0} := 1 - \alpha_{0,T} L$, $\Delta_{S/2}^{c_{S/2}} := 1 + \alpha_{S/2,T} L$, $\Delta_k^{c_k} := 1 - 2\cos[\omega_k]\, \alpha_{k,T} L + \alpha^2_{k,T} L^2$, for $k = 1, ..., S^*$ and where $\alpha_{i,T} := \exp\left(\frac{c_i}{T}\right) \cong \left(1 + \frac{c_i}{T}\right)$, $i = 0, 1, ..., \lfloor S/2 \rfloor$. For the purpose of the analysis that follows it will be convenient to introduce the vector of seasons representation. For the process under analysis in (2.3), it is observed that $\alpha(L) := 1 - \sum_{j=1}^{P} \alpha_j^* L^j$ is a P-th order, $P \leq S$, autoregressive polynomial, which we factor as $\alpha(L) := (\Delta_0^{c_0})^{h_0} \left(\Delta_{S/2}^{c_{S/2}}\right)^{h_{S/2}} \prod_{k=1}^{S^*} \left(\Delta_k^{c_k}\right)^{h_k}$. For any value $P \leq S$, (2.3) can always be represented as a first order vector of seasons process, *viz.*,

$$\Psi_0 X_n = \Psi_1 X_{n-1} + U_n^*, \; n = 1, ..., N \tag{2.4}$$

where $X_n := \left(x_{Sn-(S-1)}, x_{Sn-(S-2)}, ..., x_{Sn}\right)'$, $X_{n-1} := \left(x_{S(n-1)-(S-1)}, x_{S(n-1)-(S-2)}, ..., x_{S(n-1)}\right)'$, and where X_0 is a vector of zeros and $U_n^* := \left(u_{Sn-(S-1)}, u_{Sn-(S-2)}, ..., u_{Sn}\right)'$ is a vector white noise process with zero mean vector and variance $\sigma^2 I_S$, where I_S is an $S \times S$ identity matrix, and Ψ_0 and Ψ_1 are lower and upper triangular $S \times S$ parameter matrices, respectively, whose precise form is given in section S.5 of the supplementary appendix.

In what follows it will be convenient to consider the reduced form of (2.4); that is,

$$X_n = \Phi X_{n-1} + U_n \tag{2.5}$$

where $\Phi := \Psi_0^{-1}\Psi_1$ and $U_n := \Psi_0^{-1} U_n^*$. The error term U_n has zero mean and variance matrix $E\left(U_n U_n'\right) = \Psi_0^{-1} E\left(U_n^* U_n^{*\prime}\right) \Psi_0^{-1\prime} = \sigma^2 \Psi_0^{-1} \Psi_0^{-1\prime}$. Backward substitution in (2.5) yields that,

$$X_n = \sum_{k=0}^{n-1} \Phi^k U_{n-k} \tag{2.6}$$

with the convention that $\Phi^0 = I_S$.

As shown in the proof of Lemma 1 (which can be found in the supplementary appendix), the representation in (2.6) can be equivalently written as the sum of partial sum processes relating to the zero and seasonal frequencies, plus a lower order remainder term; viz,

$$X_n = \psi_0 C_0 \sum_{i=1}^{n} \alpha_{0,N}^{n-i} U_i + \psi_{S/2} C_{S/2} \sum_{i=1}^{n} \alpha_{S/2,N}^{n-i} U_i + \sum_{k=1}^{S^*} \left(\psi_k^- C_k^- + \psi_k^+ C_k^+\right) \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + o_p(N^{1/2}) \tag{2.7}$$

where $\alpha_{i,N} := \exp\left(\frac{c_i}{N}\right) \cong \left(1 + \frac{c_i}{N}\right)$, $i = 0, 1, ..., \lfloor S/2 \rfloor$, and

$$C_0 := \quad Circ[1, 1, 1, \ldots, 1], \;\; C_{S/2} := Circ[1, -1, 1, \ldots, -1], \tag{2.8}$$

$$C_k^- := \quad Circ\left[1, e^{-\mathrm{i}(S-1)\omega_k}, e^{-\mathrm{i}(S-2)\omega_k}, \cdots, e^{-\mathrm{i}\omega_k}\right], \tag{2.9}$$

$$C_k^+ := \quad Circ\left[1, e^{+\mathrm{i}(S-1)\omega_k}, e^{+\mathrm{i}(S-2)\omega_k}, \cdots, e^{+\mathrm{i}\omega_k}\right], \tag{2.10}$$

where C_0, $C_{S/2}$, C_k^- and C_k^+ are $S \times S$ circulant matrices of rank 1. The weights ψ_0, $\psi_{S/2}$ and, ψ_k^- and ψ_k^+, $k = 1, \ldots, S^*$, are non-zero (zero) when $\Delta_0^{c_0}$, $\Delta_{S/2}^{c_{S/2}}$ and $\Delta_k^{c_k}$, $k = 1, \ldots, S^*$, respectively are factors (not factors) of $\alpha(L)$. So, for example, if $h_k = 0$ ($h_k = 1$) for some $k \in \{1, \ldots, S^*\}$, then $\psi_k^- = \psi_k^+ = 0$ ($\psi_k^- \neq 0$ and $\psi_k^+ \neq 0$). The specific value of these weights will depend of the total number of factors present in $\alpha(L)$. In Remarks 3 and 4 below we provide some specific examples. It is seen from the representation in (2.7) that a different circulant matrix arises for each spectral frequency that admits a (near-) unit root. Being able to use this approximation to the representation in (2.6) enables us to derive Lemma 1 below and, as a result, then to demonstrate clearly the impact that aggregation has on each of these components as the time span of the data increases.

In Lemma 1 we now provide a multivariate invariance principle for X_n in (2.7). Where u_{Sn+s} is serially uncorrelated this contains, as a particular case, the result given in Lemma A.1 of del Barrio Castro, Rodrigues and Taylor (2018) (see Remark 4 below), and will form the basic building block for the asymptotic results that will be provided in this paper.

Lemma 1. *Let x_{Sn+s} be generated according to (2.3), and let Assumption 1 hold on u_{Sn+s}. Then, as $T \to \infty$, and denoting weak convergence by "$\Rightarrow$", it follows that,*

$$N^{-1/2} X_{\lfloor rN \rfloor} \Rightarrow \sigma \left[\psi_0 C_0 \mathbf{J}_{c_0}(r) + \psi_{S/2} C_{S/2} \mathbf{J}_{c_{S/2}}(r) + \sum_{k=1}^{S^*} \left(\psi_k^- C_k^- + \psi_k^+ C_k^+ \right) \mathbf{J}_{c_k}(r) \right], \; r \in [0,1] \tag{2.11}$$

where $\mathbf{J}_{c_k}(r) := (J_{c_k,1-S}(r), J_{c_k,2-S}(r), ..., J_{c_k,0}(r))'$, $k = 0, 1, ..., \lfloor S/2 \rfloor$, is an $S \times 1$ vector standard Ornstein-Uhlenbeck (OU) process such that $d\mathbf{J}_{c_k}(r) = c_k \mathbf{J}_{c_k}(r)\, dr + d\mathbf{W}(r)$, with $\mathbf{W}(r)$ an $S \times 1$ vector standard Brownian motion and where C_0, $C_{S/2}$, C_k^- and C_k^+, $k = 1, \ldots, S^*$, are the $S \times S$ circulant matrices defined in (2.8)-(2.10). The coefficients ψ_0 and $\psi_{S/2}$ will be non-zero if $\Delta_0^{c_0}$ and $\Delta_{S/2}^{c_{S/2}}$, respectively, are factors of $\alpha(L)$, while ψ_k^- and ψ_k^+ will be non-zero if $\Delta_k^{c_k}$, $k \in \{1, \ldots, S^*\}$ is a factor of $\alpha(L)$.

Remark 2: The $S \times S$ circulant matrices, C_0, $C_{S/2}$, C_k^- and C_k^+ for $k = 1, \ldots, S^*$, can be written as, $C_0 := \mathbf{v}_0 \mathbf{v}_0{}'$, where $\mathbf{v}_0{}' := (1, 1, 1, ..., 1)$, $C_{S/2} := \mathbf{v}_{S/2} \mathbf{v}_{S/2}{}'$, where $\mathbf{v}_{S/2}{}' := (-1, 1, -1, ..., 1)$, and $C_k^- := \mathcal{E}_{k1}^- \mathcal{E}_{k2}^{-\prime}$ and $C_k^+ := \mathcal{E}_{k1}^+ \mathcal{E}_{k2}^{+\prime}$, where $\mathcal{E}_{k1}^- := \left(1, e^{-\mathrm{i}\omega_k}, e^{-\mathrm{i}2\omega_k}, ..., e^{-\mathrm{i}(S-1)\omega_k}\right)'$, $\mathcal{E}_{k1}^+ := \left(1, e^{+\mathrm{i}\omega_k}, e^{+\mathrm{i}2\omega_k}, ..., e^{+\mathrm{i}(S-1)\omega_k}\right)'$, $\mathcal{E}_{k2}^- := \left(1, e^{-\mathrm{i}(S-1)\omega_k}, ..., e^{-\mathrm{i}2\omega_k}, e^{-\mathrm{i}\omega_k}\right)'$ and $\mathcal{E}_{k2}^+ := \left(1, e^{+\mathrm{i}(S-1)\omega_k}, ..., e^{+\mathrm{i}2\omega_k}, e^{+\mathrm{i}\omega_k}\right)'$. For a generic circulant matrix, say $C := Circ\,[a_1, a_2, a_3, \cdots, a_S]$, of order $S \times S$ it is always possible to write $C = F \Lambda F^*$ where F is the matrix of eigenvectors, which for all circulant matrices is defined as

$$F := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-\mathrm{i}\frac{2\pi}{S}} & e^{-\mathrm{i}\frac{4\pi}{S}} & \cdots & e^{-\mathrm{i}\frac{2(S-1)\pi}{S}} \\ 1 & e^{-\mathrm{i}\frac{4\pi}{S}} & e^{-\mathrm{i}\frac{8\pi}{S}} & \cdots & e^{-\mathrm{i}\frac{4(S-1)\pi}{S}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-\mathrm{i}\frac{2(S-1)\pi}{S}} & e^{-\mathrm{i}\frac{4(S-1)\pi}{S}} & \cdots & e^{-\mathrm{i}\frac{2(S-1)^2\pi}{S}} \end{bmatrix},$$

F^* is the conjugate transpose of F, and $\Lambda := diag\,[\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_S]$, where λ_j, $j = 1, 2, \cdots, S$, are the eigenvalues of C. The latter can be obtained as $\lambda_j := P_{\mathcal{C}}(\exp(\frac{2\pi}{S})^{j-1})$, where $P_{\mathcal{C}}(z) := \sum_{j=1}^{S} a_j z^{j-1}$ is the polynomial associated with the circulant matrix C. Hence, C_0, $C_{S/2}$, C_k^- and C_k^+, $k = 1, \ldots, S^*$, all have rank one and, from Theorem 3.1.1 in Fuller (1996), the non-zero eigenvalues of C_0, $C_{S/2}$, C_k^- and C_k^+, which are equal to S, are located in the first position for C_0, in position $S/2-1$ for $C_{S/2}$, in position $k+1$ for C_k^-, and in position $S-k+1$ for C_k^+ of the principal diagonal of Λ. For further details on circulant matrices see, for example, Davis (1979), Osborn and Rodrigues (2002), and Smith, Taylor and del Barrio Castro (2009). □

Remark 3: To take an example, consider the case where $\alpha(L) = \Delta_0^{c_0}\Delta_k^{c_k}$, for some $k \in \{1, ..., S^*\}$. Then, $h_0 = 1$ and $h_k = 1$, and it can be shown that $\psi_0 = \frac{1}{2(1-\cos(\omega_k))}$, $\psi_k^+ = b_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-1}\right)$ and $\psi_k^- = a_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-1}\right)$, where for any $k \in \{1, ..., S^*\}$, we define $b_k := \frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_k}}$ and $a_k := \frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_k}}$. As a second example, suppose $\alpha(L) = \Delta_{S/2}^{c_{S/2}}\Delta_k^{c_k}$, again for some $k \in \{1, ..., S^*\}$. In this case, $\psi_{S/2} = \frac{1}{2(1+\cos(\omega_k))}$, $\psi_k^+ = b_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}+1}\right)$ and $\psi_k^- = a_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}+1}\right)$. As a final example, consider the case where $\alpha(L) = \Delta_k^{c_k}\Delta_j^{c_j}$, for some $j, k \in \{1, ..., S^*\}$ with $j \neq k$. In this case,

$$\begin{aligned}
\psi_k^- &= a_k\left\{a_j\left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j}-e^{\mathrm{i}\omega_k}}\right)+b_j\left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j}-e^{\mathrm{i}\omega_k}}\right)\right\}\\
\psi_j^- &= a_j\left\{a_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_j}}\right)+b_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_j}}\right)\right\}\\
\psi_k^+ &= b_k\left\{a_j\left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j}-e^{-\mathrm{i}\omega_k}}\right)+b_j\left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j}-e^{-\mathrm{i}\omega_k}}\right)\right\}\\
\psi_j^+ &= b_j\left\{a_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_j}}\right)+b_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_j}}\right)\right\}.
\end{aligned}$$

Derivations for the three examples above are provided in the supplementary appendix. □

Remark 4: For the case where a (near-) integrated component is present at all of the zero and seasonal frequencies, such that $h_k = 1$, for $k = 0, ..., \lfloor S/2 \rfloor$, it follows from Lemma A.1 of del Barrio Castro, Rodrigues and Taylor (2018), noting that the circulant matrices $C_k := Circ\left[\cos[0], \cos[\omega_k], \cos[2\omega_k], \ldots, \cos[(S-1)\omega_k]\right]$, $k = 1, \ldots, S^*$, which appear in Lemma A.1 of del Barrio Castro, Rodrigues and Taylor (2018) can be written[4] as $C_k = \frac{1}{2}(C_k^- + C_k^+)$, that

$$N^{-1/2}X_{\lfloor rN \rfloor} \Rightarrow \frac{\sigma}{S}\left[C_0\mathbf{J}_{c_0}(r) + C_{S/2}\mathbf{J}_{c_{S/2}}(r) + \sum_{i=1}^{S^*}\left(C_k^- + C_k^+\right)\mathbf{J}_{c_k}(r)\right],\ r \in [0,1]. \tag{2.12}$$

The result in (2.12) is a special case of Lemma 1 above. Notice that in this example each of the ψ_0, $\psi_{S/2}$, ψ_k^- and ψ_k^+, $k = 1, \ldots, S^*$, weights is equal to $1/S$. □

Remark 5: Using the results given for the circulant matrices C_0, $C_{S/2}$, C_k^- and C_k^+, $k = 1, \ldots, S^*$, in Remark 2, the vector of seasons representation in (2.7) can also be written as,

$$X_n = \psi_0\mathbf{v}_0\mathbf{v}_0{}'\sum_{i=1}^{n}\alpha_{0,N}^{n-i}U_i + \psi_{S/2}\mathbf{v}_{S/2}\mathbf{v}_{S/2}{}'\sum_{i=1}^{n}\alpha_{S/2,N}^{n-i}U_i + \sum_{k=1}^{S^*}\left(\psi_k^-\mathcal{E}_{k1}^-\mathcal{E}_{k2}^{-\prime} + \psi_k^+\mathcal{E}_{k1}^+\mathcal{E}_{k2}^{+\prime}\right)\sum_{i=1}^{n}\alpha_{k,N}^{n-i}U_i + o_p(N^{1/2}).$$

Defining $\mathbf{v}_k^- := e^{-\mathrm{i}\omega_k}\mathcal{E}_{j1}^- = (e^{-\mathrm{i}\omega_k}, e^{-\mathrm{i}\omega_k 2}, ..., e^{-\mathrm{i}\omega_k S})'$ and $\mathbf{v}_k^+ := e^{\mathrm{i}\omega_k}\mathcal{E}_{j1}^+ = (e^{\mathrm{i}\omega_k}, e^{\mathrm{i}\omega_k 2}, ..., e^{\mathrm{i}\omega_k S})'$, $k = 1, ..., S^*$, and using the results that $e^{-\mathrm{i}\omega_k}\mathcal{E}_{j1}^- = e^{-\mathrm{i}\omega_k}\mathcal{E}_{j2}^+$ and $e^{\mathrm{i}\omega_k}\mathcal{E}_{j1}^+ = e^{\mathrm{i}\omega_k}\mathcal{E}_{j2}^-$, it is seen that

[4] Using the fact that for $\omega_k = 2\pi k/S$, $(S-1)\,\omega_k = 2\pi k - 2\pi k/S$, and $\cos(j\omega_k) = \cos((S-j)\,\omega_k) = \frac{1}{2}(e^{-\mathrm{i}(S-j)\omega_k} + e^{\mathrm{i}(S-j)\omega_k})$, $j = 0, 1, \ldots, S-1$.

the foregoing equation can be equivalently written as

$$X_n = \psi_0 \mathbf{v}_0 \mathbf{v}_0{}' \sum_{i=1}^{n} \alpha_{0,N}^{n-i} U_i + \psi_{S/2} \mathbf{v}_{S/2} \mathbf{v}_{S/2}{}' \sum_{i=1}^{n} \alpha_{S/2,N}^{n-i} U_i + \sum_{k=1}^{S^*} \left(\psi_k^- \mathbf{v}_k^- \mathbf{v}_k^{+\prime} + \psi_k^+ \mathbf{v}_k^+ \mathbf{v}_k^{-\prime} \right) \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + o_p(N^{1/2}). \tag{2.13}$$

The representation in (2.13) will prove convenient for our subsequent analysis of the impact of temporal aggregation on the unit root and stationarity properties of x_{Sn+s}. Notice in particular, that the pre-multiplication of the near-integrated processes $\mathbf{v}_0{}' \sum_{i=1}^{n} \alpha_{0,N}^{n-i} U_i$, $\mathbf{v}_{S/2}{}' \sum_{i=1}^{n} \alpha_{S/2,N}^{n-i} U_i$, $\mathbf{v}_k^{+\prime} \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i$ and $\mathbf{v}_k^{-\prime} \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i$, $k = 1, ..., S^*$, by the vectors $\mathbf{v}_0$, $\mathbf{v}_{S/2}$, $\mathbf{v}_k^-$, $\mathbf{v}_k^+$, $k = 1, ..., S^*$, respectively, transforms these to near-integrated processes whose spectral peak occurs at the zero, π and ω_k, $k = 1, ..., S^*$, frequencies respectively; cf. Definition 1.1 of Gregoir (2006, p.48). These transformation vectors, also known as demodulation operators, were introduced by Granger and Hatanaka (1964) and have also been employed by, among others, Gregoir (1999, 2006, 2010) and del Barrio Castro, Rodrigues and Taylor (2018). As we will see in the next section, temporal aggregation can alter the form of these demodulation operators, and this can lead to the presence of near-integrated behaviour at different spectral frequencies to those observed in the data before aggregation. □

Remark 6: Using the result in (2.13), the multivariate FCLT result in (2.11) of Lemma 1 can be equivalently be expressed as,

$$N^{-1/2} X_{\lfloor rN \rfloor} \Rightarrow \sigma \left[\psi_0 \mathbf{v}_0 \mathbf{v}_0{}' \mathbf{J}_{c_0}(r) + \psi_{S/2} \mathbf{v}_{S/2} \mathbf{v}_{S/2}{}' \mathbf{J}_{c_{S/2}}(r) + \sum_{k=1}^{S^*} \left(\psi_k^- \mathbf{v}_k^- \mathbf{v}_k^{+\prime} + \psi_k^+ \mathbf{v}_k^+ \mathbf{v}_k^{-\prime} \right) \mathbf{J}_{c_k}(r) \right],$$

$r \in [0,1]$. Moreover, noting that $\mathbf{v}_k^{+\prime} = \mathbf{h}_k^{\alpha\prime} + \mathrm{i}\mathbf{h}_k^{\beta\prime}$ and $\mathbf{v}_k^{-\prime} = \mathbf{h}_k^{\alpha\prime} - \mathrm{i}\mathbf{h}_k^{\beta\prime}$ where

$$\begin{bmatrix} \mathbf{h}_k^{\alpha\prime} \\ \mathbf{h}_k^{\beta\prime} \end{bmatrix} := \begin{bmatrix} \cos[\omega_k] & \cos[2\omega_k] & \cdots & \cos[S\omega_k] \\ \sin[\omega_k] & \sin[2\omega_k] & \cdots & \sin[S\omega_k] \end{bmatrix},$$

the FCLT can also be written as,

$$\begin{aligned} N^{-1/2} X_{\lfloor rN \rfloor} \Rightarrow \sigma \Big\{ & \psi_0 \mathbf{v}_0 \sqrt{S} J_{0,c_0}(r) + \psi_{S/2} \mathbf{v}_{S/2} \sqrt{S} J_{S/2,c_{S/2}}(r) + \\ & \sum_{k=1}^{S^*} \left[\psi_k^- \mathbf{v}_k^- \sqrt{S/2} \left(J_{k,c_k}^{\alpha}(r) + \mathrm{i} J_{k,c_k}^{\beta}(r) \right) + \psi_k^- \mathbf{v}_k^+ \sqrt{S/2} \left(J_{k,c_k}^{\alpha}(r) - \mathrm{i} J_{k,c_k}^{\beta}(r) \right) \right] \Big\}, \end{aligned} \tag{2.14}$$

$r \in [0,1]$, where $J_{j,c_j}(r) := S^{-1/2} \mathbf{v}_j' \mathbf{J}_{c_j}(r)$, $j = 0, S/2$, $J_{k,c_k}^{\alpha}(r) := \mathbf{h}_k^{\alpha\prime} (S/2)^{-1/2} \mathbf{J}_{c_k}(r)$ and $J_{k,c_k}^{\beta}(r) := \mathbf{h}_k^{\beta\prime} (S/2)^{-1/2} \mathbf{J}_{c_k}(r)$, $k = 1, ..., S^*$, are seen to be S mutually independent (by virtue of the fact that they are orthogonal linear combinations of the elements of vector OU processes each of which is driven by the same vector standard Brownian motion process) scalar OU processes. □

3 The Impact of Temporal Aggregation

3.1 Temporal Aggregation

Following Franses and Boswijk (1996), we consider temporal aggregation schemes that can be written such that the temporally aggregated data, when written in vector of seasons form, is defined via the original data, again in vector of seasons form, and a sample aggregation matrix $\mathbf{A}$ as

$$Y_n^A := \mathbf{A} Y_n, \quad n = 1, 2, ..., N. \tag{3.1}$$

In (3.1), $\mathbf{A}$ is an $S_A \times S$ matrix of full row rank where S is the number of seasons prior to aggregation and S_A is the number of seasons after aggregation; for example, in the case of aggregating monthly to quarterly data, $S = 12$ and $S_A = 4$, while in the case where the seasonality is completely aggregated out of the data, $S_A = 1$.

For most temporal aggregation schemes, the matrix $\mathbf{A}$, can be written in generic form as

$$\mathbf{A} := \begin{bmatrix} \mathbf{a}_Q & \mathbf{0}_Q & \mathbf{0}_Q & \cdots & \mathbf{0}_Q \\ \mathbf{0}_Q & \mathbf{a}_Q & \mathbf{0}_Q & \cdots & \mathbf{0}_Q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_Q & \mathbf{0}_Q & \mathbf{0}_Q & \cdots & \mathbf{a}_Q \end{bmatrix} \tag{3.2}$$

where $Q := S/S_A$ is an integer, and where $\mathbf{0}_Q$ is the $1 \times Q$ vector of zeros, and $\mathbf{a}_Q$ is a $1 \times Q$ vector. Both systematic (or point-in-time) sampling and average sampling satisfy the form given in (3.2). The former is used when dealing with stock variables[5] and is defined such that $\mathbf{a}_Q := (0, 0, \ldots, 1)$ so that only observations $Q, 2Q, ..., kQ, ...$ of the original time series are retained in the aggregated data. The latter is used when dealing with flow variables and is defined such that $\mathbf{a}_Q := \mathbf{1}_Q$, where $\mathbf{1}_Q$ is the $1 \times Q$ vector of ones, so that the kth element of the aggregated data series is formed as the sum of elements $(k-1)Q + 1, ..., kQ$ in the original data. Temporal aggregation can also be considered for cases where Q is non-integer, although we will not analyse such cases in any detail here. An example of this is where daily data is converted to working day data, such that $S = 7$ and $S_A = 5$. On possible way this can be done is by treating the data for Saturday and Sunday as repeated missing observations. In this case $\mathbf{A}$ is comprised of the first five rows of the seven by seven identity matrix.

In section 3.2 we will first consider the impact of temporal aggregation on the stochastic part of y_{Sn+s} of (2.1); that is, we will consider the properties of $X_n^A := \mathbf{A} X_n$. We will consider the impact of systematic sampling in section 3.2.1, and average sampling in section 3.2.2. The impact of temporal aggregation on the deterministic component, μ_{Sn+s}, in (2.1) will then be considered in section 3.3.

[5]Stocks are variables such as prices, unemployment, temperature, and the capital stock that can, in principle, be observed at any given point in time, whereas flows are variables such as rainfall, income, and consumption expenditures that are defined with respect to an interval of time.

3.2 The Impact of Aggregation on the Stochastic Component

3.2.1 Systematic Sampling

Consider again the decomposition given for the vector of seasons representation for X_n given in (2.13). Pre-multiplying through by $\mathbf{A}$, we therefore have that

$$\begin{aligned}\mathbf{A}X_n &= \psi_0 \mathbf{A}\mathbf{v}_0 \mathbf{v}_0{}' \sum_{i=1}^{n} \alpha_{0,N}^{n-i} U_i + \psi_{S/2} \mathbf{A}\mathbf{v}_{S/2} \mathbf{v}_{S/2}{}' \sum_{i=1}^{n} \alpha_{S/2,N}^{n-i} U_i \\ &\quad + \sum_{k=1}^{S^*} \left(\psi_k^- \mathbf{A}\mathbf{v}_k^- \mathbf{v}_k^{+\prime} + \psi_k^+ \mathbf{A}\mathbf{v}_k^+ \mathbf{v}_k^{-\prime} \right) \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + o_p(N^{1/2}). \qquad (3.3)\end{aligned}$$

It immediately follows from (2.14) and an application of the continuous mapping theorem that, under the conditions of Lemma 1, the following FCLT applies to X_n^A,

$$\begin{aligned}&N^{-1/2} X_{\lfloor rN \rfloor}^A = N^{-1/2} \mathbf{A} X_{\lfloor rN \rfloor} \Rightarrow \sigma \Big\{ \psi_0 \mathbf{A}\mathbf{v}_0 \sqrt{S} J_{0,c_0}(r) + \psi_{S/2} \mathbf{A}\mathbf{v}_{S/2} \sqrt{S} J_{S/2,c_{S/2}}(r) \\ &+ \sum_{k=1}^{S^*} \left[\psi_k^- \mathbf{A}\mathbf{v}_k^- \sqrt{S/2} \left(J_{k,c_k}^{\alpha}(r) + \mathrm{i} J_{k,c_k}^{\beta}(r) \right) + \psi_k^- \mathbf{A}\mathbf{v}_k^+ \sqrt{S/2} \left(J_{k,c_k}^{\alpha}(r) - \mathrm{i} J_{k,c_k}^{\beta}(r) \right) \right] \Big\}, \ r \in [0,1].\end{aligned}$$

It is immediately seen from a comparison of (3.3) with (2.13) that the demodulation operators $\mathbf{v}_0$, $\mathbf{v}_{S/2}$, $\mathbf{v}_k^-$ and $\mathbf{v}_k^+$, $k = 1, ..., S^*$, in (2.13) are transformed under data aggregation to $\mathbf{A}\mathbf{v}_0$, $\mathbf{A}\mathbf{v}_{S/2}$, $\mathbf{A}\mathbf{v}_k^-$ and $\mathbf{A}\mathbf{v}_k^+$, $k = 1, ..., S^*$, respectively, in (3.3). We will therefore need to examine the form of these demodulation operators for a given $\mathbf{A}$ matrix to establish at which spectral frequencies near-integrated behaviour will be present in the temporally aggregated data.

Under systematic sampling $\mathbf{a}_Q := (0, 0, \ldots, 1)$. In connection with the first partial sum on the right hand side of (3.3), we therefore have that $\mathbf{A}\mathbf{v}_0 = \mathbf{1}_{S_A}$. Consequently, the frequency associated with the demodulation operator is not altered by aggregation and, hence, this term will converge (on scaling) to an S_A-dimensional vector each element of which is given by $J_{0,c_0}(r)$. Therefore the temporally aggregated data will, like the original process, admit a near-unit root at the zero frequency.

Turning to the second partial sum process in (3.3), we have that $\mathbf{A}\mathbf{v}_{S/2} = (e^{\mathrm{i}\pi Q}, e^{\mathrm{i}\pi 2Q}, e^{\mathrm{i}\pi 3Q}, \ldots, e^{\mathrm{i}\pi S_A Q})'$. Where Q is odd it is seen that $\mathbf{A}\mathbf{v}_{S/2} = (-1, 1, ..., 1)'$ and, hence, the demodulation operator will again not be altered by aggregation and the scaled partial sum will converge to an S_A-dimensional vector the jth element of which is given by $(-1)^j J_{S/2,c_{S/2}}(r)$ which therefore implies the presence of a Nyquist frequency near-unit in the aggregated data, just as in the original data. However, where Q is even it is seen that $\mathbf{A}\mathbf{v}_{S/2} = \mathbf{1}_{S_A}$ and so the frequency associated with the demodulation operator is altered to the zero frequency by aggregation. Consequently, the partial sum is transformed to the zero frequency with convergence on scaling to an S_A-dimensional vector each element of which is given by $J_{S/2,c_{S/2}}(r)$ which therefore implies the presence of a zero frequency near-unit root in the aggregated data.

Finally, consider the seasonal harmonic frequencies, ω_k, $k = 1, ..., S^*$. Here we have that,

$$\begin{aligned} \mathbf{Av}_k^+ &= e^{\mathrm{i}\omega_k}(e^{\mathrm{i}\omega_k(Q-1)}, e^{\mathrm{i}\omega_k(2Q-1)}, e^{\mathrm{i}\omega_k(3Q-1)}, \ldots, e^{\mathrm{i}\omega_k(S_AQ-1)})' \\ &= (e^{\mathrm{i}\omega_k Q}, e^{\mathrm{i}\omega_k 2Q}, e^{\mathrm{i}\omega_k 3Q}, \ldots, e^{\mathrm{i}\omega_k S_A Q})', \\ \mathbf{Av}_k^- &= e^{-\mathrm{i}\omega_k}(e^{-\mathrm{i}\omega_k(Q-1)}, e^{-\mathrm{i}\omega_k(2Q-1)}, e^{-\mathrm{i}\omega_k(3Q-1)}, \ldots, e^{-\mathrm{i}\omega_k(S_AQ-1)})' \\ &= (e^{-\mathrm{i}\omega_k Q}, e^{-\mathrm{i}\omega_k 2Q}, e^{-\mathrm{i}\omega_k 3Q}, \ldots, e^{-\mathrm{i}\omega_k S_A Q})'. \end{aligned}$$

We therefore see that the frequency associated with the demodulation operator is mapped from ω_k to $\omega_k Q$, $k = 1, ..., S^*$, by systematic sampling. As a consequence, the corresponding complex conjugate pairs of partial sums in (3.3) are transformed to either a single partial sum process (if $\omega_k Q$ coincides with the zero or Nyquist frequencies), or a complex conjugate pair of partial sum processes associated with frequency $\omega_k Q$. These will converge, on scaling, to either an S_A-dimensional vector or a pair of S_A-dimensional vectors each element of which is given by an OU process or a complex-valued OU process at frequency $\omega_k Q$, which therefore implies the presence of either a near-unit root or a complex conjugate pair of near-unit roots at frequency $\omega_k Q$ in the aggregated data.

Table 1: Summary of Frequency Allocations under Systematic Sampling of Monthly Data

	k	0	1	2	3	4	5	6
Original monthly frequency	ω_k	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
Allocation in quarterly data	ω_k^A	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
Allocation in annual data	ω_k^A	0	0	0	0	0	0	0

To illustrate these effects, consider the specific example of systematic sampling from monthly ($S = 12$) observations to either quarterly ($S_A = 4$) observations, such that $Q = 3$, or to annual observations, such that $Q = 12$. Using the general results above, the implications for this example are summarised in Table 1.[6] We can see from Table 1 that a near-unit root at the zero frequency in the systematically sampled quarterly data could be attributable to either the presence in the monthly data of a zero frequency near-unit root or a complex conjugate pair of near-unit roots at frequency $\frac{2\pi}{3}$ in the monthly data, or indeed the presence of both. Similarly, a Nyquist frequency near-unit root in the quarterly data under systematic sampling can obtain from the presence in the monthly data of either a Nyquist frequency near-unit root or a pair of complex conjugate near-unit roots at frequency $\frac{\pi}{3}$, or both. A complex conjugate pair of near-unit roots will arise at the annual frequency ($\frac{\pi}{2}$) in the systematically sampled quarterly data if a complex conjugate pair of near-unit roots appears at frequency $\pi/2$ in the monthly data but will also arise if a complex conjugate pair of near-unit roots is present at the $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ frequencies in the monthly data. In the case of the systematically sampled annual data, it can be seen that a conventional near-unit root will arise from the presence of near-unit roots at any of the zero and seasonal frequencies in the monthly

[6]Recalling that the spectrum repeats with period 2π, and that ω and $2\pi - \omega$, $\omega \in (0, \pi)$, are conjugate frequencies.

data.

Remark 7: In the simplest possible case where a near-unit root at a given frequency in the temporally aggregated data arises from near-unit root behaviour at a single frequency in the original data then so it can be straightforwardly seen from the results above that it will inherit the local-to-unity parameter associated with that near-unit root in the original data. However, as the examples given in Table 1 have shown, near-unit root behaviour at a given frequency in the aggregated data can also arise from near-unit root behaviour at more than one distinct frequency in the original data. An interesting question this raises is what will the local-to-unity parameter associated with that near-unit root be in such cases? The answer is that it will be some linear combination of the relevant local-to-unity parameters present in the original process. In the supplementary appendix we present in detail the algebra underlying this result for three example DGPs where the data are systematically sampled and near-unit roots are present at more than one of the zero and seasonal frequencies: Case A, $\Delta_0^{c_0}\Delta_k^{c_k}x_{Sn+s} = u_{Sn+s}$; Case B, $\Delta_{S/2}^{c_{S/2}}\Delta_k^{c_k}x_{Sn+s} = u_{Sn+s}$, and Case C, $\Delta_k^{c_k}\Delta_j^{c_j}x_{Sn+s} = u_{Sn+s}$. Case A relates to a process which admits near-unit roots at the zero and kth harmonic seasonal frequency, ω_k, while in Case B near-unit roots are present at the Nyquist frequency and ω_k. Finally, in case C near-unit roots are present at the jth and kth harmonic frequencies, ω_j and ω_k, $j \neq k$. To illustrate, consider the case where we systematically sample from monthly to quarterly data. For Case A suppose we have $\omega_4 = \frac{2\pi}{3}$ such that both the zero frequency near-unit root and the harmonic frequency complex conjugate pair of near unit-roots are mapped to the zero frequency under aggregation. The resulting zero frequency near-unit root in the aggregated data has local-to-unity parameter $c_0^A := (c_0 + 2c_4)/3$. Notice, therefore, that even if the original data contained a pure unit root at the zero frequency ($c_0 = 0$), the temporally aggregated data would not contain a pure zero frequency unit root unless the original process also contained a pair of complex conjugate pure unit roots at frequency $\frac{2\pi}{3}$ (i.e. $c_4 = 0$). Similarly, in Case B where $\omega_k = \frac{\pi}{3}$, such that both the Nyquist frequency near-unit root and the harmonic frequency complex conjugate pair of near unit-roots are mapped to the Nyquist frequency under aggregation, the resulting Nyquist frequency near-unit root in the aggregated data has local-to-unity parameter $c_{S/2}^A := (c_{S/2} + 2c_k)/3$. □

3.2.2 Average Sampling

In contrast to systematic sampling, it is well known that average sampling will, in general, induce moving average (MA) behaviour into the aggregated data; see Wei (2006, Chapter 20) for a general discussion on this point and the Appendix of Pons (2006) for a discussion specific to the case of average sampling from monthly to quarterly data when unit roots are present in the monthly data. This will therefore also induce MA behaviour into the vector of seasons form for the aggregated data; further discussion on this issue is provided in Section S.5 of the on-line supplementary appendix. However, the resulting MA dynamics do not impact on the near-integrated properties of the aggregated process (they merely change the long-run variances associated with the terms in the FCLT). For the purposes of the main text, without loss of generality for what concerns us here, we may therefore simply disregard any MA behaviour induced in the vector of seasons form by average sampling of the data.

Under average sampling $\mathbf{a}_Q$ is a $1 \times Q$ vector of ones, $\mathbf{1}_Q$ and, therefore, $\mathbf{Av}_0 = Q\mathbf{1}_{S_A}$. In the case of the Nyquist frequency,

$$\begin{aligned}\mathbf{Av}_{S/2} &= e^{\mathrm{i}\pi} \sum_{h=0}^{S/S_A-1} e^{\mathrm{i}\pi h}(1, e^{\mathrm{i}\pi Q}, e^{\mathrm{i}\pi 2Q}, \ldots, e^{\mathrm{i}\pi(S_A-1)Q})' \\ &= \frac{\sin(Q\pi/2)}{\sin(\pi/2)} e^{\mathrm{i}(Q+1)\frac{\pi}{2}}(1, e^{\mathrm{i}\pi Q}, e^{\mathrm{i}\pi 2Q}, \ldots, e^{\mathrm{i}\pi(S_A-1)Q})'. \end{aligned} \tag{3.4}$$

We can therefore see immediately that, just as in the case of systematic sampling, a near-unit root at the zero frequency will always be retained under average sampling. Moreover, from (3.4) and using the fact that $\sum_{h=0}^{Q-1} e^{\mathrm{i}\pi h} = \frac{\sin(Q\pi/2)}{\sin(\pi/2)} e^{\mathrm{i}(Q-1)\pi/2}$, it follows that average sampling of a process which admits a near-unit root at the Nyquist frequency will lead to the presence of a near-unit root at the Nyquist frequency in the aggregated data only in the case where Q is odd. Where Q is even, such that $\sin(Q\pi/2) = 0$, the root will vanish under average sampling.

Turning to the seasonal harmonic frequencies, ω_k, $k = 1, ..., S^*$, we have that,

$$\begin{aligned}\mathbf{Av}_k^+ &= e^{\mathrm{i}\omega_k} \sum_{h=0}^{Q-1} e^{\mathrm{i}\omega_k h}(1, e^{\mathrm{i}\omega_k Q}, e^{\mathrm{i}\omega_k 2Q}, \ldots, e^{\mathrm{i}\omega_k(S_A-1)Q})' \\ &= \frac{\sin(Q\omega_k/2)}{\sin(\omega_k/2)} e^{\mathrm{i}(Q+1)\frac{\omega_k}{2}}(1, e^{\mathrm{i}\omega_k Q}, e^{\mathrm{i}\omega_k 2Q}, \ldots, e^{\mathrm{i}\omega_k(S_A-1)Q})' \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\mathbf{Av}_k^- &= e^{-\mathrm{i}\omega_k} \sum_{h=0}^{Q-1} e^{-\mathrm{i}\omega_k h}(1, e^{-\mathrm{i}\omega_k Q}, e^{-\mathrm{i}\omega_k 2Q}, \ldots, e^{-\mathrm{i}\omega_k(S_A-1)Q})' \\ &= \frac{\sin(Q\omega_k/2)}{\sin(\omega_k/2)} e^{-\mathrm{i}(Q+1)\frac{\omega_k}{2}}(1, e^{-\mathrm{i}\omega_k Q}, e^{-\mathrm{i}\omega_k 2Q}, \ldots, e^{-\mathrm{i}\omega_k(S_A-1)Q})' \end{aligned} \tag{3.6}$$

where we have used the fact that $\sum_{h=0}^{Q-1} e^{\mp \mathrm{i}\omega_k h} = \frac{\sin(Q\omega_k/2)}{\sin(\omega_k/2)} e^{\mp \mathrm{i}(Q-1)\omega_k/2}$. It can be observed that a pair of complex conjugate near-unit roots at frequency ω_k will vanish under average sampling in cases where $\sin(Q\omega_k/2) = 0$. This will happen at all frequencies ω_k for which the ratio (k/S_A) takes integer values, $k = 1, ..., S^*$. Where $\sin(Q\omega_k/2) \neq 0$, the frequency associated with the demodulation operator is, as in the case of systematic sampling, mapped from ω_k to $\omega_k Q$, $k = 1, ..., S^*$, under average sampling and so the effects here are the same as described above under systematic sampling.[7]

Paralleling the results in Table 1 for systematic sampling, Table 2 summarises the implications of average sampling for the near-unit root properties of data aggregated from monthly observations to either quarterly or annual observations. Notice that for the quarterly aggregated data, the

[7] In other words, Q is a constant that determines the outcome of the filters at a specific frequency, for instance, for the zero, Nyquist and harmonic frequencies the filters are $(1 + L + ... + L^{Q-1})$, $\sin(Q\pi/2)/\sin(\pi/2)e^{-i(Q+1)\pi/2}$, and $\sin(Q\omega_k/2)/\sin(\omega_k/2)e^{i(Q+1)\omega_k/2}$ and $\sin(Q\omega_k/2)/\sin(\omega_k/2)e^{-i(Q+1)\omega_k/2}$, respectively.

impact of average sampling is the same as for systematic sampling except for the case of a pair of complex conjugate near-unit roots at frequency $\frac{2\pi}{3}$ which are annihilated by average sampling. For the annual aggregated data, average sampling will annihilate near-unit root behaviour at any seasonal frequency. As with the case of systematic sampling discussed in Remark 7, the local-to-unity parameter associated with a near-unit root process arising at a given frequency in the average sampled data will be a weighted combination of the non-centrality parameter(s) at the frequency or frequencies in the non-aggregated data which are mapped to that frequency in the aggregate data.

Table 2: Summary of Frequency Allocations under Average Sampling of Monthly Data

	k	0	1	2	3	4	5	6
Original monthly frequency	ω_k	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
Allocation in quarterly data	ω_k^A	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	–	$\frac{\pi}{2}$	π
Allocation in annual data	ω_k^A	0	–	–	–	–	–	–

3.3 The Impact of Aggregation on the Deterministic Kernel

We now turn our attention to the impact of temporal aggregation on the deterministic component $\mu_{Sn+s} := \delta' z_{Sn+s}$ of (2.1). We can write the vector of seasons representation for Y_n as $Y_n = X_n + Z_n\delta$, $n = 1, ..., N$, where X_n is as defined just below (2.7), and where $Z_n := (z_{Sn+1-S}, z_{Sn+2-S}, ..., z_{Sn})'$. Consequently, the deterministic component in the temporally aggregated data will be given by $\mathbf{A}Z_n\delta \equiv Z_n^A \delta^A$ where Z_n^A is the matrix of resulting deterministic variables in the aggregated data and δ^A the corresponding parameter vector. The deterministic component in the aggregated data can therefore be straightforwardly evaluated for a given deterministic kernel in the original data and aggregation matrix, $\mathbf{A}$.

In what follows we will adopt the familiar trigonometric deterministic seasonal form and follow del Barrio Castro, Rodrigues and Taylor (2018) by focusing attention on the following three cases of practical relevance, although other possibilities, including seasonal mean shifts, could also be analysed using the same framework:

Case 1: Zero and seasonal frequency intercepts:

$$z_{Sn+s} = z_{Sn+s,1} := \quad (1, \cos(2\pi(Sn+s)/S), \sin(2\pi(Sn+s)/S), ... \\ ..., \cos(2\pi S^*(Sn+s)/S), \sin(2\pi S^*(Sn+s)/S), (-1)^{Sn+s})',$$

$s = 1 - S, ..., 0$, $n = 1, ..., N$, with $\delta := \left(\delta_0, \delta_1', ..., \delta_{S^*}', \delta_{S/2}\right)'$ and $\delta_k := (\delta_{k,1}, \delta_{k,2})'$, $k = 1, ..., S^*$. Notice that the first element of $z_{Sn+s,1}$ is a zero frequency (or standard) intercept variable, while the last element, $(-1)^{Sn+s}$, is a Nyquist frequency intercept variable. Finally, $\cos(2\pi k(Sn+s)/S)$

and $\sin(2\pi k(Sn+s)/S)$ together form a pair of intercepts at the kth harmonic frequency, ω_k, $k = 1, ..., S^*$.

Case 2: Zero and seasonal frequency intercepts, and zero frequency trend: $z_{Sn+s} = z_{Sn+s,2} := (z'_{Sn+s,1}, Sn+s)'$, $s = 1-S, ..., 0$, $n = 1, ..., N$, with $\delta := (\delta_0, \delta'_1, ..., \delta'_{S^*}, \delta_{S/2}, \overline{\delta}_0)'$ and $\delta_k := (\delta_{k,1}, \delta_{k,2})'$, $k = 1, ..., S^*$.

Case 3: Zero and seasonal frequency intercepts and trends:

$$z_{Sn+s} = z_{Sn+s,3} := \left(z'_{Sn+s,1}, (Sn+s)\, z'_{Sn+s,1}\right)', \; s = 1-S, ..., 0, \; n = 1, ..., N$$

with $\delta := (\delta_0, \delta'_1, ..., \delta'_{S^*}, \delta_{S/2}, \overline{\delta}_0, \overline{\delta}'_1, ..., \overline{\delta}'_{S^*}, \overline{\delta}_{S/2})'$ and $\overline{\delta}_k := \left(\overline{\delta}_{k,1}, \overline{\delta}_{k,2}\right)'$, $k = 1, ..., S^*$. Notice that the interaction of the time index, $(Sn+s)$, onto $z_{Sn+s,1}$ results in deterministic spectral (linear) trends at each of the zero and seasonal frequencies, ω_k, $k = 0, ..., \lfloor S/2 \rfloor$.

Remark 8: Notice that under Case 1, $Z_n = (\mathbf{v}_0, \mathbf{h}_1^\alpha, \mathbf{h}_1^\beta, ..., \mathbf{h}_{S^*}^\alpha, \mathbf{h}_{S^*}^\beta, \mathbf{v}_{S/2})$, where $\mathbf{v}_0$, $\mathbf{h}_k^\alpha$ and $\mathbf{h}_k^\beta$, $k = 1, ..., S^*$, and $\mathbf{v}_{S/2}$ are the $S \times 1$ vectors defined in section 2. Under Case 2, $Z_n = (\mathbf{v}_0, \mathbf{h}_1^\alpha, \mathbf{h}_1^\beta, ..., \mathbf{h}_{S^*}^\alpha, \mathbf{h}_{S^*}^\beta, \mathbf{v}_{S/2}, \mathbf{t}_n)$, where $\mathbf{t}_n := (Sn+1-S, Sn+2-S, ..., Sn)'$. Finally, under Case 3, $Z_n = (\mathbf{v}_0, \mathbf{h}_1^\alpha, \mathbf{h}_1^\beta, ..., \mathbf{h}_{S^*}^\alpha, \mathbf{h}_{S^*}^\beta, \mathbf{v}_{S/2}, \mathbf{t}_n, \mathbf{t}_n \circ \mathbf{h}_1^\alpha, \mathbf{t}_n \circ \mathbf{h}_1^\beta, ..., \mathbf{t}_n \circ \mathbf{h}_{S^*}^\alpha, \mathbf{t}_n \circ \mathbf{h}_{S^*}^\beta, \mathbf{t}_n \circ \mathbf{v}_{S/2})$, where $\circ$ denotes the Hadamard (elementwise) product. □

Remark 9: The deterministic component μ_{Sn+s} under Case 3 above can be equivalently written in terms of seasonally varying intercept and trend coefficients as $\mu_{Sn+s} := a_s^* + b_s^*(Sn+s)$. For Case 1 this holds with $b_s^* = 0$, $s = 1-S, ..., 0$, imposed. In Case 2 this holds but with $b_s^* = b^*$, $s = 1-S, ..., 0$, imposed. These representations are mathematically equivalent with a unique linear mapping existing between the elements of δ and $\{a_s^*, b_s^*\}_{s=1-S}^0$ above; see Canova and Hansen (1992, p.239) for precise details. The vector of seasons representation for the deterministic component in this case can be written as $D_n\gamma$, where: under Case 1, $D_n = I_S$, the Sth order identity matrix; under Case 2, $D_n = (I_S, \mathbf{t}_n)$, and under Case 3, $D_n = (I_S, \mathbf{t}_n \circ \mathbf{d}_1, ..., \mathbf{t}_n \circ \mathbf{d}_S)$, where $\mathbf{d}_j$, $j = 1, ..., S$, are $S \times 1$ vectors whose elements are all zero other than their jth element which is 1. □

It can be seen that the trigonometric representation for deterministic seasonality in Cases 1–3 is closely related to the demodulation operator introduced in section 2.1. In particular, the real ($Re[.]$) and imaginary ($Im[.]$) parts of $e^{-\mathrm{i}\omega_k(Sn+s)}$ for ω_k, $k = 0, \ldots, \lfloor S/2 \rfloor$ correspond to the columns of $z'_{Sn+s,1}$. Clearly for both ω_0 and $\omega_{S/2}$ only the real parts are relevant because $e^{-\mathrm{i}0(Sn+s)} = 1$ and $e^{-\mathrm{i}\pi(Sn+s)} = (-1)^{Sn+s}$. For the harmonic seasonal frequencies, ω_k, $k = 1, ..., S^*$, we have that $Re[e^{-\mathrm{i}\frac{2\pi}{S}(Sn+s)}] = \cos\left(\frac{2\pi}{S}(Sn+s)\right)$ and $Im[e^{-\mathrm{i}\frac{2\pi}{S}(Sn+s)}] = \sin\left(\frac{2\pi}{S}(Sn+s)\right)$. Bearing this in mind, the impact of temporal aggregation on these trigonometric deterministic variables through the implied change in the demodulation operator induced by the matrix $\mathbf{A}$ will parallel what we observed in section 3.1 for the spectral frequency stochastic components of the aggregated data. We consider the effects of systematic and average sampling on the deterministic component in turn.

Where systematic sampling is used it can then be seen that zero frequency intercept and trend variables for the original data will be transformed to zero frequency intercept and trend variables respectively for the aggregated data. Similarly, Nyquist frequency intercept and trend variables will be transformed to Nyquist frequency intercept and trend variables respectively for the aggregated data when Q is odd, but will shift to zero frequency intercept and trend variables respectively when Q is even. Finally for the harmonic frequencies, ω_k, $k = 1, \ldots, S^*$, the deterministic variables associated with frequency ω_k will be transformed such that together they span the space of the analogous deterministic variable(s) at frequency $Q\omega_k$ after systematic sampling. Defining the $S^A \times 1$ vectors $\mathbf{v}_0^A := (1, 1, ..., 1)'$, $\mathbf{v}_{S/2}^A := (-1, 1, ..., 1)'$, together with $\mathbf{h}_{kQ}^{\alpha\,A} := (\cos[Q\omega_k], \cos[2Q\omega_k], \ldots, \cos[S_A Q\omega_k])$ and $\mathbf{h}_{kQ}^{\beta\,A} := (\sin[Q\omega_k], \sin[2Q\omega_k], \ldots, \sin[S_A Q\omega_k])$, $k = 1, ..., S$, these transformations can be seen to occur because, under systematic sampling: $\mathbf{A}\mathbf{v}_0 = \mathbf{v}_0^A$; $\mathbf{A}\mathbf{v}_{S/2} = \mathbf{v}_{S/2}^A$ when Q is odd and $\mathbf{A}\mathbf{v}_{S/2} = \mathbf{v}_0^A$ when Q is even; and $\mathbf{A}\mathbf{h}_k^{\alpha} = \mathbf{h}_{kQ}^{\alpha\,A}$ and $\mathbf{A}\mathbf{h}_k^{\beta} = \mathbf{h}_{kQ}^{\beta\,A}$, $k = 1, ..., S^*$.[8] Taken together, these results entail that the deterministic component in the aggregated data will again contain zero and seasonal frequency intercepts in Case 1, zero and seasonal frequency intercepts and a zero frequency linear trend in Case 2, and zero and seasonal frequency intercepts and trends in Case 3. In each case the zero and seasonal frequencies are now collectively given by $\omega_k^A := 2\pi k/S_A$, $k = 0, ..., \lfloor S_A/2 \rfloor$. The coefficients on these deterministic variables will be functions of the intercept and trend parameters in the original data. To give a simple example, suppose we consider systematic aggregation from $S = 4$ to $S_A = 2$ and let Case 1 above hold on the deterministic component. Then in vector of seasons form the deterministic component for the original data is given by

$$Z_n\delta = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_{1,1} \\ \delta_{1,2} \\ \delta_2 \end{bmatrix}.$$

Here $\mathbf{A} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and so we have that $\mathbf{A}Z_n = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. Consequently,

$$\mathbf{A}Z_n\delta \equiv Z_n^A\delta^A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_0 + \delta_2 \\ \delta_{1,1} \end{bmatrix}$$

so that in the systematically sampled (biannual) data we have a zero frequency intercept with coefficient $\delta_0 + \delta_2$ and a Nyquist frequency intercept with coefficient $\delta_{1,1}$. Notice therefore that, for example, the zero frequency intercept in the aggregated data could be zero (non-zero) even where it is non-zero (zero) in the original data.

Similarly, it can be seen that under average sampling the zero frequency deterministic variables for the original data will again be transformed to the corresponding zero frequency variables for the aggregated data. Nyquist frequency deterministic variables will either be annihilated (vanish) when Q is even, or will be transformed to the corresponding Nyquist frequency variables for the aggregated data when Q is odd. For the harmonic frequencies ω_k, $k = 1, \ldots, S^*$, the deterministic variables associated with frequency ω_k will be annihilated when k/S_A is an integer, and transformed

[8]Notice that where $Q\omega_k = a\pi$ with a a positive integer, then so $\mathbf{h}_{kQ}^{\beta\,A} = \mathbf{0}$, while $\mathbf{h}_{kQ}^{\alpha\,A} = (-1, 1, ..., 1)'$ if a is odd and $\mathbf{h}_{kQ}^{\alpha\,A} = (1, 1, ..., 1)'$ if a is even.

to the analogous deterministic variable(s) at frequency $Q\omega_k$ when k/S_A is non-integer. In particular: $\mathbf{Av}_0$ is proportional to $\mathbf{v}_0^A$; $\mathbf{Av}_{S/2} = \mathbf{0}$ when Q is even and proportional to $\mathbf{v}_{S/2}^A$ when Q is odd; $\mathbf{Ah}_k^\alpha = \mathbf{Ah}_k^\alpha = \mathbf{0}$ when k/S_A is an integer, $k = 1, ..., S^*$, while $\mathbf{Ah}_{k,S}^\alpha$ and $\mathbf{Ah}_{k,S}^\beta$ are proportional to $\mathbf{h}_{kQ}^{\alpha\,A}$ and $\mathbf{h}_{kQ}^{\beta\,A}$, respectively, when k/S_A is non-integer, $k = 1, ..., S^*$. As with systematic sampling, the deterministic component in the aggregated data will therefore again contain zero and seasonal frequency intercepts in Case 1, zero and seasonal frequency intercepts and a zero frequency linear trend in Case 2, and zero and seasonal frequency intercepts and trends in Case 3, but where in each case the zero and seasonal frequencies are again collectively given by ω_k^A, $k = 0, ..., \lfloor S_A/2 \rfloor$. The coefficients on these deterministic variables will again be functions of the intercept and trend parameters in the original data but these will not be the same functions as in the systematic sampling case. For the sample example given above aggregating quarterly to biannual data under Case 1, $\mathbf{A} := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, and so we have that $\mathbf{A}Z_n = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}$. Consequently,

$$\mathbf{A}Z_n\delta \equiv Z_n^A\delta^A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2\delta_0 \\ \delta_{1,1} - \delta_{2,1} \end{bmatrix}$$

so that in the average sampled data we have a zero frequency intercept with coefficient $2\delta_0$ (which therefore depends only on the value of the zero frequency intercept in the original data), and a Nyquist frequency intercept with coefficient $\delta_{1,1} - \delta_{2,1}$.

4 Conclusions

In this paper we have built upon earlier work pertaining to monthly data in Pons (2006) and investigated the implications for the zero and seasonal frequency near-unit root properties of a time series which has been subject to temporal aggregation from S seasons per cycle to $S_A < S$ seasons. As part of these results we have shown that systematic sampling, appropriate for stock variables, can impact on the non-seasonal unit root properties of the data, while for average sampling, appropriate for flow variables, this is not the case. We have also analysed the impact of aggregation on the deterministic kernel of the series. Our results again show that for systematically sampled data the non-seasonal aspect of the deterministic kernel can be affected by temporal aggregation. The results relating to the near-unit root properties and the deterministic kernel properties of the aggregated series were shown to parallel each other. In particular, both were shown to be attributable to changes, induced by the temporal aggregation scheme, in the frequency associated with the relevant demodulation operator.

As a final comment, we have explored the impact that temporal aggregation has on the integration properties of a seasonally observed processes by using conventional asymptotic arguments based on an increasing time span. As pointed out by a referee, it might also be possible to attack this problem in a way that views the sample size as two-dimensional, using so-called infill asymptotics whereby the sampling frequency is also allowed to increase. Undertaking such an analysis is beyond the scope of the present paper, but could constitute a fruitful avenue for further research in this topic.

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Supplementary Online Appendix

to

Temporal Aggregation of Seasonally Near-Integrated Processes

by

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S.1 Introduction

The contents of this supplement are organised as follows. In section S.2 we provide a proof of Lemma 1. Next in section S.3 we provide a derivation of the results given for the three examples in Remark 3. In section S.4 we provide detailed derivations for Cases A, B and C considered in Remark 7 where near-unit roots occur at more than one distinct frequency. Finally, in section S.5, we present an additional discussion related to the moving average dynamics potentially induced by average sampling. Additional references are included at the end of the supplement.

S.2 Proof of Lemma 1

Consider first the following partial fraction decomposition,

$$\frac{1}{\Delta_k^{c_k\pm}\Delta_j^{c_j\pm}} = \left(\Delta_k^{c_k\pm}\right)^{-1}\wp_{kj}^{k\pm} - \left(\Delta_j^{c_j\pm}\right)^{-1}\wp_{kj}^{j\pm} \tag{S.1}$$

where $k, j = 0, 1, ..., S/2$, $\Delta_k^{c_k\pm} := \left(1 - e^{\pm \mathrm{i}\omega_k}\alpha_{k,T}L\right)$, $\Delta_k^{c_j\pm} := \left(1 - e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}L\right)$, and

$$\wp_{kj}^{k\pm} := \frac{e^{\pm \mathrm{i}\omega_k}\alpha_{k,T}}{\left(e^{\pm \mathrm{i}\omega_k}\alpha_{k,T} - e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}\right)}, \quad \wp_{kj}^{j\pm} := \frac{e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}}{\left(e^{\pm \mathrm{i}\omega_k}\alpha_{k,T} - e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}\right)}.$$

The terms $\wp_{kj}^{k\pm}$ and $\wp_{kj}^{j\pm}$, are such that

$$\wp_{kj}^{k\pm} = \frac{e^{\pm \mathrm{i}\omega_k}\alpha_{k,T}}{\left(e^{\pm \mathrm{i}\omega_k}\alpha_{k,T} - e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}\right)} = \frac{e^{\pm i\omega_k}}{\left(e^{\pm i\omega_k} - e^{\pm i\omega_j}\right)} + O(1/T) =: \widetilde{\wp}_{kj}^{k\pm} + O(1/T),$$

and

$$\wp_{kj}^{j\pm} = \frac{e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}}{\left(e^{\pm \mathrm{i}\omega_k}\alpha_{k,T} - e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}\right)} = \frac{e^{\pm i\omega_j}}{\left(e^{\pm i\omega_k} - e^{\pm i\omega_j}\right)} + O(1/T) =: \widetilde{\wp}_{kj}^{j\pm} + O(1/T).$$

In what follows, to simplify notation, but without loss of asymptotic generality, we will ignore the $O(1/T)$ remainders and define $\widetilde{\wp}_{kj}^{k\pm} := \frac{e^{\pm \mathrm{i}\omega_k}}{e^{\pm \mathrm{i}\omega_k} - e^{\pm \mathrm{i}\omega_j}}$ and $\widetilde{\wp}_{kj}^{j\pm} := \frac{e^{\pm \mathrm{i}\omega_j}}{e^{\pm \mathrm{i}\omega_k} - e^{\pm \mathrm{i}\omega_j}}$. The following results which will prove useful in the proofs that follow may then be stated,

$$\Delta_0^{c_0} \widetilde{\wp}_{S/2,0}^{S/2} - \Delta_{S/2}^{c_{S/2}} \widetilde{\wp}_{S/2,0}^{0} = 1 + \frac{\left(c_{S/2} - c_0\right)}{2T} L \tag{S.2}$$

$$\Delta_k^{c_k \mp} \widetilde{\wp}_{kj}^{k\pm} - \Delta_j^{c_j \mp} \widetilde{\wp}_{kj}^{j\pm} = 1 + \frac{(c_k - c_j)}{T(e^{\pm i\omega_j} - e^{\pm i\omega_k})} L \tag{S.3}$$

$$\Delta_v^{c_v} \widetilde{\wp}_{kv}^{k\pm} - \Delta_j^{c_j \pm} \widetilde{\wp}_{kv}^{v} = 1 + \frac{e^{\pm i\omega_k} e^{i\omega_v}(c_k - c_v)}{T(e^{\pm i\omega_k} - e^{i\omega_v})} L \tag{S.4}$$

where $k,\ j = 1, ..., S^*$. Recall also that $e^{\mathrm{i}\omega_v} = 1$, for $v = 0$ and $e^{\mathrm{i}\omega_v} = -1$ for $v = S/2$.

Notice that the results for the pure unit root case follow straightforwardly from (S.2) to (S.4) by considering $c_k = 0$, $k = 0, 1, ..., S/2$.

Proofs of (S.2) - (S.4)

• Consider first the left hand side of (S.2). This satisfies

$$\begin{aligned} \Delta_0^{c_0} \widetilde{\wp}_{S/2,0}^{S/2} - \Delta_{S/2}^{c_{S/2}} \widetilde{\wp}_{S/2,0}^{0} &= \widetilde{\wp}_{S/20}^{S/2} - \widetilde{\wp}_{S/20}^{0} - (\widetilde{\wp}_{S/20}^{S/2} e^{\mp \mathrm{i}\omega_0}(1 + \frac{c_0}{T}) - \widetilde{\wp}_{S/20}^{0} e^{\mp \mathrm{i}\omega_{S/2}}(1 + \frac{c_{S/2}}{T}))L \\ &= 1 - \frac{1}{2T}\left(c_0 - c_{S/2}\right) L \\ &= 1 + \frac{\left(c_{S/2} - c_0\right)}{2T} L \end{aligned} \tag{S.5}$$

with $\widetilde{\wp}_{S/2,0}^{0} = \frac{e^{\mathrm{i}0}}{e^{\mathrm{i}\pi} - e^{\mathrm{i}0}} = -\frac{1}{2}$ and $\widetilde{\wp}_{S/2,0}^{S/2} = \frac{e^{\mathrm{i}\pi}}{e^{\mathrm{i}\pi} - e^{\mathrm{i}0}} = \frac{1}{2}$.

• Consider next (S.3). For $k,\ j = 1, ..., S^*$, we observe first that,

$$\Delta_k^{c_k \mp} \widetilde{\wp}_{kj}^{k\pm} - \Delta_j^{c_j \mp} \widetilde{\wp}_{kj}^{j\pm} = \widetilde{\wp}_{kj}^{k\pm} - \widetilde{\wp}_{kj}^{j\pm} - (\widetilde{\wp}_{kj}^{k\pm} e^{\mp \mathrm{i}\omega_k}(1 + \frac{c_k}{T}) - \widetilde{\wp}_{kj}^{j\pm} e^{\mp \mathrm{i}\omega_j}(1 + \frac{c_j}{T}))L. \tag{S.6}$$

Next, observe that

$$\widetilde{\wp}_{kj}^{k\pm} - \widetilde{\wp}_{kj}^{j\pm} = \frac{e^{\pm \mathrm{i}\omega_k}}{e^{\pm \mathrm{i}\omega_k} - e^{\pm \mathrm{i}\omega_j}} - \frac{e^{\pm \mathrm{i}\omega_j}}{e^{\pm \mathrm{i}\omega_k} - e^{\pm \mathrm{i}\omega_j}} = 1 \tag{S.7}$$

and

$$\begin{aligned} (\widetilde{\wp}_{kj}^{k\pm} e^{\mp \mathrm{i}\omega_k}(1 + \frac{c_k}{T}) - \widetilde{\wp}_{kj}^{j\pm} e^{\mp \mathrm{i}\omega_j}(1 + \frac{c_j}{T})) &= \left(\frac{e^{\pm \mathrm{i}\omega_k}}{e^{\pm \mathrm{i}\omega_k} - e^{\pm \mathrm{i}\omega_j}} e^{\mp \mathrm{i}\omega_k}(1 + \frac{c_k}{T}) - \frac{e^{\pm \mathrm{i}\omega_j}}{e^{\pm \mathrm{i}\omega_k} - e^{\pm \mathrm{i}\omega_j}} e^{-\mp \mathrm{i}\omega_j}(1 + \frac{c_j}{T}) \right) \\ &= \frac{c_j - c_k}{T(e^{\pm i\omega_j} - e^{\pm i\omega_k})}, \text{ for } k \neq j \text{ and } k, j = 1, ..., S^*. \end{aligned} \tag{S.8}$$

Substituting (S.7) and (S.8) into (S.6), the result in (S.3) is obtained.

• Consider next (S.4). For the zero ($v = 0$) and Nyquist ($v = S/2$) frequencies notice that $e^{\mathrm{i}\omega_v} = e^{-\mathrm{i}\omega_v}$. Therefore, for $v = 0,\ S/2$ and $j = 1, ..., S^*$, $\widetilde{\wp}_{kv}^{k\pm} := \frac{e^{\pm \mathrm{i}\omega_k}}{e^{\pm \mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}}$ and $\widetilde{\wp}_{kv}^{v} := \frac{e^{\mathrm{i}\omega_v}}{e^{\pm \mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}}$ and, hence, $(\frac{e^{\pm \mathrm{i}\omega_k}}{e^{\pm \mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}e^{\mathrm{i}\omega_v}(1+\frac{c_v}{T}) - \frac{e^{\mathrm{i}\omega_v}}{e^{\pm \mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}e^{\pm \mathrm{i}\omega_k}(1+\frac{c_k}{T})) = -e^{vi\omega_k}e^{i\omega_v}\frac{c_k-c_v}{T(e^{\pm i\omega_k}-e^{i\omega_v})}$. Consequently, the result in (S.4) follows from,

$$\begin{aligned}
\Delta_v^{c_v}\widetilde{\wp}_{kv}^{k\pm} - \Delta_j^{c_j\pm}\widetilde{\wp}_{kv}^{v} &= \widetilde{\wp}_{kv}^{k\pm} - \widetilde{\wp}_{kv}^{v} - (\widetilde{\wp}_{kv}^{k\pm}e^{\mathrm{i}\omega_v}(1+\frac{c_v}{T}) - \widetilde{\wp}_{kv}^{v}e^{\pm \mathrm{i}\omega_j}(1+\frac{c_j}{T}))L \\
&= 1 + \frac{e^{\pm i\omega_k}e^{i\omega_v}(c_k-c_v)}{T(e^{\pm i\omega_k}-e^{i\omega_v})}L. \qquad \text{(S.9)}
\end{aligned}$$

Recall that $e^{\mathrm{i}\omega_v}=1$, for $v=0$ and $e^{\mathrm{i}\omega_v}=-1$ for $v=S/2$.

Next consider the following definitions from del Barrio Castro, Rodrigues and Taylor (2018),

$$S_{i,c_i}(Sn+s) := \sum_{j=1}^{Sn+s}\cos\left[((Sn+s)-j)\,\omega_j\right]\alpha_{i,T}^{Sn+s-j}L^{Sn+s-j}, \qquad i=0,S/2 \qquad \text{(S.10)}$$

$$S_{k,c_k}^{-}(Sn+s) := \sum_{j=1}^{Sn+s}e^{-\mathrm{i}\omega_k(Sn+s-j)}\alpha_{k,T}^{Sn+s-j}L^{Sn+s-j}, \qquad k=1,\ldots,S^* \qquad \text{(S.11)}$$

$$S_{k,c_k}^{+}(Sn+s) := \sum_{j=1}^{Sn+s}e^{\mathrm{i}\omega_k(Sn+s-j)}\alpha_{k,T}^{Sn+s-j}L^{Sn+s-j}, \qquad k=1,\ldots,S^*. \qquad \text{(S.12)}$$

Using (S.10) - (S.12), we can write (2.3) as,

$$\begin{aligned}
x_{Sn+s} &= \left[\left(S_{0,c_0}(Sn+s)\right)^{h_0}\left(S_{S/2,c_{S/2}}(Sn+s)\right)^{h_{S/2}}\prod_{k=1}^{S^*}\left\{S_{k,c_k}^{-}(Sn+s)\,S_{k,c_k}^{+}(Sn+s)\right\}^{h_k}\right]u_{Sn+s} \\
&= \left[\left(S_{0,c_0}(Sn+s)\right)^{h_0}\left(S_{S/2,c_{S/2}}(Sn+s)\right)^{h_{S/2}}\right. \\
&\quad \left.\times\prod_{k=1}^{S^*}\left\{a_k S_{k,c_k}^{-}(Sn+s) + b_k S_{k,c_k}^{+}(Sn+s)\right\}^{h_k}\right]u_{Sn+s} + O_p(1) \qquad \text{(S.13)}
\end{aligned}$$

where $b_k := \frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_k}}$ and $a_k := \frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_k}}$, $k\in\{1,...,S^*\}$ and where we have used the fact that $S_{k,c_k}^{-}(Sn+s)\,S_{k,c_k}^{+}(Sn+s) = a_k S_{k,c_k}^{-}(Sn+s) + b_k S_{k,c_k}^{+}(Sn+s)$; see Gregoir (2006). Combining (S.13) with (S.2) to (S.4) we obtain that,

$$\begin{aligned}
x_{Sn+s} &= \left\{\psi_0 S_{0,c_0}(Sn+s) + \psi_{S/2}S_{S/2,c_{S/2}}(Sn+s)\right. \\
&\quad \left.+\sum_{k=1}^{S^*}\left[\psi_k^{-}S_{k,c_k}^{-}(Sn+s) + \psi_k^{+}S_{k,c_k}^{+}(Sn+s)\right]\right\}u_{Sn+s} + O_p(1). \qquad \text{(S.14)}
\end{aligned}$$

To determine the specific values for ψ_0, $\psi_{S/2}$, ψ_k^{-} and ψ_k^{+}, $k=1,\ldots,S^*$, in (S.14) knowledge of which operators $S_{0,c_0}(Sn+s)$, $S_{S/2,c_{S/2}}(Sn+s)$, $S_{k,c_k}^{-}(Sn+s)$ and $S_{k,c_k}^{+}(Sn+s)$, $k=1,\ldots,S^*$ are present in (S.13) is required. The specific values of these weights can then be computed directly from (S.2) to (S.4).

From (S.14) it then follows that the vector of seasons representation for X_n is given by,

$$X_n = \psi_0 C_0 \sum_{i=1}^{n} \alpha_{0,N}^{n-i} U_i + \psi_{S/2} C_{S/2} \sum_{i=1}^{n} \alpha_{S/2,N}^{n-i} U_i + \sum_{k=1}^{S^*} \left\{ \left(\psi_k^- C_k^- + \psi_k^+ C_k^+ \right) \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i \right\} + O_p(1). \tag{S.15}$$

S.3 Derivation of the Examples in Remark 3

- Consider first the DGP $\Delta_v^{c_v} \Delta_k^{c_k} x_{Sn+s} = u_{Sn+s}$, $v = 0, S/2$, $k = 1, \ldots, S^*$. This corresponds to (S.13) with $h_v = h_k = 1$ and all remaining $h_j = 0$, for all $j \neq v, k$. Using (S.2) and (S.3), it is possible after some algebra to write x_{Sn+s} as,

$$\begin{aligned} x_{Sn+s} \; = \; & \left[\frac{1}{2\left(1 - e^{\mathrm{i}\omega_v} \cos(\omega_k)\right)} S_{v,c_v}(Sn+s) + a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) S_{k,c_k}^-(Sn+s) \right. \\ & \left. + \, b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) S_{k,c_k}^+(Sn+s) \right] u_{Sn+s} + O_p(1). \end{aligned} \tag{S.16}$$

It is then straightforward to show that the vector of seasons representation associated with (S.16) is given by,

$$\begin{aligned} X_n \; = \; & \frac{1}{2\left(1 - e^{\mathrm{i}\omega_v} \cos(\omega_k)\right)} C_v \sum_{i=1}^{n} \alpha_{v,N}^{n-i} U_i + a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) C_k^- \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i \\ & + \, b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) C_k^+ \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + O_p(1). \end{aligned} \tag{S.17}$$

- Consider next the DGP $\Delta_k^{c_k} \Delta_j^{c_j} x_{Sn+s} = u_{Sn+s}$, which corresponds to (S.13) with $h_k = h_j = 1$ and all remaining $h_i = 0$, $i \neq j, k$. Using (S.3) we obtain after some algebra that,

$$\begin{aligned} x_{Sn+s} \; = \; & \left[a_k \left\{ a_j \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) + b_j \left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) \right\} S_{k,c_k}^-(Sn+s) + \right. \\ & a_j \left\{ a_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_j}} \right) + b_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_j}} \right) \right\} S_{j,c_j}^-(Sn+s) + \\ & b_k \left\{ a_j \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_k}} \right) + b_j \left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_k}} \right) \right\} S_{k,c_k}^+(Sn+s) + \\ & \left. b_j \left\{ a_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_j}} \right) + b_k \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) \right\} S_{j,c_j}^+(Sn+s) \right] u_{Sn+s} + O_p(1) \end{aligned} \tag{S.18}$$

where $a_k := \frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_k}}$, $b_k := \frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_k}}$, $a_j := \frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_j}}$ and $b_j := \frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_j}}$. Consequently, the corresponding vector of seasons representation of (S.18) is given by

$$\begin{aligned}
X_n &= a_k \left\{ a_j \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) + b_j \left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) \right\} C_k^- \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + \\
& a_j \left\{ a_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_j}} \right) + b_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_j}} \right) \right\} C_j^- \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + \\
& b_k \left\{ a_j \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_k}} \right) + b_j \left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_k}} \right) \right\} C_k^+ \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + \\
& b_j \left\{ a_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_j}} \right) + b_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_j}} \right) \right\} C_j^+ \sum_{i=1}^{n} \alpha_{k,N}^{n-i} U_i + O_p(1) . \qquad \text{(S.19)}
\end{aligned}$$

S.4 Derivation of Examples in Remark 7

In what follows we make use of the multivariate FCLT (see, for example, Phillips, 1988), from which we observe for the scaled partial sums that, under Assumption 1,

$$\frac{\sigma^{-1}}{\sqrt{N}} \sum_{i=1}^{\lfloor rN \rfloor} \alpha_{k,N}^{\lfloor rN \rfloor - i} U_i \Rightarrow \mathbf{J}_{c_k}(r), \quad r \in [0,1],\ k = 0, ..., S/2$$

where $\mathbf{J}_{c_k}(r)$ is an $S \times 1$ vector of standard Ornstein-Uhlenbeck processes.

- Consider first the DGP

$$\Delta_v^{c_v} \Delta_k^{c_k} x_{Sn+s} = u_{Sn+s}. \qquad \text{(S.20)}$$

where u_{Sn+s} satisfies Assumption 1. Here x_{Sn+s} admits a near-unit root at the zero ($v = 0$) or the Nyquist ($v = S/2$) frequency and a complex conjugate pair of near-unit roots at the kth harmonic seasonal frequency, $\omega_k := 2\pi k/S$. The DGP in (S.20) therefore coincides with (2.3) for the case where $h_v = h_k = 1$ and all remaining $h_j = 0$, for $j \neq v, k$. Consequently, using Lemma 1, Remark 3, and the definition of $\mathbf{A}$ in (3.2), we have that

$$\begin{aligned}
N^{-1/2} X_{\lfloor rN \rfloor}^{A} \Rightarrow \sigma \mathbf{A} \Bigg[& \frac{1}{2\left(1 - e^{\mathrm{i}\omega_v} \cos(\omega_k)\right)} \mathbf{v}_v \mathbf{v}_v' \mathbf{J}_{c_v}(r) + a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{v}_k^- \mathbf{v}_k^{+\prime} \mathbf{J}_{c_k}(r) \\
& + b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{v}_k^+ \mathbf{v}_k^{-\prime} \mathbf{J}_{c_k}(r) \Bigg] . \qquad \text{(S.21)}
\end{aligned}$$

Recalling the definitions $\mathbf{v}_k^{+'} := \mathbf{h}_k^{\alpha\prime} + \mathrm{i}\mathbf{h}_k^{\beta\prime}$ and $\mathbf{v}_k^{-'} := \mathbf{h}_k^{\alpha\prime} - \mathrm{i}\mathbf{h}_k^{\beta\prime}$ from Remark 6, it is possible to write (S.21) as,

$$\begin{aligned}
N^{-1/2} X_{\lfloor rN \rfloor}^{A} \Rightarrow\ & \frac{\sigma}{2\left(1 - e^{\mathrm{i}\omega_v} \cos(\omega_k)\right)} \mathbf{A} \mathbf{v}_v \mathbf{v}_v' \mathbf{J}_{c_v}(r) + \sigma a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{A} \mathbf{v}_k^- \left(\mathbf{h}_k^{\alpha\prime} + \mathrm{i}\mathbf{h}_k^{\beta\prime} \right) \mathbf{J}_{c_k}(r) \\
& + \sigma b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{A} \mathbf{v}_k^+ \left(\mathbf{h}_k^{\alpha\prime} - \mathrm{i}\mathbf{h}_k^{\beta\prime} \right) \mathbf{J}_{c_k}(r) . \qquad \text{(S.22)}
\end{aligned}$$

Moreover, for $\mathbf{Av}_k^-$ and $\mathbf{Av}_k^+$ we have that

$$\begin{aligned}\mathbf{Av}_k^+ &= \left[\begin{array}{ccccc} e^{\mathrm{i}\frac{2\pi k}{S}Q} & e^{\mathrm{i}\frac{2\pi k}{S}2Q} & e^{\mathrm{i}\frac{2\pi k}{S}3Q} & \dots & e^{\mathrm{i}\frac{2\pi k}{S}S_AQ}\end{array}\right]' \\ &= \left[\begin{array}{ccccc} e^{\mathrm{i}\frac{2\pi k}{S_A}} & e^{\mathrm{i}\frac{2\pi k}{S_A}2} & e^{\mathrm{i}\frac{2\pi k}{S_A}3} & \dots & e^{\mathrm{i}2\pi k}\end{array}\right]' \end{aligned} \tag{S.23}$$

and

$$\begin{aligned}\mathbf{Av}_k^- &= \left[\begin{array}{ccccc} e^{-\mathrm{i}\frac{2\pi k}{S}Q} & e^{-\mathrm{i}\frac{2\pi k}{S}2Q} & e^{-\mathrm{i}\frac{2\pi k}{S}3Q} & \dots & e^{-\mathrm{i}\frac{2\pi k}{S}S_AQ}\end{array}\right]' \\ &= \left[\begin{array}{ccccc} e^{-\mathrm{i}\frac{2\pi k}{S_A}} & e^{-\mathrm{i}\frac{2\pi k}{S_A}2} & e^{-\mathrm{i}\frac{2\pi k}{S_A}3} & \dots & e^{-\mathrm{i}2\pi k}\end{array}\right]'. \end{aligned} \tag{S.24}$$

The results in (S.23) and (S.24) are crucial for the understanding of whether or not near-unit roots present at a given spectral frequency in the original data will shift to another frequency under temporal aggregation. Specifically, it can be seen from the general results in the main text that a pair of complex conjugate near-unit roots at the harmonic seasonal frequency $\omega_k := 2\pi k/S$ will be mapped to the zero frequency under systematic sampling if the ratio k/S_A belongs to the set of natural numbers or to the Nyquist frequency under systematic sampling if the ratio $2k/S_A$ is a positive odd integer. For the former case we have that $\mathbf{Av}_0 = \mathbf{Av}_k^- = \mathbf{Av}_k^+ = \mathbf{1}_{S_A} =: \mathbf{v}_{S_A}^0$ and for the latter $\mathbf{Av}_{S/2} = \mathbf{Av}_k^- = \mathbf{Av}_k^+ = \mathbf{v}_{S_A}^\pi$ where $\mathbf{v}_{S_A}^\pi$ is an $S_A \times 1$ vector defined as $\mathbf{v}_{S_A}^\pi := [-1, 1, -1, \ldots]'$. Hence, for these two cases ($v = 0$ and $v = S/2$), (S.22) becomes,

$$\begin{aligned} N^{-1/2}X_{\lfloor rN\rfloor}^A \Rightarrow\; & \sigma\mathbf{v}_{S_A}^f \left[\frac{\sigma}{2\left(1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right)}\mathbf{v}_v'\mathbf{J}_{c_v}(r)\right. \\ & + a_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}\right)\left(\mathbf{h}_k^{\alpha\prime}+\mathrm{i}\mathbf{h}_k^{\beta\prime}\right)\mathbf{J}_{c_k}(r) + b_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}\right)\left.\left(\mathbf{h}_k^{\alpha\prime}-\mathrm{i}\mathbf{h}_k^{\beta\prime}\right)\mathbf{J}_{c_k}(r)\right] \\ =\; & \sigma\mathbf{v}_{S_A}^f\left[\frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right)}\mathbf{v}_v'\mathbf{J}_{c_v}(r)\right. \\ & \left.+\frac{\sin\left(\omega_k\right)-e^{\mathrm{i}\omega_v}\sin\left(2\omega_k\right)}{2\left[1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right]\sin\left(\omega_k\right)}\mathbf{h}_k^{\alpha\prime}\mathbf{J}_{c_k}(r) - \frac{\cos\left(\omega_k\right)-e^{\mathrm{i}\omega_v}\cos\left(2\omega_k\right)}{2\left[1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right]\sin\left(\omega_k\right)}\mathbf{h}_k^{\beta\prime}\mathbf{J}_{c_k}(r)\right] \end{aligned} \tag{S.25}$$

where $f = 0$ when $v = 0$ and $f = \pi$ when $v = S/2$, and where we have used the following results

$$a_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}\right)+b_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}\right) = \frac{\sin\left(\omega_k\right)-e^{\mathrm{i}\omega_v}\sin\left(2\omega_k\right)}{2\left[1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right]\sin\left(\omega_k\right)} \tag{S.26}$$

and

$$a_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}\right)-b_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_v}}\right) = -\frac{\cos\left(\omega_k\right)-e^{\mathrm{i}\omega_v}\cos\left(2\omega_k\right)}{2\mathrm{i}\left[1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right]\sin\left(\omega_k\right)}. \tag{S.27}$$

Recall that the results for the Nyquist frequency are only considered when $2k/S_A$ is a positive odd integer.

To determine the local-to-unity parameter associated with the resulting zero frequency near-unit root in the systematically sampled aggregated data, consider the following lemma.

Lemma S.1. *Let x_{Sn+s} be generated as in* (S.20). *Then, the vector of seasons representation for the systematically sampled data when k/S_A takes values in the set of natural numbers (when the zero frequency is considered, $f=0$ and $v=0$) or $2k/S_A$ takes values in the set of positive odd integer numbers (when the Nyquist frequency is considered, $f=\pi$ and $v=S/2$) is given by*

$$\begin{aligned} X_n^A &= \mathbf{v}_{S_A}^f \left(\frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)} \mathbf{v}_v' \sum_{i=1}^n \alpha_{v,N}^{n-i} U_i + \frac{\sin(\omega_k) - e^{\mathrm{i}\omega_v}\sin(2\omega_k)}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)\sin(\omega_k)} \mathbf{h}_k^{\alpha\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \right. \\ & \left. - \frac{\cos(\omega_k) - e^{\mathrm{i}\omega_v}\cos(2\omega_k)}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)\sin(\omega_k)} \mathbf{h}_k^{\beta\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \right) + O_p(1). \qquad \text{(S.28)} \end{aligned}$$

Proof of Lemma S.1. To prove (S.28), note first that on pre-multiplying (S.17) by $\mathbf{A}$ we obtain

$$\begin{aligned} \mathbf{A}X_n &= \frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)} \mathbf{A}\mathbf{v}_v\mathbf{v}_v{}' \sum_{i=1}^n \alpha_{v,N}^{n-i} U_i + a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{A}\mathbf{v}_k^- \mathbf{v}_k^{+\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \\ & + b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{A}\mathbf{v}_k^- \mathbf{v}_k^{+\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i + O_p(1). \qquad \text{(S.29)} \end{aligned}$$

When the ratio k/S_A takes values in the set of natural numbers (the zero frequency case) or $2k/S_A$ takes values in the set of positive odd integer numbers (the Nyquist frequency case), (S.29) becomes

$$\begin{aligned} \mathbf{A}X_n &= \mathbf{v}_{S_A}^f \left(\frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)} \mathbf{v}_v{}' \sum_{i=1}^n \alpha_{v,N}^{n-i} U_i + a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{v}_k^{+\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \right. \\ & \left. + b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \mathbf{v}_k^{-\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \right) + O_p(1). \qquad \text{(S.30)} \end{aligned}$$

Noting that $\mathbf{v}_k^{+'} = \mathbf{h}_k^{\alpha\prime} + \mathrm{i}\mathbf{h}_k^{\beta\prime}$ and $\mathbf{v}_k^{-'} = \mathbf{h}_k^{\alpha\prime} - \mathrm{i}\mathbf{h}_k^{\beta\prime}$, it is possible to write:

$$\begin{aligned} \mathbf{A}X_n &= \mathbf{v}_{S_A}^f \left(\frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)} \mathbf{v}_v{}' \sum_{i=1}^n \alpha_{v,N}^{n-i} U_i \right. \\ & + a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \left(\mathbf{h}_k^{\alpha\prime} + \mathrm{i}\mathbf{h}_k^{\beta\prime} \right) \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \\ & \left. + b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \left(\mathbf{h}_k^{\alpha\prime} + \mathrm{i}\mathbf{h}_k^{\beta\prime} \right) \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \right) + o_p(1) \\ &= \mathbf{v}_{S_A}^f \left(\frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos(\omega_k)\right)} \mathbf{v}_v{}' \sum_{i=1}^n \alpha_{v,N}^{n-i} U_i \right. \\ & + \left[a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) + b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_v}} \right) \right] \mathbf{h}_k^{\alpha\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \\ & \left. + \left[a_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - 1} \right) - b_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - 1} \right) \right] \mathrm{i}\mathbf{h}_k^{\beta\prime} \sum_{i=1}^n \alpha_{k,N}^{n-i} U_i \right) + o_p(1) \end{aligned}$$

$$= \mathbf{v}_{S_A}^{f}\left(\frac{1}{2\left(1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right)}\mathbf{v}_v{}'\sum_{i=1}^{n}\alpha_{v,N}^{n-i}U_i + \frac{\sin\left(\omega_k\right)-e^{\mathrm{i}\omega_v}\sin\left(2\omega_k\right)}{2\left(1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right)\sin\left(\omega_k\right)}\mathbf{h}_k^{\alpha\prime}\sum_{i=1}^{n}\alpha_{k,N}^{n-i}U_i - \frac{\cos\left(\omega_k\right)-e^{\mathrm{i}\omega_v}\cos\left(2\omega_k\right)}{2\left(1-e^{\mathrm{i}\omega_v}\cos\left(\omega_k\right)\right)\sin\left(\omega_k\right)}\mathbf{h}_k^{\beta\prime}\sum_{i=1}^{n}\alpha_{k,N}^{n-i}U_i\right) + O_p\left(1\right). \quad \text{(S.31)}$$

Then from (S.26) and (S.27), (S.31) is seen to be equivalent to (S.28), as required. ∎

To illustrate how to determine the local-to-unity parameter in (S.28), we will use the following two examples:

Example 1: Consider the case shown in Table 1 for $S = 12$ and $S_A = 4$. For frequency $\omega_k = 2\pi/3$, $k = 4$ so that $k/S_A = 1$ and so the results in (S.25)-(S.28) above apply. Hence, (S.28) for this example is given by

$$X_n^A = \mathbf{1}_{S_A}\left(\frac{1}{3}\mathbf{v}_0'\sum_{i=1}^{n}\alpha_{0,N}^{n-i}U_i + \frac{2}{3}\mathbf{h}_k^{\alpha\prime}\sum_{i=1}^{n}\alpha_{k,N}^{n-i}U_i\right) + O_p\left(1\right). \quad \text{(S.32)}$$

Note that frequency $2\pi/3$ completes a full cycle every 3 periods, and hence for $\mathbf{h}_k^{\alpha\prime}$ we only need to consider the following three values that are repeated cyclically, $\cos\left(2\pi/3\right) = -1/2$, $\cos\left(4\pi/3\right) = -1/2$ and $\cos\left(6\pi/3\right) = 1$, so that $\mathbf{h}_k^{\alpha\prime} = \left[-1/2, -1/2, 1, -1/2, -1/2, 1, -1/2, -1/2, 1, -1/2, -1/2, 1\right]$. Therefore, for (S.32) we have that

$$\begin{aligned} X_n^A = \mathbf{1}_{S_A}\Bigg[& \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-11}, \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-10}, \sum_{i=1}^{n}\left(1+\frac{\left(c_0+2c_k\right)/3}{N}\right)^{n-i}u_{12i-9}, \\ & \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-8}, \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-7}, \sum_{i=1}^{n}\left(1+\frac{\left(c_0+2c_k\right)/3}{N}\right)^{n-i}u_{12i-6}, \\ & \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-5}, \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-4}, \sum_{i=1}^{n}\left(1+\frac{\left(c_0+2c_k\right)/3}{N}\right)^{n-i}u_{12i-3}, \\ & \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-2}, \frac{1}{3}\sum_{i=1}^{n}\delta_{0k,N}^{n-i}u_{12i-1}, \sum_{i=1}^{n}\left(1+\frac{\left(c_0+2c_k\right)/3}{N}\right)^{n-i}u_{12i}\Bigg] + O_p\left(1\right) \end{aligned} \quad \text{(S.33)}$$

where $\delta_{0k,N} := \left(\frac{c_0-c_k}{N}\right)$. Given that $\delta_{0k,N} = O\left(N^{-1}\right)$, (S.33) can be re-written as

$$X_n^A = \mathbf{1}_{S_A}\mathbf{v}_{\mathbf{a}}^{0\prime}\sum_{i=1}^{n}\exp\left(\frac{\left(c_0+2c_k\right)/3}{N}\right)^{n-i}U_i + O_p\left(1\right)$$

where $\mathbf{v}_{\mathbf{a}}^{0\prime} := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$. This can immediately be seen to correspond to a zero frequency near-unit root with local-to-unity parameter $c_0^A := \left(c_0 + 2c_k\right)/3$. Notice also from the multivariate FCLT that,

$$N^{-1/2}X^A_{\lfloor rN\rfloor} \Rightarrow \sigma\varpi_0\mathbf{1}_{S_A}\varpi_0^{-1}\mathbf{v}_{\mathbf{a}}^{0\prime}\mathbf{J}_{\mathbf{c}_0^A}(r) = \sigma\varpi_0\mathbf{1}_{S_A}J_{0,c_0^A}(r)$$

where c_0^A is as given above and $\varpi_0 := \left(\mathbf{v}_{\mathbf{a}}^{0\prime}\mathbf{v}_{\mathbf{a}}^{0}\right)^{1/2} = 2$.

Example 2: Consider the case shown in Table 1 for $S = 12$ and $S_A = 4$. If $\omega_k = \pi/3$, then $k = 2$ and so $2k/S_A = 1$. Here (S.28) reduces to,

$$X^A_n = \mathbf{v}^{\pi}_{S_A}\left(\frac{1}{3}\mathbf{v}'_{S/2}\sum_{i=1}^{n}\alpha^{n-i}_{S/2,N}U_i + \frac{2}{3}\mathbf{h}^{\alpha\prime}_k\sum_{i=1}^{n}\alpha^{n-i}_{k,N}U_i\right) + O_p(1). \tag{S.34}$$

Since frequency $\pi/3$ completes a full cycle every six periods, we only need to consider the following six values that are repeated cyclically for determining $\mathbf{h}^{\alpha\prime}_k$: $\cos(\pi/3) = 1/2$, $\cos(2\pi/3) = -1/2$, $\cos(3\pi/3) = -1$, $\cos(4\pi/3) = -1/2$, $\cos(5\pi/3) = 1/2$ and $\cos(6\pi/3) = 1$; viz, $\mathbf{h}^{\alpha\prime}_k = [1/2, -1/2, -1, -1/2, 1/2, 1, 1/2, -1/2, -1, -1/2, 1/2, 1]$. Therefore, from (S.34) we have that,

$$\begin{aligned} X^A_n \;=\; \mathbf{v}^{\pi}_{S_A}\Bigg[& \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-11}, \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-10}, -\sum_{i=1}^{n}\left(1+\frac{\left(c_{S/2}+2c_k\right)/3}{N}\right)^{n-i}u_{12i-9}, \\ & \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-8}, \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-7}, \sum_{i=1}^{n}\left(1+\frac{\left(c_{S/2}+2c_k\right)/3}{N}\right)^{n-i}u_{12i-6}, \\ & \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-5}, \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-4}, -\sum_{i=1}^{n}\left(1+\frac{\left(c_{S/2}+2c_k\right)/3}{N}\right)^{n-i}u_{12i-3}, \\ & \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-2}, \frac{1}{3}\sum_{i=1}^{n}\delta^{n-i}_{kS/2,N}u_{12i-1}, \sum_{i=1}^{n}\left(1+\frac{\left(c_{S/2}+2c_k\right)/3}{N}\right)^{n-i}u_{12i}\Bigg] + O_p(1) \end{aligned} \tag{S.35}$$

where $\delta_{kS/2,N} := \frac{c_k - c_{S/2}}{N}$. Given that $\delta_{kS/2,N} = O\left(N^{-1}\right)$, we can re-write (S.35) as

$$X^A_n := \mathbf{A}X_n = \mathbf{v}^{\pi}_{S_A}\mathbf{v}^{\pi\prime}_{\mathbf{a}}\sum_{i=1}^{n}\exp\left(\frac{\left(c_{S/2}+2c_k\right)/3}{N}\right)^{n-i}U_i + O_p(1)$$

where $\mathbf{v}^{\pi\prime}_{\mathbf{a}} := \left[\begin{array}{cccccccccccc} 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$. This can then be seen to correspond to a Nyquist frequency near-unit root with local-to-unity parameter $c^A_{S/2} := \left(c_{S/2}+2c_k\right)/3$. Moreover,

$$N^{-1/2}X^A_{\lfloor rN\rfloor} \Rightarrow \sigma\varpi_\pi\mathbf{v}^{\pi}_{S_A}\varpi_\pi^{-1}\mathbf{v}^{\pi\prime}_{\mathbf{a}}\mathbf{J}_{c^A_{S/2}}(r) = \sigma\varpi_\pi\mathbf{v}^{\pi}_{S_A}J_{\pi,c^A_{S/2}}(r)$$

where $\varpi_\pi := \left(\mathbf{v}^{\pi\prime}_{\mathbf{a}}\mathbf{v}^{\pi}_{\mathbf{a}}\right)^{1/2} = 2$.

- Finally, consider the case where the DGP is given by

$$\Delta_k^{c_k}\Delta_j^{c_j}x_{Sn+s} = u_{Sn+s}, \tag{S.36}$$

where u_{Sn+s} again satisfies Assumption 1. Here x_{Sn+s} admits a pair of complex conjugate near-unit roots at both the jth and kth, $j > k$, harmonic seasonal frequencies, $\omega_j := 2\pi j/S$ and $\omega_k := 2\pi k/S$, respectively. The DGP in (S.36) therefore coincides with (2.3) on setting $h_j = h_k = 1$ and all remaining $h_i = 0$, for $i \neq j, k$. Again, using Lemma 1 and Remark 3, we have that

$$\begin{aligned} N^{-1/2}X^A_{\lfloor rN \rfloor} \;\Rightarrow\; & \sigma\mathbf{A}\Bigg[a_k\left\{a_j\left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j}-e^{\mathrm{i}\omega_k}}\right)+b_j\left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j}-e^{\mathrm{i}\omega_k}}\right)\right\}\mathbf{v}_k^-\mathbf{v}_k^{+\prime}\mathbf{J}_{c_k}(r) \\ & + a_j\left\{a_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_j}}\right)+b_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_j}}\right)\right\}\mathbf{v}_j^-\mathbf{v}_j^{+\prime}\mathbf{J}_{c_j}(r) \\ & + b_k\left\{a_j\left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j}-e^{-\mathrm{i}\omega_k}}\right)+b_j\left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j}-e^{-\mathrm{i}\omega_k}}\right)\right\}\mathbf{v}_k^+\mathbf{v}_k^{-\prime}\mathbf{J}_{c_k}(r) \\ & + b_j\left\{a_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_j}}\right)+b_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_j}}\right)\right\}\mathbf{v}_j^+\mathbf{v}_j^{-\prime}\mathbf{J}_{c_j}(r)\Bigg] \end{aligned} \tag{S.37}$$

which can be re-written as

$$\begin{aligned} N^{-1/2}X^A_{\lfloor rN \rfloor} \;\Rightarrow\; & \sigma a_k\left\{a_j\left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j}-e^{\mathrm{i}\omega_k}}\right)+b_j\left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j}-e^{\mathrm{i}\omega_k}}\right)\right\}\mathbf{A}\mathbf{v}_k^-\mathbf{v}_k^{+}\mathbf{J}_{c_k}(r) \\ & + \sigma a_j\left\{a_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_j}}\right)+b_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_j}}\right)\right\}\mathbf{A}\mathbf{v}_j^-\mathbf{v}_j^{+}\mathbf{J}_{c_j}(r) \\ & + \sigma b_k\left\{a_j\left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j}-e^{-\mathrm{i}\omega_k}}\right)+b_j\left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j}-e^{-\mathrm{i}\omega_k}}\right)\right\}\mathbf{A}\mathbf{v}_k^+\mathbf{v}_k^{-}\mathbf{J}_{c_k}(r) \\ & + \sigma b_j\left\{a_k\left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_j}}\right)+b_k\left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_j}}\right)\right\}\mathbf{A}\mathbf{v}_j^+\mathbf{v}_j^{-}\mathbf{J}_{c_j}(r). \end{aligned} \tag{S.38}$$

We observe from (S.24) and (S.23) that $\mathbf{Av}_k^-$ and $\mathbf{Av}_k^+$ will correspond with frequency $\omega_k^A = 2\pi k/S_A$ and $\mathbf{Av}_j^-$ and $\mathbf{Av}_j^+$ with frequency $\omega_j^A = 2\pi j/S_A$. Hence, the only possible situation where the OU processes in (S.38) will be allocated to the same harmonic frequency is when $2\pi k/S_A$ and $2\pi j/S_A$ are such that $2\pi j/S_A = 2\pi - 2\pi k/S_A$; that is, $j = S_A - k$. For example, when aggregating from monthly to quarterly observations ($S = 12$ and $S_A = 4$), the complex pairs of OU processes associated with frequencies $\pi/6$ ($k = 1$) and $\pi/2$ ($j = 3$) in the monthly data will both be mapped under aggregation to a complex pair of OU processes at frequency $\pi/2$ (since $3\pi/2 = 2\pi - \pi/2$) in the quarterly data. Where $j = S_A - k$, we therefore have that $\mathbf{Av}_k^- = \mathbf{Av}_j^+$ and $\mathbf{Av}_k^+ = \mathbf{Av}_j^-$, and, hence,

$$\begin{aligned} N^{-1/2}X^A_{\lfloor rN \rfloor} \;\Rightarrow\; & \frac{\sigma S}{2}\Big\{\mathbf{Av}_k^-\left[a_k^*\left(\mathbf{h}_k^{\alpha\prime}+\mathrm{i}\mathbf{h}_k^{\beta\prime}\right)\mathbf{J}_{c_k}(r)+b_j^*\left(\mathbf{h}_j^{\alpha\prime}-\mathrm{i}\mathbf{h}_j^{\beta\prime}\right)\mathbf{J}_{c_j}(r)\right] \\ & + \mathbf{Av}_k^+\left[a_j^*\left(\mathbf{h}_j^{\alpha\prime}+\mathrm{i}\mathbf{h}_j^{\beta\prime}\right)\mathbf{J}_{c_j}(r)+b_k^*\left(\mathbf{h}_k^{\alpha\prime}-\mathrm{i}\mathbf{h}_k^{\beta\prime}\right)\mathbf{J}_{c_k}(r)\right]\Big\} \end{aligned} \tag{S.39}$$

where

$$
\begin{aligned}
a_k^* &:= a_k \left\{ a_j \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) + b_j \left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{\mathrm{i}\omega_k}} \right) \right\} \\
a_j^* &:= a_j \left\{ a_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_j}} \right) + b_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_j}} \right) \right\} \\
b_k^* &:= b_k \left\{ a_j \left(\frac{e^{\mathrm{i}\omega_j}}{e^{\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_k}} \right) + b_j \left(\frac{e^{-\mathrm{i}\omega_j}}{e^{-\mathrm{i}\omega_j} - e^{-\mathrm{i}\omega_k}} \right) \right\} \\
b_j^* &:= b_j \left\{ a_k \left(\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_j}} \right) + b_k \left(\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_j}} \right) \right\}.
\end{aligned}
$$

As with Cases A and B above, it can be shown that the resulting complex pair of OU processes at frequency ω_k^A will have a drift (local-to-unity) parameter given by a linear combination of the drifts defining the OU processes at frequencies ω_j and ω_k prior to aggregation, and, hence, that the systematically sampled process has a complex pair of near-unit roots at frequency ω_k^A with this local-to-unity parameter.

S.5 Additional Material relating to Average Sampling

For any value $P \leq S$, (2.3) can always be represented as a first order vector of seasons process as in (2.4), replicated below for convenience

$$\Psi_0 X_n = \Psi_1 X_{n-1} + U_n^*, \; n = 1, ..., N$$

where $X_n := \left(x_{Sn-(S-1)}, x_{Sn-(S-2)}, ..., x_{Sn}\right)'$, $X_{n-1} := \left(x_{S(n-1)-(S-1)}, x_{S(n-1)-(S-2)}, ..., x_{S(n-1)}\right)'$, X_0 is a vector of zeros, and $U_n^* := \left(u_{Sn-(S-1)}, u_{Sn-(S-2)}, ..., u_{Sn}\right)'$ is a vector white noise process with zero mean vector and variance $\sigma^2 I_S$, and where Ψ_0 and Ψ_1 are lower and upper triangular $S \times S$ parameter matrices, respectively. In particular, Ψ_0 is a lower triangular Toeplitz matrix which for $P < S$ takes the form

$$
\Psi_0 = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\alpha_1^* & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & -\alpha_1^* & 1 & \cdots & 0 & 0 & 0 \\
-\alpha_P^* & \vdots & -\alpha_1^* & \cdots & \vdots & \vdots & \vdots \\
0 & -\alpha_P^* & \vdots & \cdots & 1 & 0 & 0 \\
\vdots & 0 & -\alpha_P^* & \cdots & -\alpha_1^* & 1 & 0 \\
0 & \vdots & \vdots & \cdots & -\alpha_2^* & -\alpha_1^* & 1
\end{bmatrix}
$$

and for $P = S$ takes the form

$$\Psi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\alpha_1^* & 1 & 0 & 0 & 0 & \cdots & 0 \\ -\alpha_2^* & -\alpha_1^* & 1 & 0 & 0 & \cdots & 0 \\ -\alpha_3^* & -\alpha_2^* & -\alpha_1^* & 1 & 0 & \cdots & 0 \\ -\alpha_4^* & -\alpha_3^* & -\alpha_2^* & -\alpha_1^* & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_S^* & -\alpha_{S-1}^* & -\alpha_{S-2}^* & -\alpha_{S-3}^* & -\alpha_{S-4}^* & \cdots & 1 \end{bmatrix}.$$

For $P < S$, the upper triangular matrix Ψ_1 takes the form

$$\Psi_1 = \begin{bmatrix} 0 & \cdots & 0 & \alpha_P^* & \cdots & \alpha_1^* \\ \cdots & \cdots & 0 & 0 & \cdots & \alpha_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_3^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and for $P = S$ takes the form

$$\Psi_1 = \begin{bmatrix} 0 & \alpha_P^* & \cdots & \alpha_2^* & \alpha_1^* \\ 0 & \cdots & \cdots & \alpha_3^* & \alpha_2^* \\ 0 & \cdots & \cdots & \alpha_5^* & \alpha_4^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

As we will see, the vector of seasons representation has clear advantages compared to univariate approaches which have appeared in the literature because the effects of temporal aggregation in terms of processes moving from one frequency to another or vanishing after aggregation can be seen clearly, as can the form of any moving average dynamics induced by average sampling.

As discussed in the main text, it is convenient in what follows to consider the reduced form of (2.4) given in (2.5), again replicated for convenience below,

$$X_n = \Phi X_{n-1} + U_n$$

where[9] $\Phi := \Psi_0^{-1}\Psi_1$ and $U_n := \Psi_0^{-1}U_n^*$, where the error term U_n has zero mean and variance matrix $E\left(U_n U_n'\right) = \Psi_0^{-1}E\left(U_n^* U_n^{*\prime}\right)\Psi_0^{-1\prime} = \sigma^2\Psi_0^{-1}\Psi_0^{-1\prime}$.

In the body of the paper we are concerned only with the impact of average sampling on the autoregressive dynamics of (2.5). However, when we aggregate (2.5) the structure of Ψ_0^{-1} has the potential to generate moving average behaviour. In general, for any seasonal unrestricted AR(p)

[9]The elements of Ψ_0^{-1} are determined by the parameters of the inverse of $\alpha(L)$; see Pollock (1999,pp.46-48) for details. An example is given in (S.46) below for the DGP in (S.40).

process, with $p \leq S$, Ψ_0^{-1} will always be lower triangular, which means that average sampling of such a process, will result in the presence of moving average dynamics in the aggregated process. An exception occurs in cases where the AR(p) process is a restricted seasonal process, with $p = S$, such as $\left(1 - \alpha_T L^S\right) x_{Sn+s} = u_{Sn+s}$. In this case, in the vector of seasons representation the matrix $\Psi_0 = \Psi_0^{-1} = I_S$ and therefore no moving average dynamics will be generated by average sampling.

As an illustration of the result in (2.5) consider the general AR(1) process,

$$\left(1 - e^{\pm \mathrm{i}\omega_j} \alpha_{j,T} L\right) x_{Sn+s} = u_{Sn+s} \tag{S.40}$$

which admits the following first order vector of seasons representation, *viz.*,

$$\Psi_0^{j\pm} X_n = \Psi_1^{j\pm} X_{n-1} + U_n^*, \; n = 1, ..., N, \; j = 0, 1, ..., S/2 \tag{S.41}$$

where $X_n := \left(x_{Sn-(S-1)}, x_{Sn-(S-2)}, ..., x_{Sn}\right)'$, $X_{n-1} := \left(x_{S(n-1)-(S-1)}, x_{S(n-1)-(S-2)}, ..., x_{S(n-1)}\right)'$, X_0 is again taken for simplicity to be a vector of zeros, and $U_n^* := \left(u_{Sn-(S-1)}, u_{Sn-(S-2)}, ..., u_{Sn}\right)'$ is a vector white noise process with zero mean vector and variance $\sigma^2 I_S$, and $\Psi_0^{j\pm}$ is a lower triangular $S \times S$ parameter matrix with the following form: $\Psi_0^{j\pm} = (I_S - M_S e^{\pm \mathrm{i}\omega_j} \alpha_{j,T})$ where M_S is an $S \times S$ matrix with elements $m_{ij} = \begin{cases} 1 & if \quad i - j = 1 \\ 0 & \text{otherwise} \end{cases}$ and $\Psi_1^{j\pm}$ is an $S \times S$ matrix of zeros with the exception of the element in position $(1, S)$ which is equal to $e^{\pm \mathrm{i}\omega_j} \alpha_{j,T}$. Notice that because $e^{\mathrm{i}\omega_0} = e^{-\mathrm{i}\omega_0}$ and $e^{\mathrm{i}\omega_{S/2}} = e^{-\mathrm{i}\omega_{S/2}}$, whenever the zero or Nyquist frequencies are under consideration we will simply use $e^{\mathrm{i}\omega_0}$ and $e^{\mathrm{i}\omega_{S/2}}$.

It will prove convenient to consider the reduced form of (S.41); that is,

$$X_n = \Phi^{k\pm} X_{n-1} + \left(\Psi_0^{k\pm}\right)^{-1} U_n^*, \; k = 0, 1, ..., S/2 \tag{S.42}$$

where $\Phi^{k\pm} := \left(\Psi_0^{k\pm}\right)^{-1} \Psi_1^{k\pm}$. When $k = 0$ or $k = S/2$ are under consideration, we will therefore denote the resulting parameter matrices by simply Ψ_0^k and Ψ_1^k, $k = 0, S/2$. Notice that the error term $U_n = \left(\Psi_0^{k\pm}\right)^{-1} U_n^*$ has zero mean and variance matrix $E\left(U_n U_n'\right) = \left(\Psi_0^{k\pm}\right)^{-1} E\left(U_n^* U_n^{*\prime}\right) \left(\Psi_0^{k\pm}\right)^{-1\prime} = \sigma^2 \left(\Psi_0^{k\pm}\right)^{-1} \left(\Psi_0^{k\pm}\right)^{-1\prime}$. Backward substitution in (S.42) yields that,

$$X_n = \sum_{j=0}^{n-1} \left(\Phi^{k\pm}\right)^j \left(\Psi_0^{k\pm}\right)^{-1} U_{n-j}^* \tag{S.43}$$

with the convention that $\left(\Phi^{j\pm}\right)^0 = I_S$. Hence, the vector of seasons representation of (S.40) can be expressed as,

$$\begin{aligned} X_n &= \delta_0 \sum_{j=0}^{n-1} \left(\Phi^0\right)^j \left(\Psi_0^{S/2}\right)^{-1} U_{n-j}^* + \delta_{S/2} \sum_{j=0}^{n-1} \left(\Phi^{S/2}\right)^j \left(\Psi_0^{S/2}\right)^{-1} U_{n-j}^* + \\ & \sum_{k=1}^{S^*} \left(\delta_k^- \sum_{j=0}^{n-1} \left(\Phi^{k-}\right)^j U_{n-j}^* + \delta_k^+ \sum_{j=0}^{n-1} \left(\Phi^{k+}\right)^j \left(\Psi_0^{k+}\right)^{-1} U_{n-j}^* \right). \end{aligned} \tag{S.44}$$

The effects of temporal aggregation on (S.40) can be explored by pre-multiplying (S.44) by the $S \times S_A$ matrix $\mathbf{A}$ defined in (3.2); *viz*,

$$\begin{aligned} \mathbf{A}X_n &= \delta_0 \left(\mathbf{A} \left(\Psi_0^0\right)^{-1} U_n^* + \mathbf{A} \sum_{j=1}^{n-1} \left(\Phi^0\right)^j \left(\Psi_0^0\right)^{-1} U_{n-j}^* \right) + \\ & \delta_{S/2} \left(\mathbf{A} \left(\Psi_0^{S/2}\right)^{-1} U_n^* + \mathbf{A} \sum_{j=1}^{n-1} \left(\Phi^{S/2}\right)^j \left(\Psi_0^{S/2}\right)^{-1} U_{n-j}^* \right) + \\ & \sum_{k=1}^{S^*} \left(\delta_k^- \left[\mathbf{A} \left(\Psi_0^{k-}\right)^{-1} U_n^* + \mathbf{A} \sum_{j=0}^{n-1} \left(\Phi^{k-}\right)^j \left(\Psi_0^{k-}\right)^{-1} U_{n-j}^* \right] \right. \\ & \left. + \delta_k^+ \left[\mathbf{A} \left(\Psi_0^{k+}\right)^{-1} U_n^* + \mathbf{A} \sum_{j=0}^{n-1} \left(\Phi^{k+}\right)^j \left(\Psi_0^{k+}\right)^{-1} U_{n-j}^* \right] \right). \end{aligned} \tag{S.45}$$

Temporal aggregation of these frequency specific processes induces MA(1) dynamics in the error process which is observed from $\mathbf{A}\left(\Psi_0^0\right)^{-1} U_n^*$, $\mathbf{A}\left(\Psi_0^{S/2}\right)^{-1} U_n^*$, $\mathbf{A}\left(\Psi_0^{k-}\right)^{-1} U_n^*$ and $\mathbf{A}\left(\Psi_0^{k+}\right)^{-1} U_n^*$, $k = 1, ..., S^*$, where the coefficients of the infinite order moving average representation of (S.40) are collected as the elements of the resulting matrices $\mathbf{A}\left(\Psi_0^0\right)^{-1}$, $\mathbf{A}\left(\Psi_0^{S/2}\right)^{-1}$ and $\mathbf{A}\left(\Psi_0^{k\pm}\right)^{-1}$. To illustrate, consider frequency $\omega_k = 2\pi k/S$ for the case of $Q = S/S_A = 3$. The coefficients of the infinite order moving average representation of (S.40) for this frequency are given by

$$\left(\Psi_0^{k\pm}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_{k,T} e^{\pm i\omega_k} & 1 & 0 & 0 & \cdots & 0 \\ \alpha_{k,T}^2 e^{\pm i2\omega_k} & \alpha_{k,T} e^{\pm i\omega_k} & 1 & 0 & \cdots & 0 \\ \alpha_{k,T}^3 e^{\pm i3\omega_k} & \alpha_{k,T}^2 e^{\pm i2\omega_k} & \alpha_{k,T} e^{\pm i\omega_k} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{k,T}^{(S-1)} e^{\pm i(S-1)\omega_k} & \alpha_{k,T}^{(S-2)} e^{-i(S-2)\omega_k} & \alpha_{k,T}^{(S-3)} e^{-i(S-3)\omega_k} & \alpha_{k,T}^{(S-4)} e^{-i(S-4)\omega_k} & \cdots & 1 \end{bmatrix}. \tag{S.46}$$

Hence, the elements of $\mathbf{A}\left(\Psi_0^{k\pm}\right)^{-1}$, which follow directly from (11), characterize the moving average dynamics. □

Remark S.1: It is interesting to contrast the foregoing results with the alternative univariate approach to determining the impact of temporal aggregation considered in Silvestrini and Veredas (2008, Section 3.1) and Teles, Wei and Hodgess (2008). These authors show that the effects of temporal aggregation of the process in (S.40) from S observations per year to S_A per year, that is $S/S_A = Q$, can be obtained by multiplying both sides of (S.40) by $\left[\frac{\left(1-e^{\pm \mathrm{i} Q\omega_j}\alpha_{j,T}^{Q}L^{Q}\right)}{\left(1-e^{\pm \mathrm{i}\omega_j}\alpha_{j,T}L\right)}\right]\left[\frac{1-L^Q}{1-L}\right]$, so that,

$$
\begin{aligned}
&\left(1-e^{\pm \mathrm{i}Q\omega_j}\alpha_{j,T}^{Q}L^{Q}\right)\left(1+L+\cdots+L^{Q-1}\right)x_{Sn+s} = \\
&\left(1+e^{\pm i\omega_j}\alpha_{j,T}L+\cdots e^{\pm i(Q-1)\omega_j}\alpha_{j,T}^{Q-1}L^{Q-1}\right)\left(1+L+\cdots+L^{Q-1}\right)u_{Sn+s}.
\end{aligned} \tag{S.47}
$$

From (S.47) we observe that after aggregation of the frequency specific AR(1) process we obtain an AR(1) process for the temporally aggregated process $\left(1+L+\cdots+L^{Q-1}\right)x_{Sn+s}$ with the autoregressive parameter given by $e^{\pm \mathrm{i}Q\omega_j}\alpha_{j,T}^{Q}$. Moreover, Silvestrini and Veredas (2008) and Teles, Wei and Hodgess (2008) show that the left hand side of (S.47) implies the presence of an MA(1) disturbance, such that the temporally aggregated time series follows an ARMA(1,1) process. □

Remark S.2: An advantage of the demodulator operator is that we clearly see that the AR(1) process vanishes when k/S_A is an integer. However, this result cannot be clearly appreciated from (S.47). To illustrate, consider the case where $S = 12$, $S_A = 4$ and $Q = 3$. Here the coefficients of the MA(∞)

$$
\psi(L) = 1+\sum_{i=1}^{\infty}\psi_i L^i = \left(1+e^{\mp \mathrm{i}\omega_k}\alpha_{k,T}L+e^{\mp \mathrm{i}\omega_k 2}\alpha_{k,T}^2L^2\right)\left(1+L+L^2\right)/\left(1-e^{\mp \mathrm{i}\omega_k 3}\alpha_{k,T}^3L^3\right)
$$

are such that

$$
\begin{aligned}
\psi_1 &= 1+e^{\mp \mathrm{i}\omega_k}\alpha_{k,T} \\
\psi_2 &= 1+e^{\mp \mathrm{i}\omega_k}\alpha_{k,T}+e^{\mp \mathrm{i}\omega_k 2}\alpha_{k,T}^2 \\
\psi_3 &= e^{\mp \mathrm{i}\omega_k}\alpha_{k,T}+e^{\mp \mathrm{i}\omega_k 2}\alpha_{k,T}^2+e^{\mp \mathrm{i}\omega_k 3}\alpha_{k,T}^3 \\
\psi_4 &= e^{\mp \mathrm{i}\omega_k 2}\alpha_{k,T}^2+e^{\mp \mathrm{i}\omega_k 3}\alpha_{k,T}^3+e^{\mp \mathrm{i}\omega_k 4}\alpha_{k,T}^4 \\
&\vdots \\
\psi_j &= e^{\mp \mathrm{i}\omega_k (j-2)}\alpha_{k,T}^{(j-2)}+e^{\mp \mathrm{i}\omega_k (j-1)}\alpha_{k,T}^{(j-1)}+e^{\mp \mathrm{i}\omega_k j}\alpha_{k,T}^{j}
\end{aligned} \tag{S.48}
$$

which correspond precisely to the coefficients we would obtain from aggregating (S.46), i.e., from $A\left(\Psi_0^{k\pm}\right)^{-1}$. For example, consider the Nyquist frequency process $\left(1-e^{\mathrm{i}\pi}\alpha_{S/2,T}L\right)x_{Sn+s} = u_{Sn+s}$. Using the same approach as in (S.47) it follows that

$$
\begin{aligned}
\left(1-\left(e^{\mathrm{i}\pi}\alpha_{S/2,T}\right)^{Q}L^{Q}\right)\left(1+L+\cdots+L^{Q-1}\right)x_{Sn+s} &= \left(1+e^{\mathrm{i}\pi}\alpha_{S/2,T}L+\cdots+\left(e^{\mathrm{i}\pi}\alpha_{S/2,T}\right)^{Q-1}L^{Q-1}\right) \\
&\quad \left(1+L+\cdots+L^{Q-1}\right)u_{Sn+s}.
\end{aligned} \tag{S.49}
$$

From (S.49) see that after aggregation the resulting process is an ARMA(1,1) with the MA(1) component induced by the temporal aggregation. Clearly when $Q = S/S_A$ is odd the AR(1) process remains associated with the Nyquist frequency because $\left(1-e^{\mathrm{i}\pi Q}\alpha_{S/2,T}^{Q}L^{Q}\right) = \left(1+\alpha_{S/2,T}^{Q}L^{Q}\right)$.

However, for Q even the resulting AR(1) dynamics from temporal aggregation are now associated with the zero frequency because in this case $\left(1-e^{\mathrm{i}\pi Q}\alpha_{S/2,T}^{Q}L^{Q}\right)=\left(1-\alpha_{S/2,T}^{Q}L^{Q}\right)$. However, as shown in the main text, when using the demodulator operator for Q even the AR(1) process $\left(1-e^{\mathrm{i}\pi Q}\alpha_{S/2,T}^{Q}L^{Q}\right)$ will vanish. This situation can also be illustrated with the following example for $Q=2$, where (S.49) becomes:

$$\left(1-\alpha_{S/2,T}^{2}L^{2}\right)(1+L)\,x_{Sn+s}=\left(1-\alpha_{S/2,T}L\right)(1+L)\,u_{Sn+s}. \tag{S.50}$$

Regarding the coefficients of the resulting MA(∞) representation of (S.50), i.e., $\psi(L)=1+\sum_{i=1}^{\infty}\psi_i L^i=\left(1-\alpha_{S/2,T}L\right)(1+L)/\left(1-\alpha_{S/2,T}^{2}L^{2}\right)$, it can be shown that

$$\begin{aligned}
\psi_1 &= 1-\alpha_{S/2,T}=-\frac{c_{S/2}}{T}\\
\psi_2 &= \alpha_{S/2,T}^{2}-\alpha_{S/2,T}\cong\left(1+\frac{c_{S/2}}{T}\right)^{2}-\left(1+\frac{c_{S/2}}{T}\right)\cong\frac{c_{S/2}}{T}\\
\psi_3 &= \alpha_{S/2,T}^{2}-\alpha_{S/2,T}\cong\left(1+\frac{c_{S/2}}{T}\right)^{2}-\left(1+\frac{c_{S/2}}{T}\right)^{3}\cong-\frac{c_{S/2}}{T}\\
\psi_4 &= \alpha_{S/2,T}^{2}-\alpha_{S/2,T}\cong\left(1+\frac{c_{S/2}}{T}\right)^{4}-\left(1+\frac{c_{S/2}}{T}\right)^{3}\cong\frac{c_{S/2}}{T}\\
&\vdots\\
\psi_j &\cong (-1)^j\,\frac{c_{S/2}}{T}.
\end{aligned}$$

As $\alpha_{S/2,T}\cong\left(1+\frac{c_{S/2}}{T}\right)$, in this case the AR(1) process originally associated with the Nyquist frequency vanishes after aggregation when moving from S to S_A seasons per year and $Q=2$. This is clearly seen when we use the approach based on circulant matrices and the demodulator operator. □

Additional References

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