

Is Geometry Analytic?

1. Introduction

In the fourth chapter of *Language, Truth and Logic*, Ayer undertakes the task of showing how *a priori* knowledge of mathematics and logic is possible. In doing so, he argues that only if we understand mathematics and logic as analytic¹, by which he memorably meant “devoid of factual content”², do we have a justified account of *a priori* knowledge of these disciplines.³ In this chapter, it is not clear whether Ayer gives an argument *per se* for the analyticity of mathematics and logic. For when one reads that chapter, one sees that Ayer is mainly criticizing the views held by Kant and Mill with respect to arithmetic.⁴ Nevertheless, I believe that the positive argument is present. Ayer’s discussion of geometry⁵ in this chapter shows that it is this discussion that constitutes his positive argument for the thesis that analytic sentences are true in virtue of the definitions of the terms in them and are thus “devoid of factual content”.

Here’s my summary of the argument Ayer makes in his discussion of geometry.

Suppose, as Kant believed in his *Critique of Pure Reason*, that Euclidean geometry is

¹ Now, I am aware that ‘analytic’ was understood differently by Kant, Carnap, Ayer, Quine and Putnam. It is not even clear whether there is even an agreed definition of ‘analytic’ today. For the purposes of my paper, the meaning of ‘analytic’ is Carnap’s sense as described in Michael, Friedman, *Reconsidering Logical Positivism* (New York: Cambridge University Press, 1999) in terms of the *relativized a priori*.

² Alfred Jules, Ayer, *Language, Truth and Logic* (New York: Dover Publications, 1952), p. 79, 87. In these sections of the book, I think the reason Ayer chose to characterize analytic statements as “devoid of factual” content is in order to give an account of why analytic statements could not be shown to be false on the basis of observation. In other words, the reason why the analytic statements were necessary truths, according to Ayer, is that they did not assert that which required *further facts or observation in order to establish their truth*. One just needed to know the “meanings” or definitions of the terms in those statements in order to “see” that they are true independent of further observation or empirical data. For example it is a fact that ‘Euclidean triangle’ refers to a plane three-sided figure. And it is a fact that every Euclidean triangle has angles adding up to 180 degrees. But given these facts, it follows that every plane three-sided figure has angles adding up to 180 degrees *without need of any further observation*. So ‘every plane three-sided figure has angles adding up to 180 degrees’ is an analytic statement in Ayer’s sense of ‘analytic’.

³ Ayer, *Language, Truth and Logic*, p. 73

⁴ Kant believed arithmetic is synthetic *a priori* while Mill believed that we arrive at mathematical beliefs on the basis of scientific induction. See Ayer, *Language, Truth and Logic*, p. 74 – 75; 77 – 78.

⁵ Ayer, *Language, Truth and Logic*, p. 82.

synthetic *a priori*. On the one hand, Euclidean geometry is synthetic because one could not by conceptual analysis alone arrive at the truths of Euclidean geometry. Moreover, that geometry is synthetic because unlike analytic judgments, which, according to Kant, do not amplify (or increase) our knowledge, geometry does increase our knowledge. On the other hand, Euclidean geometry is *a priori* for it is grounded in our *a priori* idea of space, which for Kant was the pure form of sensible intuition. The idea of space itself is *a priori*, in the transcendental sense, insofar as it is only if we have that idea, due to the unfathomable constitution of our mind, is any experience possible. But, since Kant's *Critique of Pure Reason*, other consistent (and practically useful) non-Euclidean geometries⁶ have been developed. Thus, it cannot be case that Euclidean geometry is synthetic *a priori* in Kant's sense and that its *a priori* status is due to the constitution of our mind. "We see now," Ayer says, "that the axioms of a geometry are simply definitions, and that the theorems of a geometry are simply the logical consequences of these definitions. A geometry is not in itself about physical space; in itself it cannot be said to be "about" anything. But we can use a geometry to reason about physical space."⁷

Fast-forward about 30 years later to Hilary Putnam. In a chapter entitled *Analytic and Synthetic* in his book *Mind, Language and Reality*, Putnam, (somewhat) following Quine⁸, argues that principles of geometry are not analytic if the paradigmatic analytic sentence is 'all bachelors are unmarried'. Apart from the separate reasons that Putnam gives for the analyticity of 'all bachelors are unmarried' that I shall not speak of here, there are other reasons that he uses to argue that principles of geometry are not analytic.

⁶ For example Einstein used a non-Euclidean geometry to describe space-time in his formulation of the general theory of relativity.

⁷ Ayer, *Language, Truth and Logic*, p. 82

⁸ Cf. Hilary, Putnam, "Analytic and Synthetic" printed in *Mind, Language, and Reality* (Hilary, Putnam Philosophical papers Vol. 2) (New York: Cambridge University Press, 1975) p. 40

First of all, he rejects the “linguistic convention” account of analyticity that some philosophers were using to argue that the principles of geometry, like physical definitions (e.g. $e = \frac{1}{2}mv^2$), are true by (linguistic) convention or stipulation, hence analytic.⁹

Putnam argues that these definitions introduced by stipulation lose their conventional character and acquire systematic import within our conceptual system in such a way that it would be a mistake to construe them as analytic if analyticity is understood to mean true by linguistic convention or stipulation.¹⁰ Secondly (and this is where Putnam’s acceptance Duhemian and Quinean holism¹¹ is very evident) mathematics and principles of geometry are characterized by their centrality, as *framework principles*, within our web of beliefs – revisable but only after holistic considerations. Thus, principles of Euclidean geometry, once thought to be analytic, in the sense of immune from revision, were abandoned because a rival theory was available.¹²

What is intriguing, especially in light of Ayer’s argument, is Putnam’s conclusion that Euclidean geometry is *false* (my emphasis):

If the paradigm for an analytic sentence is “all bachelors are unmarried” - and it is - then it is of course absurd to say that the principles of geometry are analytic. Indeed we cannot any longer say that the principles of geometry are analytic; because analytic sentences are true; *and we no longer say that principles of Euclidean geometry are true.*¹³

In this paper I want to attempt to answer the questions: is geometry analytic? and in what sense? In doing so, I will begin by critically evaluating Ayer and Putnam’s arguments on the analyticity (or lack thereof) of geometry on the basis of historico-philosophical work

⁹ Putnam, “Analytic and Synthetic”, pp. 36, 38ff

¹⁰ Ibid.

¹¹ See Friedman, *Reconsidering Logical Positivism* p. 70 for further discussion of Duhemian and Quinean holism.

¹² Putnam, “Analytic and Synthetic”, pp. 47ff

¹³ Ibid.

on the foundations of geometry by Roberto Torretti *Philosophy of Geometry from Riemann to Poincaré* (1978) and Michael Friedman *Reconsidering Logical Positivism* (1999). My critical evaluations of Ayer and Putnam will show that in their arguments against Kant and Linguistic Conventionalism respectively, they fail to distinguish clearly between what Einstein called “pure axiomatic geometry” and “practical geometry.”¹⁴ On the one hand, I will show that Ayer fails to notice that Kant could have been talking about pure geometry not applied geometry when Kant argued that geometry is synthetic *a priori*. On the other hand, I will show that Putnam fails to distinguish between *applied* Euclidean geometry and *pure* Euclidean geometry in the quoted passage. After my critical evaluations of Ayer and Putnam’s arguments, I will conclude by suggesting how someone could plausibly think that *applied* Euclidean geometry is analytic in Carnap’s sense. I will be drawing from Friedman, *Reconsidering Logical Positivism* in my presentation of Carnap’s sense of ‘analytic’. For Friedman argues in that book that Carnap can accept Duhemian holism while still rejecting Quinean holism. “To obtain Quinean holism,” Friedman says, “we must exhibit the incoherence of Carnap’s *Logical Syntax* program, and only this, I suggest, demonstrates the ultimate failure of the logical positivists’ version of the *relativized a priori*”¹⁵(my emphasis)

2. Pure Geometry and Applied Geometry

On the one hand, what I choose to call pure geometry is geometry studied as a branch of pure mathematics. Specifically, I take pure geometry to include classical

¹⁴ See Albert, Einstein, *Sidelights on Relativity* (New York: Dover, 1983), p. 32. I thank Peter Koellner for suggesting this approach to my paper. I think pure geometry is a better term than pure axiomatic geometry. Pure axiomatic geometry implies that all the pure geometries are approached axiomatically which is not necessarily the case. Also applied geometry sounds more accurate than practical geometry; for it suggests that what we are talking about is pure geometry as it is applied to the study of physical space or in physics e.g. in optics. In this paper I may sometimes use the terms ‘physical geometry’ and ‘applied geometry’ interchangeably

¹⁵ Cf. Friedman, *Reconsidering Logical Positivism*, p. 70

Euclidean geometry, say as was pursued in Classical Antiquity and through the Middle Ages in *Euclid's Elements*; the analytical coordinate geometry invented by René Descartes in the 17th century, the non-Euclidean geometries developed independently by Bolyai and Lobachevsky in the 1820s; Gauss's intrinsic geometry of surfaces in his *Disquisitiones* of 1827, which together with the earlier non-Euclidean geometries of Bolyai and Lobachevsky influenced Riemann in his 1854 lecture: *On the Hypotheses that Lie at the Foundations of Geometry* to come up with a generalized conception of space as a n-fold extended quantity i.e. an n-dimensional differentiable manifold – a conception which could accommodate *both* the Euclidean and non-Euclidean geometries.¹⁶ I would also include under pure geometry the work of Felix Klein and Sophus Lie on transformation groups; projective geometry, which was axiomatized by Moritz Pasch in 1882; and the axiomatic conception of Euclidean geometry and *both* Euclidean and non-Euclidean geometry found in David Hilbert's 1899 *Foundations of Geometry* and 1902 article "On The Foundations of Geometry" respectively.¹⁷ Topology could also be classed within the study of pure geometry as well. Pure geometry is *a priori* in both senses of the term: "known independent of experience" and "[demonstrated] from the grounds."¹⁸ But

¹⁶ Cf. Roberto, Torretti, *Philosophy of Geometry from Riemann to Poincaré* (Dordrecht-Holland; Boston: D. Reidel Pub., 1978) Chapter 1 and 2 and 3. See Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 185 for Hilbert's works.

¹⁷ Torretti also points out the work of Levi-Civita into the geometric meaning of curvature and that of Weyl with the idea of an affine structure. So, we may include their contributions as contributions to pure geometry.

¹⁸ The "from the grounds" sense of the *a priori* comes from the theory of demonstration. In Peter Koellner's seminar class: *Topics in the Philosophy of Mathematics: The Concept of Apriority (Philosophy 243, Fall 2015)* for which I wrote this paper, Koellner had shown that '*a priori*' did not always mean what it means today i.e. "known independent of experience". '*A priori* [demonstration]' was used earlier by William of Ockham (ca. 13 Century) when he distinguishes: (1) Demonstration from what is prior i.e. why it is so from explanatory grounds; from (2) Demonstration from what is posterior i.e. that it is so. He cited Ockham's *Summa Logicae* Part III Tractate II Chapter 17. This sense of '*a priori*' Peter argued in seminar can serve to illuminate certain accounts of justifications in mathematics.

as we shall see subsequently in this paper, saying *why* pure geometry is *a priori* generates a lot of controversy.

On the other hand, what I call applied geometry is geometry in the literal sense of the term i.e. “earth measuring.”¹⁹ Applied geometry arises whenever the formal terms in pure geometry receive a physical interpretation i.e. when their designata are specified for use in the exact sciences like physics. In classical mechanics, for example, Euclidean geometry was used in kinematics by “building bridge equations”²⁰ from pure Euclidean geometry to physics. These bridge equations are such as those that grew out of Descartes’s analytic geometry, where he introduced the coordinate systems and algebra to geometry e.g. the equation of a straight line ($ax + by + c = 0$), which found fruitful application in analysis and subsequently in kinematics. I think that physical geometry in classical mechanics was still *a priori*. For I do not think that ‘straight line’ had received a physical interpretation in optics. However, in General Relativity, the apriority of physical geometry gets lost as physical geometry becomes entangled with the physical structure of the universe, especially with the distribution of mass and energy in the universe. For in General Relativity, the metric of the underlying topology of space-time depends on the distribution of mass-energy across the universe.²¹ Specifically, if a geodesic (which is the non-Euclidean equivalent of the straight line in Euclidean geometry) in a 4-dimensional semi-Riemannian manifold of non-constant curvature is interpreted as the path of an unimpeded beam of light, the physical structure of the universe is such that two

¹⁹ See Einstein, *Sidelights on Relativity* p. 31

²⁰ This is a term that Peter Koellner explicitly used in his presentation during this seminar class. Since I was submitting this paper for that class, I wanted to acknowledge that it is not my own.

²¹ Cf. Friedman, *Reconsidering Logical Positivism*, p. 62

unimpeded parallel beams of light can converge globally (if not locally or infinitesimally) as they go past a star.²²

3. Critical Evaluation of Ayer

Armed with this distinction between pure geometry and applied geometry, I would now like to engage in critical evaluation of Ayer's argument in the fourth chapter of *Language Truth and Logic*. One way of critically evaluating Ayer's argument is to ask whether the argument works in refuting Kant's epistemology of geometry as Ayer intended. With respect to this aspect of critical evaluation, the first point I want to make is that Ayer is correct in saying, "A geometry is not in itself about physical space²³; in itself it cannot be said to be "about" anything." He is correct insofar as he is talking about pure geometry as I have distinguished it above. But I do not believe that this particular argument succeeds in refuting Kant's thesis that geometry is synthetic *a priori*. Here's why.

First, for Kant the 'synthetic' had a primary sense and a secondary sense. The primary sense of 'synthetic' is well known. It is roughly the idea that a judgment is synthetic whenever the predicate B of a judgment is not contained (covertly) in the subject A of the judgment. So, one can think of A without necessarily thinking of B. Or put another way, mere conceptual analysis does not reveal that the predicate B was contained in subject A all along. The secondary sense of the synthetic is that while analytic judgments are explicative, which means that they reveal the concepts already

²² I thank Peter Koellner for fruitful discussion of these points as I was writing this paper in Fall 2015.

²³ Cf. Friedman, *Reconsidering Logical Positivism*, pp. 46f for the distinctions that Carnap drew between formal, intuitive and physical space.

contained in the subject albeit confusedly or less clearly; synthetic judgments are ampliative, which means that synthetic judgments extend our knowledge.²⁴

Secondly, the ‘*a priori*’ also had two distinct senses for Kant. First, whatever was *a priori* was necessary and had universal validity. Secondly, whatever was *a priori* was so in a transcendental sense, namely that it arose out of the unfathomable constitution of our mind insofar as it made experience possible. The implication here being that whatever makes experience possible cannot itself be known through experience.²⁵

On the basis of these clarifications, I want to argue that Ayer’s argument that seeks to refute Kant’s epistemology of geometry in terms of geometry not being “about” anything does not succeed. To see why, recall that for Ayer a proposition has factual content if and only if it “provides information about matters of fact”.²⁶ Elsewhere, he says that propositions that have factual content are empirical hypotheses.²⁷ So, when Ayer says (emphasis mine), “[It] is natural for us to think, as Kant thought, that geometry is the study of the properties of physical space, and *consequently that its propositions have factual content,*” I am at a loss to know why. For there is *no evidence* in Kant’s *Critique* to suggest that Kant’s view of geometry was such that the propositions of geometry expressed empirical hypotheses. In fact, Kant is very clear:

Geometrical propositions are one and all apodeitic...such propositions cannot be empirical or, in other words, judgments of experience, nor can they be derived from any such judgments.²⁸

²⁴ Cf. Immanuel, Kant, *Critique of Pure Reason* (Smith Norman K., Trans.) (China: Palgrave Macmillan, 2007) (2nd edition: Original work published in German 1787) Introduction Part IV p. 48f

²⁵ Cf. Kant, *Critique of Pure Reason* (Smith Norman K., Trans.) Introduction Part II p. 43f and p. 70f and Friedman, *Reconsidering Logical Positivism*, p. 7

²⁶ Ayer, *Language, Truth and Logic*, p. : 79

²⁷ Ayer, *Language, Truth and Logic*, p. : 15 - 17

²⁸ Cf. Kant, *Critique of Pure Reason* (Smith Norman K., Trans.) p. 70

It may be objected that in the context of the quoted passage Kant says, “Geometry is a science which determines the properties of space synthetically, and yet *a priori*.” This objection may be put forward to suggest that the phrase ‘science’ and the clause ‘determines the properties of space synthetically’ together imply that for Kant geometry was empirical and that it was *about something* (that it had factual content and that this content was physical space). Couldn’t this, after all, be what Ayer is objecting to? My response is yes. Ayer is objecting to this view. But his objection does not work for the following reasons. First, the quotation from Kant does not establish that the propositions of geometry are empirical hypotheses. That geometry is a “science” could just mean that it is a systematically organized body of knowledge, which is just what a science means. Pure mathematics is also sometimes viewed as a science in the sense of being a systematically organized body of knowledge. Secondly, the quoted passage does not imply that the space in question is the actual *physical* space. In fact, that the passage does not imply physical space is strongly suggested by the fact that part of what Kant has in mind throughout the Transcendental Aesthetic is arriving at our idea of space. This idea of space – as a form of our sensible intuition – is *a priori* in the transcendental sense. So, if geometry is about this *a priori* idea, then it cannot be the case that in the quoted passage the space in question is physical space. Lastly, ‘synthetically’ for Kant does not mean that synthetic judgments are known by experience; for after all there are synthetic *a priori* judgments. The right way of interpreting ‘synthetically’ is in the primary and secondary senses of synthetic judgments that I have described above.²⁹

What the foregoing aspect of critical evaluation shows is that one cannot explain away Ayer’s overthrow of Kant’s epistemology of geometry by saying that Ayer is

²⁹ Cf. Kant, *Critique of Pure Reason* (Smith Norman K., Trans.) p. 53

talking about pure geometry while Kant is talking about applied geometry. For the passages from Kant I have quoted above strongly suggest that Kant possibly had pure geometry in mind. The way that Ayer's argument *does* succeed is in challenging the *grounds* of the apriority of geometry. While Kant grounds the apriority of geometrical propositions on the *a priori* idea of space that arises as a result of the constitution of our mind, Ayer believes that a geometry is *a priori* because it is analytic (although not in Kant's sense of the term). Moreover, since Kant's time there have been other consistent theories of pure geometry other than the classical Euclidean one that was the only one known to him. For within these geometrical theories the theorems are logical consequences of the axioms and so these theories are all *a priori* because they are analytic. It is on this basis that Ayer is able to endorse the conventionalism of Poincaré in the quoted passage: roughly the idea that the choice of a pure geometry to be applied in physical theory is a matter of convention based on expediency and the overall fruitfulness and simplicity of working with the said geometry.³⁰

The conventionalism of Poincaré will be better appreciated in the context of critical evaluation of Putnam's essay. For in that essay, Putnam targets linguistic conventionalism. Although some logical positivists (especially Moritz Schlick) were inspired by Poincaré's conventionalism, the linguistic conventionalism advocated by Carnap is unique and has important differences from the conventionalism of Poincaré.

4. Critical Evaluation of Putnam

4.1. Why was Euclidean Geometry abandoned?

³⁰ Ayer, *Language, Truth and Logic*, p. 83

I begin my critical evaluation of Putnam's essay by looking at where Putnam says that laws of geometry were abandoned because there was a rival theory.³¹ Here we make use of our distinction between pure geometry and applied geometry. While it is true to say that when Euclidean geometry is *applied* to our physical space, it turns out to be incorrect, it does not follow that pure Euclidean geometry itself is false and that it was abandoned. Viewed historico-philosophically, what actually happened is that geometers doubted that Postulate 5 was self-evident and hence they doubted that it ought to be included with the rest of Euclid's axioms. Torretti points out that mathematicians like Proclus, John Wallis, Girolamo Saccheri, John Heinrich Lambert and Adrien Legendre made attempts to prove Postulate 5 from the other axioms.³² Although Gauss thought that there were no mathematical reasons for preferring Euclidean geometry to the non-Euclidean one³³, it was in the 1820s that Bolyai and Lobachevsky working independently of each other developed non-Euclidean geometry, which was constructed by denying postulate 5 and using the rest of Euclid's axioms that do not depend on Postulate 5.³⁴ Lobachevsky indeed did not view his system as contrary to Euclidean geometry. He viewed both systems as equally consistent.³⁵ Then Riemann, by building on Gauss's work on the intrinsic geometry of surfaces, produced a generalized geometry capable of accommodating *both* the Euclidean and non-Euclidean geometries – Euclidean space is a special case of the genus of the manifold.³⁶

³¹ Putnam, "Analytic and Synthetic", pp. 46, 48

³² Torretti, *Philosophy of Geometry from Riemann to Poincaré*, pp. 43, 44, 50

³³ Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 53

³⁴ Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 40

³⁵ Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 66

³⁶ Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 101

Moreover, where Putnam says, “before the development of non-Euclidean geometry by Riemann and Lobachevski, the best philosophic minds regarded the principles of geometry as virtually analytic. The human mind could not conceive their falsity.” Putnam’s remarks need to be corrected in light of what actually happened historico-philosophically. If by the best philosophic minds he counts Kant, then Kant did not regard the principles of geometry as analytic. For Kant, principles of geometry were synthetic *a priori*. It is important to distinguish analyticity from apriority. Analyticity is one way of explaining the apriority of mathematics and geometry. In fact, given my discussion in section 3 above, Kant’s epistemology of geometry is compatible with the existence of non-Euclidean geometries. Friedman points out that the only difference between Poincaré and Kant is that the former was familiar with non-Euclidean geometry while the latter was not.³⁷

4.2. The Rationale of the Analytic-Synthetic Distinction

A second way we may critically evaluate Putnam’s argument is with the preliminary remarks leading up to his discussion of the analyticity (or lack thereof) of principles of geometry. One such remark he makes is intended to defend the Quinean insight that he thinks is underappreciated by the philosophers who undertake to challenge Quine’s views. Putnam thinks that citing garden variety examples of analyticity will not do as a reply to Quine. Instead he insists that what is needed is the nature and rationale of the analytic-synthetic distinction: “what point is there to having a separate class of statements called analytic statements?”³⁸ So, we may begin our critical evaluation of Putnam’s view here by responding to this question historico-philosophically.

³⁷ Friedman, *Reconsidering Logical Positivism*, p. 83

³⁸ Putnam, “Analytic and Synthetic”, p. 35

On the one hand, I believe that the nature and rationale of the analytic-synthetic distinction lay in the fact that the logical positivists in general, and Carnap in particular, wanted to respond to the Kantian problem: how is pure mathematics possible?³⁹ In doing so, they also wanted to avoid the Kantian synthetic *a priori* doctrine of the transcendental aesthetic.⁴⁰ Since Putnam directs some of his criticisms of the analyticity of geometry to Reichenbach⁴¹, whose conventionalism he says grew out of the Viennese circle, seeing what Reichenbach *actually* thought at one time in his career will serve us well as we seek to understand the nature and rationale of the analytic-synthetic distinction.

4.3. The Relativized *A priori*

As Friedman points out, Reichenbach had a unique conception of the relationship between the *a priori* and empirical science that was neither strictly Kantian nor radically empiricist.⁴² Friedman argues that in his 1920 book *General Relativity and A Priori Knowledge*, Reichenbach distinguished between *axioms of coordination* and *axioms of connection*.⁴³ On the one hand, the axioms of coordination preserved *part* of the Kantian sense of the *a priori*, namely that they make science, through the physical theory, possible. He held that whatever the axioms of coordination were, they were *a priori* relative to the scientific theory that was employing them. Friedman notes that for Reichenbach, “these nonempirical axioms of coordination – which include, paradigmatically, the principles of physical geometry – are ‘constitutive of the object of knowledge.’”⁴⁴ Axioms of connection, on the other hand, are scientific inductive

³⁹ Friedman, *Reconsidering Logical Positivism*, p. 165

⁴⁰ Cf. Ayer, *Language, Truth and Logic*, p. Chapter 4 and Friedman, *Reconsidering Logical Positivism*, pp. 165 and Kant, *Critique of Pure Reason* (Smith Norman K., Trans.) p. 56

⁴¹ Putnam, “Analytic and Synthetic”, pp. 33, 47

⁴² Friedman, *Reconsidering Logical Positivism*, p. 7

⁴³ Friedman, *Reconsidering Logical Positivism*, p. 7

⁴⁴ Friedman, *Reconsidering Logical Positivism*, p. 7

generalizations. What is important to note, according to Friedman, is that for Reichenbach, the axioms of coordination, *relative to a given scientific theory* (Newtonian Mechanics, Special Relativity or General Relativity), are *a priori* and as such are not subject to any empirical confirmation or disconfirmation. But whichever axioms of coordination are in fact used depends on the scientific theory. So, the axioms of coordination could in principle be revised as a result of scientific advances, e.g. physical Euclidean geometry is *a priori* in classical mechanics but is not *a priori* in General Relativity, where “topology (sufficient to admit a Riemannian structure)” is instead *a priori*.⁴⁵ This is the *relativized a priori* – the idea that relative to a scientific theory, a geometry, for example, is *a priori*.

Friedman notes that Schlick, in an exchange of letters, rebuked Reichenbach for holding on to elements of the Kantian doctrine. Instead, Schlick wanted Reichenbach to adopt the conventionalism of Poincaré. By acquiescing to Schlick, the notion of *relativized a priori* was lost.⁴⁶ For later, as Friedman points out, Reichenbach devoted his writing to reconciling post-general relativity science to the conventionalism of Schlick and Poincaré.⁴⁷

Schlick’s conventionalism grew out of that of Poincaré and the work of David Hilbert. From the former, Schlick was persuaded that the question of which of the geometries is to be applied to space is a matter to be settled by experience and requirements of overall simplicity of our scientific conceptual system. Applying the

⁴⁵ Friedman, *Reconsidering Logical Positivism*, p. 7 and especially pp. 66 – 68

⁴⁶ Friedman, *Reconsidering Logical Positivism*, pp. 64 This is important because Putnam, “Analytic and Synthetic” in a footnote on p. 47 cites Reichenbach’s 1928 *Philosophy of Space and Time* and (Reichenbach, 1956) which suggest *publication dates much later* than the ones which Friedman, *Reconsidering Logical Positivism*, p. 7 is drawing on to explain Reichenbach’s conception of the *relativized a priori*.

⁴⁷ Friedman, *Reconsidering Logical Positivism*, pp. 63

Helmholtz-Lie theorem, which states that based on the experience (or idealization in the case of Poincaré) that rigid motion is possible in our space, our space must be either Euclidean or characterized by one of the manifolds of constant curvature⁴⁸, Poincaré thought that the choice between the non-Euclidean and Euclidean geometries was conventional based on which was the most expedient to work with.⁴⁹ From Hilbert, Schlick got the idea of “implicit definitions”, namely that the axioms of a geometry implicitly define the geometry’s primitive terms. Different geometries differ in so far as they employ different implicit definitions of “point”, “line”, “between” and so on.

But as Friedman points out that the conventionalism of Poincaré and Schlick is in no way different from Duhemian holism, which is the idea that the theoretical components of our conceptual system face the tribunal of experience not in isolation but as a whole.⁵⁰ So, Schlick’s conventionalism does not do justice to the apriority and analyticity of the principles of geometry and differs from the linguistic conventionalism of Carnap that I shall now proceed to explain.

4.4. Linguistic Conventionalism

According to Friedman, where Reichenbach had failed to give a clear explication of the difference between the axioms of coordination and the axioms of connection, Carnap in the *Logical Syntax of Language* had the machinery to do so: L-rules⁵¹ (the analytic) and P-rules (the synthetic) of the physical language of science. The choice of L-rules and the

⁴⁸ Cf. Friedman, *Reconsidering Logical Positivism*, p. 77

⁴⁹ Since he died in 1912, Poincaré was not aware that Einstein would in fact use a non-Euclidean geometry of variable curvature. See Friedman, *Reconsidering Logical Positivism*, p. 79

⁵⁰ Cf. The foregoing discussion with Friedman, *Reconsidering Logical Positivism*, pp. 64 – 65 and pp. 78 - 79

⁵¹ Cf. Friedman, *Reconsidering Logical Positivism*, p. 13. “The L-rules or logical rules represent the purely formal, non-empirical part of our scientific theory, whereas P-rules or physical rules represent its material or empirical content.”

interpretation that made them true is a matter of convention, for these rules are purely formal and one has a lot of leeway in selecting the L-rules in the formulation of a language. In Classical Mechanics, for example, the L-rules include the principles of Euclidean geometry while the P-rules are the general principles and laws of physics. In General Relativity, the L-rules include a topology sufficient to admit the 4-dimensional semi-Riemannian manifold of non-constant curvature, while the P-rules now include *applied* geometry through optics and the other general laws of physics. What is important to note for Carnap, says Friedman, is that the L-rules (or analytic sentences) are distinguished from the P-rules *at a given stage* in the development of the scientific enterprise.⁵² That they are revisable in light of the development of the scientific enterprise is exemplified by applied or physical geometry in the move from classical mechanics to general relativity. But Carnap can still say this and hold on to the idea that in a physical language there is a distinction between analytic sentences and synthetic sentences. For it is the analytic sentences, axioms of coordination in the case of Reichenbach and conventions in the case of Poincaré and Schlick, that are constitutive of any endeavor to gain any objective scientific knowledge.⁵³ In other words, only if we are armed with the axioms of coordination or L-rules can we begin to generate hypotheses and test them.⁵⁴ So, it is useful to have the distinction between the analytic and the synthetic even in science.

⁵² Friedman, *Reconsidering Logical Positivism*, p. 13.

⁵³ Friedman, *Reconsidering Logical Positivism*, p. 13.

⁵⁴ Cf. Rudolf Carnap, *Foundations of Mathematics and Logic* (International Encyclopedia of Unified Science ; Vol. 1, no. 3) (Chicago: The University of Chicago Press, 1939) p. 1 where he says, "In these theoretical activities, deduction plays an important part; this includes calculation, which is a special form of deduction applied to numerical expressions."

On the other hand, I would argue that holists, like Putnam, who oppose the distinction *in science*, have a hard time responding to the Kantian question, “How is pure mathematics possible?” That is, they have a hard time explaining why mathematics and logic seem to be necessarily true, and therefore *a priori*. While they would say that mathematics and logic are framework principles which are to be identified with centrality and priority within our conceptual scheme such that if revisions become necessary due to advances in science they are the last to be considered for revision⁵⁵, I’d still argue that the holists would be hard pressed to explain away our belief that there is a marked difference between mathematics on the one hand and physical laws on the other. The former cannot be otherwise while the latter can.

4.5. Applied Geometry and the Physical Interpretation of a straight line

Lastly we may critically evaluate Putnam’s essay with respect to the physical interpretation of a straight line. Hume, Putnam says, would rather deny that light rays travel in straight lines than conclude that Euclid’s postulate 5 is false.⁵⁶ In other words, optical theory was synthetic for Hume and (pure) Euclidean geometry analytic. In a footnote to a remark on Reichenbach, Putnam points out that Reichenbach actually claimed that there were other possible alternative coordinative definitions of ‘straight line’.⁵⁷ I take this to imply that for Reichenbach, the optical theory was analytic (in the sense of stipulated true by convention) while the principles of geometry in general relativity were synthetic.⁵⁸ Putnam says that both Hume and Reichenbach are wrong:

The principle that light travels in straight lines is a law of optics, nothing more or less serious than that. We test the conjunction of geometry and

⁵⁵ Putnam, “Analytic and Synthetic”, p. 40

⁵⁶ Putnam, “Analytic and Synthetic”, pp. 46 - 47

⁵⁷ Putnam, “Analytic and Synthetic”, p. 47.

⁵⁸ Putnam, “Analytic and Synthetic”, p. 49.

optics indeed, and if we get into trouble, then we can alter either the geometry or the optics, depending on the nature of the trouble.⁵⁹

It is possible that Reichenbach was right in Putnam's footnote discussion.⁶⁰ For in a very enlightening discussion on the empiricism with respect to geometry of Ernst Mach, Torretti points out that *there are* alternative physical interpretations of the straight line. In fact, interpreting straight line as the path of a light ray may not be the best approximation of a straight line. In this discussion from Mach on how planes are constructed in practice:

Physically a plane is constructed by rubbing three bodies together until three surfaces A, B, C, are obtained, each of which exactly fits the others – a result which can be accomplished [...] with neither convex nor concave surfaces, but with plane surfaces only.⁶¹

Torretti observes, “If you construct two adjacent planes by this method, their common edge will provide a better approximation of the straight line than any taut string or light ray.”⁶²

5. Conclusion

What this observation by Torretti on the passage from Mach suggests to me is that we have a choice of what physical interpretation to give our pure geometrical concepts. Therefore, Reichenbach could have been right in saying that the optical theory is analytic, hence *a priori* relative to pre-General Relativity physics. Recall that in presenting Carnap's view, I said that Friedman argues that for Carnap the choice of L-rules *and the interpretation* that made them true is a matter of convention.⁶³ Now when we stipulate the interpretation of a straight line as the path of a ray of light, the stipulation is a matter of convention since there are alternatives that are equally good physical interpretations.

⁵⁹ Putnam, “Analytic and Synthetic”, p. 49f

⁶⁰ Putnam, “Analytic and Synthetic”, p. 47.

⁶¹ Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 283

⁶² Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 283

⁶³ Friedman, *Reconsidering Logical Positivism*, p. 13.

Now, given that in the discussion of kinetic energy, Putnam seems to grant that *conventions are valid only if there are alternatives*,⁶⁴ and given that there are alternative interpretations of a straight line, it follows that statements in the optical theory that interpret geometrical concepts are *valid* conventions hence analytic. So, using the idea of the *relativized a priori* from Friedman, I want to suggest that Reichenbach could have been right in saying that the optical theory, which is a kind of *applied* geometry, was analytic (in the sense of stipulated true by convention) hence *a priori relative* to pre-General Relativity science. But optical theory (applied geometry) is now synthetic and hence *not a priori, relative* to General Relativity physics. But we can say this without having to deny the view that *pure* Euclidean geometry is analytic in Carnap's sense.

⁶⁴ Putnam, "Analytic and Synthetic", p. 45 He says, "[$e = \frac{1}{2}mv^2$] would be true by stipulation, yes, but only in a context which is defined by the fact that the only alternative principle is ' $e = mv$ '"

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