TEMPORAL INTERVALS AND TEMPORAL ORDER*

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I

In philosophical discussions about the nature of time, *instants* have long been a source of controversy, a good example being Aristotle's problem about the state of motion of a body at the instant of change between rest and motion. For the foundations of physics it is perhaps unclear, but by no means a forgone conclusion, that instants really are necessary in the basic ontology. On the other hand, in the analysis of natural language it is doubtful whether instants have any basic role to play. In the simple and straightforward cases, it seems that predicates of natural language do not hold relative to instants of time. Consider an activity predicate like 'runs', for example. The activity can only occur during an *interval* of time sufficiently long to encompass a minimum of the necessary movements involved. To say that it is true of an instant must involve some convention according to which the instant is a member of an interval during which the running takes place. There are other predicates, like moving at a certain velocity for which there does not seem to be a minimum period during which they can hold. These might be classified as states, with the common feature that they are true of every subinterval of any interval during which they hold. And again there seems to be no need to call upon instants; but if one does, then being true of an instant would involve reference to the containing interval.

It would take me beyond the scope of this paper to pursue the details of this classification of predicates of times, with a corresponding treatment of tense and aspect, which I believe possible on the basis of temporal intervals. My aim here is just to provide an account of the fundamental notion of a temporal interval underlying the analysis. (1) I put forward a system the universe of discourse of which

(*) I am grateful to Wlodzimierz Rabinowicz, Lars Lindahl and Kaj Børge Hansen for helpful comments and discussions of earlier versions of this paper.

(1) It will perhaps forestall misunderstanding on this point to say that no attempt is made to develope a corresponding propositional tense logic where the intervals would

is intended to comprise intervals and no more, so that they can be conceived as fundamental rather than defined as sets of instants. There is just one primitive, the symmetric relation of separation of Lesniewski's mereology, in terms of which all other necessary concepts can be defined. The paper might be of independent interest to students of mereology in as much as the sum operation is restricted so that not any arbitrary pair of individuals can be summed, and in a subsequent development the relation of separation is distinguished from that of non-overlapping.

A logic of intervals akin to the one proposed here has already been published (Hamblin (1969) and (1971)). Hamblin's system is also based on a single primitive, the two-place relation *abuts at the earlier end.* However, the work presented here differs from Hamblin's in a number of respects. Most importantly, the present system is explicitly based on mereological ideas in such a way that not only are the two notions of abutment and temporal order involved in Hamblin's basic notion distinguished; the temporal ordering itself is built up from a symmetric mereological primitive. It seems to me there is something to be said for building up the temporal order in this way, and a comparison with the ordering of the real numbers may help to make this clear.

A dyadic order relation always has a converse which is also an order relation. Therefore, in order to give a sense or direction to a linear order something must be said in addition to distinguish between these two relations. In the case of the real numbers, there is an algebraic structure over and above the basic order properties which imply that the relation usually signified by '>' receives the interpretation 'greater than' and not 'less than'. This is brought out nicely in Lang's formulation of the real number system (Lang (1969)), where a definition of the 'greater than' relation is provided which takes advantage of this algebraic structure and distinguishes it from a mere order relation. A subset of the real numbers P of *positive elements* is selected satisfying

be confined to the metalanguage. My view is that this sort of tense logic is irrelevant to the problems of the analysis of natural language; but of course the task remains open to anyone who considers it a sufficiently well motivated one.

(i) Either $x \varepsilon P$ or $-x \varepsilon P$ or $x = 0$, and these alternatives are mutually exclusive.
(ii) If $x, y \varepsilon P$, then both $x + y \varepsilon P$ and $xy \varepsilon P$.

The 'greater than' relation can then be defined by stipulating that '$x > 0$' stands for '$x \varepsilon P$' and '$y > x$' stands for '$y - x \varepsilon P$'. Now, on the basis of the other axioms for the real numbers, 1 exists, and it can easily be shown that 1 is positive. For by (i), either $1 \varepsilon P$ or $-1 \varepsilon P$, but not both. But if $-1 \varepsilon P$, then $(-1)(-1) \varepsilon P$ by (ii); i.e. $1 \varepsilon P$ – a contradiction. Hence, $1 \varepsilon P$ and so $1 > 0$. Then $2 = 1 + 1 \varepsilon P$, $3 = 2 + 1 \varepsilon P$, and so on.

Similarly with time, it seems that the 'later than' relation involves a sense or direction over and above the basic properties involved in establishing just a linear ordering. There are, it seems, ways of distinguishing 'earlier than' and 'later than' which imply that they are more than mere order relations. But in virtue of what additional structure, by analogy with the algebraic structure of the real numbers, is this the case? A set of times corresponding to Lang's positive elements cannot be selected by choosing a time as the origin (as in the dating system) and defining the set as all those times in the future of this fixed time. This just means «all times later than the origin», and relies upon the relation we are trying to define. Nor can indexical distinctions of past, present and future be used for this purpose since they do not denote fixed times. Perhaps the answer lies in the concept of causation, but I have no such theory to present on this occasion. In the absence of an account of this sort, a theory confining itself to the linear order of time should be presented in such a way that it is clear that it has nothing to say about direction and the distinctions of past and future. (Hopefully, it might then provide a framework within which such an account could be developed.) Such a theory of linear order expresses no more than that certain times lie between others, the fundamental order concept being that of a three-place betweenness relation. The procedure adopted in the theory presented here is to define a betweenness relation in terms of which the linear ordering of time is expressed. A dyadic order relation can then be defined on the basis of the betweenness relation, but this is not analogous to Lang's definition of 'greater than', the relation so defined having only certain necessary properties of the 'earlier than' and 'later than' relations, not sufficient to distinguish the two.

Finally, the last section deals briefly with instants. It has often been suggested that instants are involved in so-called achievments and success verbs. Although this is a point on which I have considerable doubts, it is perhaps appropriate to indicate how instants are related to the intervals discussed here. Accordingly, some modifications of the basic system which allow instants to enter the universe of discourse are outlined in the final section.

II

In the 1920's Tarski developed a geometry which was not based on the notion of a point (Tarski (1956)). His geometry of solids, as he called it, was based on Lesniewski's mereology with a primitive dyadic 'part of' relation. Just one specifically geometric primitive was required in addition, and that was the notion of a sphere. The logic of intervals presented here follows a pattern of development similar to Tarski's, though there is some modification of approach which allows that the only primitive concept required is a mereological relation. The relation of separation has been chosen following Leonard and Goodman's treatment of mereology (Leonard and Goodman (1940)), part of which has been adopted here. All other relevant notions can be defined within the first order predicate calculus with identity.

The system of mereology has not been adopted in its entirety because the operation of summation is too broad. There the sum of a class of individuals is an individual containing all the component individuals of the class as parts and having no part separate from all the component individuals. The existence of a unique sum is guaranteed by the axioms without any restriction on the relations between the individuals in the class summed. Such sums could well have unconnected parts, and this is a feature characteristically lacking in intervals as ordinarily conceived. The sum operation is restricted here, and for this reason no analogue of Tarski's geometric primitive 'sphere' is required.

To begin with, two definitions in terms of the primitive relation of separation are followed by two postulates corresponding to the first two postulates of Leonard and Goodman.

DEFINITION 1 *An interval s is* part of *an interval t if every interval separate from t is separate from s. s is a* proper part of *t if s is part of t but not identical with t. If s is part of t, t is said to* contain *s.*

DEFINITION 2 *An interval s* overlaps *an interval t if they have some part in common.*

POSTULATE 1 *If s is a part of t and t is a part of s, then s and t are identical.*

POSTULATE 2 *s overlaps t iff s and t are not separate.*

It follows that 'part of' is transitive and reflexive, 'proper part of' is transitive, irreflexive and asymmetric, 'is separate from' is symmetric and irreflexive and 'overlaps' is symmetric and reflexive.

The sum, product and difference of two intervals do not always exist, there being no null interval. They exist only when certain conditons are fulfilled. In the case of the product of two intervals, no further concepts need be introduced and the conditional definition can be given directly, together with a postulate establishing the existence of the product under the appropriate conditions. Uniqueness is easily established so that reference to the product is justified.

DEFINITION 3 *If the intervals s and t overlap, then an interval r is the* product of *s and t if every interval separate from r is separate from s or t, and conversely.*

POSTULATE 3 *If the intervals s and t overlap, then there exists an interval r such that every interval separate from it is separate from s or t, and conversely.*

The sum of two intervals also exists when they overlap, but following Hamblin (1971), in addition when the two intervals just come into contact with one another, or *abut*. As was noted in the introductory section, Hamblin's notion of abutment carries an implication of precedence not intended here. The concept of interest here is symmetric, and can be defined as follows:

DEFINITION 4 *Two intervals* abut *if they are separate and there is an interval overlapping both but not any other separate from both.*

The sum of two intervals can now be defined, and a postulate is laid down guaranteeing its existence under the appropriate conditions.

DEFINITION 5 *If the intervals s and t either abut or overlap, then an interval r is the* sum of *s and t if every interval separate from r is separate from both s and t, and conversely,*

POSTULATE 4 *If the intervals s and t either abut or overlap, there exists an interval r such that every interval separate from r is separate from both s and t, and conversely.*

The difference between two intervals remains to be considered. It is not sufficient simply to say that an interval *s* is a proper part of another interval *t* as a condition of the existence of the difference between *s* and *t*, because there might be parts of *t* separate from *s* lying on both sides of *s*. What is required is that *s* is both a proper part of *t* and shares a boundary with *t*; i.e., *s abuts t internally*:

DEFINITION 6 *Two intervals* abut internally *if one is part of the other and there is an interval abutting both.*

DEFINITION 7 *If the intervals s and t abut internally and s is a proper part of t, then the interval r is the* difference between *t and s if every part of r is a part of t and seperate from s, and conversely.*

It is not necessary to lay down a postulate to the effect that the difference exists under the appropriate conditons because this is provable on the basis of the other postulates, as will be shown in due course.

The possibility of two intervals abutting internally at the beginning or end of time, or at the edge of some break in time, which would raise difficulties for definition 6, is excluded by later postulates guaranteeing that there are no intervals limiting the extent of time and no gaps occur not occupied by intervals. To express these properties of time a concept of order among intervals is necessary. The fundamental order relation here is to be a three-place betweenness relation, defined as follows:

DEFINITION 8 *An interval s* lies between *the intervals r and t if s* *is separate from both r and t, and every two intervals, each separate from s and such that one contains r and the other t, are separate from one another.*

Some basic relations between the mereological properties and the order relation can be established at this point.

THEOREM *If an interval s lies between two parts of an interval t, then s is also part of t.*

This follows immediately from definition 8 and the irreflexivity of the separation relation.

THEOREM *There is no interval between two abutting intervals.*

Proof: If u and v abut, then by definition 4 an interval r separate from every interval separate from both u and v exists which overlaps both u and v. By postulate 4, sum (r, u) and sum (r, v) exist. Consider any s separate from u and v, which must be separate from r. By definition 5, s is then separate from sum (r, u) and from sum (r, v). But since u is part of sum (r, u) and v is part of sum (r, v), s cannot lie between u and v given definition 8 because obviously sum (r, u) and sum (r, v) overlap.

The choice of the postulates of order adopted here is based on the extensive work of Huntington and Kline (Huntington and Kline (1917) and Huntington (1924)). From the definition of betweenness follow two immediate consequences which correspond to postulates A and D of Huntington and Kline:

THEOREM *A triple of intervals standing in the betweenness relation is symmetric in the sense that the interchange of the first and the third intervals is always allowed.*

THEOREM *The members of a triple of intervals standing in the betweenness relation are mutually separate and therefore distinct.*

The remaining three postulates needed for the present correspond, respectively, to Huntington and Kline's postulates B and C, and Huntington's postulate 9 in his 1924 paper.

POSTULATE 5 *If r, s and t are three pairwise separate intervals, one lies between the other two.*

POSTULATE 6 *It is never the case that an interval s lies between two intervals r and t and that t also lies between r and s.*

POSTULATE 7 *If an interval s lies between the intervals r and t, and u is an interval separate from all three, then s lies either between r and u or between u and t.*

In the 1924 paper Huntington showed his five postulates to be independent and deduced from them eight theorems concerned with four distinct individuals, the antecedents of which can be thought of as all the possible ways in which two individuals can stand to two fixed individuals on a line. Abbreviating 's lies between r and t' to 'B (rst)', and thinking of s and t as the two fixed intervals, these theorems are as follows. (It is assumed that all four intervals are mutually separate.)

T1 $B(ust) \& B(stv) \supset B(usv)$
T2 $B(ust) \& B(svt) \supset B(usv)$
T3 $B(ust) \& B(svt) \supset B(uvt)$
T4 $B(sut) \& B(svt) \supset B(suv) \vee B(svu)$
T5 $B(sut) \& B(svt) \supset B(suv) \vee B(vut)$
T6 $B(ust) \& B(vst) \supset B(uvt) \vee B(vut)$
T7 $B(ust) \& B(vst) \supset B(uvs) \vee B(vus)$
T8 $B(ust) \& B(vst) \supset B(uvs) \vee B(vut)$

A further abbreviation can now be introduced, writing 'B ($ustv$)' for 'B (ust) & B (stv)', since it is determined how all four triples chosen from u, s, t and v stand in the betweenness relation by T1 and T3.

Three existential postulates now follow.

POSTULATE 8 *If the intervals s and t are separate, either they abut or there exists an interval r between them which abuts both.*

POSTULATE 9 *For every interval s there exist intervals r and t between which s lies.*

POSTULATE 10 *Every interval has a proper part.*

Postulate 8 is a non-metric form of the Archimedian axiom guaranteeing that (if, as intended, intervals are thought of as finite) no two intervals lie infinitely far apart, and excludes gaps in time. Postulate 9 says that time extends indefinitely in both directions and postulate 10 ensures the infinite divisibility of intervals.

Tense logicians have sometimes been concerned to work with a system of temporal order which makes no specific commitment on these matters, establishing completeness of a propositional tense logic relative to a minimal system of temporal order. Of course, specific modifications can be made for the purpose of any particular application. And if no commitment is to be made on whether time is linear or circular, there might be something to be said for adopting Grünbaum's procedure (Grünbaum (1973)) of working, instead of a three-place betweenness relation, with a four-place relation of separation closure which could be defined as follows:

The intervals r and s separate *the intervals t and u if all four intervals are mutually separate (in the mereological sense) and every interval containing s and separate from t and u is separate from r.*

But the central concern here has been to adopt assumptions which certainly seem highly plausible, in the absence of good counter arguments, in order to obtain a complete axiomatisation (see section IV).

However, it is interesting to note in this connection that in the usual treatments of time as a system of structureless points in tense logic, the introduction of a condition analogous to postulate 9, to the effect that there is no first nor last moment of time, is consistent with time being circular (so long as it remains infinitely divisible, which means denseness in the case of instants). But this is not the case with postulate 9 in the present system (where infinite divisibility is also maintained). If time is circular, then given three intervals standing in the betweenness relation, each interval either abuts two of the others or there exist abutting intervals between to fill the gaps. In any case, successive applications of the sum operation yield an interval which exhausts the whole of time. But this is impossible, for according to postulate 9, there exist two intervals lying on either side of every interval and there just cannot be an interval which is the whole of time.

Circular time, together with other closed topologies, is also excluded by postulate 6, which is now deducible as a theorem from postulates 8 and 4 given definition 8.

It is also possible to establish the result mentioned earlier in connection with the discussion of the mereological properties of intervals.

THEOREM *For every proper part s of an interval t which abuts t internally, the difference between t and s exists and abuts s.*

To facilitate the presentation of the proof, an ancilliary lemma is first stated

LEMMA *If r abuts s then there is an interval t abutting s such that B(rst).*

Proof of theorem: Assume *s* is a proper part of *t* and abuts *t* internally. Then there is an interval *u* abutting both, by definition 6. This implies that there is an interval *y* abutting *t* such that B(*uty*) (lemma), and therefore B(*usy*) and *s* is separate from *y*. Now let *z* be any part of *t* separate from *s* (at least one such exists by the first assumption). Clearly, *z* is separate from *y* and *s*, and so by postulate 5, either B(*zsy*) or B(*zys*) or B(*szy*). Suppose B(*zsy*). This, together with B(*usy*), implies B(*uzs*) or B(*zus*) by T7 (the separation prerequisites are satisfied). But since *u* and *s* abut, B(*uzs*) is impossible and therefore B(*zus*). But this is also impossible by definition 8 since both *z* and *s* are parts of *t* and *t* is not separate from its parts. Suppose B(*zys*). This together with B(*usy*) implies B(*zyu*) by Tl. But this is inconsistent, given postulate 6, with the result established from the lemma, namely B(*uty*), which implies B(*uzy*). This leaves the third alternative, B(*szy*), which holds for any *z* separate from *s* which is part of *t*. Consider any two such intervals. By postulate 9, either they abut, or there is an interval between them abutting both, which is also a part of *t* by an earlier theorem. In any case, they can be summed to yield another interval separate from *s* which is part of *t*. The difference between *t* and *s* is the largest such interval (containing all the intervals separate from *s* and part of *t*). To show that *r* abuts *s*, suppose not. Then either *r* and *s* overlap, or there is an interval between them (postulate 8). *r* does not overlap *s* since all its parts are separate from *s*. And any interval *x* between *s* and *r* must be a part of *t* by an earlier theorem. But *x* is separate from *s*, and therefore must be part of *r* – an impossibility if it is also between *s* and *r*.

Corollary *Every interval can be divided into an abutting pair.*

This follows directly given postulates 8, 9 and 10.

III

An asymmetric, dyadic order relation is not definable in the present system (this is easily shown using Padua's method). But it is possible, following the basic idea of Huntington's (Huntington (1938)), to define one in terms of betweenness together with an ordered pair of intervals stipulating a direction. For want of a better term, I shall refer to the 'earlier than' relation, and its converse as the 'later than' relation. In fact, as explained in the introductory section, these definitions do not suffice to define the 'earlier than' and 'later than' relations proper, but only necessary conditions to be satisfied by them.

DEFINITION 9 *Let p and q be two fixed, separate intervals and define p as* earlier than *q. Suppose u and v are two separate intervals. Put s = p if neither u nor v overlap p. Otherwise, form a sum of p with u and v and any intervals lying between them. Then let s be one of the intervals lying on either side of this sum such that B(spq). Similarly, put t = q if neither u nor v overlap q, and otherwise let t be an interval separate from u and v such that B(pqt) as before. Then u is* earlier than *v provided B(stuv) or B(sutv) or B(suvt) or B(ustv) or B(usvt) or B(uvst).*

DEFINITION 10 *An interval u is* later than *an interval v if v is earlier than u.*

The fundamental properties of the dyadic order relation are established in the following theorem.

THEOREM *The 'earlier than' relation is irreflexive, connected (in the sense that, given two separate intervals, one is earlier than the other) and transitive.*

Irreflexivity follows immediately from the requirement that only separate intervals stand in the 'earlier than' relation. For connectivity, suppose *u* and *v* are two separate intervals and that *s* and *t* are defined as in definition 9. Then these four intervals are pairwise separate and by postulate 5, of the following twelve possible permutations of betweenness orderings, at least one from each row must be true; and at most one, by postulate 9.

1.	B (*ust*)	2.	B (*sut*)	3.	B (*uts*)
4.	B (*suv*)	5.	B (*uvs*)	6.	B (*usv*)
7.	B (*vst*)	8.	B (*svt*)	9.	B (*vts*)
10.	B (*utv*)	11.	B (*uvt*)	12.	B (*vut*)

Consider 2, for example, which may be combined with exactly one of 4, 5 and 6. Suppose it is combined with 4; then by T6, either 8 or 9 must be chosen from the third row. If 8 is chosen, 8 together with 4 implies 11 by T2 and the ordering of the quadruple is B (*suvt*). If 9 rather than 8 is chosen, this and 2 imply 10 by T2 and the overall ordering is B (*sutv*). If instead 2 is combined with 5, 8 follows by T3 and then 12 by T2, yielding B (*svut*). And combining 2 with 6 yields B (*tusv*). Proceeding similarly from 1 and 3, eight additional possible orderings are obtained giving a total of twelve. These twelve possibilities can be considered to divide into a disjunction of twelve disjuncts equivalent by definition 9 to '*u* is earlier than *v* or *v* is earlier than *u*'.

Transitivity is established by repeated applications of De Morgan's laws, the details of which are omitted here.

IV

Rational numbers can be used to construct models for the logic of temporal intervals in several ways. Amongst the simplest involves interpreting the temporal intervals as intervals of rational numbers with distinct end points open above and closed below, separation being null intersection. Equally, the rational intervals with distinct end points open below and closed above would serve just as well, the asymmetry of these rational intervals not being a feature of the temporal intervals.

If symmetric, open or closed rational intervals are to be used instead, other steps must be taken to avoid singleton and null intervals. The temporal intervals can be interpreted as closed intervals of rationals with distinct end points provided separation is now defined as the absence of such an interval as a common subset of both the separate intervals. If open intervals are chosen, separation can now be defined in terms of null intersection as before, but some account must be taken of the sums of abutting intervals, which do not include the common end point. This can be done by not distinguishing an open interval of rationals with distinct end points from any set differing by lacking at most a finite number of members. Temporal intervals are then interpreted as sets of such sets, any one of which can represent the temporal interval in the separation relation.

The basic result of this section concerns denumerable models of this sort.

THEOREM *The logic of temporal intervals is* $\aleph_0$*-categorical.*

Without going into all the details of the proof, a 1-1 onto mapping which preserves separation can be defined along the following lines.

Let A be the set of rational intervals of the model first outlined at the beginning of this section, and let $a_0, a_1, a_2, \ldots$ be an enumeration of them. Suppose Q is an arbitrary, denumerably infinite model and $q_0, q_1, q_2, \ldots$ an enumeration of the members. A mapping f from A to Q can be defined as follows. Put $f(a_0) = q_0$. Now consider how a_0 and a_1 are related. One of them might contain the other, they might overlap whilst retaining parts separate from the other, they might abut or, finally, there might be another interval lying between them. In every case, a sequence of abutting intervals is found which can be summed to yield an interval a_1^* containing a_0 and a_1. For example, if there is an interval between a_0 and a_1, then there is an interval abutting both, and a_1^* is the sum of these three. If a_0 and a_1 overlap but neither contains the other, then the difference between the product of a_0 and a_1 and a_0 abuts the product, which in turn abuts the difference between the product and a_1. a_1^* is in this case the sum of these last three intervals. The procedure is similar in the other cases, though sometimes requiring some results not explicitly mentioned in section II but which are easily established as ancilliary lemmas. Then $f(a_1)$ is defined as the first member of Q in the enumeration which bears the same relation to q_0 that a_1 bears to a_0. Moreover, a_1^* and the abutting intervals from which a_1^* is constructed occur later in the enumeration of A and are assigned, respectively, the first members of Q which bear the corresponding relations to q_0. The first member in the enumeration of Q not yet assigned is then considered in relation to q_0 and a sum q_1^* of abutting intervals constructed in similar fashion. These intervals are then assigned, respectively, under the mapping f, to the first members in the enumeration of A bearing the same relation to a_0 that they bear to q_0. The relation to a_1^* of the first member of A not yet assigned an image under f is then considered, and a sum a_2^* constructed from an abutting sequence of intervals as before, partitioning overlapping intervals by the difference operation where necessary. Assignments are then made as before. Alternating between A and Q in this way ensures that all members of each set are correlated with distinct members of the other. It is easily verified that for any a_i, $a_j \in A$, a_i is separate from a_j iff $f(a_i)$ is separate from $f(a_j)$.

COROLLARY *The logic of temporal intervals is complete.*

This follows immediately by Vaught's criterion since finite models are precluded by postulates 9 and 10.

V

In the papers referred to in section I, Hamblin showed how instants could be defined in terms of his intervals within a wider, set-theoretical setting. The entities so defined satisfy the axioms of dense linear order without endpoints. It is natural to ask whether the assumptions of set theory can be dispensed with and the same results achieved within the confines of a first-order theory. The purpose of this final section is to say sufficient to show how the logic of temporal intervals presented in section II can be modified to achieve this.

The basic move is to distinguish the mereological relations of separation and non-overlapping. As before, non-overlapping is to be understood as the absence of a common interval, definitions 1 and 2 remaining unchanged. But the primitive relation of separation now receives a stronger interpretation, excluding any contact whatsoever including common instants. Accordingly, postulate 2 is dropped in favour of the weaker postulate

POSTULATE 2′ *If two intervals overlap, they are not separate,*

which is just the left-to-right implication of postulate 2. The symmetry of the relation of separation must now be established by a distinct postulate:

POSTULATE 2″ *If s is separate from t, then t is separate from s,*

but irreflexivity follows directly from postulate 2′.

The definitions and postulates of the former system must now be reviewed from this new perspective and some further modifications are required. In particular, it is now possible to distinguish between abutment in the sense of non-overlapping with no intervening interval, and a stronger relation of touching defined as neither overlapping nor being separate. If the former definition of the product of two times is then modified so that it exists not only when they overlap, but also when they touch (and the corresponding existence postulate changed accordingly), an instant can be defined as the product of two touching times, and has the property of having no proper parts. (Postulate 10 must be restricted to non-instants.) The axioms of dense linear order without end points are satisfied by instants defined in this way.

An intermediate logic is also possible here, the general idea of which is to modify postulate 2 as above, but to leave the existence postulate for the product unchanged. This opens up the possibility of distinguishing between what can be called open and closed intervals whilst at the same time not recognising the existence of instants explicitly. However, the details will not be pursued in this paper. (2)

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(2) There is an interesting technical exercise here for tense logicians in the style of A.N. Prior. What systems of tense logic correspond to the various interval logics reflecting these subtle differences among temporal intervals? And are there tense-logical postulates corresponding to particular postulates in the interval logics?

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