TI 2001-081/4
Tinbergen Institute Discussion Paper

Introduction to Convex and Quasiconvex Analysis

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Introduction to Convex and Quasiconvex Analysis

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August 27, 2001

Abstract

In this paper the basic ideas of convex and quasiconvex analysis are discussed.

1 Introduction.

In this paper[1] some of the fundamental questions studied within the field of convex and quasiconvex analysis are discussed. Although some of these questions can also be answered within infinite dimensional real topological vector spaces our universe will be the finite dimensional real linear space $\mathbb{R}^n$ equipped with the well-known Euclidean norm $\|.\|$. Since convex and quasiconvex analysis can be seen as the study of certain sets, we consider in section 2 the basic sets studied in this field and list with or without proof the most important algebraic and topological properties of those sets. In this section a proof based on elementary calculus of the important separation result for convex sets in $\mathbb{R}^n$ will be given. In section 3 we introduce the functions studied within this field and show that this study can be reduced to the study of the sets considered in the first section. As such, the equivalent formulation of the separation result for convex sets is now given by the dual representation of a function. In section 4 we will consider some important applications of convex and quasiconvex analysis to optimization theory, game theory and positively homogeneous evenly quasiconvex functions. Finally in

[1] To appear in Handbook of Generalized Convexity, Kluwer Academic Publishers.

section 5 we discuss a part of the historical development within the field of convex and quasiconvex analysis.

2 Sets studied within convex and quasiconvex analysis.

In this section the basic sets studied within convex and quasiconvex analysis in $\mathbb{R}^n$ are discussed. At the same time their most important properties are listed and since in some cases these properties are well-known we often list them without any proof. First we introduce in subsection 2.1 the definition of linear subspaces, affine sets, cones and convex sets in $\mathbb{R}^n$ together with their so-called primal representations. Also the important concept of a hull operation applied to arbitrary sets is considered. In subsection 2.2 the topological properties of those sets are listed and in subsection 2.3 we prove the well-known separation result for convex sets. Finally in subsection 2.4 this separation result is applied to derive so-called dual representations of convex sets. In case proofs are included we have tried to make these proofs as transparent and simple as possible. Also in some cases these proofs can be easily adapted if our universe is an infinite dimensional real topological vector space. Observe most of the material in this section together with the proofs can be found in Rockafellar (cf.[44]) and Hiriart-Urruty and Lemarechal (cf.[26]).

2.1 Algebraic properties of sets.

As already observed our universe will always be the n-dimensional Euclidean space $\mathbb{R}^n$ and any element of $\mathbb{R}^n$ is denoted by the vector $\mathbf{x} = (x_1, ..., x_n), x_i \in \mathbb{R}$ or $\mathbf{y} = (y_1, ..., y_n), y_i \in \mathbb{R}$. The innerproduct $< ., . >$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is then given by

$$< \mathbf{x}, \mathbf{y} >:= \sum_{i=1}^n x_i y_i = \mathbf{x}^\top \mathbf{y},$$

while the Euclidean norm $\|.\|$ is defined by

$$\|\mathbf{x}\| := \sqrt[2]{< x, x >}.$$

To simplify the notation, we also introduce for the sets $A, B \subseteq \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ the *Minkowsky sum* $\alpha A + \beta B$ given by

$$\alpha A + \beta B := \{\alpha \mathbf{x} + \beta \mathbf{y} : \mathbf{x} \in A, \mathbf{y} \in B\}.$$

The first sets to be introduced are the main topic of study within *linear algebra* (cf.[44]).

Definition 1 *A set $L \subseteq \mathbb{R}^n$ is called a linear subspace if L is nonempty and $\alpha L + \beta L \subseteq L$ for every α, $\beta \in \mathbb{R}$. Moreover, a set $M \subseteq \mathbb{R}^n$ is called affine if $\alpha M + (1-\alpha)M \subseteq M$ for every $\alpha \in \mathbb{R}$.*

The empty set $\varnothing$ and $\mathbb{R}^n$ are extreme examples of an affine set. Also it can be shown relatively easy that the set M is affine and $\mathbf{0} \in M$ if and only if M is a linear subspace and for each nonempty affine set M there exists a unique linear subspace L_M satisfying $M = L_M + \mathbf{x}$ for any given $\mathbf{x} \in M$ (cf.[44]).

Since $\mathbb{R}^n$ is a linear subspace, we can apply to any nonempty set $S \subseteq \mathbb{R}^n$ the so-called *linear hull operation* and construct the set

$$lin(S) := \cap\{L : S \subseteq L \text{ and } L \text{ a linear subspace}\}. \tag{1}$$

For any collection of linear subspaces $L_i, i \in I$ containing S it is obvious that the intersection $\cap_{i\in I} L_i$ is again a linear subspace containing S and this shows that the set $\text{lin}(S)$ is the smallest linear subspace containing S. The set $\text{lin}(S)$ is called the *linear hull generated by the set S* and if S has a finite number of elements the linear hull is called *finitely generated*. By a similar argument one can construct using the so-called *affine hull operation* the smallest affine set containing S. This set, denoted by $aff(S)$, is called the *affine hull generated by the set S* and is given by

$$aff(S) := \cap\{M : S \subseteq M \text{ and } M \text{ an affine set}\}. \tag{2}$$

If the set S has a finite number of elements, the affine hull is called *finitely generated*. Since any linear subspace is an affine set it is clear that $\text{aff}(S) \subseteq \text{lin}(S)$. To give a so-called *primal representation* of these sets we introduce the next definition.

Definition 2 *A vector $\mathbf{x}$ is a linear combination of the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ if $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{x}_i, \alpha_i \in \mathbb{R}, 1 \le i \le k$. A vector $\mathbf{x}$ is an affine combination of the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ if $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{x}_i$ and $\sum_{i=1}^k \alpha_i = 1$. A linear combination of the nonempty set S is given by the set $\sum_{i=1}^k \alpha_i S$ with $\alpha_i \in \mathbb{R}, 1 \le i \le k$, while an affine combination of the same set is given by the set $\sum_{i=1}^k \alpha_i S$ with $\sum_{i=1}^k \alpha_i = 1$.*

A trivial consequence of Definitions 1 and 2 is given by the next result which also holds in infinite dimensional vector spaces.

Lemma 3 *A nonempty set $L \subseteq \mathbb{R}^n$ is a linear subspace if and only if it contains all linear combinations of the set L. Moreover, a nonempty set $M \subseteq \mathbb{R}^n$ is an affine set if and only if it contains all affine combinations of the set M.*

The result in Lemma 3 yields a primal representation of a linear subspace and an affine set. In particular, we easily obtain from Lemma 3 that the set $lin(S)$ $(aff(S))$ with $S \subseteq \mathbb{R}^n$ nonempty equals all linear (affine) combinations of the set S. This means

$$lin(S) = \cup_{m=1}^{\infty} \{\sum_{i=1}^{m} \alpha_i S : \alpha_i \in \mathbb{R}\} \tag{3}$$

and

$$aff(S) = \cup_{m=1}^{\infty} \{\sum_{i=1}^{m} \alpha_i S : \alpha_i \in \mathbb{R} \text{ and } \sum_{i=1}^{m} \alpha_i = 1\}. \tag{4}$$

Using relation (4) one can show

$$aff(S_1 \times S_2) = aff(S_1) \times aff(S_2) \tag{5}$$

for any nonempty sets $S_1 \subseteq \mathbb{R}^n, S_2 \subseteq \mathbb{R}^m$ and

$$A(aff(S)) = aff(A(S)) \tag{6}$$

for any affine mapping $A : \mathbb{R}^n \to \mathbb{R}^m$. Observe a mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if $A(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) = \alpha A(\mathbf{x}) + (1-\alpha)A(\mathbf{y})$ for every $\alpha \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Moreover, in case we apply relation (5) to the affine mapping $A : \mathbb{R}^{2n} \to \mathbb{R}^n$ given by $A(\mathbf{x}, \mathbf{y}) = \alpha \mathbf{x} + \beta \mathbf{y}$ with $\alpha, \beta \in \mathbb{R}$ and use relation (6) the following rule for the affine hull of the sum of sets is easy to verify.

Lemma 4 *For any nonempty sets $S_1, S_2 \subseteq \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ it follows that $aff(\alpha S_1 + \beta S_2) = \alpha aff(S_1) + \beta aff(S_2)$.*

Another application of relations (3) and (4) yields the next result.

Lemma 5 *For any nonempty set $S \subseteq \mathbb{R}^n$ and $\mathbf{x}_0$ belonging to $aff(S)$ it follows that $aff(S) = \mathbf{x}_0 + lin(S - \mathbf{x}_0)$.*

An improvement of Lemma 3 is given by the observation that any linear subspace (affine set) of $\mathbb{R}^n$ can be written as the linear or affine hull of a *finite* subset $S \subseteq \mathbb{R}^n$. To show this improvement one needs to introduce the next definition (cf.[41]).

Definition 6 *The vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ are called linear independent if*

$$\sum_{i=1}^{k} \alpha_i \mathbf{x}_i = \mathbf{0} \text{ and } \alpha_i \in \mathbb{R} \Rightarrow \alpha_i = 0,\ 1 \leq i \leq k$$

Moreover, the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ are called affinely independent if

$$\sum_{i=1}^{k} \alpha_i \mathbf{x}_i = \mathbf{0} \text{ and } \sum_{i=1}^{k} \alpha_i = 0 \Rightarrow \alpha_i = 0,\ 0 \leq i \leq k.$$

For $k \geq 2$ an equivalent characterization of affinely independent vectors is given by the observation that the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ are affinely independent if and only if the vectors $\mathbf{x}_2 - \mathbf{x}_1, ..., \mathbf{x}_k - \mathbf{x}_1$ are linear independent (cf.[26]). To explain the name linear and affinely independent we observe that the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ are linear independent if and only if any vector $\mathbf{x}$ belonging to the linear hull $lin(\{\mathbf{x}_1, ..., \mathbf{x}_k\})$ can be written as a unique linear combination of the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$. Moreover, the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$ are affinely independent if and only if any vector $\mathbf{x}$ belonging to the affine hull $aff(\{\mathbf{x}_1, ..., \mathbf{x}_k\})$ can be written as a unique affine combination of the vectors $\mathbf{x}_1, ..., \mathbf{x}_k$. The improvement of Lemma 3 is given by the following result well-known within linear algebra (cf.[41]).

Lemma 7 *For any linear subspace $L \subseteq \mathbb{R}^n$ there exists a set of linear independent vectors $\mathbf{x}_1, ..., \mathbf{x}_k$, $k \leq n$ satisfying $L = lin(\{\mathbf{x}_1, ..., \mathbf{x}_k\})$. Also for any nonempty affine set $M \subseteq \mathbb{R}^n$ there exists a set of affinely independent vectors $\mathbf{x}_0, ..., \mathbf{x}_k, k \leq n$ satisfying $M = aff(\{\mathbf{x}_0, ..., \mathbf{x}_k\})$.*

By Lemma 7 any linear subspace of $\mathbb{R}^n$ can be represented as the linear hull of $k \leq n$ linear independent vectors and if this holds the dimension $\dim(L)$ is given by k. Since any $\mathbf{x}$ belonging to L can be written as a unique linear combination of linear independent vectors this shows that $\dim(L)$ is well defined. To extend this to affine sets we observe by the remark after Definition 1 that any nonempty affine set M is parallel to its unique subspace L_M and the dimension $\dim(M)$ of a nonempty affine set M is now given by $\dim(L_M)$. By definition the dimension of the empty set $\varnothing$ equals 0. Finally, the dimension $\dim(S)$ of an arbitrary set $S \subseteq \mathbb{R}^n$ is given by $\dim(aff(S))$. In the next definition we will introduce the sets which are the main topic of study within the field of convex and quasiconvex analysis.

Definition 8 *A set $C \subseteq \mathbb{R}^n$ is called convex if $\alpha C + (1 - \alpha)C \subseteq C$ for every $0 < \alpha < 1$. Moreover, a set $K \subseteq \mathbb{R}^n$ is called a cone if $\alpha K \subseteq K$ for every $\alpha > 0$.*

The empty set $\varnothing$ is an extreme example of a convex set and a cone. An affine set is clearly a convex set but it is obvious that not every convex set is an affine set. This shows that convex analysis is an extension of linear algebra. Moreover, it is easy to show for every cone K that

$$K \text{ convex} \Leftrightarrow K + K \subseteq K. \tag{7}$$

Finally, for $A : \mathbb{R}^n \to \mathbb{R}^m$ an affine mapping and $C \subseteq \mathbb{R}^n$ a nonempty convex set it follows that the set $A(C)$ is convex, while for $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping and $K \subseteq \mathbb{R}^n$ a nonempty cone the set $A(K)$ is a cone.

To relate convex sets to convex cones we observe for $\mathbb{R}_+ := [0, \infty)$ and any nonempty set $S \subseteq \mathbb{R}^n$ that the set $\mathbb{R}_+(S \times \{1\}) := \{(\alpha\mathbf{x}, \alpha) : \alpha \geq 0, \mathbf{x} \in S\} \subseteq \mathbb{R}^{n+1}$ is a cone. This implies by relation (7) that the set $\mathbb{R}_+(C \times \{1\})$ is a convex cone for any convex set $C \subseteq \mathbb{R}^n$. It is now clear for any set $S \subseteq \mathbb{R}^n$ that

$$\mathbb{R}_+(S \times \{1\}) \cap (\mathbb{R}^n \times \{1\}) = S \times \{1\} \tag{8}$$

and so any convex set C can be seen as an intersection of the convex cone $\mathbb{R}_+(C \times \{1\})$ and the affine set $\mathbb{R}^n \times \{1\}$. This shows that convex sets are closely related to convex cones and by relation (8) one can study convex sets by only studying affine sets and convex cones containing $\mathbf{0}$. We will not pursue this approach but only remark that the above relation is sometimes useful. Introducing an important subclass of convex sets let $\mathbf{a}$ be a nonzero vector belonging to $\mathbb{R}^n$ and $b \in \mathbb{R}$ and

$$H^{<}(\mathbf{a}, b) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\intercal\mathbf{x} < b\}. \tag{9}$$

The set $H^{<}(\mathbf{a}, b)$ is called a halfspace and clearly this halfspace is a convex set. Moreover, the set $H^{\leq}(\mathbf{a}, b) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\intercal\mathbf{x} \leq b\}$ is also called a halfspace and this set is also a convex set. Another important subclass of convex sets is now given by the following definition (cf.[55]).

Definition 9 *A set $C_e \subseteq \mathbb{R}^n$ is called evenly convex if $C_e = \mathbb{R}^n$ or C_e is the intersection of a collection of halfspaces $H^{<}(\mathbf{a}, b)$.*

Clearly the empty set $\varnothing$ is evenly convex and since any halfspace $H^{\leq}(\mathbf{a}, b)$ can be obtained by intersecting the halfspaces $H^{<}(\mathbf{a}, b+\frac{1}{n}), n \geq 1$ it also follows that any halfspace $H^{\leq}(\mathbf{a}, b)$ is evenly convex. In subsection 2.3 it will be shown that any closed or open convex set is evenly convex. However, there exist convex sets which are not evenly convex.

Example 10 *If $C := \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 < 1\} \cup \{(1, 1)\}$ it follows that C is convex but not evenly convex.*

Since $\mathbb{R}^n$ is a convex set, we can apply to any nonempty set $S \subseteq \mathbb{R}^n$ the so-called *convex hull operation* and construct the set

$$co(S) := \cap\{C : S \subseteq C \text{ and } C \text{ a convex set}\}. \tag{10}$$

For any collection of convex sets $C_i, i \in I$ containing S it is obvious that the intersection $\cap_{i \in I} C_i$ is again a convex set containing S and this shows that the set $co(S)$ is the smallest convex set containing S. The set $co(S)$ is called the *convex hull generated by the set* S and if S has a finite number of elements the convex hull is called *finitely generated*. Since $\mathbb{R}^n$ is by definition evenly convex one can construct by a similar argument using the so-called *evenly convex hull operation* the smallest evenly convex set containing the nonempty set S. This set, denoted by $\mathrm{eco}(S)$, is called the *evenly convex hull generated by the set* S and is given by

$$eco(S) := \cap \{C_e : S \subseteq C_e \text{ and } C_e \text{ an evenly convex set }\}. \tag{11}$$

Since any evenly convex set is convex it follows that $co(S) \subseteq eco(S)$.

By the so-called *canonic hull operation* one can also construct the smallest convex cone containing the nonempty set S, and the smallest convex cone containing $S \cup \{\mathbf{0}\}$. The last set is given by

$$cone(S) := \cap \{K : S \cup \{\mathbf{0}\} \subseteq K \text{ and } K \text{ a convex cone}\}. \tag{12}$$

Unfortunately this set is called the *convex cone generated by* S (cf.[44]). Clearly the set $cone(S)$ is in general not equal to the smallest cone containing S unless the zero element belongs to S. To give an alternative characterization of the above sets we introduce the next definition.

Definition 11 *A vector* $\mathbf{x}$ *is a canonical combination of the vectors* $\mathbf{x}_1, ..., \mathbf{x}_k$ *if* $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{x}_i, \alpha_i \geq 0, 1 \leq i \leq k$. *The vector* $\mathbf{x}$ *is called a strict canonical combination of the same vectors if* $\alpha_i > 0, i = 1, ..., k$. *A vector* $\mathbf{x}$ *is a convex combination of the vectors* $\mathbf{x}_1, ..., \mathbf{x}_k$ *if* $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{x}_i$ *and* $\sum_{i=1}^k \alpha_i = 1, \alpha_i > 0$. *A canonical combination of the nonempty set* S *is given by the set* $\sum_{i=1}^k \alpha_i S$ *with* $\alpha_i \geq 0, 1 \leq i \leq k$, *while a strict canonical combination of the same set is given by* $\sum_{i=1}^k \alpha_i S$ *with* $\alpha_i > 0, 1 \leq i \leq k$. *Finally a convex combination of the set* S *is given by the set* $\sum_{i=1}^k \alpha_i S$ *with* $\sum_{i=1}^k \alpha_i = 1, \alpha_i > 0, i = 1, ..., k$.

A trivial consequence of definitions 8 and 11 is given by the next result which also holds in infinite dimensional vector spaces.

Lemma 12 *A nonempty set* $K \subseteq \mathbb{R}^n$ *is a convex cone if and only if it contains all strict canonical combinations of the set* K. *Moreover, a nonempty set* $C \subseteq \mathbb{R}^n$ *is a convex set if and only if it contains all convex combinations of the set* C.

The result in Lemma 12 yields a primal representation of a convex cone and a convex set. In particular, we easily obtain from Lemma 12 that the set *cone*(S) (*co*(S)) with $S \subseteq \mathbb{R}^n$ nonempty equals all canonical (convex) combinations of the set S. This means

$$cone(S) = \cup_{m=1}^{\infty}\{\sum_{i=1}^{m} \alpha_i S : \alpha_i \geq 0\} \tag{13}$$

and

$$co(S) = \cup_{m=1}^{\infty}\{\sum_{i=1}^{m} \alpha_i S : \sum_{i=1}^{m} \alpha_i = 1, \alpha_i > 0\}. \tag{14}$$

We observe that the above representations are the "convex equivalence" of the representation for $lin(S)$ and $aff(S)$ given by relations (3) and (4). Moreover, to relate the above representations, it is easy to see that

$$cone(S) = \mathbb{R}_+(co(S)). \tag{15}$$

Since a convex cone containing $\mathbf{0}$ (convex set) can be seen as a generalization of a linear subspace (affine set) one might wonder whether a similar result as in Lemma 7 holds. Hence we wonder whether any convex cone containing $\mathbf{0}$ (convex set) can be seen as a canonical (convex) combination of a finite set S.

Example 13 *Contrary to linear subspaces it is not true that any convex cone containing* $\mathbf{0}$ *is a canonical combination of a finite set. An example is given by the so-called convex ice-cream cone* $K = \{(\mathbf{x}, t) : \parallel \mathbf{x} \parallel \leq t\} \subseteq \mathbb{R}^{n+1}$.

Despite this negative result it is possible in finite dimensional linear spaces to improve for convex cones and convex sets the representation given by relations (13) and (14). In the next result it is shown that any element belonging to cone(S) can be written as a canonical combination of at most n linear independent vectors belonging to S. This is called *Caratheodory's theorem* for convex cones. Using this result and relation (8) a related result holds for convex sets and in this case linear independent is replaced by affine independent and at most n is replaced by at most $n + 1$. Clearly this result (cf.[44]) is the "convex equivalence" of Lemma 7.

Lemma 14 *If* $S \subseteq \mathbb{R}^n$ *is a nonempty set then for any* $\mathbf{x}$ *belonging to* *cone*(S) *there exists a set of linear independent vectors* $\mathbf{x}_1, ..., \mathbf{x}_k, k \leq n$ *belonging to* S *such that* $\mathbf{x}$ *can be written as a canonical combination of* *these vectors. Moreover, for any $\mathbf{x} \in co(S)$ there exists a set of affinely independent vectors $\mathbf{x}_1, ..., \mathbf{x}_k, k \leq n+1$ belonging to S such that $\mathbf{x}$ can be written as a convex combination of these vectors.*

Proof. Assuming the result holds true for convex cones (cf.[44]) it follows for any $\mathbf{x} \in co(S)$ that $(\mathbf{x}, 1)$ belongs to $co(S) \times \{1\} \subseteq \mathbb{R}^+(co(S) \times \{1\}\} \subseteq \mathbb{R}^{n+1}$. By relation (15) the set $\mathbb{R}^+(co(S) \times \{1\}\}$ is the convex cone generated by $S \times \{1\}$ and so one can apply the first part. Using now relation (8) the second part follows. $\square$

Although in the above lemma $k \leq n$ for cones and $k \leq n+1$ for convex hulls it is easy to see that n can be replaced by $\dim(S) \leq n$. This concludes our discussion on algebraic properties of linear subspaces, affine sets, convex sets, and convex cones. In the next subsection we investigate topological properties of the above sets.

2.2 Topological properties of sets.

For a given set $S \subseteq \mathbb{R}^n$ let $int(S)$ and $cl(S)$ denote the interior and the closure of the set S, respectively. The following result holds for affine sets (linear subspaces) and this can be easily verified using Lemma 7 (cf.[4]).

Lemma 15 *Any affine set $M \subseteq \mathbb{R}^n$ is closed.*

By Lemma 15 we obtain $cl(S) \subseteq aff(S) \subseteq lin(S)$ and this yields by the monotonicity of the hull operation that

$$aff(cl((S)) = aff(S) \text{ and } lin(cl(S)) = lin(S). \tag{16}$$

Opposed to affine sets it is not true that any convex cone is closed. The same holds for convex sets. It will be shown for *closed* convex sets and *closed* convex cones that it is relatively easy to give a *dual representation* of those sets and this is the main reason why we like to identify which classes of convex sets and convex cones are closed. Since affine sets can always be generated by a *finite set of affinely independent vectors* (and this guarantees that affine sets are closed) and we know by Example 13 that this is not true for convex sets one might now wonder which property replacing finiteness should be imposed on S to guarantee that $co(S)$ is closed. Looking at the following counterexample it is not sufficient to impose that the set S is closed and this implies that we need a stronger property.

Example 16 *If* $S = \mathbf{0} \cup \{(x,1) : x \geq 0\}$ *then* S *is closed and its convex hull given by* $co(S) = \{(x_1, x_2) : 0 < x_2 \leq 1, x_1 \geq 0\} \cup \{\mathbf{0}\}$ *is clearly not closed.*

In the above counterexample the closed set S is unbounded and this prevents $co(S)$ to be closed. Imposing now the additional property that the closed set S is bounded or equivalently compact one can show that $co(S)$ is indeed closed and even compact. At the same time this yields a way to identify for which sets S the set $cone(S)$ is closed. So finiteness of the generator S for affine sets should be replaced by compactness of S for convex hulls. To prove the next result we first introduce the so-called *unit simplex* $\Delta_{n+1} := \{\alpha : \sum_{i=1}^{n+1} \alpha_i = 1 \text{ and } \alpha_i \geq 0\} \subseteq \mathbb{R}^{n+1}$. By Lemma 14 it follows that

$$co(S) = f(\Delta_{n+1} \times S^{n+1}) \tag{17}$$

with S^m denoting the m-fold Cartesian product of the set $S \subseteq \mathbb{R}^n$ and the function f is given by $f(\alpha, \mathbf{x}_1, ..., \mathbf{x}_{n+1}) = \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i$. A related observation holds for convex cones and using the above observations one can now show the following result (cf.[26]).

Lemma 17 *If the nonempty set* $S \subseteq \mathbb{R}^n$ *is compact then the set* $co(S)$ *is compact. Moreover, if* S *is compact and* $\mathbf{0}$ *does not belong to* $co(S)$ *then the set* $cone(S)$ *is closed.*

Proof. It is well known that the set $\Delta_{n+1} \times S^{n+1}$ is compact (cf. [56]) and this shows by relation (17) and f is a continuous function that $\mathrm{co}(S)$ is compact. To verify the second part we observe by relation (15) that it is sufficient to show that the set $\mathbb{R}_+(\mathrm{co}(S))$ is closed. Consider now an arbitrary sequence $t_n \mathbf{x}_n, n \in \mathbb{N}$ belonging to $\mathbb{R}_+(co(S))$ with $\lim_{n\uparrow\infty} t_n \mathbf{x}_n = \mathbf{y}$. This implies $\lim_{n\uparrow\infty} t_n \parallel \mathbf{x}_n \parallel = \parallel \mathbf{y} \parallel$ and since $\mathbf{0} \notin co(S)$ and $co(S)$ is compact there exists a subsequence $N_0 \subseteq \mathbb{N}$ satisfying $\lim_{n\in N_0\uparrow\infty} \mathbf{x}_n = \mathbf{x}_\infty \in co(S)$ and $\mathbf{x}_\infty \neq \mathbf{0}$. Hence we obtain

$$\lim_{n\in N_0\uparrow\infty} t_n = \lim_{n\in N_0\uparrow\infty} \frac{t_n \parallel \mathbf{x}_n \parallel}{\parallel \mathbf{x}_n \parallel} = \frac{\|\mathbf{y}\|}{\parallel \mathbf{x}_\infty \parallel} := t_\infty < \infty.$$

and this yields $\mathbf{y} = t_\infty \mathbf{x}_\infty \in \mathbb{R}_+(co(S))$, showing the desired result. $\square$

The following example shows that the condition $\mathbf{0} \notin co(S)$ cannot be omitted in Lemma 17.

Example 18 *If the condition* $\mathbf{0} \notin S$ *is omitted in Lemma 17 then the set* $cone(S)$ *might not be closed as shown by the following example. Let* $S = \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 \leq 1\}$. *Clearly* S *is compact and* $\mathbf{0} \in S$. *Moreover, by relation (15) it follows that* $cone(S) = \{(x_1, x_2) : x_1 > 0\} \cup \{\mathbf{0}\}$ *and this set is not closed.*

An immediate consequence of Caratheodory's theorem and Lemma 17 is given by the next result for convex cones generated by some nonempty set S.

Lemma 19 *If the set* $S \subseteq \mathbb{R}^n$ *contains a finite number of elements then the set* $cone(S)$ *is closed.*

Proof. For any finite set S we consider the finite set $V := \{I : I \subseteq S$ and the elements of I are linear independent$\}$. By Lemma 14 it follows that $cone(S) = \cup_{I \in V} cone(I)$. Since each I belonging to V is a finite set of linear independent vectors the set I is compact and $\mathbf{0}$ does not belong to $co(I)$. This shows by Lemma 17 that $cone(I)$ is closed for every I belonging to V and since V is a finite set the result follows. □

Next we introduce the definition of a relative interior point. This generalizes the notion of an interior point.

Definition 20 *If* $E := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < 1\}$, *a vector* $\mathbf{x} \in \mathbb{R}^n$ *is called a relative interior point of the set* $S \subseteq \mathbb{R}^n$ *if* $\mathbf{x}$ *belongs to* $aff(S)$ *and there exists some* $\epsilon > 0$ *such that*

$$(\mathbf{x} + \epsilon E) \cap aff(S) \subseteq S.$$

The relative interior $ri(S)$ *of any set* S *is given by* $ri(S) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}$ *is a relative interior point of* $S\}$. *The set* $S \subseteq \mathbb{R}^n$ *is called relatively open if* S *equals* $ri(S)$ *and it is called regular if* $ri(S)$ *is nonempty.*

As shown by the next example it is quite natural to assume that $\mathbf{x}$ belongs to $aff(S)$. This assumption implies that $ri(S) \subseteq S$.

Example 21 *Consider the set* $S = \{0\} \times [-1, 1]$ *and let* $\mathbf{x} = (1, 0)$. *Clearly the set* $aff(S)$ *is given by* $\{0\} \times \mathbb{R}$ *and for* $\epsilon = 1$ *it follows that* $(\mathbf{x} + E) \cap aff(S) \subseteq S$. *If in the definition of a relative interior point one would delete the condition that* $\mathbf{x}$ *must belong to* $aff(S)$ *then according to this, the vector* $(1, 0)$ *would be a relative interior point of the set* S. *However, the vector* $(1, 0)$ *is not an element of* S *and so this definition would not be natural.*

By the above definition it is clear for $S \subseteq \mathbb{R}^n$ full dimensional or equivalently $aff(S) = \mathbb{R}^n$ that relative interior means interior and hence relative refers to relative with respect to $aff(S)$. By the same definition we also obtain that every affine manifold is relatively open. Moreover, since by Lemma 15 the set $aff(S)$ is closed it follows that $cl(S) \subseteq aff(S)$ and so it is useless to introduce closure relative to the affine hull of a given set S. Contrary to the different hull operations the relative interior operator is not a monotone operator. This means that $S_1 \subseteq S_2$ does not imply that $ri(S_1) \subseteq ri(S_2)$.

Example 22 *If $C_1 = \{0\}$ and $C_2 = [0, 1]$ then both sets are convex and $ri(C_1) = \{0\}$ and $ri(C_2) = (0, 1)$. This shows $C_1 \subseteq C_2$ and $ri(C_1) \nsubseteq ri(C_2)$.*

To guarantee that the relative interior operator is monotone we need to impose the additional condition that $aff(S_1) = aff(S_2)$. If this holds it is easy to check that

$$S_1 \subseteq S_2 \Rightarrow ri(S_1) \subseteq ri(S_2). \tag{18}$$

By the above observation it is important to known which different sets cannot be distinguished by the affine operator. The next result shows that this holds for the sets S, $cl(S)$, $co(S)$ and $cl(co(S))$ and this result can be easily verified using $cl(co(S)) \subseteq aff(S)$.

Lemma 23 *It follows for every nonempty set $S \subseteq \mathbb{R}^n$ that*

$$aff(S) = aff(cl(S)) = aff(co(S)) = aff(cl(co(S))).$$

By relation (18) and Lemma 23 we obtain $ri(S) \subseteq ri(cl(S)) \subseteq ri(cl(co(S)))$ and $ri(S) \subseteq ri(co(S))$ for arbitrary sets $S \subseteq \mathbb{R}^n$. Moreover, by relation (5) it is easy to verify that $ri(S_1 \times S_2) = ri(S_1) \times ri(S_2)$. An alternative definition of a relative interior point which is needed to show that the relative interior operator is invariant when applied to a relatively open set is given by the next lemma.

Lemma 24 *If the set $S \subseteq \mathbb{R}^n$ is regular then the vector $\mathbf{x}$ is a relative interior point of the set S if and only if $\mathbf{x}$ belongs to $aff(S)$ and there exists some $\epsilon > 0$ such that $(\mathbf{x} + \epsilon E) \cap aff(S) \subseteq ri(S)$.*

Proof. We only need to verify the if implication. Let $\mathbf{x}$ be a relative interior point of the set S. This means $\mathbf{x} \in aff(S)$ and there exists some

$\epsilon > 0$ such that $(\mathbf{x} + \epsilon E) \cap aff(S) \subseteq S$. Since $\mathbf{x} \in aff(S)$ we obtain that $(\mathbf{x}+\delta E) \cap aff(S)$ is nonempty for every $\delta > 0$. Consider now some $\mathbf{y} \in (\mathbf{x} + \frac{\epsilon}{2}E) \cap aff(S)$. Clearly $\mathbf{y} \in aff(S)$ and $\mathbf{y}+\frac{\epsilon}{2}E \subseteq \mathbf{x}+\epsilon E$ and this implies $(\mathbf{y}+\frac{\epsilon}{2}E) \cap aff(S) \subseteq S$. Hence $\mathbf{y}$ belongs to $ri(S)$ and so the inclusion $(\mathbf{x} + \frac{\epsilon}{2}E) \cap aff(S) \subseteq ri(S)$ holds. $\square$

The next result shows that for regular sets $S \subseteq \mathbb{R}^n$ the affine hull operation cannot distinguish the sets $ri(S)$ and S and so this lemma can be seen as an extension of Lemma 23.

Lemma 25 *If the set $S \subseteq \mathbb{R}^n$ is regular then it follows that $aff(ri(S)) = aff(S)$.*

Proof. It is clear that $aff(ri(S)) \subseteq aff(S)$ and to show the converse inclusion it is sufficient to verify that $S \backslash ri(S) \subseteq aff(ri(S))$. Let $\mathbf{x} \in S \backslash ri(S)$. Since the set S is regular one can find some $\mathbf{y} \in ri(S) \subseteq S$ and so by Lemma 24 there exists some $\epsilon > 0$ satisfying

$$(\mathbf{y} + \epsilon E) \cap aff(S) \subseteq ri(S). \tag{19}$$

Clearly the set $[\mathbf{y}, \mathbf{x}] := \{(1-\alpha)\mathbf{y} + \alpha\mathbf{x} : 0 \leq \alpha \leq 1\}$ belongs to $co(S) \subseteq aff(S)$ and this implies by relation (19) that $(\mathbf{y} + \epsilon E) \cap [\mathbf{y}, \mathbf{x}] \subseteq ri(S)$. This means that the halfline starting in $\mathbf{y}$ and passing through $\mathbf{x}_1 \in (\mathbf{y} + \epsilon E) \cap [\mathbf{y}, \mathbf{x}]$ is contained in $aff(ri(S))$ and contains $\mathbf{x}$. Hence $\mathbf{x}$ belongs to $aff(ri(S))$ and so $S \backslash ri(S) \subseteq aff(ri(S))$. $\square$

An immediate consequence of Lemmas 25 and 24 is given by the observation that for any regular set $S \subseteq \mathbb{R}^n$ it follows that $\mathbf{x}$ is a relative interior point of S if and only if $\mathbf{x}$ belongs to $aff(ri(S))$ and there exists some $\epsilon > 0$ satisfying $(\mathbf{x} + \epsilon E) \cap aff(ri(S)) \subseteq ri(S)$. This implies for every regular set $S \subseteq \mathbb{R}^n$ that $ri(ri(S)) = ri(S)$, and since by definition $ri(\varnothing) = \varnothing$ we obtain for any set S that

$$ri(ri(S)) = ri(S). \tag{20}$$

Keeping in mind the close relationship between affine hulls and convex sets and the observation that nonempty affine manifolds are regular (in fact $\mathrm{ri}(M) = M$!) we might wonder whether convex sets are regular. This is indeed the case as the following result shows (cf.[44]).

Lemma 26 *Every nonempty convex set $C \subseteq \mathbb{R}^n$ is regular.*

Observe the proof of Lemma 26 uses that C is a subset of a finite dimensional linear space and in general this result does not hold in infinite dimensional topological vector spaces. Returning to finite dimensional linear spaces the set $\mathbb{Q} \subseteq \mathbb{R}$ of rational numbers is not regular and so there exist sets in finite dimensional linear spaces which are not regular. We will now list some important properties of relative interiors. To start with this we first verify the following technical result.

Lemma 27 *If $S_1, S_2 \subseteq \mathbb{R}^n$ are nonempty sets then it follows for every $0 < \alpha < 1$ that*

$$(\alpha S_1 + (1-\alpha)S_2) \cap aff(S_1) \subseteq \alpha S_1 + (1-\alpha)(S_2 \cap aff(S_1)).$$

Proof. Consider for $0 < \alpha < 1$ the vector $\mathbf{y} = \alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ with $\mathbf{x}_i \in S_i$, $i = 1, 2$ and $\mathbf{y} \in aff(S_1)$. It is now necessary to verify that $\mathbf{x}_2$ belongs to $S_2 \cap aff(S_1)$. By the definition of $\mathbf{y}$ and $0 < \alpha < 1$ we obtain that

$$\mathbf{x}_2 = \frac{1}{1-\alpha}\mathbf{y} - \frac{\alpha}{1-\alpha}\mathbf{x}_1 \in \frac{1}{1-\alpha} aff(S_1) - \frac{\alpha}{1-\alpha} S_1,$$

and so it follows that $\mathbf{x}_2$ belongs to $aff(S_1)$. Hence the vector $\mathbf{x}_2$ belongs to $S_2 \cap aff(S_1)$ and this shows the desired result. □

Applying now Lemma 27, the next important result for convex sets can be shown. This result will play an important role in the proof of the subsequent results.

Lemma 28 *If $C \subseteq \mathbb{R}^n$ is a nonempty convex set then it follows for every $0 \leq \alpha < 1$ that*

$$\alpha cl(C) + (1-\alpha) ri(C) \subseteq ri(C).$$

Proof. To prove the above result it is sufficient to show that $\alpha cl(C) + (1-\alpha)\mathbf{x}_2 \subseteq ri(C)$ for any fixed $\mathbf{x}_2 \in ri(C)$ and $0 < \alpha < 1$. Clearly this set is a subset of $aff(C)$ and since $\mathbf{x}_2$ belongs to $ri(C) \subseteq C$ there exists some $\epsilon > 0$ satisfying

$$(\mathbf{x}_2 + \frac{(1+\alpha)\epsilon}{1-\alpha} E) \cap aff(C) \subseteq C. \tag{21}$$

Moreover, it is easy to see that $cl(C) \subseteq C + \epsilon E$, and this implies

$$\alpha cl(C) + (1-\alpha)\mathbf{x}_2 + \epsilon E \subseteq \alpha C + (1-\alpha)(\mathbf{x}_2 + \frac{1+\alpha}{1-\alpha}\epsilon E).$$

Applying now Lemma 27 and relation (21) we obtain by the convexity of the set C that

$$(\alpha cl(C) + (1-\alpha)\mathbf{x}_2 + \epsilon E) \cap aff(C) \subseteq \alpha C + (1-\alpha)C \subseteq C$$

and this shows the result. $\square$

By Lemmas 26 and 28 it follows immediately for any nonempty convex set C that the set $ri(C)$ is nonempty and convex. Also, since $cl(C) = \cap_{\epsilon>0}(C + \epsilon E)$ it is easy to verify that $cl(C)$ is a convex set. An easy and important consequence of Lemma 28 is given by the observation that the relative interior operator cannot distinguish the convex sets C and $cl(C)$. A similar observation holds for the closure operator applied to the convex sets $ri(C)$ and C. The next result also plays an important role in the proof of the weak separation result.

Lemma 29 *If $C \subseteq \mathbb{R}^n$ is a nonempty convex set then it follows that*

$$cl(ri(C)) = cl(C) \text{ and } ri(C) = ri(cl(C)).$$

Proof. To prove the first formula we only need to check that $cl(C) \subseteq cl(ri(C))$. To verify this we consider $\mathbf{x} \in cl(C)$ and select some $\mathbf{y}$ belonging to ri(C). By Lemma 28 the half-open line segment $[\mathbf{y}, \mathbf{x})$ belongs to $ri(C)$ and this implies that the vector $\mathbf{x}$ belongs to $cl(ri(C)$. Hence $cl(C) \subseteq cl(ri(C))$ and the first formula is verified. To prove the second formula, it follows immediately by relation (18) that $ri(C) \subseteq ri(cl(C))$. To verify $ri(cl(C)) \subseteq ri(C)$ consider an arbitrary $\mathbf{x}$ belonging to $ri(cl(C))$ and so one can find some $\epsilon > 0$ satisfying

$$(\mathbf{x} + \epsilon E) \cap aff(cl(C)) \subseteq cl(C). \tag{22}$$

Moreover, since $ri(C)$ is nonempty construct for some $\mathbf{y} \in ri(C)$ the line $M := \{(1-t)\mathbf{x} + t\mathbf{y} : t \in \mathbb{R}\}$ through the points $\mathbf{x}$ and $\mathbf{y}$. Since $\mathbf{x} \in ri(cl(C))$ and $\mathbf{y} \in ri(C)$ it follows that $M \subseteq aff(cl(C))$ and so by relation (22) there exists some $\mu < 0$ satisfying $\mathbf{y}_1 := (1-\mu)\mathbf{x} + \mu\mathbf{y} \in cl(C)$. This shows

$$\mathbf{x} = \frac{1}{1-\mu}\mathbf{y}_1 - \frac{\mu}{1-\mu}\mathbf{y}, \tag{23}$$

and since $\mathbf{y}_1 \in cl(C)$ and $\mathbf{y} \in ri(C)$ this implies by Lemma 28 and relation (23) that $\mathbf{x} \in ri(C)$. Hence it follows that $ri(cl(C)) \subseteq ri(C)$, and this proves the second formula. $\square$

In the above lemma one might wonder whether the convexity of the set C is necessary. In the following example we present a regular set

S with $ri(S)$ and $cl(S)$ convex and S not convex and this set does not satisfy the result of Lemma 29.

Example 30 *Let $S = [0,1] \cup ((1,2] \cap \mathbb{Q})$. This set is clearly not convex and $ri(S) = (0,1)$ while $cl(S) = [0,2]$. Moreover, $ri(cl(S)) \neq ri(S)$ and $cl(ri(S)) \neq cl(S)$.*

We will now give a primal representation of the relative interior of a convex set S (cf.[44]).

Lemma 31 *If $S \subseteq \mathbb{R}^n$ is a nonempty convex set then it follows that*

$$ri(S) = \{\mathbf{x} \in \mathbb{R}^n : \forall_{\mathbf{y} \in cl(S)}\ \exists_{\mu < 0} \text{ such that } (1-\mu)\mathbf{x} + \mu\mathbf{y} \in S\}.$$

The above result is equivalent to the geometrically obvious fact that for S a convex set and any $\mathbf{x} \in ri(S)$ and $\mathbf{y} \in S$ the line segment $[\mathbf{y}, \mathbf{x}]$ can be extended beyond $\mathbf{x}$ without leaving S. Also, by relation (20) and Lemma 29 another primal representation of $ri(S)$ with S a convex set is given by

$$ri(S) = \{\mathbf{x} \in \mathbb{R}^n : \forall_{\mathbf{y} \in cl(S)}\ \exists_{\mu < 0} \text{ such that } (1-\mu)\mathbf{x} + \mu\mathbf{y} \in ri(S)\}.$$

Since affine mappings preserve convexity it is also of interest to know how the relative interior operator behaves under an affine mapping. Using Lemma 31 one can show the next result (cf.[44]).

Lemma 32 *If $A : \mathbb{R}^n \to \mathbb{R}^m$ is an affine mapping and $C \subseteq \mathbb{R}^n$ is a nonempty convex set then it follows that $A(ri(C)) = ri(A(C))$. Moreover, if $C \subseteq \mathbb{R}^m$ is a nonempty convex set satisfying $A^{-1}(ri(C)) := \{\mathbf{x} \in \mathbb{R}^n : A(\mathbf{x}) \in ri(C)\}$ is nonempty then $ri(A^{-1}(C)) = A^{-1}(ri(C))$.*

As shown by the following counterexample the condition $A^{-1}(ri(C))$ is nonempty cannot be omitted in the previous lemma.

Example 33 *Let $A : \mathbb{R} \to \mathbb{R}$ given by $A(x) = 1$ for all $x \in \mathbb{R}$ and let $C := [0,1] \subset \mathbb{R}$. Then clearly $ri(C) = (0,1)$, $A^{-1}(ri(C)) = \emptyset$ and $ri(A^{-1}(C)) = \mathbb{R}$.*

An immediate consequence of Lemma 32 is given by the observation that

$$ri(\alpha S_1 + \beta S_2) = \alpha ri(S_1) + \beta ri(S_2), \tag{24}$$

for any $\alpha, \beta \in \mathbb{R}$ and $S_i \subseteq \mathbb{R}^n, i = 1,2$ convex sets. To conclude our discussion on topological properties for sets we finally mention the following result (cf.[44]).

Lemma 34 *If the sets C_i, $i \in I$ are convex and $\cap_{i\in I} ri(C_i)$ is nonempty then it follows that $cl(\cap_{i\in I} C_i) = \cap_{i\in I} cl(C_i)$. Moreover, if the set I is finite, we obtain $ri\,(\cap_{i\in I} C_i) = \cap_{i\in I} ri(C_i)$.*

As shown by the next counterexample it is necessary to assume in Lemma 34 that the intersection $\cap_{i\in I}\mathrm{ri}(C_i)$ is nonempty.

Example 35 *Let $C_1 = \{\mathbf{x} \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \cup \{\mathbf{0}\}$ and $C_2 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}$. It is obvious that $ri(C_1) = \{\mathbf{x} : x_1 > 0, x_2 > 0\}$ and $ri(C_2) = C_2$, and so we obtain $ri(C_1) \cap ri(C_2) = \varnothing$ and $ri(C_1 \cap C_2) \neq ri(C_1) \cap ri(C_2)$. For the same example it is also easy to see that $cl(C_1 \cap C_2) \neq cl(C_1) \cap cl(C_2)$.*

In the following counterexample we show that the last result of Lemma 34 does not hold if the set I is not finite.

Example 36 *Let $I = (0, \infty)$ and $C_\alpha = [0, 1+\alpha], \alpha > 0$. For this example it follows $ri(\cap_{\alpha>0} C_\alpha) = ri([0,1]) = (0,1)$, and since $ri(C_\alpha) = (0, 1+\alpha)$ for each $\alpha > 0$, we obtain $\cap_{\alpha>0} ri(C_\alpha) = (0,1]$.*

This last example concludes our discussion of topological properties with respect to sets. In the next subsection we will discuss basic separation results for convex sets.

2.3 Separation of convex sets.

For a nonempty convex set $C \subseteq \mathbb{R}^n$ consider for any $\mathbf{y} \in \mathbb{R}^n$ the so-called minimum norm problem given by

$$v(\mathbf{y}) := \inf\{\frac{1}{2} \parallel \mathbf{y} - \mathbf{x} \parallel^2 : \mathbf{x} \in C\}. \tag{25}$$

If additionally C is closed, a standard application of the Weierstrass theorem (cf.[56]) shows that the infimum $v(\mathbf{y})$ in the above optimization problem is attained. To show that the minimum norm problem has a unique solution, observe for any $\mathbf{y}_1, \mathbf{y}_2$ belonging to $\mathbb{R}^n$ that

$$\frac{1}{2} \parallel \mathbf{y}_1 + \mathbf{y}_2 \parallel^2 + \frac{1}{2} \parallel \mathbf{y}_1 - \mathbf{y}_2 \parallel^2 = \parallel \mathbf{y}_1 \parallel^2 + \parallel \mathbf{y}_2 \parallel^2 . \tag{26}$$

For every $\mathbf{x}_1 \neq \mathbf{x}_2$ belonging to C it follows by relation (26) with $\mathbf{y}_i$ replaced by $\mathbf{y} - \mathbf{x}_i$ for $i = 1, 2$ that

$$\frac{1}{2}\|\mathbf{y} - \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)\|^2 < \frac{1}{4}\|\mathbf{y} - \mathbf{x}_1\|^2 + \frac{1}{4}\|\mathbf{y} - \mathbf{x}_2\|^2,$$

and so $\frac{1}{2}\|\mathbf{y}-\frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)\|^2 < v(\mathbf{y})$ for $\mathbf{x}_i, i = 1,2$ different optimal solutions. Since the set C is convex and hence $\frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)$ belongs to C this yields a contradiction and so the optimal solution is unique. Denote now this optimal solution by $p_C(\mathbf{y})$. The next result can be found in [26].

Lemma 37 *For any $\mathbf{y} \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ a nonempty closed convex set it follows that*

$$\mathbf{z} = p_C(\mathbf{y}) \Leftrightarrow \forall_{\mathbf{x}\in C}\ (\mathbf{z}-\mathbf{y})^\intercal(\mathbf{x}-\mathbf{z}) \geq 0 \textit{ and } \mathbf{z} \in C.$$

Proof. To show the only if implication we observe that $(\mathbf{y}-\mathbf{z})^\intercal(\mathbf{x}-\mathbf{z}) = \|\mathbf{y}-\mathbf{z}\|^2 - (\mathbf{y}-\mathbf{z})^\top(\mathbf{y}-\mathbf{x})$ and this shows by the Cauchy-Schwarz inequality (cf.[4])

$$0 \geq (\mathbf{y}-\mathbf{z})^\intercal(\mathbf{x}-\mathbf{z}) \geq \|\mathbf{y}-\mathbf{z}\|^2 - \|\mathbf{y}-\mathbf{z}\|\|\mathbf{y}-\mathbf{x}\| \tag{27}$$

for every $\mathbf{x} \in C$. If $\mathbf{y} \in C$ we obtain by relation (27) for $\mathbf{x} = \mathbf{y}$ that $0 \geq \|\mathbf{y}-\mathbf{z}\|^2$ and this shows $\mathbf{z} = \mathbf{y} = p_C(\mathbf{y})$. Moreover, if $\mathbf{y} \notin C$ then $\|\mathbf{y}-\mathbf{z}\| > 0$ and this implies by relation (27) that $\|\mathbf{y}-\mathbf{z}\| \leq \|\mathbf{y}-\mathbf{x}\|$ for every $\mathbf{x} \in C$. Hence $\mathbf{z}$ is an optimal solution and by the uniqueness of this solution we obtain $\mathbf{z} = p_C(\mathbf{y})$. To verify the if implication, it follows for $\mathbf{z} = p_C(\mathbf{y})$ that $\mathbf{z} \in C$ and since C is convex this shows

$$\|\mathbf{y}-\mathbf{z}\|^2 \leq \|\mathbf{y}-(\alpha\mathbf{x}+(1-\alpha)\mathbf{z})\|^2 = \|\mathbf{y}-\mathbf{z}-\alpha(\mathbf{x}-\mathbf{z})\|^2 \tag{28}$$

for every $\mathbf{x} \in C$ and $0 < \alpha < 1$. Rewriting relation (28) we obtain for every $0 < \alpha < 1$ that $2(\mathbf{z}-\mathbf{y})^\top(\mathbf{x}-\mathbf{z}) + \alpha\|\mathbf{x}-\mathbf{z}\|^2 \geq 0$ and letting $\alpha \downarrow 0$ this implies the desired inequality. $\square$

Actually the above result is nothing else than the first order necessary and sufficient condition for a minimum of a convex function on a closed convex set. We will now prove one of the most fundamental results in convex analysis. This result has an obvious geometric interpretation and serves as a basic tool in deriving dual representations. Observe in infinite dimensional locally convex topological vector spaces the next result is known as the Hahn Banach theorem (cf.[57]).

Theorem 38 *If $C \subseteq \mathbb{R}^n$ is a nonempty convex set and $\mathbf{y}$ does not belong to the set $cl(C)$, then there exists some nonzero vector $\mathbf{y}_0 \in \mathbb{R}^n$ and $\epsilon > 0$ satisfying $\mathbf{y}_0^\top\mathbf{x} \geq \mathbf{y}_0^\top\mathbf{y} + \epsilon$ for every $\mathbf{x}$ belonging to $cl(C)$. In particular the vector $\mathbf{y}_0$ can be chosen equal to $p_{cl(C)}(\mathbf{y}) - \mathbf{y}$.*

Proof. Since $\mathbf{y} \in cl(C)$ it follows that $\epsilon :=\| \mathrm{p}_{\mathrm{cl}(C)}(\mathbf{y}) - \mathbf{y} \|^2 > 0$. Also by Lemma 37 we obtain for every $\mathbf{x} \in cl(C)$ that

$$(p_{cl(C)}(\mathbf{y}) - \mathbf{y})^\top \mathbf{x} - \|p_{cl(C)}(\mathbf{y}) - \mathbf{y}\|^2 - (p_{cl(C)}(\mathbf{y}) - \mathbf{y})^\top \mathbf{y} \geq 0$$

and this yields $(p_{cl(C)}(\mathbf{y}) - \mathbf{y})^\intercal \mathbf{x} \geq (p_{cl(C)}(\mathbf{y}) - \mathbf{y})^\intercal \mathbf{y} + \epsilon$ for every $\mathbf{x}$ belonging to $\mathrm{cl}(C)$. □

The nonzero vector $\mathbf{y}_0 \in \mathrm{cl}(C) - \mathbf{y}$ is called the *normal vector* of the *separating hyperplane* $H^{=}(\mathbf{a}, a) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} = a\}, \mathbf{a} = \mathbf{y}_0$ and $a = \mathbf{y}_0{}^\intercal \mathbf{y} + \frac{\epsilon}{2}$, and this hyperplane strongly separates the closed convex set $\mathrm{cl}(C)$ and $\mathbf{y}$. Without loss of generality we may take as a normal vector of the hyperplane the vector $\mathbf{y}_0 \parallel \mathbf{y}_0 \|^{-1}$ and this vector has norm 1 and belongs to $\mathrm{cone}(\mathrm{cl}(C) - \mathbf{y})$.

The strong separation result of Theorem 38 can be used to prove the following "weaker" separation result valid under a weaker condition on the point $\mathbf{y}$. In this weaker form we assume that $\mathbf{y}$ does not belong to $\mathrm{ri}(C)$. By Theorem 38 it is clear that we may assume without loss of generality that $\mathbf{y}$ belongs to the *relative boundary* $rbd(C) := cl(C) \backslash ri(C)$ of the convex set $C \subseteq \mathbb{R}^n$.

Theorem 39 *If $C \subseteq \mathbb{R}^n$ is a nonempty convex set and $\mathbf{y}$ does not belong to $ri(C)$, then there exists some nonzero vector $\mathbf{y}_0$ belonging to the unique linear subspace $L_{\mathit{aff}(C)}$ satisfying $\mathbf{y}_0^\top \mathbf{x} \geq \mathbf{y}_0^\top \mathbf{y}$ for every $\mathbf{x} \in C$. Moreover, for the vector $\mathbf{y}_0$ there exists some $\mathbf{x}_0 \in C$ such that $\mathbf{y}_0^\intercal \mathbf{x}_0 > \mathbf{y}_0^\top \mathbf{y}$.*

Proof. Consider for every $n \in \mathbb{N}$ the set $(\mathbf{y} + n^{-1}E) \cap aff(cl(C))$. By Lemma 29 it follows that $\mathbf{y}$ does not belong to $ri(cl(C))$ and so there exists some vector $\mathbf{y}_n$ satisfying

$$\mathbf{y}_n \notin cl(C) \text{ and } \mathbf{y}_n \in (\mathbf{y} + n^{-1}E) \cap aff(cl(C)). \tag{29}$$

The set $cl(C)$ is a closed convex set and by relation (29) and Theorem 38 one can find some vector $\mathbf{y}_n^* \in \mathbb{R}^n$ satisfying

$$\| \mathbf{y}_n^* \| = 1 \text{ , } \mathbf{y}_n^* \in cone(cl(C) - \mathbf{y}_n) \subseteq L_{aff(C)} \text{ and } \mathbf{y}_n^{*\intercal} \mathbf{x} \geq \mathbf{y}_n^{*\intercal} \mathbf{y}_n \tag{30}$$

for every $\mathbf{x} \in cl(C)$. The sequence $\{\mathbf{y}_n^* : n \in \mathbb{N}\}$ belongs to a compact set and so there exists a convergent subsequence $\{\mathbf{y}_n^* : n \in N_0\}$ with

$$\lim_{n \in N_0 \to \infty} \mathbf{y}_n^* = \mathbf{y}_0. \tag{31}$$

This implies by relations (29), (30) and (31) that

$$\mathbf{y}_0^{\intercal}\mathbf{x} = \lim_{n\in N_0\to\infty} \mathbf{y}_n^{*\intercal}\mathbf{x} \geq \lim_{n\in N_0\to\infty} \mathbf{y}_n^{*\top}\mathbf{y}_n = \mathbf{y}_0^{\intercal}\mathbf{y} \tag{32}$$

for every $\mathbf{x} \in cl(C)$ and

$$\mathbf{y}_0 \in L_{aff(C)} \text{ and } \parallel \mathbf{y}_0 \| = 1. \tag{33}$$

Suppose now that there does not exist some $\mathbf{x}_0 \in C$ satisfying $\mathbf{y}_0^{\intercal}\mathbf{x}_0 > \mathbf{y}_0^{\intercal}\mathbf{y}$. By relation (32) this implies that $\mathbf{y}_0^{\intercal}(\mathbf{x}-\mathbf{y}) = 0$ for every $\mathbf{x} \in C$ and since $\mathbf{y}$ belongs to $\mathrm{cl}(C) \subseteq \mathrm{aff}(C)$ we obtain by relation (3) and Lemma 5 that $\mathbf{y}_0^{\intercal}\mathbf{z} = 0$ for every $\mathbf{z}$ belonging to $L_{aff(C)}$. Since by relation (33) the vector $\mathbf{y}_0$ belongs to $L_{aff(C)}$ we obtain $\parallel \mathbf{y}_0 \parallel^2 = 0$ and this contradicts $\parallel \mathbf{y}_0 \parallel = 1$. Hence it must follow that there exists some $\mathbf{x}_0 \in C$ satisfying $\mathbf{y}_0^{\intercal}\mathbf{x}_0 > \mathbf{y}_0^{\top}\mathbf{y}$ and this proves the desired result. □

The separation of Theorem 39 is called a *proper separation* between the set C and $\mathbf{y}$. One can also introduce proper separation between two convex sets.

Definition 40 *The convex sets $C_1, C_2 \subseteq \mathbb{R}^n$ are called properly separated if there exist some $\mathbf{y}_0 \in \mathbb{R}^n$ satisfying*

$$\inf_{x\in C_1} \mathbf{y}_0^{\top}\mathbf{x} \geq \sup_{\mathbf{x}\in C_2} \mathbf{y}_0^{\top}\mathbf{x} \text{ and } \mathbf{y}_0^{\top}\mathbf{x}_1 > \mathbf{y}_0^{\top}\mathbf{x}_2$$

for some $\mathbf{x}_1 \in C_1$ and $\mathbf{x}_2 \in C_2$.

An immediate consequence of Theorem 39 is given by the next result.

Theorem 41 *If the convex sets C_1,$C_2 \subseteq \mathbb{R}^n$ satisfy $ri(C_1) \cap ri(C_2) = \varnothing$ then the two sets can be properly separated.*

Proof. By relation (24) we obtain for $\alpha = 1$ and $\beta = -1$ that $ri(C_1 - C_2) = ri(C_1) - ri(C_2)$, and this shows $ri(C_1) \cap ri(C_2) = \varnothing$ if and only if $\mathbf{0} \notin ri(C_1 - C_2)$. Applying now Theorem 39 with $\mathbf{y} = \mathbf{0}$ and the convex set given by $C_1 - C_2$, the result follows. □

The above separation results are the corner stones of convex and quasiconvex analysis. Observe in infinite dimensional locally convex topological vector spaces one can show similar separation results under stronger assumptions on the convex sets C_1 and C_2 (cf.[57], [43]). An easy consequence of the separation results is given by the observation that closed convex sets and relatively open convex sets are evenly convex. These convex sets play an important role in duality theory for quasiconvex functions.

Lemma 42 *If the nonempty convex set $C \subseteq \mathbb{R}^n$ is closed or relatively open then C is evenly convex.*

Proof. If $C = \mathbb{R}^n$, the result follows by definition and so we may suppose that C is a proper subset of $\mathbb{R}^n$. Hence there exists some $\mathbf{y} \notin C$ and this implies by Theorem 38 that there exists some $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ satisfying $C \subseteq H^<(\mathbf{a}, b)$. This shows that the set $\mathcal{H}_C$ of all open halfspaces H satisfying $C \subseteq H$ is nonempty and by the definition of $\mathcal{H}_C$ it is clear that $C \subseteq \cap\{H : H \in \mathcal{H}_C\}$. Again by Theorem 38 one can show using contradiction that C equals $\cap\{H : H \in \mathcal{H}_C\}$ and this shows that every closed convex set is evenly convex. To verify the second result, we observe $\mathcal{H}_{\mathrm{cl}(C)} \subseteq \mathcal{H}_C$ and by the the first part $\mathcal{H}_{\mathrm{cl}(C)}$ is nonempty. This implies $C \subseteq \cap\{H : H \in \mathcal{H}_C\}$ and to show that C equals $\cap\{H : H \in \mathcal{H}_C\}$ we assume by contradiction that there exists some

$$\mathbf{y} \in \cap\{H : H \in \mathcal{H}_C\} \text{ and } \mathbf{y} \notin C. \tag{34}$$

Applying Theorem 39 one can find some nonzero $\mathbf{y}_0 \in L_{aff(C)}$ satisfying

$$\mathbf{y}_0^\top \mathbf{x} \geq \mathbf{y}_0^\top \mathbf{y} \text{ for every } \mathbf{x} \in C \tag{35}$$

and since C is relatively open there exists for every $\mathbf{x} \in C$ some $\epsilon > 0$ satisfying $\mathbf{x} - \epsilon\mathbf{y}_0 \in C$. This yields by relation (35)applied to $\mathbf{x} - \epsilon\mathbf{y}_0$ that $\mathbf{y}_0^\top \mathbf{x} = \mathbf{y}_0^\top(\mathbf{x} - \epsilon\mathbf{y}_0) + \epsilon\|\mathbf{y}_0\| > \mathbf{y}_0^\top \mathbf{y}$ for every $\mathbf{x} \in C$ and so there is an open halfspace containing C which does not contain $\mathbf{y}$. This contradicts relation (34) and we are done. □

This concludes our discussion of separation results of convex sets. In the next subsection we will use these separation results to derive dual representations for convex sets.

2.4 Dual representations of convex sets

In contrast to the primal representation of a linear subspace, affine set, convex cone and convex set discussed in subsection 2.1 we can also give a so-called *dual representation* of these sets. From a geometrical point of view a primal representation is a representation from "within" the set, while a dual representation turns out to be a representation from "outside" the set. Such a characterization can be seen as an improvement of the hull operation given by relations (1) and (2). We start with linear subspaces or affine sets (cf.[41]).

Definition 43 *If $S \subseteq \mathbb{R}^n$ is some nonempty set then the nonempty set $S^\perp \subseteq \mathbb{R}^n$ given by $S^\perp := \{\mathbf{x}^* \in \mathbb{R}^n : \mathbf{x}^\intercal \mathbf{x}^* = 0$ for every $\mathbf{x} \in S\}$ is called the orthogonal complement of the set S.*

It is easy to verify that the orthogonal complement $S^\perp$ of the set S is a linear subspace. Moreover, a basic result (cf.[41]) in linear algebra is given by the following.

Lemma 44 *For any linear subspace L it follows that $(L^\perp)^\perp = L$.*

By the above lemma a so-called dual representation of any linear hull $\text{lin}(S)$ with S nonempty can be given using the following procedure. Since $S \subseteq \text{lin}(S)$ we obtain by Lemma 44 that $(S^\perp)^\perp \subseteq (lin(S)^\perp)^\perp = lin(S)$. The set $(S^\perp)^\perp$ is clearly a linear subspace containing S and since $lin(S)$ denotes the smallest linear subspace containing S the dual representation $lin(S) = (S^\perp)^\perp$ holds. For affine hulls it follows by Lemma 5 and the dual representation of a linear hull that $aff(S) = \mathbf{x}_0 + ((S - \mathbf{x}_0)^\perp)^\perp$ for every $\mathbf{x}_0$ belonging to $aff(S)$. Since it is easy to verify that $\lambda^\top(\mathbf{x}_1 - \mathbf{x}_0) = 0$ for every λ belonging to $(S - \mathbf{x}_0)^\perp$ and $\mathbf{x}_1 \in aff(S)$, we obtain that $(S - \mathbf{x}_0)^\perp \subseteq (S - \mathbf{x}_1)^\perp$ for every $\mathbf{x}_0, \mathbf{x}_1 \in aff(S)$. By a similar argument the reverse inclusion also holds and so it follows that $(S - \mathbf{x}_0)^\perp = (S - \mathbf{x}_1)^\perp$ for every $\mathbf{x}_0, \mathbf{x}_1$ belonging to $aff(S)$. Therefore a dual representation of the affine hull of a set S is given by $aff(S) = \mathbf{x}_0 + ((S - \mathbf{x}_1)^\perp)^\perp$ for every $\mathbf{x}_0, \mathbf{x}_1 \in aff(S)$.

Next we discuss the dual representation for a closed convex set and a closed convex cone. This dual representation will be proved by means of the strong separation result given by Theorem 38. Recall first the definition of a support function.

Definition 45 *If $S \subseteq \mathbb{R}^n$ is some nonempty set then the function $\sigma_S : \mathbb{R}^n \to (-\infty, \infty]$ given by $\sigma_S(\mathbf{s}) := \sup\{\mathbf{s}^\intercal \mathbf{x} : \mathbf{x} \in S\}$ is called the support function of the set S.*

Another formulation of Theorem 38 involving the support function of the closed convex set C is given by the following result. Observe this result can be seen as a dual representation of a closed convex set.

Theorem 46 *If $C \subseteq \mathbb{R}^n$ is a proper nonempty convex set then it follows that $\mathbf{x}_0 \in cl(C)$ if and only if $\mathbf{s}^\top \mathbf{x}_0 \leq \sigma_{cl(C)}(\mathbf{s})$ for every $\mathbf{s} \in \mathbb{R}^n$.*

Proof. Clearly $\mathbf{x}_0 \in cl(C)$ implies that $\mathbf{s}^\intercal \mathbf{x} \leq \sigma_{cl(S)}(\mathbf{s})$ for every $\mathbf{s}$ belonging to $\mathbb{R}^n$. To show the reverse implication let $\mathbf{s}^\intercal \mathbf{x}_0 \leq \sigma_{cl(C)}(\mathbf{s})$

for every $\mathbf{s} \in \mathbb{R}^n$ and suppose by contradiction that $\mathbf{x}_0 \notin cl(C)$. By Theorem 38 there exists some nonzero vector $\mathbf{y}_0 \in \mathbb{R}^n$ and $\epsilon > 0$ satisfying $-\mathbf{y}_0^\top \mathbf{x} \leq -\mathbf{y}_0^\top \mathbf{x}_0 - \epsilon$ for every $\mathbf{x}$ belonging to $cl(C)$. This implies $\sigma_{cl(C)}(-\mathbf{y}_0) \leq -\mathbf{y}_0^\top \mathbf{x}_0 - \epsilon < -\mathbf{y}_0^\top \mathbf{x}_0$, contradicting our initial assumption and so it must follow that $\mathbf{x}_0$ belongs to $cl(C)$. $\square$

To generalize the orthogonality relation for linear subspaces we introduce the so-called polarity relation for convex cones.

Definition 47 *If $K \subseteq \mathbb{R}^n$ is a nonempty convex cone then the set K^0 given by $K^0 := \{\mathbf{x}^* \in \mathbb{R}^n : \mathbf{x}^\intercal \mathbf{x}^* \leq 0$ for every $\mathbf{x} \in K\}$ is called the polar cone of K.*

In case L is a linear subspace it is easy to verify that $L^0 = L^\perp$ and so the *polar operator* applied to a linear subspace reduces to the *orthogonal operator.* Moreover, it is also easy to verify that the nonempty set K^0 is a closed convex cone. Applying now the *polar operator* twice the bipolar cone K^{00} is given by

$$K^{00} := (K^0)^0 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\intercal \mathbf{x}^* \leq 0 \text{ for every } \mathbf{x}^* \in K^0\}.$$

Using Theorem 46 an important dual representation for closed convex cones can be verified. This result is known as the *bipolar theorem* and is a generalization of Lemma 44.

Theorem 48 *If $K \subseteq \mathbb{R}^n$ is a nonempty convex cone then it follows that $K^{00} = cl(K)$.*

Proof. Since the set K is a convex cone we obtain that $cl(K)$ is a closed convex cone and this implies by the definition of a support function that $\sigma_{cl(K)}(\mathbf{s}) = 0$ for every $\mathbf{s} \in K^0$ and ∞ otherwise. Applying now Theorem 46 it follows that $\mathbf{x} \in cl(K) \Leftrightarrow \mathbf{s}^\intercal \mathbf{x} \leq 0$ for every $\mathbf{s} \in K^0 \Leftrightarrow \mathbf{x} \in K^{00}$, and this shows the desired result. $\square$

By means of similar proof techniques (cf.[28]) it is also possible to give a dual representation of the relative interior $\text{ri}(K)$ of a convex cone K. Without proof we now list the following result.

Theorem 49 *For any nonempty convex cone $K \subseteq \mathbb{R}^m$ it follows that*

$$\mathbf{x} \in ri(K) \Leftrightarrow \mathbf{x} \in (K^\perp)^\perp \text{ and } \mathbf{x}^{*\top} \mathbf{x} < 0 \text{ for } \mathbf{x}^* \in K^0 \cap (K^\perp)^\perp \backslash \{0\}.$$

This concludes our section on sets. In the next section we will consider functions studied within convex and quasiconvex analysis.

3 Functions studied within convex and quasiconvex analysis.

In this section we first introduce in subsection 3.1 the different classes of functions studied within convex and quasiconvex analysis and derive their algebraic properties. These algebraic properties are an easy consequence of two important relations between functions and sets and the properties of sets derived in subsection 2.1. Also from subsection 2.1 we know how to apply hull operations to sets and using this it is also possible to construct so-called hull functions. These different hull functions are also introduced in subsection 3.1 and their properties will be derived. In subsection 3.2 topological properties of functions are introduced together with some of the "topological" hull functions. It will turn out that especially the class of lower semicontinuous functions is extremely important in this field. Finally in subsections 3.3 and 3.4 dual characterizations of the considered functions will be derived. The key results in these sections are the Fenchel-Moreau theorem within convex analysis and its generalization to so-called evenly quasiconvex and lower semicontinuous quasiconvex functions.

3.1 Algebraic properties of functions.

In this subsection we relate functions to sets and use the algebraic properties of sets given in subsection 2.1 to derive algebraic properties of functions. To start with this approach, let $f : \mathbb{R}^n \to [-\infty, \infty]$ be an extended real valued function and associate with f its so-called *epigraph* $epi(f) := \{(\mathbf{x}, r) \in \mathbb{R}^{n+1} : f(\mathbf{x}) \leq r\}$. A related set is the strict *epigraph* $epi_S(f) := \{(\mathbf{x}, r) \in \mathbb{R}^{n+1} : f(\mathbf{x}) < r\}$. Within convex analysis it is now useful to represent a function f by the obvious relation (cf.[44])

$$f(\mathbf{x}) = \inf\{r : (\mathbf{x}, r) \in epi(f)\}. \tag{36}$$

By definition $\inf\{\varnothing\} = \infty$ and this only happens if the vector $\mathbf{x}$ does not belong to the so-called *effective domain* $dom(f) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < \infty\}$ of the function f. By this observation it follows that $dom(f)$ nonempty if and only if $epi(f)$ is nonempty and if this holds we obtain

$$dom(f) = A(epi(f)) \tag{37}$$

with A the projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ given by $A(\mathbf{x}, r) = \mathbf{x}$. As shown by the following definition, the representation of the function f given by relation (36) is useful in the study of convex functions.

Definition 50 *The function* $f : \mathbb{R}^n \to [-\infty, \infty]$ *is called convex if the set* $epi(f)$ *is convex. Moreover, the function* $f : \mathbb{R}^n \to [-\infty, \infty]$ *is called positively homogeneous if the set* $epi(f)$ *is a cone.*

An equivalent definition of a convex function is given by the next result, which is easy to verify.

Lemma 51 *A function* $f : \mathbb{R}^n \to [-\infty, \infty]$ *is convex if and only if the set* $epi_S(f)$ *is convex.*

Using Lemma 51 we obtain that a function $f : \mathbb{R}^n \to [-\infty, \infty]$ is convex if and only if

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < \alpha r_1 + (1-\alpha) r_2 \tag{38}$$

whenever $f(\mathbf{x}_i) < r_i \in \mathbb{R}$. In case we know additionally that $f > -\infty$ we obtain by relation (36) that f is convex if and only if

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \tag{39}$$

and so we recover the more familiar definition of a convex function. An important special case satisfying relation (39) is given by $f > -\infty$ and $dom(f)$ nonempty. If this holds the function f is called proper. Also the next result is easy to verify.

Lemma 52 *The function* $f : \mathbb{R}^n \to [-\infty, \infty]$ *is positively homogeneous if and only if* $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$ *for every* $\mathbf{x} \in \mathbb{R}^n$ *and* $\alpha > 0$.

To investigate under which operations on convex functions this property is preserved we observe for any collection of functions f_i, $i \in I$ that

$$epi(\sup{}_{i\in I} f_i) = \cap_{i\in I} epi(f_i). \tag{40}$$

Since the intersection of convex sets is again convex we obtain by relation (40) that the function $\sup_{i\in I} f_i$ is convex if f_i is convex for every $i \in I$. Moreover, by relation (39) it follows that any strict canonical combination of the convex functions $f_i > -\infty$, $i = 1, 2$ is again convex.

In case we use the representation of a function f given by relation (36) and the different hull operations on a set defined in subsection 2.1 it is easy to introduce the different so-called hull functions of f. The first hull function is given by the next definition (cf.[44]).

Definition 53 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the function $co(f) : \mathbb{R}^n \to [-\infty, \infty]$ given by $co(f)(\mathbf{x}) := \inf\{r : (\mathbf{x}, r) \in co(epi(f))\}$ is called the convex hull function of the function f.*

The next result yields an interpretation of the convex hull function of a function f. Observe the convex hull of the empty set is again the empty set.

Lemma 54 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the convex hull function $co(f)$ is the greatest convex function majorized by f. Moreover, it follows that $epi_S(co(f)) \subseteq co(epi(f)) \subseteq epi(co(f))$ and $dom(co(f)) = co(dom(f))$.*

Proof. Without loss of generality we may assume that $epi(f)$ or equivalently $dom(f)$ is nonempty. For every $r_i > co(f)(\mathbf{x}_i), i = 1, 2$ it follows by Definition 53 that $co(f)(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < \alpha r_1 + (1-\alpha)r_2$ for every $0 < \alpha < 1$ and so by relation (38) the hull function $co(f)$ is convex. Moreover, for any convex function $h \leq f$ we obtain $epi(f) \subseteq epi(h)$ and this shows, since $epi(h)$ is a convex set, that $co(epi(f)) \subseteq epi(h)$. Hence by Definition 53 it follows that $h \leq co(f)$ and so $co(f)$ is the greatest convex function majorized by f. Also by Definition 53 it is easy to verify that $epi_S(co(f)) \subseteq co(epi(f)) \subseteq epi(co(f))$. To show the last result, we observe for $\mathbf{x} \in co(dom(f))$ that by relation (14) there exists some points $\mathbf{x}_i \in dom(f), 1 \leq i \leq m$ satisfying $\mathbf{x} = \sum_{i=1}^m \alpha_i \mathbf{x}_i$ with $\alpha_i > 0$ and $\sum_{i=1}^m \alpha_i = 1$. Hence for $r_i > f(\mathbf{x}_i)$ the vectors $(\mathbf{x}_i, r_i), 1 \leq i \leq m$ belong to $epi(f)$ and so the vector $(\sum_{i=1}^m \alpha_i \mathbf{x}_i, \sum_{i=1}^m \alpha_i r_i)$ belongs to $co(epi(f))$. This implies using $r_i < \infty$, $1 \leq i \leq m$ that $co(f)(\mathbf{x}) \leq \sum_{i=1}^m \alpha_i r_i < \infty$ and so $\mathbf{x} \in dom(co(f))$. To verify the reverse inclusion it follows for $\mathbf{x} \in dom(co(f))$ that by Definition 53 the vector $(\mathbf{x}, r) \in co(epi(f))$ for every $r > co(f)(\mathbf{x})$. Hence we obtain by relation (37) that $\mathbf{x}$ belongs to $A(co(epi(f)) = co(A(epi(f)) = co(dom(f))$ with A denoting the projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ and this shows the desired result. $\square$

A direct consequence of Lemma 54 and $\sup_{i \in I} f_i$ is convex for f_i, $i \in I$ a collection of convex functions yields an often used representation of $co(f)$ given by

$$co(f) = \sup\{h : h \leq f \text{ and } h \text{ is a convex function}\}. \tag{41}$$

Next to the epigraph of a function $f : \mathbb{R}^n \to [-\infty, \infty]$ one also considers the so-called *lower-level set* $L(f, r), r \in \mathbb{R}$ given by $L(f, r) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq r\}$. A related set is the *strict lower-level set* $L_S(f, r) :=$

$\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < r\}$. Within quasiconvex analysis it is now useful to represent a function f by the obvious relation (cf.[18])

$$f(\mathbf{x}) = \inf\{r : \mathbf{x} \in L(f, r)\}. \tag{42}$$

As shown by the following definition, the representation of the function f given by relation (42) is useful in the study of quasiconvex functions.

Definition 55 *The function $f : \mathbb{R}^n \to [-\infty, \infty]$ is called quasiconvex if for every $r \in \mathbb{R}$ the lower-level set $L(f, r)$ is convex. Moreover, the function f is called evenly quasiconvex if for every $r \in \mathbb{R}$ the lower level set $L(f, r)$ is evenly convex.*

To derive the relation between convex and quasiconvex functions we observe that $epi(f) \cap (\mathbb{R}^n \times \{r\}) = L(f, r) \times \{r\}$ for every $r \in \mathbb{R}$. This implies that a convex function is also a quasiconvex function. Since each monotonic (increasing or decreasing) function $f : \mathbb{R} \to \mathbb{R}$ is quasiconvex, but not necessarily convex, the converse is not true. For quasiconvex functions a similar result as in Lemma 51 can be easily verified.

Lemma 56 *A function $f : \mathbb{R}^n \to [-\infty, \infty]$ is quasiconvex if and only if the set $L_S(f, r)$ is convex for every $r \in \mathbb{R}$.*

To recover a more familiar representation of a quasiconvex function it can be shown easily (cf.[34]) that a function $f : \mathbb{R}^n \to [-\infty, \infty]$ is quasiconvex if and only if $f(\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2) \leq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$.

As for convex functions, one is interested under which operations on quasiconvex functions this property is preserved. Clearly for any collection of functions $f_i, i \in I$ it follows that

$$L(\sup{}_{i \in I} f_i, r) = \cap_{i \in I} L(f_i, r) \tag{43}$$

and this shows that the function $\sup_{i \in I} f_i$ is quasiconvex if f_i is quasiconvex for every $i \in I$. Opposed to convex functions, it is not true that a strict canonical combination of quasiconvex functions is quasiconvex and this is shown by the following example.

Example 57 *Let $f_i : \mathbb{R} \to \mathbb{R}, i = 1, 2$ be given by $f_1(x) = x$ and*

$$f_2(x) = x^2 \text{ for } |x| \leq 1 \text{ and } f_2(x) = 1 \text{ otherwise.}$$

These functions are quasiconvex, but it is easy to verify by means of a picture that the sum of both functions is not quasiconvex.

Using relation (42) one can apply the different hull operations to the lower level set. The first hull function constructed in this way is listed in the next definition (cf.[18]).

Definition 58 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the function $qc(f) : \mathbb{R}^n \to [-\infty, \infty]$ given by $qc(f)(\mathbf{x}) := \inf\{r : \mathbf{x} \in co(L(f, r))\}$ is called the quasiconvex hull function of the function f.*

The next result ([18]) yields an interpretation of the quasiconvex hull function of a function f.

Lemma 59 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the quasiconvex hull function $qc(f)$ is the greatest quasiconvex function majorized by f. Moreover, it follows that $L(qc(f), r) = \cap_{\beta>r} co(L(f, \beta))$ for every $r \in \mathbb{R}$.*

Proof. Again we may assume without loss of generality that $dom(f)$ is nonempty. By Definition 58 it follows that $L(qc(f), r) \subseteq \cap_{\beta>r} co(L(f, \beta))$. Since it is obvious that the reverse inclusion holds we obtain $L(qc(f), r) = \cap_{\beta>r} co(L(f, \beta))$. By this relation it is clear that the function $qc(f)$ is quasiconvex and applying a similar argument as in Lemma 54 to lower level sets it can be shown that this function is the greatest quasiconvex function majorized by f. $\square$

A direct consequence of Lemma 59 and $\sup_{i \in I} f_i$ is quasiconvex for f_i, $i \in I$ a collection of quasiconvex functions yields an often used representation of $qc(f)$ given by

$$qc(f) = \sup\{h : h \leq f \text{ and } h \text{ is a quasiconvex function}\}. \tag{44}$$

To conclude this subsection, we consider a hull function based on evenly convex sets (cf.[51]). It will turn out that this function plays an important role in duality theory for quasiconvex functions.

Definition 60 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the function $eqc(f) : \mathbb{R}^n \to [-\infty, \infty]$ given by $eqc(f)(\mathbf{x}) := \inf\{r : \mathbf{x} \in eco(L(f, r))\}$ is called the evenly quasiconvex hull function of the function f.*

As done for the quasiconvex hull function one can show by a similar proof the following result (cf.[51]).

Lemma 61 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the evenly quasiconvex hull function $eqc(f)$ is the greatest evenly quasiconvex function majorized by f. Moreover, it follows that $L(eqc(f), r) = \cap_{\beta>r} eco(L(f, \beta))$ for every $r \in \mathbb{R}$.*

A direct consequence of Lemma 61 and $\sup_{i \in I} f_i$ is evenly quasiconvex for f_i, $i \in I$ a collection of evenly quasiconvex functions yields an often used representation of $eqc(f)$ given by

$$eqc(f) = \sup\{h : h \leq f,\, h \text{ is an evenly quasiconvex function}\}. \quad (45)$$

Since an evenly quasiconvex function is clearly a quasiconvex function it holds that $eqc(f) \leq qc(f)$. This concludes our discussion of algebraic properties of convex and quasiconvex functions. In the next subsection we will consider topological properties of functions.

3.2 Topological properties of functions.

In this subsection we first introduce the class of lower semicontinuous functions. These functions play an important role within the theory of convex functions.

Definition 62 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is some function then this function is called lower semicontinuous at $\mathbf{x} \in \mathbb{R}^n$ if $\liminf_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$ with*

$$\liminf_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y}) := \sup_{\epsilon > 0} \inf\{f(\mathbf{y}) : \mathbf{y} \in \mathbf{x} + \epsilon E\}. \quad (46)$$

Moreover, the function $f : \mathbb{R}^n \to [-\infty, \infty]$ is called upper semicontinuous at $\mathbf{x} \in \mathbb{R}^n$ if the function $-f$ is lower semicontinuous at $\mathbf{x}$ and it is called continuous at $\mathbf{x}$ if it is both lower and upper semicontinuous at $\mathbf{x}$. The function $f : \mathbb{R}^n \to [-\infty, \infty]$ is called lower semicontinuous (upper semicontinuous) if f is lower semicontinuous (upper semicontinuous) at every $\mathbf{x} \in \mathbb{R}^n$ and it is called continuous if it is both upper semicontinuous and lower semicontinuous.

We mostly abbreviate lower semicontinuous by l.s.c. To relate the above definition of liminf to the liminf of a sequence we observe for every sequence $\mathbf{y}_k, k \in \mathbb{N}$ that $\liminf_{k \uparrow \infty} f(\mathbf{y}_k) := \lim_{n \uparrow \infty} \inf_{k \geq n} f(\mathbf{y}_k)$. Using this definition one can easily show the following result.

Lemma 63 *The function $f : \mathbb{R}^n \to [-\infty, \infty]$ is l.s.c at $\mathbf{x} \in \mathbb{R}^n$ if and only if $\liminf_{k \uparrow \infty} f(\mathbf{y}_k) \geq f(\mathbf{x})$ for every sequence $\mathbf{y}_k, k \in \mathbb{N}$ satisfying* $\lim_{k \uparrow \infty} \mathbf{y}_k = \mathbf{x} \in \mathbb{R}^n$.

Using Lemma 63 the following important characterization of l.s.c functions can be proved (cf.[44], [20]).

Theorem 64 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an extended real valued function then it follows that f is l.s.c.$\Leftrightarrow$ $epi(f)$ closed $\Leftrightarrow L(f, r)$ closed for every $r \in \mathbb{R}$.*

It is useful to know under which operations on l.s.c functions this property is preserved. Since $epi(\sup_{i\in I} f_i) = \cap_{i\in I} epi(f_i)$ and the intersection of closed sets is again a closed set we obtain by Theorem 64 that the function $\sup_{i\in I} f_i$ is l.s.c if each function $f_i, i \in I$ is l.s.c. Also it can be checked easily for every finite set I that $epi(\min_{i\in I} f_i) = \cup_{i\in I} epi(f_i)$ and this shows by Theorem 64 and the finite union of closed sets is closed, that the function $\min_{i\in I} f_i$ is l.s.c if each f_i, $i \in I$ is l.s.c. Finally, if for a given $\alpha, \beta \in \mathbb{R}$ the value $\alpha \liminf_{k\uparrow\infty} f_1(\mathbf{y}_k) + \beta \liminf_{k\uparrow\infty} f_2(\mathbf{y}_k)$ is well defined then it follows that

$$\liminf_{k\uparrow\infty} (\alpha f_1 + \beta f_2)(\mathbf{y}_k) \geq \alpha \liminf_{k\uparrow\infty} f_1(\mathbf{y}_k) + \beta \liminf_{k\uparrow\infty} f_2(\mathbf{y}_k).$$

By this inequality and Lemma 63 it is clear that every strict canonical combination of the l.s.c functions $f_i > -\infty$, $i = 1, 2$ is again lower semicontinuous.

In the next result we prove that l.s.c functions satisfying some additional property can be seen as a pointwise limit of an increasing sequence of real valued continuous functions (cf.[25]). Before showing this result we introduce for any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the (possibly empty) set of continuous minorants $\mathcal{C}_f$ of f given by $\mathcal{C}_f := \{h : h \leq f$ and h is a continuous real valued function$\}$. Also in the next proof we need for any set $A \subseteq \mathbb{R}^n$ the function $\chi_A : \mathbb{R}^n \to \{0, 1\}$ defined by $\chi_A(\mathbf{x}) := 1$ for $\mathbf{x} \in A$ and zero for $\mathbf{x} \notin A$.

Theorem 65 *For any function $f : \mathbb{R}^n \to (-\infty, \infty]$ the following conditions are equivalent:*

1. The function f is l.s.c and the set $\mathcal{C}_f$ is nonempty.

2. There exists an increasing sequence of continuous function $(h_m)_{m\in\mathbb{N}}$ satisfying $f(x) = \lim_{m\uparrow\infty} h_m(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$.

3. It follows $f = \sup\{h : h \leq f$ and h is a continuous function$\}$.

Proof. We only give a proof of $1 \Rightarrow 2$ since the other implications are obvious. Without loss of generality we may assume, since the set $\mathcal{C}_f$ is nonempty and hence $f - h \geq 0$ for $h \in \mathcal{C}_f$ and $f - h$ l.s.c for f l.s.c, that the function $f \geq 0$ and $\mathrm{dom}(f)$ is nonempty. Introduce now the sets $A(km^{-1}) := \{\mathbf{x} \in \mathbb{R}^n : f(x) > km^{-1}\} = L^c(f, km^{-1})$, and consider the sequence of functions $g_m : \mathbb{R}^n \to [0, \infty]$, $m \in \mathbb{N}$ given by

$$g_m(\mathbf{x}) := \frac{1}{m} \sum\nolimits_{k=1}^{\infty} \chi_{A(km^{-1})}(\mathbf{x}).$$

It is easy to verify for every $\mathbf{x}$ belonging to $\mathrm{dom}(f)$ and $m \in \mathbb{N}$ that $f(\mathbf{x}) - m^{-1} < g_m(\mathbf{x}) < f(\mathbf{x})$ and since $g_m(\mathbf{x}) = \infty$ for every $\mathbf{x}$ not belonging to $\mathrm{dom}(f)$ we obtain for every $\mathbf{x} \in \mathbb{R}^m$ that

$$\sup\nolimits_{m \in \mathbb{N}} g_m(\mathbf{x}) = f(\mathbf{x}). \tag{47}$$

Since the function f is l.s.c it follows by Theorem 64 that the set $L(f, km^{-1})$ is closed and this implies for any nonempty set $L(f, km^{-1})$ that the distance function $d_{L(f,\ km^{-1})} : \mathbb{R}^n \to \mathbb{R}$ given by

$$d_{L(f,\ km^{-1})}(\mathbf{x}) := \min\{\|\mathbf{x} - \mathbf{z}\| : \mathbf{z} \in L(f, km^{-1})\}$$

is continuous. If the set $L(f, km^{-1})$ is empty we set $d_{L(f,\ km^{-1})} \equiv \infty$. For each $k \in \mathbb{N}$ and $m \in \mathbb{N}$ the collection of functions $\psi_{k,m,q} : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathbb{N}$ given by $\psi_{k,m,q}(\mathbf{x}) := \min\{q d_{L(f,\ km^{-1})}(\mathbf{x}), 1\}$ is therefore continuous. Moreover, since clearly

$$\chi_{A(km^{-1})}(\mathbf{x}) \geq \psi_{k,m,q+1}(\mathbf{x}) \geq \psi_{k,m,q}(\mathbf{x}) \geq 0 \tag{48}$$

for every $\mathbf{x} \in \mathbb{R}^n$ and $d_{L(f,\ km^{-1})}(\mathbf{x}) > 0$ if and only if $\mathbf{x} \in A(km^{-1})$ we obtain

$$\sup\nolimits_{q \in \mathbb{N}} \psi_{k,m,q}(\mathbf{x}) = \lim\nolimits_{q \uparrow \infty} \psi_{k,m,q}(\mathbf{x}) = \chi_{A(km^{-1})}(\mathbf{x}) \tag{49}$$

for every $\mathbf{x} \in \mathbb{R}^n$. Introducing now for every $m, q \in \mathbb{N}$ the continuous function $\phi_{m,q} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\phi_{m,q}(\mathbf{x}) := \frac{1}{m} \sum\nolimits_{k=1}^{q^2} \psi_{k,m,q}(\mathbf{x}) \tag{50}$$

we obtain by relations (47) and (48) that

$$\phi_{m,q}(\mathbf{x}) \leq \phi_{m,q+1}(\mathbf{x}) \leq g_m(\mathbf{x}) \leq f(\mathbf{x}) \tag{51}$$

for every $\mathbf{x} \in \mathbb{R}^n$. Moreover, by relation (49) and (51) it follows for every $\mathbf{x} \in \mathbb{R}^n$ that

$$\sup\nolimits_{q \in \mathbb{N}} \phi_{m,q}(\mathbf{x}) = \lim\nolimits_{q \uparrow \infty} \phi_{m,q}(\mathbf{x}) = g_m(\mathbf{x}). \tag{52}$$

Applying now relation (47) and (52) this implies $\sup_{m,q\in\mathbb{N}\times\mathbb{N}} \phi_{m,q}(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$. Taking now the collection of continuous functions $\{h_m\}_{m\in\mathbb{N}}$ given by $h_m(\mathbf{x}) = \max\{\phi_{i,j}(\mathbf{x}) : i + j \leq m\}$ yields the desired result. $\square$

Since it is well known (cf.[20]) that any l.s.c. function $f : \mathbb{R} \to (-\infty, \infty]$ attains its minimum on a compact subset $A \subseteq \mathbb{R}^n$ it follows that $\mathcal{C}_f$ is nonempty and so by Theorem 65 we obtain that the set of l.s.c. functions $f : A \to (-\infty, \infty]$ on a compact set A is the smallest set of functions closed under taking the sup operation to a collection of functions belonging to this set and containing the set of continuous real valued functions on A.

As in the previous subsection, we are going to introduce hull operations related to functions. In this case topological properties will be involved. First we consider the so called l.s.c hull function of a function f (cf.[44]).

Definition 66 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the function $\overline{f} : \mathbb{R}^n \to [-\infty, \infty]$ given by $\overline{f}(\mathbf{x}) := \inf\{r : (\mathbf{x}, r) \in cl(epi(f))\}$ is called the l.s.c hull function of the function f.*

In the next result an interpretation of the l.s.c hull function of a function f is given.

Lemma 67 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the l.s.c hull function $\overline{f}$ is the greatest l.s.c function majorized by f. Moreover, its epigraph equals $cl(epi(f))$ and $dom(f) \subseteq dom(\overline{f}) \subseteq cl(dom(f))$. If additionally $dom(f)$ is a convex set it follows that $ri(dom(\overline{f})) = ri(dom(f)$.*

Proof. By Definition 66 we obtain $(\mathbf{x}, r) \in epi(\overline{f}) \Leftrightarrow \forall_{\epsilon>0}\ (\mathbf{x}, r + \epsilon) \in cl(epi(f)) \Leftrightarrow (\mathbf{x}, r) \in cl(epi(f))$. This means that the epigraph of $\overline{f}$ equals $cl(epi(f))$ and so by Theorem 64 the function $\overline{f}$ is l.s.c. Moreover, if $h \leq f$ is l.s.c then $epi(f) \subseteq epi(h)$. This implies by Theorem 64 that $cl(epi(f)) \subseteq epi(h)$ and again by Definition 66 we obtain $h \leq \overline{f}$. To verify the last part we may assume without loss of generality that $dom(f)$ is nonempty. Since $\overline{f} \leq f$ it follows that $dom(f) \subseteq dom(\overline{f})$ and by relation (37) we obtain $dom(\overline{f}) = A(cl(epi(f)) \subseteq cl(A(epi(f)) = cl(dom(f))$. Finally, if $dom(f)$ is a nonempty convex set it follows by Lemma 29 that $ri(dom(f)) = ri(cl(dom(f))$ and since $dom(f) \subseteq dom(\overline{f}) \subseteq cl(dom(f))$ we obtain $ri(dom(\overline{f})) = ri(dom(f)$. $\square$

A direct consequence of Lemma 67 and $\sup_{i\in I} f_i$ is l.s.c for f_i, $i \in I$ a collection of l.s.c functions yields an often used representation of $\overline{f}$ given by

$$\overline{f} = \sup\{h : h \leq f \text{ and } h \text{ is a l.s.c function }\}. \tag{53}$$

For nondecreasing functions $f : \mathbb{R} \to [-\infty, \infty]$ it is possible using relation (53) to give a more detailed description of $\overline{f}$. This result is needed in the proof of a dual representation of a l.s.c quasiconvex funnction.

Lemma 68 *For any nondecreasing function $f : \mathbb{R} \to [-\infty, \infty]$ it follows that $\overline{f}(t) = \sup_{s<t} f(s)$.*

Proof. Introducing the function $f^{\diamondsuit}(t) := \sup_{s<t} f(s)$ it is easy to verify using f is nondecreasing that $f^{\diamondsuit}$ is nondecreasing and $f^{\diamondsuit} \leq f$. We first show that the function $f^{\diamondsuit}$ is l.s.c and so by Theorem 64 we have to check that the lower level set $L(f^{\diamondsuit}, r)$ is closed for every $r \in \mathbb{R}$. Assume now by contradiction that there exists some $r_0 \in \mathbb{R}$ such that the set $L(f^{\diamondsuit}, r_0)$ is not closed. Hence there exists a sequence $\{t_n : n \in \mathbb{N}\} \subseteq L(f^{\diamondsuit}, r_0)$ with limit $t_\infty \notin L(f^{\diamondsuit}, r_0)$ and this implies by the definition of $f^{\diamondsuit}$ that one can find some $s_0 < t_\infty$ satisfying $f(s_0) > r_0$. Also by the monotonicity of $f^{\diamondsuit}$ and $f^{\diamondsuit}(t_{\infty)} > r_0$ it follows that $t_n < t_\infty$ for every $n \in \mathbb{N}$ and since $t_n \to t_\infty$ there exists some $s_0 < t_n < t_\infty$. This yields $f^{\diamondsuit}(t_n) \geq f(s_0) > r_0$ and we obtain a contradiction. Therefore $f^{\diamondsuit}$ is l.s.c and since $f^{\diamondsuit} \leq f$ it follows by relation (53) that $f^{\diamondsuit} \leq \overline{f}$. Suppose now by contradiction that $f^{\diamondsuit}(t_0) < \overline{f}(t_0)$ for some t_0. This implies using $\overline{f}$ is l.s.c that by relation (46) one can find some $\epsilon > 0$ satisfying $\overline{f}(t) > f^{\diamondsuit}(t_0)$ for every $t_0 - \epsilon \leq t \leq t_0 + \epsilon$. Hence it follows that $f(t_0 - \epsilon) \geq \overline{f}(t_0 - \epsilon) > f^{\diamondsuit}(t_0) \geq f(t_0 - \epsilon)$ and this yields a contradiction. $\square$

The next result relates $\overline{f}$ to f and this result is nothing else than a "function value translation" of the original definition of $\overline{f}$.

Lemma 69 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ and $\mathbf{x} \in \mathbb{R}^n$ it follows that $\overline{f}(\mathbf{x}) = \liminf_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y})$.*

Proof. Since $\overline{f} \leq f$ and $\overline{f}$ is a l.s.c function we obtain that $\overline{f}(\mathbf{x}) = \liminf_{\mathbf{y} \to \mathbf{x}} \overline{f}(\mathbf{y}) \leq \liminf_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y})$. Suppose now by contradiction that $\overline{f}(\mathbf{x}) < \liminf_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y})$. If this holds then clearly $\overline{f}(\mathbf{x}) < \infty$ and by the definition of liminf there exists some finite γ and $\epsilon > 0$ satisfying $f(\mathbf{x}+\mathbf{y}) > \gamma > \overline{f}(\mathbf{x})$ for every $\mathbf{y} \in \epsilon E$. This implies that the open set $(\mathbf{x}+\epsilon E) \times (-\infty, \gamma)$ containing the point $(\mathbf{x}, \overline{f}(\mathbf{x}))$ has an empty intersection with $\text{epi}(f)$. However, by Lemma 67 it follows that $(\mathbf{x}, \overline{f}(\mathbf{x}))$ belongs to $cl(epi(f))$ and so every open set containing $(\mathbf{x}, \overline{f}(\mathbf{x}))$ must have a nonempty intersection with $epi(f)$. Hence we obtain a contradiction and so the result is proved. $\square$

By Lemma 69 and Definition 62 it follows immediately that f is l.s.c at $\mathbf{x}$ if and only if $\overline{f}(\mathbf{x}) = f(\mathbf{x})$. To improve the above result for convex functions f we need to give a representation of the relative interior of the epigraph of a convex function. Before showing this representation we observe for dom(f) nonempty and f convex that epi(f) is a nonempty convex set and so by relation (37) and Lemma 32 we obtain

$$ri(dom(f)) = ri(A(epi(f)) = A(ri(epi(f)) \tag{54}$$

with $A : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denoting the projection on $\mathbb{R}^n$.

Lemma 70 *If the function $f : \mathbb{R}^n \to [-\infty, \infty]$ is convex and $dom(f)$ is nonempty then the set $ri(epi(f))$ is nonempty and*

$$ri(epi(f)) = \{(\mathbf{x}, r) \in \mathbb{R}^{n+1} : f(\mathbf{x}) < r, \mathbf{x} \in ri(dom(f))\}.$$

Proof. Since the set dom(f) is nonempty it follows by relation (54) that $\mathbf{x} \in ri(dom(f))$ if and only if $(\{\mathbf{x}\} \times \mathbb{R}) \cap ri(epi(f)) \neq \varnothing$. This implies, using Lemma 34, $\{\mathbf{x}\} \times \mathbb{R}$ is relatively open and $\{\mathbf{x}\} \times (f(\mathbf{x}), \infty) = ri((\{\mathbf{x}\} \times \mathbb{R}) \times epi(f))$ that

$$\{\mathbf{x}\} \times (f(\mathbf{x}), \infty) = (\{\mathbf{x}\} \times \mathbb{R}) \cap ri(epi(f) \tag{55}$$

for every $\mathbf{x} \in ri(dom(f))$. Hence for $(\mathbf{x}, r)$ satisfying $\mathbf{x} \in ri(dom(f))$ and $f(\mathbf{x}) < r$ it follows by relation (55) that $(\mathbf{x}, r) \in ri(epi(f))$. Moreover, if $(\mathbf{x}, r) \in ri(epi(f))$ then by relation (54) we obtain $\mathbf{x} \in ri(dom(f))$ and this shows by relation (55) that $f(\mathbf{x}) < r$. $\square$

In case f is a convex function with $dom(f)$ nonempty the result of Lemma 69 can be improved as follows.

Lemma 71 *If the function $f : \mathbb{R}^n \to [-\infty, \infty]$ is convex and $dom(f)$ is nonempty then $\overline{f}(\mathbf{x}) = \lim_{t \downarrow 0} f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$ for every $\mathbf{y} \in ri(dom(f))$. Moreover, if $\mathbf{x} \in ri(dom(f))$ then the function f is l.s.c at $\mathbf{x}$.*

Proof. By Lemma 69 it is obvious that $\overline{f}(\mathbf{x}) \leq \liminf_{t \downarrow 0} f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$. If $\overline{f}(\mathbf{x}) = \infty$ then the result holds by the previous inequality and so we assume $\overline{f}(\mathbf{x}) < \infty$. This implies that $(\mathbf{x}, r) \in epi(\overline{f}) = cl(epi(f))$ for every $r > \overline{f}(\mathbf{x})$ and since $\mathbf{y} \in ri(dom(f)$ it follows by Lemma 70 that $(\mathbf{y}, r_1) \in ri(epi(f))$ for every $r_1 > f(\mathbf{y})$. Applying now Lemma 28 we obtain for every $0 < t < 1$ that $((1-t)\mathbf{x} + t\mathbf{y}, (1-t)r + tr_1) \in epi(f)$ and this shows $f(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) = f(t\mathbf{y} + (1-t)\mathbf{x}) \leq tr_1 + (1-t)r$. Therefore $\limsup_{t\downarrow 0} f(\mathbf{x} + t(\mathbf{x}_1 - \mathbf{x})) \leq r$ and since $r > \overline{f}(\mathbf{x})$ it follows that $\limsup_{t\downarrow 0} f(\mathbf{x} + t(\mathbf{x}_1 - \mathbf{x})) \leq \overline{f}(\mathbf{x})$. This verifies the first part and to prove the second part we assume that the set $ri(dom(f)$ is nonempty. By Lemma 29 and f convex it follows that

$$ri(epi(\overline{f})) = ri(cl(epi(f)) = ri(epi(f) \subseteq epi(f), \tag{56}$$

and this shows using $\overline{f}$ convex and Lemma 67 and 70 that

$$\{(\mathbf{x}, r) \in \mathbb{R}^{n+1} : \overline{f}(\mathbf{x}) < r, \mathbf{x} \in ri(dom(f))\} = ri(epi(\overline{f})) \tag{57}$$

Applying now relations (56) and (57) it follows by contradiction for every $\mathbf{x} \in ri(dom(f))$ that $\overline{f}(\mathbf{x}) \geq f(\mathbf{x})$ and since $\overline{f}(\mathbf{x}) \leq f(\mathbf{x})$ the desired result follows. □

We now introduce the most important hull function used within the field of convex analysis (cf.[44]).

Definition 72 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the function $\overline{co(f)} : \mathbb{R}^n \to [-\infty, \infty]$ given by $\overline{co(f)}(\mathbf{x}) := \inf\{r : (\mathbf{x}, r) \in cl(co(epi(f)))\}$ is called the l.s.c convex hull function of the function f.*

By a similar approach as in Lemma 67 one can prove the following result.

Lemma 73 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the l.s.c convex hull function $\overline{co(f)}$ is the greatest l.s.c convex function majorized by f. Its epigraph equals $cl(co(epi(f)))$, $dom(co(f)) \subseteq dom(\overline{co(f)}) \subseteq cl(dom(co(f)))$ and $ri(dom(co(f))) = ri(dom(\overline{co(f)}))$.*

A direct consequence of Lemma 73 and $\sup_{i \in I} f_i$ is a l.s.c convex function for f_i, $i \in I$ a collection of l.s.c convex functions yields an often used representation of $\overline{\text{co}(f)}$ given by

$$\overline{co(f)} = \sup\{h : h \leq f \text{ and } h \text{ is a l.s.c convex function}\}. \tag{58}$$

To relate the different hull functions based on relation (36) it follows by relations (46), (53) and (58) that $\overline{co(f)} \leq co(f) \leq f$ and $\overline{co(f)} \leq \overline{f} \leq f$.

We now consider hull functions based on the lower level set (cf.[18]).

Definition 74 *For any function* $f : \mathbb{R}^n \to [-\infty, \infty]$ *the function* $\overline{qc(f)} : \mathbb{R}^n \to [-\infty, \infty]$ *given by* $\overline{qc(f)}(\mathbf{x}) := \inf\{r : \mathbf{x} \in cl(co(L(f, r)))\}$ *is called the l.s.c quasiconvex hull function of the function* f*.*

Using a similar approach as in Lemma 59 one can show the following result.

Lemma 75 *For any function* $f : \mathbb{R}^n \to [-\infty, \infty]$ *the l.s.c quasiconvex hull function* $\overline{qc(f)}$ *is the greatest l.s.c quasiconvex function majorized by* f*. Moreover, it follows that* $L(\overline{qc(f)}, r) = \cap_{\beta > r} cl(co(L(f, \beta)))$ *for every* $r \in \mathbb{R}$*.*

A direct consequence of Lemma 75 and $\sup_{i \in I} f_i$ is a l.s.c quasiconvex function for $f_i,\ i \in I$ a collection of l.s.c quasiconvex functions yields an often used representation of $\overline{qc(f)}$ given by

$$\overline{qc(f)} = \sup\{h : h \leq f \text{ and } h \text{ is a l.s.c quasiconvex function}\}. \tag{59}$$

Since in Lemma 42 we have shown that every closed convex set is evenly convex we finally observe that $\overline{qc(f)} \leq eqc(f) \leq qc(f) \leq f$. The above representations of the hull functions do not depend on the fact that the domain is finite dimensional and so we can also introduce the same hull functions in linear topological vector spaces (cf.[40]). In the next two subsections we consider the dual representations of some of the hull functions.

3.3 Dual representations of convex functions.

In this subsection we will consider in detail properties of convex functions which can be derived using the strong and weak separation results for nonempty convex sets. In particular, we will discuss a dual representation of a l.s.c convex function f satisfying $f > -\infty$. As always in mathematics one likes to approximate complicated functions by simpler functions. For convex functions these simpler functions are given by the so-called affine minorants.

Definition 76 *For any function* $f : \mathbb{R} \to [-\infty, \infty]$ *the affine function* $a : \mathbb{R}^n \to \mathbb{R}$ *given by* $a(\mathbf{x}) = \mathbf{a}^\intercal \mathbf{x} + \alpha$ *with* $\mathbf{a} \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$ *is called an affine minorant of the function* f *if* $f(\mathbf{x}) \geq a(\mathbf{x})$ *for every* $\mathbf{x}$ *belonging to* $\mathbb{R}^n$*. Moreover, the possibly empty set of affine minorants of the function* f *is denoted by* $\mathcal{A}_f$*.*

Since any affine minorant a of a function f is continuous and convex it is easy to verify the following result.

Lemma 77 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ it follows that $\mathcal{A}_f = \mathcal{A}_{co(f)} = \mathcal{A}_{\overline{co(f)}}$.*

Proof. We only give a proof of the above result for $\mathcal{A}_f$ nonempty. Since $\overline{co(f)} \leq co(f) \leq f$ it follows immediately that $\mathcal{A}_{\overline{co(f)}} \subseteq \mathcal{A}_{co(f)} \subseteq \mathcal{A}_f$. Moreover, if the function a belongs to $\mathcal{A}_f$ then clearly $a \leq f$ and a is continuous and convex. This implies by relation (58) that $a \leq \overline{\text{co}(f)}$ and hence the affine function a belongs to $\mathcal{A}_{\overline{co(f)}}$. $\square$

Since an affine function is always finite valued the set $\mathcal{A}_f$ is empty if there exists some $\mathbf{x} \in \mathbb{R}^n$ satisfying $f(\mathbf{x}) = -\infty$ and so it is necessary to consider functions $f : \mathbb{R}^n \to (-\infty, \infty]$. In Theorem 79 a necessary and sufficient condition is given for $\mathcal{A}_f$ to be nonempty. To prove this result we first need to verify the next important lemma.

Lemma 78 *If $f : \mathbb{R} \to [-\infty, \infty]$ is an arbitrary function and $\overline{co(f)}(\mathbf{x}_0)$ is finite for some $\mathbf{x}_0$ then the set $\mathcal{A}_f$ is nonempty.*

Proof. It follows that $(\mathbf{x}_0, \overline{co(f)}(\mathbf{x}_0) - 1)$ does not belong to the set $epi(\overline{co(f)})$. By Lemma 73 this nonempty set is convex and closed and applying Theorem 38, there exists some nonzero vector $(\mathbf{y}_0, \beta)$ satisfying $\mathbf{y}_0^\top \mathbf{x} + \beta r > \mathbf{y}_0^\top \mathbf{x}_0 + \beta(\overline{co(f)}(\mathbf{x}_0) - 1)$ for every $(\mathbf{x},r) \in epi(\overline{co(f)})$. Since $(\mathbf{x}_0, \overline{co(f)}(\mathbf{x}_0)) \in epi(\overline{co(f)})$ this implies $\beta > 0$ and so for every $(\mathbf{x},r) \in epi(\overline{co(f)})$ the inequality

$$r > -\beta^{-1}\mathbf{y}_0^\top(\mathbf{x} - \mathbf{x}_0) + \overline{co(f)}(\mathbf{x}_0) - 1 \tag{60}$$

holds. By relation (60) it follows by contradiction that $\overline{co(f)}(\mathbf{x})$ is finite for every $\mathbf{x} \in dom(\overline{co(f)})$ and this yields using $dom(f) \subseteq dom(\overline{co(f)})$ that $(\mathbf{x}, \overline{co(f)}(\mathbf{x})) \in epi(\overline{co(f)})$ for every $\mathbf{x} \in dom(f)$. Substituting this into relation (60) we obtain

$$f(\mathbf{x}) \geq \overline{co(f)}(\mathbf{x}) > -\beta^{-1}\mathbf{y}_0^\top(\mathbf{x} - \mathbf{x}_0) + \overline{co(f)}(\mathbf{x}_0) - 1$$

for every $\mathbf{x} \in dom(f)$. Since the previous inequality trivially holds for $\mathbf{x} \notin dom(f)$ the function $a(\mathbf{x}) := -\mathbf{y}_0^\top(\mathbf{x} - \mathbf{x}_0) + \overline{co(f)}(\mathbf{x}_0) - 1$ is an affine minorant of f and the desired result is proved. $\square$

Using Lemma 78 one can show the following theorem.

Theorem 79 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ it follows that the set $\mathcal{A}_f$ is nonempty if and only if $co(f) > -\infty$.*

Proof. If the set $\mathcal{A}_f$ is nonempty then for any $a \in \mathcal{A}_f$ we obtain by relation (41) that $co(f)(\mathbf{x}) \geq a(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$, and this shows the first part. To show the reverse implication we consider some f satisfying $co(f)(\mathbf{x}) > -\infty$ for every $\mathbf{x} \in \mathbb{R}^n$. In case $dom(co(f))$ is empty it follows that $f \equiv \infty$ and so trivially $\mathcal{A}_f$ is nonempty. Therefore assume that $dom(co(f))$ is nonempty. By Lemma 54 this is a nonempty convex set and so by Lemma 26 one can find some $\mathbf{x}_0 \in ri(dom(co(f)))$. Since $co(f) > -\infty$ is a convex function it follows by Lemma 71 that $-\infty < co(f)(\mathbf{x}_0) = \overline{co(f)}(\mathbf{x}_0) < \infty$ and so we have found some $\mathbf{x}_0$ satisfying $\overline{co(f)}(\mathbf{x}_0)$ is finite. Applying now Lemma 78 yields $\mathcal{A}_f$ is nonempty and the result is proved. $\square$

As shown by the following example it is not true that $\mathcal{A}_f$ is nonempty for $f > -\infty$.

Example 80 *For the concave function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x^2$ it is easy to verify that $co(epi(f)) = \mathbb{R}^2$ and $f > -\infty$. Hence we obtain that $\mathcal{A}_{co(f)}$ is empty and this yields by Lemma 77 that $\mathcal{A}_f$ is empty.*

To prove an important representation for a subclass of convex functions we introduce the following definition.

Definition 81 *The function $f : \mathbb{R}^n \to [-\infty, \infty]$ belongs to the set $\overline{conv}(\mathbb{R}^n)$ if f is convex and l.s.c and $f > -\infty$.*

It is now possible to prove the following representation for the set $\overline{conv}(\mathbb{R}^n)$. This result is known as Minkowski's theorem.

Theorem 82 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ it follows that*

$$f \in \overline{conv}(\mathbb{R}^n) \Leftrightarrow f = \sup\{a : a \in \mathcal{A}_f\} \text{ and } \mathcal{A}_f \text{ nonempty.}$$

Proof. If the function $f : \mathbb{R}^n \to [-\infty, \infty]$ has the representation $f = \sup\{a : a \in \mathcal{A}_f\}$ and $\mathcal{A}_f$ nonempty, then clearly the function f is l.s.c, convex and $f > -\infty$ and so $f \in \overline{conv}(\mathbb{R}^n)$. To prove the reverse implication we observe for $f \in \overline{conv}(\mathbb{R}^n)$ that $co(f) = f > -\infty$ and this shows by Theorem 79 that the set $\mathcal{A}_f$ is nonempty and hence $f(\mathbf{x}) \geq \sup\{a(\mathbf{x}) : a \in \mathcal{A}_f\} > -\infty$. Suppose now by contradiction that

$f(\mathbf{x}_0) > \sup\{a(\mathbf{x}_0) : a \in \mathcal{A}_f\}$ for some $\mathbf{x}_0 \in \mathbb{R}^n$. Hence one can find some $\gamma \in \mathbb{R}$ satisfying

$$f(\mathbf{x}_0) > \gamma > \sup\{a(\mathbf{x}_0) : a \in \mathcal{A}_f\}, \tag{61}$$

and so $(\mathbf{x}_0, \gamma) \notin epi(f)$. If $epi(f)$ is empty then the affine function $a(\mathbf{x}) = \gamma$ is an affine minorant of f and this contradicts relation (61). Therefore we assume that $epi(f)$ is nonempty and since this set is closed and convex there exists by Theorem 38 a nonzero vector $(\mathbf{y}_0, \beta)$ and $\epsilon > 0$ satisfying

$$\mathbf{y}_0^\top \mathbf{x} + \beta r \geq \mathbf{y}_0^\top \mathbf{x}_0 + \beta\gamma + \epsilon \tag{62}$$

for every $(\mathbf{x}, r) \in epi(f)$. Since for $(\mathbf{x}, r) \in epi(f)$ and $h > 0$ the vector $(\mathbf{x}, r{+}h)$ belongs to $epi(f)$ it follows by relation (62) that $\beta \geq 0$. Consider now the two cases $f(\mathbf{x}_0) < \infty$ and $f(\mathbf{x}_0) = \infty$. If $f(\mathbf{x}_0) < \infty$ we obtain by relation (62) replacing $(\mathbf{x}, r)$ by $(\mathbf{x}_0, f(\mathbf{x}_0))$ that $\beta(f(\mathbf{x}_0){-}\gamma) \geq \epsilon$ and this implies using relation (61) that $\beta > 0$. Hence by relation (62) it holds that $f(\mathbf{x}) \geq -\frac{1}{\beta}\mathbf{y}_0^\top(\mathbf{x} - \mathbf{x}_0) + \gamma := a(\mathbf{x})$ for every $\mathbf{x}$ belonging to $dom(f)$ and we have found some $a \in \mathcal{A}_f$ satisfying $a(\mathbf{x}_0) = \gamma$ contradicting relation (61). If $f(\mathbf{x}_0) = \infty$ and $\beta > 0$ in relation (62) then by the same proof we obtain a contradiction and so we consider the last case $f(\mathbf{x}_0) = \infty$ and $\beta = 0$. Introduce now the affine function $a_0 : \mathbb{R}^n \to \mathbb{R}$ given by $a_0(\mathbf{x}) = -\mathbf{y}_0^\top(\mathbf{x} - \mathbf{x}_0) + \epsilon$. By relation (62) $a_0(\mathbf{x}) \leq 0$ for every $\mathbf{x} \in dom(f)$ and $a_0(\mathbf{x}_0) > 0$. Since $\mathcal{A}_f$ is nonempty select some $a \in \mathcal{A}_f$ and by relation (61) it follows that $\lambda_0 := \frac{\gamma - a(\mathbf{x}_0)}{a_0(\mathbf{x}_0)} > 0$ and for $a_{\lambda_0}(\mathbf{x}) := a(\mathbf{x}) + \lambda_0 a_0(\mathbf{x})$ we obtain $a_{\lambda_0}(\mathbf{x}_0) = \gamma$. Moreover, since we know that $a_0(\mathbf{x}) \leq 0$ for every $\mathbf{x} \in dom(f)$ and $a \in \mathcal{A}_f$ we also obtain $a_{\lambda_0} \in \mathcal{A}_f$ and so a_{λ_0} is an affine minorant of f satisfying $a_{\lambda_0}(\mathbf{x}_0) = \gamma$. This contradicts relation (61) and the result is proved. □

An immediate consequence of Minkowski's theorem and Lemma 77 is listed in the next result.

Theorem 83 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is a function satisfying $\overline{co(f)} > -\infty$ then it follows that $\overline{co(f)} = \sup\{a : a \in \mathcal{A}_f\}$ and $\mathcal{A}_f$ is nonempty.*

Proof. By Theorem 82 we obtain that $\overline{co(f)} = \sup\{a : a \in \mathcal{A}_{\overline{co(f)}}\}$ with $\mathcal{A}_{\overline{co(f)}}$ is nonempty and since by Lemma 77 $\mathcal{A}_f = \mathcal{A}_{\overline{co(f)}}$ the desired result follows. □

In Theorem 83 we only guarantee that any function $\overline{co(f)} > -\infty$ can be approximated from below by affine functions. However, it is sometimes useful to derive an approximation formula in terms of the original function f. This formula was first constructed in its general form by Fenchel (cf.[53]) and it has an easy geometrical interpretation.

Definition 84 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ the function $f^* : \mathbb{R}^n \to [-\infty, \infty]$ given by $f^*(\mathbf{a}) := \sup\{\mathbf{a}^\top \mathbf{x} - f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ is called the conjugate function of the function f. The function $f^{**} : \mathbb{R}^n \to [-\infty, \infty]$ given by $f^{**}(\mathbf{x}) := \sup\{\mathbf{a}^\top \mathbf{x} - f^*(\mathbf{a}) : \mathbf{a} \in \mathbb{R}^n\}$ is called the biconjugate function of f.*

By the above definition it is immediately clear that the conjugate function f^* is convex and l.s.c. Moreover, if the function $f : \mathbb{R}^n \to [-\infty, \infty]$ is proper and the set $\mathcal{A}_f$ of affine minorants is nonempty then it is easy to verify that the function f^* is also proper. As shown by the next result the biconjugate function has a clear geometrical interpretation.

Lemma 85 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an arbitrary function satisfying $\mathcal{A}_f$ is nonempty then it follows that $(\mathbf{a}, r) \in epi(f^*)$ if and only if $a \in \mathcal{A}_f$ with $a(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} - r$. Additionally, it holds that $f^{**} = \sup\{a : a \in \mathcal{A}_f\}$.*

Proof. To verify the equivalence relation we observe for $a(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} - r \leq f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ that $r \geq f^*(\mathbf{a}) = \sup\{\mathbf{a}^\top \mathbf{x} - f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ or $(\mathbf{a}, r) \in epi(f^*)$. Moreover, if $(\mathbf{a}, r) \in epi(f^*)$ we obtain $r \geq f^*(\mathbf{a})$ and this implies for every $\mathbf{x} \in \mathbb{R}^n$ that $a(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} - r \leq f(\mathbf{x})$. To prove the relation for the biconjugate function it follows by the definition of $epi(f^*)$ that $f^{**}(\mathbf{x}) = \sup\{\mathbf{a}^\top \mathbf{x} - r : (\mathbf{a}, r) \in epi(f^*)\}$. Since by the first part $(\mathbf{a}, r) \in epi(f^*)$ if and only if $a(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} - r$ is an affine minorant of the function f this shows that $f^{**}(\mathbf{x}) = \sup\{a(\mathbf{x}) : a \in \mathcal{A}_f\}$ and hence the equality for the biconjugate function is verified. □

To prove one of the most important theorems in convex analysis we need to introduce the next definition.

Definition 86 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an arbitrary function then the closure $cl(f) : \mathbb{R}^n \to [-\infty, \infty]$ of the function f is given by $cl(f) = \overline{f}$ if $\overline{f} > -\infty$ and $cl(f) = -\infty$ otherwise.*

Clearly the function $cl(f)$ is l.s.c and satisfies $cl(f) \leq \overline{f}$. The next result is known as the Fenchel-Moreau theorem and is one of the most important results in convex analysis.

Theorem 87 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ it follows that $f^{**} = cl(co(f))$.*

Proof. If $\overline{co(f)}(\mathbf{x}_0) = -\infty$ for some $\mathbf{x}_0 \in \mathbb{R}^n$ then $f^* \equiv \infty$. To show this, suppose by contradiction that $f^*(\mathbf{a}_0) < \infty$ for some $\mathbf{a}_0$. This implies the existence of some $r \in \mathbb{R}$ satisfying $r \geq \mathbf{a}_0^\top \mathbf{x} - f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and so the function $a(\mathbf{x}) = \mathbf{a}_0^\top \mathbf{x} - r$ is an affine minorant of f. Hence by relation (58) we obtain that $\overline{co(f)}(\mathbf{x}_0) > -\infty$ and this contradicts our initial assumption. Since $f^* \equiv \infty$ we obtain $f^{**} \equiv -\infty$ and by Definition 86 we obtain $f^{**} = cl(co(f))$. In case $\overline{co(f)} > -\infty$ the result follows by Theorem 83 and Lemma 85. $\square$

An important consequence of the Fenchel-Moreau theorem is given by the following result. Observe a function is called sublinear if it is positively homogeneous and convex.

Lemma 88 *Any l.s.c sublinear function $f : \mathbb{R}^n \to (-\infty, \infty]$ is the support function of a nonempty closed convex set $C = \{\mathbf{a} \in \mathbb{R}^n : f^*(\mathbf{a}) \leq 0\}$ or equivalently $f(\mathbf{x}) = \sup\{\mathbf{a}^\top \mathbf{x} : \mathbf{a} \in C\}$.*

Proof. By the Fenchel Moreau theorem we obtain $f(\mathbf{x}) = f^{**}(\mathbf{x}) = \sup_{\mathbf{a} \in \mathbb{R}^n}\{\mathbf{a}^\top \mathbf{x} - f^*(\mathbf{a})\}$. Since $f > -\infty$ is positively homogeneous we obtain by Lemma 52 for every $\alpha > 0$ and $\mathbf{a} \in \mathbb{R}^n$ that $\alpha f^*(\mathbf{a}) = \sup_{\mathbf{x} \in \mathbb{R}^n}\{\mathbf{a}^\top(\alpha \mathbf{x}) - f(\alpha \mathbf{x})\} = f^*(\mathbf{a})$ and this shows that $f^*(\mathbf{a})$ belongs to the set $\{\infty, -\infty, 0\}$. If $f^* \equiv \infty$ then $f^{**} \equiv -\infty$ and since by the Fenchel Moreau theorem $f = f^{**}$ we obtain a contradiction with $f > -\infty$. Therefore the set C is not empty and $f(\mathbf{x}) = f^{**}(\mathbf{x}) = \sup_{\mathbf{a} \in \mathbb{R}^n}\{\mathbf{a}^\top \mathbf{x} - f^*(\mathbf{a})\} = \sup_{\mathbf{a} \in C} \mathbf{a}^\top \mathbf{x}$. Due to f^* is l.s.c and convex the set C is closed and convex. $\square$

Finally we introduce the so-called subgradient set of a function at a point.

Definition 89 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ and $\mathbf{x}_0 \in \mathbb{R}^n$ the subset of $\mathbb{R}^n$ consisting of those vectors $\mathbf{a}_0$ satisfying $f(\mathbf{x}) \geq f(\mathbf{x}_0) + \mathbf{a}_0^\intercal(\mathbf{x} - \mathbf{x}_0)$ for every $\mathbf{x} \in \mathbb{R}^n$ is called the subgradient set of the function f at the point $\mathbf{x}_0$. This set is denoted by $\partial f(\mathbf{x}_0)$ and its elements are called subgradients.*

If $f(\mathbf{x}_0) = -\infty$ then clearly $\partial f(\mathbf{x}_0) = \mathbb{R}^n$ and so it is sufficient to consider those $\mathbf{x} \in \mathbb{R}^n$ satisfying $f(\mathbf{x}) > -\infty$. Moreover, if $f(\mathbf{x}_0) > -\infty$ and $\text{dom}(f)$ is empty then again $\partial f(\mathbf{x}_0) = \mathbb{R}^n$ and hence we only need to consider $f(\mathbf{x}_0) > -\infty$ and $\text{dom}(f)$ is not empty. If $\mathbf{x}_0 \notin dom(f)$ or $f(\mathbf{x}_0) = \infty$ then this implies using $\text{dom}(f)$ is nonempty that $\partial f(\mathbf{x}_0) = \varnothing$

and so the only interesting case which remains is given by $f(\mathbf{x}_0)$ finite. It is now relatively easy to prove for $f(\mathbf{x}_0)$ finite that $\partial f(\mathbf{x}_0) \neq \varnothing$ is equivalent to another condition.

Lemma 90 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an arbitrary function satisfying $f(\mathbf{x}_0)$ is finite for some $\mathbf{x}_0$ then it follows that $\mathbf{a}_0 \in \partial f(\mathbf{x}_0)$ if and only if $f(\mathbf{x}_0) + f^*(\mathbf{x}_0) = \mathbf{a}_0^\top \mathbf{x}_0$.*

Proof. If $\mathbf{a}_0 \in \partial f(\mathbf{x}_0)$ then by definition $f(\mathbf{x}) \geq f(\mathbf{x}_0) + \mathbf{a}_0^\top(\mathbf{x} - \mathbf{x}_0)$ for every $\mathbf{x}$ and this implies using $f(\mathbf{x}_0)$ is finite that $\mathbf{a}_0^\top \mathbf{x}_0 - f(\mathbf{x}_0) \geq \mathbf{a}_0^\top \mathbf{x} - f(\mathbf{x})$ for every $\mathbf{x}$. Hence we obtain that $\mathbf{a}_0^\top \mathbf{x}_0 - f(\mathbf{x}_0) = f^*(\mathbf{a}_0)$ and this shows the equality. To verify the reverse implication is trivial and so we omit its proof. $\square$

Up to now we did not show any existence result for the subgradient set of f at $\mathbf{x}_0$ in case $f(\mathbf{x}_0)$ is finite. Such a result will be given by the next theorem.

Theorem 91 *If the function $f : \mathbb{R}^n \to [-\infty, \infty]$ is convex and $f(\mathbf{x}_0)$ is finite for some $\mathbf{x}_0 \in ri(dom(f))$ then $\partial f(\mathbf{x}_0) \neq \varnothing$.*

Proof. If $\mathbf{x}_0 \in ri(dom(f))$ and $f(\mathbf{x}_0)$ is finite it follows by Lemma 26 that $\overline{co(f)}(\mathbf{x}_0) = \overline{f}(\mathbf{x}_0) = f(\mathbf{x}_0)$ and this implies by Lemma 78 that $f > -\infty$. Moreover, by Lemma 70 we obtain $(\mathbf{x}_0, f(\mathbf{x}_0)) \notin ri(epi(f))$ and since $epi(f)$ is a nonempty convex set it follows by Theorem 39 that there exists some nonzero vector $(\mathbf{y}_0, \beta) \in L_{aff(epi(f))}$ satisfying

$$\mathbf{y}_0^\intercal \mathbf{x} + \beta r \geq \mathbf{y}_0^\intercal \mathbf{x}_0 + \beta f(\mathbf{x}_0) \tag{63}$$

for $(\mathbf{x}, r) \in epi(f)$. Since $(\mathbf{x}, r+h)$ also belongs to epi(f) we obtain $\beta \geq 0$ and to show that $\beta > 0$ we assume by contradiction that $\beta = 0$. Using $aff(epi(f)) = aff(dom(f)) \times \mathbb{R}$ and so

$$L_{aff(epi(f))} = L_{aff(dom(f))} \times \mathbb{R}$$

it follows that $\mathbf{y}_0$ belongs to $L_{aff(dom(f))}$. Since $\mathbf{x}_0 \in ri(dom(f))$ this implies that there exists some $\epsilon > 0$ satisfying $\mathbf{x}_0 - \epsilon \mathbf{y}_0 \in dom(f)$. Replacing now $\mathbf{x}$ by $\mathbf{x}_0 - \epsilon \mathbf{y}_0$ in relation (63) and using $\beta = 0$ yields $-\epsilon \|\mathbf{y}_0\|^2 \geq 0$ and so $(\mathbf{y}_0, \beta) = \mathbf{0}$. Hence we contradict $(\mathbf{y}_0, \beta) \neq \mathbf{0}$ and therefore $\beta > 0$. Dividing now the inequality in relation (63) by $\beta > 0$ and using $f(\mathbf{x})$ is finite for every $\mathbf{x} \in dom(f)$ yields

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) - \beta^{-1} \mathbf{y}_0^\intercal (\mathbf{x} - \mathbf{x}_0)$$

for every $\mathbf{x} \in dom(f)$. Hence the vector $\mathbf{a}_0 = -\beta^{-1}\mathbf{y}_0$ belonging to $L_{aff(dom(f))}$ is a subgradient of the function f at the point $\mathbf{x}_0$ and the proof is completed. $\square$

In case $\mathbf{x}_0$ does not belong to $ri(dom(f))$ for some convex function f it might happen that f does not have a subgradient at the point $\mathbf{x}_0$. This is shown by the following example.

Example 92 *Consider the convex function $f : \mathbb{R} \to (-\infty, \infty]$ given by $f(x) = -\sqrt{x}$ for $x \geq 0$ and $f(x) = \infty$ otherwise.Clearly 0 belongs to the relative boundary of $dom(f)$ but $\partial f(0)$ is empty.*

In case the function $f > -\infty$ is a sublinear function then one can show the following improvement of Theorem 91.

Theorem 93 *If the function $f : \mathbb{R}^n \to (-\infty, \infty]$ is sublinear and $\mathbf{0} \in dom(f)$ then $\partial f(\mathbf{0})$ is nonempty. In particular it follows that $\partial f(\mathbf{0}) = \{\mathbf{a} \in \mathbb{R}^n : \overline{f}^*(\mathbf{a}) \leq 0\}$.*

Proof. Since f is convex it follows that $co(f) = f > -\infty$ and this implies by Theorem 79 that $\mathcal{A}_f$ is nonempty . Hence we obtain that the l.s.c hull $\overline{f}$ of f satisfies$\overline{f} > -\infty$ and since by Definition 50 and Lemma 67 the function $\overline{f}$ is also sublinear one may apply Lemma 88. This shows that the set $C = \{\mathbf{a} \in \mathbb{R}^n : \overline{f}^*(\mathbf{a}) \leq 0\}$ is nonempty and by the definition of $\overline{f}^*$ we obtain $f(\mathbf{x}) \geq \overline{f}(\mathbf{x}) \geq \mathbf{a}^\top\mathbf{x}$ for every $\mathbf{a} \in C$. Since $f(0)$ is finite and f positively homogeneous it follows that $f(0) = 0$ and so it follows that $C \subseteq \partial f(\mathbf{0})$. To verify the reverse inclusion we observe for every $\mathbf{a} \in \partial f(\mathbf{0})$ that $f(\mathbf{x}) \geq \mathbf{a}^\top\mathbf{x}$ for every $\mathbf{x}$ and by relation (58) we obtain $\overline{f}(\mathbf{x}) \geq \mathbf{a}^\top\mathbf{x}$ for every $\mathbf{x}$. Since $dom(f) \subseteq dom(\overline{f})$ and $\overline{f} > -\infty$ we obtain $\overline{f}(\mathbf{0}) = 0$ and this shows $\partial f(\mathbf{0}) \subseteq C$. Hence $C = \partial f(\mathbf{0})$ is nonempty and the proof is completed. $\square$

In Theorem 93 we actually show for $f : \mathbb{R}^n \to (-\infty, \infty]$ sublinear and $\mathbf{0} \in dom(f)$ that

$$\overline{f}(\mathbf{x}) = \sup\{\mathbf{a}^\top\mathbf{x} : \mathbf{a} \in \partial f(\mathbf{0})\} \text{ and } \partial f(\mathbf{0}) \neq \varnothing. \tag{64}$$

A nice implication of Theorem 91 is the observation that convex functions have remarkable continuity properties. Before showing this result we need the following lemma.

Lemma 94 *If the function* $f : \mathbb{R}^n \to (-\infty, \infty]$ *is convex and for some* $\mathbf{x}_0$ *there exists some finite constants* m, M *satisfying* $m \leq f(\mathbf{x}) \leq M$ *for every* $\mathbf{x} \in (\mathbf{x}_0 + 2\epsilon E) \cap dom(f)$ *then it follows that*

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leq \frac{M-m}{\epsilon} \|\mathbf{x}_1 - \mathbf{x}_2\|$$

for every $\mathbf{x}_1, \mathbf{x}_2 \in (\mathbf{x}_0 + 2\epsilon E) \cap ri(dom(f))$.

Proof. Let $\mathbf{x}_1 \neq \mathbf{x}_2$ belong to $(\mathbf{x}_0 + \epsilon E) \cap ri(dom(f))$. This implies that $\mathbf{x}_1 - \mathbf{x}_2$ belongs to $L_{aff(dom(f)}$ and so one can find some $0 < \epsilon_1 < \epsilon$ satisfying

$$\mathbf{x}_3 := \mathbf{x}_1 + \frac{\epsilon_1(\mathbf{x}_1 - \mathbf{x}_2)}{\|\mathbf{x}_1 - \mathbf{x}_2\|} \in dom(f).$$

Hence it follows that $\mathbf{x}_3$ belongs to $(\mathbf{x}_0 + 2\epsilon E) \cap dom(f)$ and since

$$\mathbf{x}_1 = \frac{\|\mathbf{x}_1 - \mathbf{x}_2\|}{\|\mathbf{x}_1 - \mathbf{x}_2\| + \epsilon_1}\mathbf{x}_3 + \frac{\epsilon_1}{\|\mathbf{x}_1 - \mathbf{x}_2\| + \epsilon_1}\mathbf{x}_2$$

we obtain by relation (39) and our assumption that

$$f(\mathbf{x}_1) - f(\mathbf{x}_2) \leq \frac{\|\mathbf{x}_1 - \mathbf{x}_2\|}{\|\mathbf{x}_1 - \mathbf{x}_2\| + \epsilon_1}(f(\mathbf{x}_3) - f(\mathbf{x}_2)) \leq \frac{M-m}{\epsilon}\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

Reversing the roles of $\mathbf{x}_1$ and $\mathbf{x}_2$ yields a similar bound for $f(\mathbf{x}_2) - f(\mathbf{x}_1)$ and the desired result follows. □

The above property of the function f is called Lipschitz continuous on $(\mathbf{x}_0 + 2\epsilon E) \cap ri(dom(f))$. Using Lemma 94 and Theorem 91 one can now show the next result which is an improvement of Lemma 71.

Theorem 95 *If* $f : \mathbb{R}^n \to (-\infty, \infty]$ *is a convex function then it follows that* f *is continuous on* $ri(dom(f))$ *and Lipschitz continuous on every compact subset of* $ri(dom(f)$.

Proof. If $\mathbf{x}_0 \in ri(dom(f)$ then one can find some $\epsilon > 0$ such that

$$(\mathbf{x}_0 + 2\epsilon E) \cap aff(dom(f)) \subseteq dom(f).$$

Also by Lemma 7 there exists a set of $k \leq n$ linear independent vectors $\mathbf{y}_1, .., \mathbf{y}_k$ with $\|\mathbf{y}_i\| = 1, 1 \leq i \leq k$ satisfying $L_{aff(dom(f))} = lin\{\mathbf{y}_1, .., \mathbf{y}_k\}$ and so $\mathbf{x}_0 + \epsilon\mathbf{y}_i$ and $\mathbf{x}_0 - \epsilon\mathbf{y}_i, i = 1, .., k$ is a subset of dom(f). This implies using f is a convex function that the convex hull P generated by $\mathbf{x}_0 + \epsilon\mathbf{y}_i$ and $\mathbf{x}_0 - \epsilon\mathbf{y}_i, i = 1, .., k$ belongs to dom(f) and $f(\mathbf{x}_0) \leq M$ with $M := \max\{f(\mathbf{x}_0 + \epsilon\mathbf{y}_i), f(\mathbf{x}_0 - \epsilon\mathbf{y}_i), 1 \leq i \leq k\} < \infty$. Since $(\mathbf{x}_0 + \frac{\epsilon}{2}E) \cap dom(f) \subseteq P$ it follows that f is bounded from above on $(\mathbf{x}_0 + \frac{\epsilon}{2}E) \cap dom(f)$. Also by Theorem 91 we obtain that $\partial f(\mathbf{x}_0)$ is nonempty and so f is bounded from below on $(\mathbf{x}_0 + \frac{\epsilon}{2}E) \cap dom(f)$. Applying now Lemma 94 yields the desired result. $\square$

This concludes our discussion on dual representations and conjugation for convex functions. In the next subsection we consider the same topic for quasiconvex functions.

3.4 Dual representations of quasiconvex functions.

In this section we study dual representations of evenly quasiconvex and l.s.c quasiconvex functions. Most of the results of this section can be found in [40]. Unfortunately in [40] no geometrical interpretation of the results are given and for a such an interpretation the reader should consult [21]. In [40] it is shown that one can use the same approach as in convex analysis and this results in proving that certain subsets of quasiconvex functions can be approximated from below by so-called c-affine functions with $c : \mathbb{R} \to [-\infty, \infty]$ belonging to a given class $\mathcal{C}$ of extended real valued univariate functions. Observe a function is called univariate if its domain is given by $\mathbb{R}$. As in convex analysis the used approximations and the generalized biconjugate functions have a clear geometrical interpretation (cf.[21]). To start with this approach we introduce in the next definition the class of c-affine functions.

Definition 96 *For a given univariate function $c : \mathbb{R} \to [-\infty, \infty]$ the function $a : \mathbb{R}^n \to [-\infty, \infty]$ is called a c-affine function if there exist some $\mathbf{a} \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that $a(\mathbf{x}) = c(\mathbf{a}^\top \mathbf{x}) + r$ for every $\mathbf{x} \in \mathbb{R}^n$. If $\mathcal{C}$ denotes a subset of the set of extended real valued univariate functions the function a is called a $\mathcal{C}$-affine function if for some $c \in \mathcal{C}$ the function a is a c-affine function. The function a is called a $\mathcal{C}$-affine minorant of the function $f : \mathbb{R}^n \to [-\infty, \infty]$ if $a \leq f$ and a is a $\mathcal{C}$-affine function. The set $\mathcal{C}\mathcal{A}_f$ denotes now the (possibly empty) set of $\mathcal{C}$-affine minorants of f.*

To specify the set $\mathcal{C}$ we first consider the set $\mathcal{C}_0$ of extended real valued nondecreasing univariate functions $c : \mathbb{R} \to [-\infty, \infty]$ and the proper subset $\mathcal{C}_1 \subseteq \mathcal{C}_0$ of extended real valued nondecreasing l.s.c univariate functions. Since for any $c \in \mathcal{C}_i, i = 0, 1$ and $r \in \mathbb{R}$ also the function c^* given by $c^*(t) = c(t) + r$ belongs to $\mathcal{C}_i$, $i = 0, 1$ we observe for these classes of extended real valued univariate functions that the class of $\mathcal{C}_i$-affine functions, $i = 0, 1$ reduces to the set of functions a given by $a(\mathbf{x}) = c(\mathbf{a}^\top \mathbf{x})$ for some $\mathbf{a} \in \mathbb{R}^n$ and $c \in \mathcal{C}_i$. Clearly $\mathcal{C}_1\mathcal{A}_f \subseteq \mathcal{C}_0\mathcal{A}_f$ and since the function $c : \mathbb{R}^n \to [-\infty, \infty]$ with $c \equiv -\infty$ belongs to the set $\mathcal{C}_1$ we obtain that $\mathcal{C}_1\mathcal{A}_f$ is nonempty for every $f : \mathbb{R}^n \to [-\infty, \infty]$. This is a major difference with the set of affine minorants of a function f since this set might be empty. Observe in Theorem 79 we showed that this set is nonempty if and only if $co(f) > -\infty$. One can now show the following result for $\mathcal{C}$-affine functions with $\mathcal{C}$ either equal to $\mathcal{C}_1$ or $\mathcal{C}_0$.

Lemma 97 *If $a : \mathbb{R}^n \to [-\infty, \infty]$ is $\mathcal{C}_0$-affine then the function a is evenly quasiconvex. Moreover, if a is $\mathcal{C}_1$-affine then the function a is l.s.c and quasiconvex.*

Proof. If a is a $\mathcal{C}_0$-affine function then there exists some $c \in \mathcal{C}_0$ and $\mathbf{a} \in \mathbb{R}^n$ such that $L(a, r) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} \in L(c, r)\}$ for every $r \in \mathbb{R}$ with $L(c, r)$ the lower level set of the function c. Since c is nondecreasing this lower level set is either empty or an interval given by $(-\infty, \beta_r)$ or $(-\infty, \beta_r]$ with $\beta_r := \sup\{t \in \mathbb{R} : c(t) \leq r\}$. Hence the set $L(a, r)$ is either empty or an open or closed halfspace and this shows that $L(a, r)$ is evenly convex. Similarly for $c \in \mathcal{C}_1$ we obtain using Theorem 64 that $L(c, r)$ is empty or $(-\infty, \beta_r]$ and hence $L(a, r)$ is empty or a closed halfspace. This shows a is quasiconvex and by Theorem 64 it is also l.s.c. $\square$

By Lemma 59, 61, 75 and 97 and $\overline{qc(f)} \leq eqc(f) \leq qc(f) \leq f$ (see observation after relation (59)) one can show, applying a similar proof as in Lemma 77, that the following result holds.

Lemma 98 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ it follows that $\mathcal{C}_0\mathcal{A}_f = \mathcal{C}_0\mathcal{A}_{eqc(f)}$ and $\mathcal{C}_1\mathcal{A}_f = \mathcal{C}_1\mathcal{A}_{qc(f)} = \mathcal{C}_1\mathcal{A}_{\overline{qc(f)}}$.*

Contrary to functions studied in convex analysis we do not have to determine for which extended real valued functions the sets $\mathcal{C}_i\mathcal{A}_f, i = 1, 2$ are nonempty and so we can start generalizing Minkowsky's theorem (see Theorem 82) to evenly quasiconvex and l.s.c quasiconvex functions. In the proof of this generalization and in the remainder of this subsection an important role is played by the following function.

Definition 99 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ and $\mathbf{a} \in \mathbb{R}^n$ let $c_\mathbf{a} : \mathbb{R} \to [-\infty, \infty]$ denote the function $c_\mathbf{a}(t) := \inf\{f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} \geq t\}$.*

It is now possible to show the following result.

Theorem 100 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an evenly quasiconvex function then $f = \sup\{a : a \in \mathcal{C}_0\mathcal{A}_f\}$. Moreover, if f is an l.s.c quasiconvex function then $f = \sup\{a : a \in \mathcal{C}_1\mathcal{A}_f\}$.*

Proof. Since the set $\mathcal{C}_0\mathcal{A}_f$ is nonempty we obtain by the definition of $\mathcal{C}_0\mathcal{A}_f$ that $f \geq \sup\{a : a \in \mathcal{C}_0\mathcal{A}_f\}$. Suppose now by contradiction that $f(\mathbf{x}_0) > \sup\{a(\mathbf{x}_0) : a \in \mathcal{C}_0\mathcal{A}_f\}$ for some $\mathbf{x}_0$ and so there exists some $\gamma \in \mathbb{R}$ satisfying

$$f(\mathbf{x}_0) > \gamma > \sup\{a(\mathbf{x}_0) : a \in \mathcal{C}_0\mathcal{A}_f\}. \tag{65}$$

If the set $L(f, \gamma)$ is empty it follows that $f(\mathbf{x}) > \gamma$ for every $\mathbf{x} \in \mathbb{R}^n$ and choosing $c(t) = \gamma$ for every $t \in \mathbb{R}$ and $a(\mathbf{x}) = c(\mathbf{a}^\top\mathbf{x})$ with $\mathbf{a} \in \mathbb{R}^n$ arbitrary we obtain that $a \in \mathcal{C}_1\mathcal{A}_f \subseteq \mathcal{C}_0\mathcal{A}_f$ contradicting relation (65). Therefore the set $L(f, \gamma)$ is nonempty and since the function f is evenly quasiconvex one can find a collection of vectors $(\mathbf{a}_i, b_i)_{i \in I}$ satisfying

$$L(f, \gamma) = \cap_{i \in I} H^{<}(\mathbf{a}_i, b_i). \tag{66}$$

By relation (65) the vector $\mathbf{x}_0 \notin L(f, \gamma)$ and this shows by relation (66) that there exists some $i \in I$ with a nonzero $\mathbf{a}_i$ satisfying $\mathbf{a}_i^\top\mathbf{x}_0 \geq b_i$. This implies again by relation (66) that

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{a}_i^\top\mathbf{y} \geq \mathbf{a}_i^\top\mathbf{x}_0\} \subseteq \{\mathbf{y} \in \mathbb{R}^n : f(\mathbf{y}) > \gamma\} \tag{67}$$

Since $\mathbf{a}_i$ is nonzero the function $c_{\mathbf{a}_i}$ given in Definition 99 is nondecreasing and so the function $a(\mathbf{x}) := c_{\mathbf{a}_i}(\mathbf{a}_i^\top\mathbf{x})$ is $\mathcal{C}_0$-affine and by relation (67) it satisfies $a(\mathbf{x}_0) \geq \gamma$. Also for every $\mathbf{x} \in \mathbb{R}^n$ we obtain $a(\mathbf{x}) \leq f(\mathbf{x})$ and so we have constructed a $\mathcal{C}_0$-affine minorant a of the function f satisfying $a(\mathbf{x}_0) \geq \gamma$. This contradicts relation (65) and hence we have shown that $f = \sup\{a : a \in \mathcal{C}_0\mathcal{A}_f\}$. To verify the representation for f quasiconvex and l.s.c we again assume by contradiction that there exists some $\gamma \in \mathbb{R}$ satisfying

$$f(\mathbf{x}_0) > \gamma > \sup\{a(\mathbf{x}_0) : a \in \mathcal{C}_1\mathcal{A}_f\} \tag{68}$$

for some $\mathbf{x}_0$. If $L(f, \gamma)$ is empty then as in the first part we obtain a contradiction. Therefore the closed set $L(f, \gamma)$ is nonempty and since by relation (68) it holds that $\mathbf{x}_0 \notin L(f, \gamma)$ there exist by Theorem 64 some nonzero vector $\mathbf{a}_0 \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ satisfying $\mathbf{a}_0^\top\mathbf{x} < \beta < \mathbf{a}_0^\top\mathbf{x}_0$ for every

$\mathbf{x} \in L(f,\gamma)$. This implies for every $\mathbf{y}$ satisfying $\mathbf{a}_0^\top \mathbf{y} \geq \beta$ that $f(\mathbf{y}) > \gamma$ and so $c_{\mathbf{a}_0}(\beta) \geq \gamma$. Introducing now the functions $c_{\mathbf{a}_0}^\diamondsuit(t) := \sup_{s<t} c_{\mathbf{a}_0}(s)$ and $a(\mathbf{x}) := c_{\mathbf{a}_0}^\diamondsuit(\mathbf{a}_0^\top \mathbf{x})$ this implies

$$a(\mathbf{x}_0) = c_{\mathbf{a}_0}^\diamondsuit(\mathbf{a}_0^\top \mathbf{x}_0) = \sup_{s<\mathbf{a}_0^\top \mathbf{x}_0} c_{\mathbf{a}_0}(s) \geq c_{\mathbf{a}_0}(\beta) \geq \gamma.$$

By Lemma 68 the function $c_{\mathbf{a}_0}^\diamondsuit$ is l.s.c and $c_{\mathbf{a}_0}^\diamondsuit(\mathbf{a}_0^\top \mathbf{x}) \leq c_{\mathbf{a}_0}(\mathbf{a}_0^\top \mathbf{x}) \leq f(\mathbf{x})$ for every $\mathbf{x}$. Hence we have constructed a $\mathcal{C}_1$-affine minorant a of the function f satisfying $a(\mathbf{x}_0) \geq \gamma$ and this contradicts relation (68). Therefore $f = \sup\{a : a \in \mathcal{C}_1\mathcal{A}_f\}$ and the proof is completed. □

By Theorem 100 it is clear that the set of $\mathcal{C}_1$-affine ($\mathcal{C}_0$-affine functions) play the same role for l.s.c quasiconvex functions (evenly quasiconvex functions) as the affine functions do for l.s.c convex functions. However, besides this observation it is also interesting to investigate the question whether these sets of $\mathcal{C}$-affine minorants are the smallest possible class satisfying the above property. In this section we will also pay attention to this question. An immediate consequence of Theorem 100 and Lemma 98 is given by the next result.

Theorem 101 *For any function $f : \mathbb{R}^n \to [-\infty,\infty]$ it follows that $eqc(f) = \sup\{a : a \in \mathcal{C}_0\mathcal{A}_f\}$ and $\overline{qc(f)} = \sup\{a : a \in \mathcal{C}_0\mathcal{A}_f\}$.*

Proof. By Theorem 100 we obtain $eqc(f) = \sup\{a : a \in \mathcal{C}_0\mathcal{A}_{eqc(f)}\}$ and since by Lemma 98 it holds that $\mathcal{C}_0\mathcal{A}_f = \mathcal{C}_0\mathcal{A}_{eqc(f)}$ the first formula follows. The second formula can be verified similarly. □

Studying the proof of Theorem 100 for evenly quasiconvex functions one can actually show the following improvement of Theorem 101.

Theorem 102 *If $f : \mathbb{R}^n \to [-\infty,\infty]$ is an arbitrary function and for every $\mathbf{a} \in \mathbb{R}^n$ the function $c_{\mathbf{a}} : \mathbb{R} \to [-\infty,\infty]$ is given by $c_{\mathbf{a}}(t) := \inf\{f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} \geq t\}$ then it follows for every $\mathbf{x} \in \mathbb{R}^n$ that*

$$eqc(f)(\mathbf{x}) = \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}).$$

Proof. It follows for every $\mathbf{a}$ and $\mathbf{x} \in \mathbb{R}^n$ that $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) \leq f(\mathbf{x})$. Since $c_{\mathbf{a}} \in \mathcal{C}_0$ this implies by Lemma 97 that the function $\mathbf{x} \to c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x})$ is evenly quasiconvex and so by Lemma 61 we obtain for every $\mathbf{x} \in \mathbb{R}^n$ that $eqc(f)(\mathbf{x}) \geq \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x})$. Suppose now by contradiction that $eqc(f)(\mathbf{x}_0) > \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}_0)$ for some $\mathbf{x}_0$ and so there exists some $\gamma \in \mathbb{R}$ satisfying

$$eqc(f)(\mathbf{x}_0) > \gamma > \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}_0). \tag{69}$$

If the set $L(eqc(f),\gamma)$ is empty we obtain $f(\mathbf{x}) \geq eqc(f)(\mathbf{x}) > \gamma$ for every $\mathbf{x} \in \mathbb{R}^n$ and this implies $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}_0) \geq \gamma$ for every $\mathbf{a} \in \mathbb{R}^n$ contradicting relation (69). Therefore the set $L(eqc(f),\gamma)$ is nonempty and since by Lemma 61 the function $eqc(f)$ is evenly quasiconvex one can find a collection of vectors $(\mathbf{a}_i, b_i)_{i\in I}$ satisfying

$$L(eqc(f),\gamma) = \cap_{i\in I} H^{<}(\mathbf{a}_i, b_i). \tag{70}$$

By relation (69) we know $\mathbf{x}_0 \notin L(eqc(f),\gamma)$ and so by relation (70) there exists some $i \in I$ and a nonzero vector $\mathbf{a}_i$ satisfying $\mathbf{a}_i^\top \mathbf{x}_0 \geq b_i$. This implies using $f \geq eqc(f)$ and relation (70) that

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{a}_i^\top \mathbf{y} \geq \mathbf{a}_i^\top \mathbf{x}_0\} \subseteq \{\mathbf{y} \in \mathbb{R}^n : f(\mathbf{y}) > \gamma\}$$

and so it follows that $c_{\mathbf{a}_i}(\mathbf{a}_i^\top \mathbf{x}_0) \geq \gamma$. This yields $\sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) \geq c_{\mathbf{a}_i}(\mathbf{a}_i^\top \mathbf{x}_0) \geq \gamma$ contradicting relation (69). This shows the desired representation and our proof is completed. $\square$

Also for l.s.c quasiconvex functions one can show the following improvement of Theorem 100. Observe this formula is more complicated then the corresponding formula for evenly quasiconvex functions.

Theorem 103 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an arbitrary function then it follows for every $\mathbf{x} \in \mathbb{R}^n$ that*

$$\overline{qc(f)}(\mathbf{x}) = \sup_{\mathbf{a}\in\mathbb{R}^n} \overline{c_{\mathbf{a}}}(\mathbf{a}^\top \mathbf{x}) = \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x})$$

with $\overline{c_{\mathbf{a}}}$ denoting the l.s.c hull of the function $c_{\mathbf{a}}$ and $c_{\mathbf{a}}^{\Diamond}$ given in Lemma 68.

Proof. By Lemma 68 it is sufficient to show for every $\mathbf{x} \in \mathbb{R}^n$ that $\overline{qc(f)}(\mathbf{x}) = \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x})$. To verify this we first observe for every $\mathbf{a}$ and $\mathbf{x} \in \mathbb{R}^n$ that $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) \leq f(\mathbf{x})$ and so we obtain $c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x}) \leq f(\mathbf{x})$ for every $\mathbf{x}$. By Lemma 68 the function $c_{\mathbf{a}}^{\Diamond} : \mathbb{R} \to [-\infty, \infty]$ is l.s.c and nondecreasing and this implies by Lemma 97 that $\mathbf{x} \to c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x})$ is quasiconcex and l.s.c. Therefore we obtain for every $\mathbf{x}$ that

$$\overline{qc(f)}(\mathbf{x}) \geq \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x}).$$

Suppose now by contradiction that $\overline{qc(f)}(\mathbf{x}_0) > \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x}_0)$ for some $\mathbf{x}_0$ and so there exists some $\gamma \in \mathbb{R}$ satisfying

$$\overline{qc(f)}(\mathbf{x}_0) > \gamma > \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}^{\Diamond}(\mathbf{a}^\top \mathbf{x}_0). \tag{71}$$

If the set $L(\overline{qc(f)},\gamma)$ is empty we obtain $f(\mathbf{x}) \geq \overline{eqc(f)}(\mathbf{x}) > \gamma$ and we obtain a contradiction with relation (71). Therefore, the set $L(\overline{eqc(f)},\gamma)$ is nonempty and since by relation (71) it holds that $\mathbf{x}_0 \notin L(\overline{eqc(f)},\gamma)$ there exist by Theorem 64 some nonzero vector $\mathbf{a}_0 \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ satisfying $\mathbf{a}_0^\top \mathbf{x} < \beta < \mathbf{a}_0^\top \mathbf{x}_0$ for every $\mathbf{x} \in L(\overline{eqc(f)},\gamma)$. Hence it follows for every $\mathbf{y}$ satisfying $\mathbf{a}_0^\top \mathbf{y} \geq \beta$ that $f(\mathbf{y}) \geq \overline{eqc(f)}(\mathbf{y}) > \gamma$ and this yields $c_{\mathbf{a}_0}(\beta) \geq \gamma$. Using this observation we obtain

$$c_{\mathbf{a}_0}^{\lozenge}(\mathbf{a}_0^\top \mathbf{x}_0) = \sup_{s<\mathbf{a}_0^\top \mathbf{x}_0} c_{\mathbf{a}_0}(s) \geq c(\beta) \geq \gamma$$

and this contradicts relation (71) completing the proof. □

It is also possible to show for every $\mathbf{a} \in \mathbb{R}^n$ that the function $c_{\mathbf{a}}^{\lozenge}$ is actually the inverse of another function.

Lemma 104 *It $f : \mathbb{R}^n \to [-\infty,\infty]$ is a function with $dom(f)$ nonempty and the function $h_{\mathbf{a}} : \mathbb{R} \to [-\infty,\infty], \mathbf{a} \in \mathbb{R}^n$ is given by $h_{\mathbf{a}}(\alpha) := \sup\{\mathbf{a}^\top \mathbf{y} : \mathbf{y} \in L(f,\alpha)\}$ then it follows for every $t \in \mathbb{R}$ that*

$$c_{\mathbf{a}}^{\lozenge}(t) = \inf\{\alpha \in \mathbb{R} : h_{\mathbf{a}}(\alpha) \geq t\}.$$

Proof. Since $\mathrm{dom}(f)$ is nonempty there exists some $\alpha \in \mathbb{R}$ satisfying $L(f,\alpha)$ is nonempty. If for some $\alpha_0 \in \mathbb{R}$ it follows that $h_{\mathbf{a}}(\alpha_0) \geq t$ then for every $s < t$ there exists some $\mathbf{y}_0$ satisfying $\alpha_0 \geq f(\mathbf{y}_0)$ and $\mathbf{a}^\top \mathbf{y}_0 \geq s$. This shows $\alpha_0 \geq f(\mathbf{y}_0) \geq c_{\mathbf{a}}(s)$ and hence $\inf\{\alpha \in \mathbb{R} : h_{\mathbf{a}}(\alpha) \geq t\} \geq c_{\mathbf{a}}(s)$. Since $s < t$ we obtain $\inf\{\alpha \in \mathbb{R} : h_{\mathbf{a}}(\alpha) \geq t\} \geq \sup_{s<t} c_{\mathbf{a}}(s) = c_{\mathbf{a}}^{\lozenge}(t)$ and to show equality we assume by contradiction that there exists some t_0 satisfying

$$\inf\{\alpha \in \mathbb{R} : h_{\mathbf{a}}(\alpha) \geq t_0\} > c_{\mathbf{a}}^{\lozenge}(t_0).$$

If this holds one can find some α_0 satisfying $\alpha_0 > c_{\mathbf{a}}^{\lozenge}(t_0)$ and $h_{\mathbf{a}}(\alpha_0) < t_0$. Hence there exists some $\epsilon > 0$ satisfying $\alpha_0 > c_{\mathbf{a}}^{\lozenge}(t_0)$ and $h_{\mathbf{a}}(\alpha_0) < t_0 - \epsilon$. Since $h_{\mathbf{a}}(\alpha_0) < t_0 - \epsilon$ we obtain for every $\mathbf{y}$ satisfying $\mathbf{a}^\top \mathbf{y} \geq t_0 - \epsilon$ that $f(\mathbf{y}) > \alpha_0$. This implies $c_{\mathbf{a}}(t_0 - \epsilon) \geq \alpha_0$ and it follows $\alpha_0 > c_{\mathbf{a}}^{\lozenge}(t_0) \geq c_{\mathbf{a}}(t_0 - \epsilon) \geq \alpha_0$. This is clearly a contradiction and so the proof is completed. □

In case $\mathrm{dom}(f)$ is empty and so $f \equiv \infty$ and we use the well-known convention that $\sup\{\varnothing\} = -\infty$ and $\inf\{\varnothing\} = \infty$ then it is easy to verify that the above relation still holds. The next result first verified in [18] is an immediate consequence of Lemma 104 and Theorem 103.

Theorem 105 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is an arbitrary function then it follows that $\overline{qc(f)}(\mathbf{x}) = \sup_{\mathbf{a} \in \mathbb{R}^n} \inf\{\alpha \in \mathbb{R} : \sup_{\mathbf{y} \in L(f,\, \alpha)} \mathbf{a}^\top \mathbf{y} \geq \mathbf{a}^\top \mathbf{x}\}$ for every $\mathbf{x}$.*

Actually the result in Theorem 102 and 103 can be seen as a generalization of the Fenchel-Moreau theorem for l.s.c convex hulls. To show this we need to generalize the notion of conjugate and biconjugate functions used within convex analysis. Since we are dealing with extended real valued functions we use the convention that $(-\infty)+(+\infty) = (+\infty) + (-\infty) = -\infty$ and $-(-\infty) = \infty$.

Definition 106 *Let $\mathcal{C}$ be a nonempty collection of extended real valued univariate functions. For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ and $c \in \mathcal{C}$ the function $f^c(\mathbf{a}) := \sup\{c(\mathbf{a}^\top \mathbf{x}) - f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ is called the c-conjugate function of the function f. The function $f^{\mathcal{C}\mathcal{C}}(\mathbf{x}) := \sup\{c(\mathbf{x}^\top \mathbf{a}) - f^c(\mathbf{a}) : \mathbf{a} \in \mathbb{R}^n, c \in \mathcal{C}\}$ is called the bi-$\mathcal{C}$-conjugate function of f.*

By a similar proof as in Lemma 85 it is easy to give a geometrical interpretation of the biconjugate function.

Lemma 107 *For $\mathcal{C}$ a nonempty collection of extended real valued univariate functions and $f : \mathbb{R}^n \to [-\infty, \infty]$ an arbitrary function it follows that $(\mathbf{a}, r) \in epi(f^c)$ if and only if $a \in \mathcal{C}\mathcal{A}_f$ with $a(\mathbf{x}) = c(\mathbf{a}^\top \mathbf{x}) - r$ and $c \in \mathcal{C}$. Additionally, it holds that $f^{\mathcal{C}\mathcal{C}}(\mathbf{x}) = \sup\{a(\mathbf{x}) : a \in \mathcal{C}\mathcal{A}_f\}$.*

Combining now Lemma 107 and Theorem 101 we immediately obtain for the sets $\mathcal{C}_i, i = 0, 1$ the following generalization of the Fenchel-Moreau theorem.

Theorem 108 *For any function $f : \mathbb{R}^n \to [-\infty, \infty]$ it follows that $f^{\mathcal{C}_0\mathcal{C}_0} = eqc(f)$ and $f^{\mathcal{C}_1\mathcal{C}_1} = \overline{qc(f)}$.*

Proof. By Lemma 107 we obtain $f^{\mathcal{C}_i\mathcal{C}_i} = \sup\{a : a \in \mathcal{C}_i\mathcal{A}_f\}, i = 0, 1$ and this shows by Theorem 101 the desired result. □

By Theorem 102, 103 and 108 we obtain the formulas

$$f^{\mathcal{C}_0\mathcal{C}_0}(\mathbf{x}) = eqc(f)(\mathbf{x}) = \sup_{\mathbf{a} \in \mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x})$$

and

$$f^{\mathcal{C}_1\mathcal{C}_1}(\mathbf{x}) = \overline{qc(f)}(\mathbf{x}) = \sup_{\mathbf{a} \in \mathbb{R}^n} c_{\mathbf{a}}^{\diamondsuit}(\mathbf{a}^\top \mathbf{x}) \tag{72}$$

for every $\mathbf{x} \in \mathbb{R}^n$. Considering these formulas we now wonder whether it is possible to achieve the same result using a smaller set of extended real valued univariate functions.

Definition 109 *For any $r \in \mathbb{R}$ the function $c_r : \mathbb{R} \to [-\infty, \infty]$ is given by $c_r(t) = -\infty$ for $t < r$ and $c_r(t) = r$ for every $t \geq r$. The set $\mathcal{C}_r \subseteq \mathcal{C}_0$ consists now of all functions $c_r, r \in \mathbb{R}$, while the set $\overline{\mathcal{C}_r}$ consists of all functions $\overline{c_r}$, $r \in \mathbb{R}$ with $\overline{c_r}$ the l.s.c hull of the function c_r.*

If $f : \mathbb{R}^n \to (-\infty, \infty]$ is an arbitrary function then for $r \in \mathbb{R}$ and $\mathbf{a} \neq \mathbf{0}$ we obtain

$$f^{c_r}(\mathbf{a}) = \max\{-\infty, \sup\{r - f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} \geq r\}\} = r - c_{\mathbf{a}}(r) \tag{73}$$

with c_a defined in Theorem 102. Moreover, for $\mathbf{a} = \mathbf{0}$ and $r \leq 0$, it follows that $f^{c_r}(\mathbf{0}) = \sup\{c_r(0) - f(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\} = r - \inf\{f(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\}$ and this shows

$$f^{c_r}(\mathbf{0}) = r - c_{\mathbf{0}}(r) = r - c_{\mathbf{0}}(0), r \leq 0. \tag{74}$$

Also for $r > 0$ it is easy to verify that $f^{c_r}(\mathbf{0}) = -\infty$ and so we have computed for every $r \in \mathbb{R}$ the c_r-conjugate function of the function f. To evaluate the $\overline{c_r}$-conjugate function of f we observe by Lemma 68 that $\overline{c_r}(t) = -\infty$ for every $t \leq r$ and $\overline{c_r}(t) = r$ for every $t > r$. Again considering $\mathbf{a} \neq \mathbf{0}$ it follows that

$$\begin{aligned} f^{\overline{c_r}}(\mathbf{a}) &= \max\{-\infty, \sup\{r - f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} > r\}\} \\ &= r - \inf\{f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} > r\}. \end{aligned} \tag{75}$$

Moreover, for $\mathbf{a} = 0$ and $r < 0$, we obtain that

$$\begin{aligned} f^{\overline{c_r}}(\mathbf{0}) &= \sup\{\overline{c_r}(0) - f(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\} \\ &= r - \inf\{f(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\} = r - c_{\mathbf{0}}(r), \end{aligned} \tag{76}$$

while for $r \geq 0$ it is easy to verify that $f^{\overline{c_r}}(\mathbf{0}) = -\infty$. Using the above computations we will first evaluate in the proof of Lemma 110 the bi-$\mathcal{C}_r$-conjugate function of a function $f : \mathbb{R}^n \to (-\infty, \infty]$, while in the proof of Lemma 111 the same computation will be carried out for a bi-$\overline{\mathcal{C}_r}$-conjugate function of the same function f.

Lemma 110 *For every $\mathbf{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to (-\infty, \infty]$ it follows that*

$$f^{\mathcal{C}_r\mathcal{C}_r}(\mathbf{x}) = \sup\nolimits_{\mathbf{a} \in \mathbb{R}^n} \inf\{f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} \geq \mathbf{a}^\top \mathbf{x}\} = eqc(f)(\mathbf{x}).$$

Proof. By relation (74) and $f^{c_r}(\mathbf{0}) = -\infty$ for every $r > 0$ we obtain using the convention $-\infty - (-\infty) = -\infty + \infty = -\infty$ that

$$\sup\nolimits_{r \in \mathbb{R}}\{c_r(0) - f^{c_r}(\mathbf{0})\} = \sup\nolimits_{r \leq 0} c_{\mathbf{0}}(0) = c_{\mathbf{0}}(0). \tag{77}$$

Also by relation (73) and $(-\infty)-(-\infty)=(-\infty)+\infty=-\infty$ it follows for every $\mathbf{x}$ that

$$\sup_{\mathbf{a}\neq\mathbf{0},\ r\in\mathbb{R}}\{c_r(\mathbf{a}^\top\mathbf{x})-f^{c_r}(\mathbf{a})\}=\sup_{\mathbf{a}\neq\mathbf{0},\ r\leq\mathbf{a}^\top\mathbf{x},\ r\in\mathbb{R}} c_{\mathbf{a}}(r).$$

This shows, using $c_{\mathbf{a}}$ is nondecreasing for every $\mathbf{a}\neq\mathbf{0}$, that

$$\sup_{\mathbf{a}\neq\mathbf{0},\ r\in\mathbb{R}}\{c_r(\mathbf{a}^\top\mathbf{x})-f^{c_r}(\mathbf{a})\}=\sup_{\mathbf{a}\neq\mathbf{0}} c_{\mathbf{a}}(\mathbf{a}^\top\mathbf{x}) \tag{78}$$

and so $f^{\mathcal{C}_r\mathcal{C}_r}(\mathbf{x})=\sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top\mathbf{x})$ using relations (77) and (78). This shows the first equality and the second one is already listed in Theorem 102. $\square$

The next result yields a similar result as Lemma 110 for a quasiconvex and l.s.c function.

Lemma 111 *For every $\mathbf{x}\in\mathbb{R}^n$ and $f:\mathbb{R}^n\to(-\infty,\infty]$ it follows that*

$$f^{\overline{\mathcal{C}_r}\,\overline{\mathcal{C}_r}}(\mathbf{x})=\sup_{\mathbf{a}\in\mathbb{R}^n}\sup_{s<\mathbf{a}^\top\mathbf{x}}\inf\{f(\mathbf{y}):\mathbf{a}^\top\mathbf{y}\geq s\}=\overline{qc(f)}(\mathbf{x}).$$

Proof. By relation (75) and $f^{\overline{c_r}}(\mathbf{0})=-\infty$ for every $r\geq 0$ we obtain using $-\infty-(-\infty)=-\infty+\infty=-\infty$ that

$$\sup_{r\in\mathbb{R}}\{\overline{c_r}(0)-f^{\overline{c_r}}(\mathbf{0})\}=\sup_{r<0} c_{\mathbf{0}}(r)=c_{\mathbf{0}}^{\diamondsuit}(0). \tag{79}$$

Also by relation (74) and $(-\infty)-(-\infty)=(-\infty)+\infty=-\infty$ it follows with $h(\mathbf{x}):=\sup_{\mathbf{a}\neq\mathbf{0},\ r\in\mathbb{R}}\{\overline{c_r}(\mathbf{a}^\top\mathbf{x})-f^{\overline{c_r}}(\mathbf{a})\}$ that

$$h(\mathbf{x})=\sup_{\mathbf{a}\neq\mathbf{0},\ r<\mathbf{a}^\top\mathbf{x},\ r\in\mathbb{R}}\inf\{f(\mathbf{y}):\mathbf{a}^\top\mathbf{y}>r\}. \tag{80}$$

Since $\inf\{f(\mathbf{y}):\mathbf{a}^\top\mathbf{y}>r\}\geq c_{\mathbf{a}}(r)$ for every $r\in\mathbb{R}$ and $\mathbf{a}\neq\mathbf{0}$ we obtain by relation (80) that

$$h(\mathbf{x})\geq\sup_{\ \mathbf{a}\neq\mathbf{0},\ r<\mathbf{a}^\top\mathbf{x},\ r\in\mathbb{R}} c_{\mathbf{a}}(r)=\sup_{\mathbf{a}\neq\mathbf{0}} c_{\mathbf{a}}^{\diamondsuit}(\mathbf{a}^\top\mathbf{x}). \tag{81}$$

Applying now relations (72), (79) and (81) it holds for every $\mathbf{x}\in\mathbb{R}^n$ that

$$f^{\overline{\mathcal{C}_r\mathcal{C}_r}}(\mathbf{x})\geq\sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}^{\diamondsuit}(\mathbf{a}^\top\mathbf{x})=\overline{qc(f)}(\mathbf{x})=f^{\mathcal{C}_1\mathcal{C}_1}(\mathbf{x}). \tag{82}$$

Since $\overline{\mathcal{C}_r}\subseteq\mathcal{C}_1$ it follows that $f^{\mathcal{C}_1\mathcal{C}_1}\geq f^{\overline{\mathcal{C}_r\mathcal{C}_r}}$ and this shows by relation (82) the desired result. $\square$

In the last two lemmas we have shown that it is sufficient for any function f satisfying $f>-\infty$ to consider the class of $\mathcal{C}_r$-affine minorants and the class of $\overline{\mathcal{C}_r}$-affine minorants for approximating $eqc(f)$, respectively $\overline{qc(f)}$. This concludes the section on quasiconvex duality. In the next section we will discuss some important applications.

4 On applications of convex and quasiconvex analysis.

In this section we will discuss different applications of the theory of convex and quasiconvex analysis. In subsection 4.1 we consider applications to noncooperative game theory, while in subsection 4.2 we discuss its applications to optimization problems and in particular to Lagrangian duality. Finally in subsection 4.3 we will use the duality representation of evenly quasiconvex functions to show that every positively homogeneous evenly quasiconvex function satisfying $f(\mathbf{0}) = 0$ and $f > -\infty$ is actually the minimum of two positively homogeneous l.s.c convex functions. This result was first verified by Crouzeix (cf.[18]) for a slightly smaller class of quasiconvex functions and serves as a very nice application of quasiconvex duality.

4.1 Minimax theorems and noncooperative game theory.

To introduce the field of infinite antagonistic game theory (cf.[39]) we assume that the set of pure strategies of player 1 is given by some nonempty set $A \subseteq \mathbb{R}^n$, while the set of pure strategies of player 2 is given by $B \subseteq \mathbb{R}^m$. If player 1 chooses the pure strategy $\mathbf{a} \in A$ and player 2 chooses the pure strategy $\mathbf{b} \in B$ then player 2 has to pay to player 1 an amount $f(\mathbf{a}, \mathbf{b})$ with $f : A \times B \to [0, \infty]$ a given function. This function is called the payoff function and for simplicity this function is taken to be nonnegative. Since player 1 likes to gain as much profit as possible but at the moment he does not know how to achieve this he first decides to compute a lower bound on his profit. To compute this lower bound player 1 argues as follows. If he decides to choose action $\mathbf{a} \in A$ then it follows that he wins at least $\inf_{\mathbf{b} \in B} f(\mathbf{a}, \mathbf{b})$ irrespective of the action of player 2. Therefore a lower bound on the profit for player 1 is given by

$$u := \sup_{\mathbf{a} \in A} \inf_{\mathbf{b} \in B} f(\mathbf{a}, \mathbf{b}).$$

Similarly player 2 likes to minimize his losses but since he does not know how to achieve this he also decides to compute first an upper bound on his losses. To compute this upper bound player 2 argues as follows. If he decides to choose action $\mathbf{b}$ it follows that he loses at most $\sup_{\mathbf{a} \in A} f(\mathbf{a}, \mathbf{b})$ and this is independent of the action of player 1. Therefore an upper bound on his losses is given by

$$w := \inf_{\mathbf{b} \in B} \sup_{\mathbf{a} \in A} f(\mathbf{a}, \mathbf{b})$$

Since the profit of player 1 is at least u and the losses of player 2 is at most w and the losses of player 2 are the profits of player 1 it follows directly that $u \leq w$. In general $u < w$ but under some properties on the action set and payoff function it holds that $u = w$. Moreover, if additionally inf and sup are attained this is called a minimax result. If this happens an optimal strategy for both players is immediately clear. For player 1 it is possible to achieve at least profit u independent of the action of player 2 while for player 2 it is possible to achieve at most loss w independent of the action of player 1. Since $w = u := v$ and both players have opposite interests they will choose an action which achieves the value v and so player 1 will choose that action $\mathbf{a}_0 \in A$ satisfying

$$\inf_{\mathbf{b}\in B} f(\mathbf{a}_0, \mathbf{b}) = \max_{\mathbf{a}\in A} \inf_{\mathbf{b}\in B} f(\mathbf{a}, \mathbf{b}).$$

Moreover, player 2 will choose that strategy $\mathbf{b}_0 \in B$ satisfying

$$\sup_{\mathbf{a}\in A} f(\mathbf{a}, \mathbf{b}_0) = \min_{\mathbf{b}\in B} \sup_{\mathbf{a}\in A} f(\mathbf{a}, \mathbf{b}).$$

Since for $u = w$ and the additional assumption that inf and sup are attained it is clear how the optimal strategies should be chosen we will investigate in this subsection for which payoff functions and strategies the equality $u = w$ holds. Before discussing this we give the following example for which this equality does not hold.

Example 112 *Consider the continuous payoff function* $f : [0,1]\times[0,1] \to [0,\infty)$ *given by* $f(a,b) = (a-b)^2$. *For this function it holds for every* $0 \leq a \leq 1$ *that* $\inf_{b\in[0,1]}(a-b)^2 = 0$ *and so* $u := \sup_{0\leq a\leq 1}\inf_{0\leq b\leq 1}(a-b)^2 = 0$. *Moreover, it follows that* $\sup_{0\leq a\leq 1}(a-b)^2 = (1-b)^2$ *for every* $0 \leq b < \frac{1}{2}$ *and* $\sup_{0\leq a\leq 1}(a-b)^2 = b^2$ *for every* $\frac{1}{2} \leq b \leq 1$. *This shows* $w := \inf_{0\leq b\leq 1}\sup_{0\leq a\leq 1}(a-b)^2 = \frac{1}{4}$ *and so* u *does not equal* w. *For this example it is not obvious which strategies should be selected by the two players.*

By extending the sets of the so-called pure strategies of each player it is possible to show under certain conditions that the extended game satisfies a minimax result. In the next definition we introduce the set of mixed strategies.

Definition 113 *For a nonempty set* D *of pure strategies and* $\mathbf{d} \in D$ *let* $\epsilon_{\mathbf{d}}$ *denote the one-point probability measure concentrated on the set* $\{\mathbf{d}\}$ *and denote by* $\mathcal{P}_D$ *the set of all probability measures on* D *with a finite support. This means* $\mathcal{P}_D := \{\lambda : \lambda = \sum_{i=1}^k \lambda_i \epsilon_{\mathbf{d}_i}, \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \mathbf{d}_i \in D, d_i \text{ distinct}, n \in \mathbb{N}\}$.

Clearly the set $\mathcal{P}_D$ is convex and a game theoretic interpretation of a strategy $\lambda \in \mathcal{P}_D$ is now given by the following. If a player with pure strategy set D select $\lambda = \sum_{i=1}^k \lambda_i \epsilon_{\mathbf{d}_i} \in \mathcal{P}_D$ then with probability $\lambda_i, 1 \leq i \leq k$ this player will use the pure strategy $d_i \in D$. By this interpretation it is clear that the set D can be identified with $\{\epsilon_{\mathbf{d}} : \mathbf{d} \in D\}$. We now assume that player 1 uses the set $\mathcal{P}_A$ of mixed strategies and the same holds for player 2 using the set $\mathcal{P}_B$. This also means that the payoff function f should now be extended to a function $f_e : \mathcal{P}_A \times \mathcal{P}_B \to \mathbb{R}$ and this extension is given by

$$f_e(\lambda, \mu) := \sum_{i=1}^k \sum_{j=1}^l \lambda_i \mu_j f(\mathbf{a}_i, \mathbf{b}_j)$$

with $\lambda = \sum_{i=1}^k \lambda_i \epsilon_{\mathbf{a}_i} \in \mathcal{P}_A$ and $\mu = \sum_{j=1}^l \mu_j \epsilon_{\mathbf{b}_j} \in \mathcal{P}_B$. This extension represents the expected profit for player 1 or expected loss of player 2 if player 1 selects $\lambda \in \mathcal{P}_A$ and player 2 selects $\mu \in \mathcal{P}_B$. Without any conditions on the pure strategy sets A and B and the function f one can show the next result.

Lemma 114 *It follows that*

$$\inf_{\mu \in \mathcal{P}_B} \sup_{\lambda \in \mathcal{P}_A} f_e(\lambda, \mu) = \inf_{\mu \in \mathcal{P}_B} \sup_{\mathbf{a} \in A} f_e(\epsilon_{\mathbf{a}}, \mu)$$

and

$$\sup_{\lambda \in \mathcal{P}_A} \inf_{\mu \in \mathcal{P}_B} f_e(\lambda, \mu) = \sup_{\lambda \in \mathcal{P}_A} \inf_{\mathbf{b} \in B} f_e(\lambda, \epsilon_{\mathbf{b}}).$$

Proof. To prove the first equality it is clear that the inequality $\geq$ is obvious. To verify the inequality $\leq$ we observe for every $\mu \in \mathcal{P}_B$ that

$$\begin{aligned} \sup_{\lambda \in \mathcal{P}_A} f_e(\lambda, \mu) &\leq \sup\{\sum_{i=1}^k \lambda_i \sup_{\mathbf{a} \in A} f_e(\epsilon_{\mathbf{a}}, \mu) : \lambda \in \Delta_k, k \in \mathbb{N}\} \\ &= \sup_{\mathbf{a} \in A} f_e(\epsilon_{\mathbf{a}}, \mu). \end{aligned}$$

This completes the proof of the first formula. The second one follows by symmetry. □

It is now possible to show that the extended game given by f_e and the mixed strategy sets $\mathcal{P}_A$ and $\mathcal{P}_B$ satisfies a minimax result under some topological conditions on the function f and the sets A and B of pure strategies. The next result was first verified by Ville (cf.[32], [42], [39]) using a much more complicated proof. In the next alternative proof we only use the separation result for convex sets listed in Theorem 41 and the well-known result that a continuous function on a compact set is uniformly continuous. For a survey on some equivalent minimax theorems related to game theory and proved by finite dimensional separation of convex sets the reader should consult [24].

Theorem 115 *If the pure strategy sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact and the function $f : A \times B \to \mathbb{R}$ is continuous then it follows that*

$$\inf_{\mu \in \mathcal{P}_B} \sup_{\lambda \in \mathcal{P}_A} f_e(\lambda, \mu) = \sup_{\lambda \in \mathcal{P}_A} \inf_{\mu \in \mathcal{P}_B} f_e(\lambda, \mu).$$

Proof. It is well-known that the inequality $\geq$ holds and so we only need to verify the inequality $\leq$. By Lemma 114 and the observation after this lemma it is now sufficient to show that $\inf_{\mu \in \mathcal{P}_B} \sup_{\mathbf{a} \in A} f_e(\epsilon_{\mathbf{a}}, \mu) \leq \sup_{\lambda \in \mathcal{P}_A} \inf_{\mathbf{b} \in B} f_e(\lambda, \epsilon_{\mathbf{b}})$. By scaling we may assume that

$$\sup_{\lambda \in \mathcal{P}_A} \inf_{\mathbf{b} \in B} f_e(\lambda, \epsilon_b) = 1 \tag{83}$$

and suppose now by contradiction that there exists some $\gamma > 0$ satisfying $\sup_{\mathbf{a} \in A} f_e(\epsilon_{\mathbf{a}}, \mu) \geq 1 + \gamma$ for every $\mu \in \mathcal{P}_B$. Since the sets A and B are compact and the function $f : A \times B \to \mathbb{R}$ is continuous it is well-known (cf.[56], [1]) that the function f is uniformly continuous on the compact set $A \times B$. Hence there exists some $\delta > 0$ such that for every $\mathbf{x}, \mathbf{y} \in A$ satisfying $\|\mathbf{x} - \mathbf{y}\| \leq \delta$ it follows that $\sup_{\mathbf{b} \in B} |f(\mathbf{x}, \mathbf{b}) - f(\mathbf{y}, \mathbf{b}| \leq \frac{\gamma}{2}$. This implies for every $\mathbf{x}, \mathbf{y} \in A$ satisfying $\|\mathbf{x} - \mathbf{y}\| \leq \delta$ that

$$\sup_{\mu \in \mathcal{P}_B} |f_e(\epsilon_{\mathbf{x}}, \mu) - f_e(\epsilon_{\mathbf{y}}, \mu)| \leq \frac{\gamma}{2}. \tag{84}$$

Since A is also compact one can find a finite set $\{\mathbf{a}_1, .., \mathbf{a}_p\} \subseteq A$ satisfying $A \subseteq \cup_{i=1}^p (\mathbf{a}_i + \delta E)$ and this shows by relation (84) that

$$\max_{\mathbf{a}_i \in A, 1 \leq i \leq p} f_e(\epsilon_{\mathbf{a}_i}, \mu) \geq \sup_{\mathbf{a} \in A} f_e(\epsilon_{\mathbf{a}}, \mu) - \frac{\gamma}{2} \geq 1 + \frac{\gamma}{2} \tag{85}$$

for every $\mu \in \mathcal{P}_{B.}$ Introducing the convex set $V \subseteq \mathbb{R}^p$ given by

$$V := co(\{(f_e(\epsilon_{\mathbf{a}_1}, \epsilon_{\mathbf{b}}), ..., f_e(\epsilon_{\mathbf{a}_p}, \epsilon_{\mathbf{b}}))^\top \in \mathbb{R}^p, \mathbf{b} \in B\})$$

we obtain by relation (85) that $\max_{1 \leq i \leq p} z_i \geq 1 + \frac{\gamma}{2}$ for every vector $\mathbf{z} = (z_1, ..., z_p) \in V$. Hence we can separate the convex set V from the open convex set $\{\mathbf{z} \in \mathbb{R}^p : \max_{i \leq i \leq p} z_i < 1 + \frac{\gamma}{2}\}$ and so there exists by Theorem 41 some vector $\lambda \in \Delta_p$ satisfying $\inf_{\mathbf{b} \in B} \sum_{i=1}^p \lambda_i f_e(\epsilon_{\mathbf{a}_i}, \epsilon_b) =$

$\inf_{\mathbf{b}\in B} f_e(\lambda, \epsilon_{\mathbf{b}}) \geq 1 + \frac{\gamma}{2}$. Applying relation (83) yields a contradiction and the minimax result is verified. $\square$

Actually the result in Theorem 115 holds under weaker topological conditions. However the proof of that result uses the Riesz representation theorem for the set of continuous functions on a compact Hausdorff space and infinite dimensional separation (cf.[25]) and is beyond the scope of this chapter. In the next subsection we will now consider applications of convex analysis into optimization theory.

4.2 Optimization theory and duality.

In this subsection we will show how the tools of convex analysis can be used within optimization theory. In particular we introduce the dual of an optimization problem and derive some important properties of this dual problem. To start with a general introduction to optimization theory let $f : \mathbb{R}^m \to [-\infty, \infty]$ be an arbitrary function and consider the so-called primal optimization problem given by

$$v(P) := \inf\{f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}. \tag{P}$$

In this optimization problem the infimum need not be attained. Since f represents an extended real valued function the above optimization problem also covers optimization problems with constraints. Associate now with the function f a function $F : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty]$ satisfying $F(\mathbf{x}, \mathbf{0}) = f(\mathbf{x})$ for every $\mathbf{x}$ and consider the so-called perturbation function $p : \mathbb{R}^n \to [-\infty, \infty]$ given by

$$p(\mathbf{y}) := \inf\{F(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \mathbb{R}^m\}. \tag{86}$$

It is easy to verify that

$$epi_S(p) = A(epi_S(F)) \text{ and } dom(p) = A(dom(F)) \tag{87}$$

with $A : \mathbb{R}^{m+n} \to \mathbb{R}^n$ the projection of $\mathbb{R}^{m+n}$ onto $\mathbb{R}^n$ given by $A(\mathbf{x}, \mathbf{y}) = \mathbf{y}$. Also by the definition of the function F we obtain that $p(\mathbf{0}) = v(P)$. In the next definition we introduce the dual of the optimization problem (P) (cf.[44]).

Definition 116 *The so-called dual problem of optimization problem (P) is given by*

$$v(D) := \sup\{-p^*(\mathbf{a}) : \mathbf{a} \in \mathbb{R}^n\} \tag{D}$$

with p^* the conjugate function of p listed in Definition 84.

By Definitions 116 and 84 it follows that $v(D) = p^{**}(\mathbf{0})$ and since $p^{**}(\mathbf{0}) \leq p(\mathbf{0})$ the inequality $v(D) \leq v(P)$ always holds. We are now interested under which conditions on the perturbation function p it follows that $v(D) = v(P)$. If $v(P) = -\infty$ then the inequality $v(D) \leq v(P)$ implies $v(D) = v(P) = -\infty$ and every $\mathbf{a} \in \mathbb{R}^n$ is an optimal solution of the dual problem (D). Therefore we only need to consider $v(P) > -\infty$. Consider now the cases $v(P)$ is finite and $v(P) = \infty$. Observe the last case only happens if $\text{dom}(f)$ is empty. For $v(P)$ finite one can now show the following result. This result is a direct consequence of Theorem 87 giving a dual characterization of a convex function (Fenchel-Moreau theorem) and Theorem 91.

Theorem 117 *If $p : \mathbb{R}^n \to [-\infty, \infty]$ is convex and $p(\mathbf{0})$ is finite then it follows that $v(P) = v(D)$ if and only if p is l.s.c at $\mathbf{0}$. Moreover, if $\mathbf{0}$ belongs to $ri(dom(p))$ then the dual problem has an optimal solution and $v(D) = v(P)$.*

Proof. Since the function p is convex, l.s.c at $\mathbf{0}$ and $p(\mathbf{0})$ finite it follows that $\overline{co(p)}(\mathbf{0}) = \overline{p}(\mathbf{0}) = p(\mathbf{0})$ is finite and this implies by Lemma 78 that $\mathcal{A}_p$ is nonempty. Therefore $\overline{p} > -\infty$ and by the Fenchel-Moreau theorem (Theorem 87) we obtain $v(P) = p(\mathbf{0}) = \overline{p}(\mathbf{0}) = p^{**}(\mathbf{0}) = v(D)$. To prove the reverse implication we observe by Theorem 87 and $v(P) = v(D)$ is finite that $p(\mathbf{0}) = p^{**}(\mathbf{0}) = cl(p)(\mathbf{0})$ is finite. Hence it must follow by Definition 86 that $\overline{p}(\mathbf{0}) = p(\mathbf{0})$ and therefore p is l.s.c at $\mathbf{0}$. To show the second part it follows by Theorem 91 that $\partial p(\mathbf{0})$ is nonempty and by Lemma 90 it is now easy to verify that any $\mathbf{a}_0 \in \partial p(\mathbf{0})$ is an optimal solution of the dual problem. Moreover, by Lemma 71 we obtain that $p(\mathbf{0}) = \overline{p}(\mathbf{0})$ and we can apply the first part. $\square$

Finally we consider the case $v(P) = \infty$. In general it does not hold even for p convex and l.s.c in $\mathbf{0}$ that $v(P) = v(D)$. To show this we will discuss in Example 122 a linear programming problem satisfying $v(P) = \infty$ and $v(D) = -\infty$.

If $f : \mathbb{R}^m \to \mathbb{R}$ is some real valued function and $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^n$ a vector valued function represented by $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), ..., g_n(\mathbf{x}))$, $g_i : \mathbb{R}^m \to \mathbb{R}$ then an important special case of optimization problem (P) is given by

$$\inf\{f(\mathbf{x}) : \mathbf{g}(\mathbf{x}) \in -K, \mathbf{x} \in D\} \tag{P_1}$$

with $K \subseteq \mathbb{R}^n$ a nonempty convex cone and $D \subseteq \mathbb{R}^m$ some nonempty set. The above optimization problem includes some important classes of optimization problems listed in the following example.

Example 118

1. If the nonempty convex cone $K \subseteq \mathbb{R}^n$ is given by $K = \mathbb{R}^p_+ \times \{\mathbf{0}\}$ with $\mathbf{0} \in \mathbb{R}^{n-p}, p \leq n$ and the set $D = \mathbb{R}^m$ then optimization problem (P_1) reduces to the classical nonlinear optimization problem $\inf\{f(\mathbf{x}) : g_i(\mathbf{x}) \leq 0, i = 1, ..., p, g_i(\mathbf{x}) = 0, p+1 \leq i \leq n\}$ (cf.[35], [31]).

2. If $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$ and $\mathbf{g}(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$ with A some $n \times m$ matrix, $K = \{\mathbf{0}\} \subseteq \mathbb{R}^n$ and $D = \mathbb{R}^m_+$ then optimization problem (P_1) reduces to the linear programming problem $\inf\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ (cf.[36], [31]).

3. If $m = n$, $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$, $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ and $D = L + \mathbf{b}$ with $L \subseteq \mathbb{R}^n$ a linear subspace then we obtain the conic convex programming problem $\inf\{\mathbf{c}^\top \mathbf{x} : \mathbf{x} \in (-K) \cap (\mathbf{b} + L)\}$ (cf.[59]).

For optimization problem (P_1) the so-called Lagrangian perturbation scheme is used and this means that the function $F : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty]$ is given by $F(\mathbf{x}, \mathbf{y}) = f_0(\mathbf{x})$ for $\mathbf{x} \in D$ and $\mathbf{g}(\mathbf{x}) \in -K + \mathbf{y}$ and $+\infty$ otherwise. For this specific choice of F we obtain by relation (86) that

$$p(\mathbf{y}) = \inf\{f(\mathbf{x}) : \mathbf{x} \in D, \mathbf{y} \in \mathbf{g}(\mathbf{x}) + K\}. \tag{88}$$

Using the representation of p listed in relation (88) one can give a more detailed expression of the dual problem. Observe this dual problem is called the Lagrangian dual problem.

Lemma 119 *If the function $\theta : K^0 \to [-\infty, \infty]$ is given by $\theta(\mathbf{a}) = \inf\{f(\mathbf{x}) - \mathbf{a}^\top \mathbf{g}(\mathbf{x}) : \mathbf{x} \in D\}$ then the Lagrangian dual of optimization problem (P_1) equals*

$$v(D_L) := \sup\{\theta(\mathbf{a}) : \mathbf{a} \in K^0\}. \tag{D_L}$$

Proof. By the definition of the function p it follows for every $\mathbf{a} \in \mathbb{R}^n$ that

$$\begin{aligned} -p^*(\mathbf{a}) &= -\sup_{\mathbf{y} \in \mathbb{R}^n}\{\mathbf{a}^\top \mathbf{y} - \inf\{f(\mathbf{x}) : \mathbf{y} \in \mathbf{g}(\mathbf{x}) + K, \mathbf{x} \in D\}\} \\ &= -\sup_{\mathbf{y} \in \mathbb{R}^n} \sup\{\mathbf{a}^\top \mathbf{y} - f(\mathbf{x}) : \mathbf{y} \in \mathbf{g}(\mathbf{x}) + K, \mathbf{x} \in D\} \\ &= \inf\{f(\mathbf{x}) - \mathbf{a}^\top \mathbf{y} : \mathbf{y} \in \mathbf{g}(\mathbf{x}) + K, \mathbf{x} \in D\} \end{aligned}$$

and so we obtain

$$-p^*(\mathbf{a}) = \inf\{f(\mathbf{x}) - \mathbf{a}^\top(\mathbf{g}(\mathbf{x}) + \mathbf{k}) : \mathbf{k} \in K, \mathbf{x} \in D\}.$$

To simplify the above expression we first consider $\mathbf{a} \in K^0$. Since by definition $\mathbf{a}^\top \mathbf{k} \leq 0$ for every $\mathbf{k} \in K$ and $\mathbf{0} \in cl(K)$ this implies

$$-p^*(\mathbf{a}) = \inf\{f(\mathbf{x}) - \mathbf{a}^\top \mathbf{g}(\mathbf{x}) \colon \mathbf{x} \in D\} = \theta(\mathbf{a}).$$

Moreover, if $\mathbf{a} \notin K^0$ it follows that one can find some $\mathbf{k}_0 \in K$ satisfying $\mathbf{a}^\top \mathbf{k}_0 > 0$ and since $\alpha \mathbf{k}_0 \in K$ for every $\alpha > 0$ and D not empty we obtain $-p^*(\mathbf{a}) = -\infty$. This shows the desired result and we are done. $\square$

By Lemmas 117 and 119 the following result about Lagrangian duality is easy to derive.

Theorem 120 *If the primal problem is represented by (P_1) and the vector valued function $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^{n+1}$ is given by $\mathbf{h}(\mathbf{x}) := (\mathbf{g}(\mathbf{x}), f(\mathbf{x}))$ and satisfies $\mathbf{h}(D) + K \times (0, \infty)$ is convex and $\mathbf{0} \in ri(\mathbf{g}(D) + K)$ then it follows that $\infty > v(P_1) = v(D_L)$ and the Lagrangian dual problem (D_L) has an optimal solution.*

Proof. Since by assumption $\mathbf{0} \in ri(\mathbf{g}(D)+K) \subseteq \mathbf{g}(D)+K$ we obtain that the feasible region of the optimization problem (P_1) is not empty and this shows $v(P_1) < \infty$. For $v(P_1) = -\infty$ the result follows immediately and so we only consider $v(P_1)$ is finite. To apply Theorem 117 we first need to verify whether the function p is convex. It is easy to check that

$$epi_S(F) = \{(\mathbf{x}, \mathbf{y}, r) \in \mathbb{R}^{m+n+1} : \mathbf{y} \in \mathbf{g}(\mathbf{x}) + K, \mathbf{x} \in D \text{ and } r > f(\mathbf{x})\}$$

and this implies by relation (87) that $epi_S(p) = \mathbf{h}(D) + K \times (0, \infty)$. By assumption this set is convex and hence by Lemma 51 the perturbation function p is convex. Also by relation (87) we obtain $ri(dom(p)) = ri(\mathbf{g}(D) + K)$ and applying Lemma 119 and Theorem 117 the desired result follows. $\square$

The condition $\mathbf{0} \in ri(\mathbf{g}(D) + K)$ is known in the literature as the generalized Slater condition. Observe, if f is a convex function and $\mathbf{g}$ is a so-called K-convex vector valued function (cf.[15], [58]) then it follows that $epi_S(F)$ is a convex set and hence also $\mathbf{h}(D) + K \times (0, \infty)$ is convex. Also it is possible to prove related results under slightly weaker conditions (cf.[23],[22]). As shown by the next lemma the Lagrangian dual (D_L) of a conic convex programming problem is again a conic convex programming problem. Due to the recent developments in interior point methods this class of optimization problems became very important (cf.[59]).

Lemma 121 *If the primal problem (P_1) is a conic convex programming problem given by* $\inf\{\mathbf{c}^\top\mathbf{x} : \mathbf{x} \in (-K)\cap(\mathbf{b}+L)\}$ *and* $(L+\mathbf{b})\cap ri(-K) \neq \varnothing$ *then it follows that* $\infty > v(P_1) = \mathbf{b}^\top\mathbf{c} - \inf\{\mathbf{b}^\top\mathbf{a} : \mathbf{a} \in K^0 \cap (\mathbf{c}+L^\perp)\}$ *and the last dual conic convex optimization problem has an optimal solution.*

Proof. Since the vector valued function $\mathbf{h}$ reduced to $\mathbf{h}(\mathbf{x}) = (\mathbf{x}, \mathbf{c}^\top\mathbf{x})$ and $D = L + \mathbf{b}$ it follows that $\mathbf{h}(D) + K \times (0, \infty)$ is a convex set. Also using $(L+\mathbf{b})\cap ri(-K) \neq \varnothing$ and L is relatively open we obtain by relation (24) that $\mathbf{0} \in L + \mathbf{b} + ri(K) = ri(L + \mathbf{b} + K)$ and this is the generalized Slater condition for a conic convex programming problem. Therefore the above result is an immediate consequence of Theorem 120 once we have evaluated $\theta(\mathbf{a})$ for $\mathbf{a} \in K^0$. Observe now $\theta(\mathbf{a}) = \inf\{(\mathbf{c} - \mathbf{a})^\top\mathbf{x} : \mathbf{x} \in L + \mathbf{b}\} = (\mathbf{c} - \mathbf{a})^\top\mathbf{b} - \inf\{(\mathbf{c} - \mathbf{a})^\top\mathbf{x} : \mathbf{x} \in L\}$ and since the last optimization problem equals 0 for $\mathbf{c} - \mathbf{a} \in L^\perp$ and $-\infty$ otherwise we obtain the desired result. $\square$

Finally we observe that by a similar computation as in Lemma 121 it is easy to check that the Lagrangian dual of the linear programming problem $\inf\{\mathbf{c}^\top\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is given by $\sup\{\mathbf{b}^\top\mathbf{a} : A^\top\mathbf{a} \leq \mathbf{c}\}$ and so this dual problem reduces to the ordinary dual listed in many text books (cf.[36]). To conclude this section we consider the following example of a linear programming problem satisfying $v(P) = \infty$ and $v(D) = -\infty$.

Example 122 *Consider the linear programming problem*

$$\inf\{-x_1 - x_2 : x_1 - x_2 \geq 1, -x_1 + x_2 \geq 1, \mathbf{x} \in \mathbb{R}^2_+\}.$$

Clearly this optimization problem has an empty feasible region and so $v(P_1) = \infty$. *Penalizing the constraints* $x_1 - x_2 - 1 \geq 0$ *and* $-x_1 + x_2 - 1 \geq 0$ *using the nonpositive Lagrangian multipliers* a_1 *and* a_2 *we obtain that the Lagrangian function* $\theta : \mathbb{R}^2_+ \to [-\infty, \infty)$ *is given by*

$$\theta(\mathbf{a}) = \inf\{x_1(\lambda_1 - \lambda_2 - 1) + x_2(\lambda_2 - \lambda_1 - 1) : \mathbf{x} \in \mathbb{R}^2_+\}$$

Observe now for every $\mathbf{a} \in \mathbb{R}^2_+$ *that*

$$a_1 - a_2 - 1 \geq 0 \Rightarrow a_2 - a_1 - 1 \leq -2$$

and

$$a_2 - a_1 - 1 \geq 0 \Rightarrow a_1 - a_2 - 1 \leq -2$$

and by this observation it follows that $\theta(\mathbf{a}) = -\infty$ *for every* $\mathbf{a} \in \mathbb{R}^2_+$ *or equivalently* $v(D) = -\infty$.

One can also use the same Lagrangian perturbation scheme and the dual representation of an evenly quasiconvex function and the corresponding c_r-conjugate function to introduce the so-called surrogate dual. Due to limited space we will not discuss the properties of such a dual but refer the reader to the literature cited in [21]. This concludes our discussion on duality and optimization problems. In the next subsection we will consider the structure of positively homogeneous evenly quasiconvex functions.

4.3 Positively homogeneous evenly quasiconvex functions and dual representations.

In this subsection we will use the dual representation of an evenly quasiconvex function to analyze the class of positively homogeneous evenly quasiconvex functions. To start with this investigation we consider an arbitrary positively homogeneous evenly quasiconvex function $f : \mathbb{R}^n \to (-\infty, \infty]$ satisfying $\mathbf{0} \in dom(f)$. Since f is positively homogeneous and $f > -\infty$ it follows by Lemma 52 that $\mathbf{0} \in dom(f)$ if and only if $f(\mathbf{0}) = 0$. Considering for every $\mathbf{a} \in \mathbb{R}^n$ the function $c_{\mathbf{a}}(t) := \inf\{f(\mathbf{y}) : \mathbf{a}^\top \mathbf{y} \geq t\}$ introduced in Definition 99 it is easy to verify the next result.

Lemma 123 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is positively homogeneous then for every $\mathbf{a} \in \mathbb{R}^n$ it follows that the function $c_{\mathbf{a}} : \mathbb{R}^n \to [-\infty, \infty]$ is positively homogeneous and nondecreasing.*

Proof. For $\mathbf{a} \neq \mathbf{0}$ it is easy to verify that $c_{\mathbf{a}}$ is nondecreasing and by Lemma 52 we obtain for every $\alpha > 0$ and $t \in \mathbb{R}$ that $c_{\mathbf{a}}(\alpha t) = \inf\{f(\alpha \mathbf{y}) : \mathbf{a}^\top \mathbf{y} \geq t\} = \alpha c_{\mathbf{a}}(t)$. Moreover, for $\mathbf{a} = \mathbf{0}$ we obtain for every $\alpha > 0$ and $t \leq 0$ that $c_{\mathbf{0}}(\alpha t) = \inf\{f(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\} = \inf\{f(\alpha \mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\} = \alpha c_{\mathbf{0}}(t)$, while for $\alpha > 0$ and $t > 0$ it follows by the convention $\inf\{\varnothing\} = \infty$ that $c_{\mathbf{0}}(\alpha t) = \inf\{\varnothing\} = \infty = \alpha c_{\mathbf{0}}(t)$. Trivially $c_{\mathbf{0}}$ is nondecreasing and the proof is completed. $\square$

To analyze the behaviour of a positively homogeneous evenly quasiconvex function f satisfying $f > -\infty$ and $\mathbf{0} \in dom(f)$ we first decompose this function. Using a slightly different decomposition as done by Crouzeix (cf.[18]) we introduce the function f_+ given by

$$f_+(\mathbf{x}) := 0, \ \mathbf{x} \in L_S(f, 0) \text{ and } f_+(\mathbf{x}) = f(\mathbf{x}) \text{ otherwise.} \qquad (89)$$

For this function it follows trivially that $f_+ \geq 0$ and using $f > -\infty$ and $\mathbf{0} \in dom(f)$ that $f_+(\mathbf{0}) = 0$. Moreover, the function f_- is given by

$$f_-(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in cl(L_S(f,0)) \text{ and } f_-(\mathbf{x}) = \infty \text{ otherwise.} \tag{90}$$

To analyze the function f_- it is only interesting to consider positively homogeneous evenly quasiconvex functions f satisfying $L_S(f,0) \neq \varnothing$. If this holds we obtain by Lemma 52 that $L_S(f,0)$ is a nonempty convex cone and since $\mathbf{0} \in cl(L_S(f,0))$ it follows that $f_-(\mathbf{0}) = f(\mathbf{0}) = 0$. Also for every $r \in \mathbb{R}$ we obtain that $L_S(f_-,r) = cl(L_S(f,0)) \cap L_S(f,r)$ and this yields for $r = 0$ that $L_S(f_-,0) = L_S(f,0)$. By relation (90) we therefore obtain

$$L_S(f,0) \neq \varnothing \Rightarrow dom(f_-) \subseteq cl(L_S(f,0)) = cl(L_S(f_-,0)). \tag{91}$$

Since trivially $f_- \geq f$ it is easy to verify considering the cases $f(\mathbf{x}) \geq 0$ and $f(\mathbf{x}) < 0$ that

$$f(\mathbf{x}) = \min\{f_+(\mathbf{x}), f_-(\mathbf{x})\} \tag{92}$$

for every $\mathbf{x} \in \mathbb{R}^n$ and this relation turns out to be very important. For the functions f_+ and f_- one can now show the following result.

Lemma 124 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is a positively homogeneous evenly quasiconvex function then f_+ and f_- are positively homogenous and evenly quasiconvex.*

Proof. Since f is positively homogeneous and evenly quasiconvex we obtain by Lemma 52 and 56 that $L_S(f,0)$ is a (possibly empty) convex cone. This implies again by Lemma 52 that f_+ is positively homogeneous. To show that f_+ is evenly quasiconvex we observe that $L(f_+,r) = L(f,r)$ for every $r > 0$. Also by the definition of f_+ we obtain

$$L(f_+,0) = L_S(f,0) \cup \{\mathbf{x} : \mathbf{x} \notin L_S(f,0) \text{ and } f(\mathbf{x}) \leq 0\} = L(f,0)$$

and since $f_+ \geq 0$ this implies using f is evenly quasiconvex that also f_+ is evenly quasiconvex. To verify the same result for f_- we observe since $cl(L_S(f,0))$ is also a (possibly empty) convex cone that f_- is positively homogeneous. Moreover, for every $r \in \mathbb{R}$ we obtain $L(f_-,r) = cl(L_S(f,0)) \cap L(f,r)$ and applying Lemma 42 and f is evenly quasiconvex it follows that f_- is evenly quasiconvex. $\square$

We will now show the following result for nonnegative positively homogeneous evenly quasiconvex functions f with $\mathbf{0} \in dom(f)$. Observe by Lemma 124 the next result also holds for f_+ in case $f > -\infty$ is a positively homogeneous evenly quasiconvex function and $\mathbf{0} \in dom(f)$. Remember a function is called sublinear if it is positively homogeneous and convex.

Lemma 125 *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is a nonnegative positively homogeneous evenly quasiconvex function with $\mathbf{0} \in dom(f)$ then f is a nonnegative l.s.c sublinear function.*

Proof. By the dual representation of an evenly quasiconvex function (Theorem 102) we obtain

$$f(\mathbf{x}) = \sup_{\mathbf{a}\in\mathbb{R}^n} c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}). \tag{93}$$

Since $f \geq 0$ it follows that $c_{\mathbf{a}} \geq 0$ for every $\mathbf{a} \in \mathbb{R}^n$. Moreover, using $f(\mathbf{0}) = 0$ and $0 \leq c_{\mathbf{a}}(0) \leq f(\mathbf{0})$ we obtain $c_{\mathbf{a}}(0) = 0$. Also for a given $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$ satisfying $\mathbf{a}^\top \mathbf{x} \leq 0$ it is easy to see that $0 \leq c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) \leq c_{\mathbf{a}}(0) = 0$ and this implies $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = 0$ for every $\mathbf{a}^\top \mathbf{x} \leq 0$. Moreover, for $\mathbf{a}^\top \mathbf{x} > 0$ we obtain by Lemma 123 that $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = r_{\mathbf{a}} \mathbf{a}^\top \mathbf{x} \geq 0$ with $r_{\mathbf{a}} := c_{\mathbf{a}}(1)$ and hence we have shown that $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = \max\{r_{\mathbf{a}} \mathbf{a}^\top \mathbf{x}, 0\}$ for every $\mathbf{a} \in \mathbb{R}^n$. Applying now relation (93) yields

$$f(\mathbf{x}) = \sup_{\mathbf{a}\in\mathbb{R}^n} \max\{r_{\mathbf{a}} \mathbf{a}^\top \mathbf{x}, 0\} = \max\{\sup_{\mathbf{a}\in\mathbb{R}^n} r_{\mathbf{a}} \mathbf{a}^\top \mathbf{x}, 0\} \tag{94}$$

and since clearly $\mathbf{x} \to \sup_{\mathbf{a}\in\mathbb{R}^n} r_{\mathbf{a}} \mathbf{a}^\top \mathbf{x}$ is a l.s.c sublinear function the desired result follows by relation (94). $\square$

Finally we will show the following result for a positively homogeneous evenly quasiconvex function f satisfying $f > -\infty$, $L_S(f, 0)$ nonempty and $dom(f) \subseteq cl(L_S(f, 0))$. By Lemma 124 this result also applies to the function f_- given in relation (90).

Lemma 126 *If $f : \mathbb{R}^n \to (-\infty, \infty]$ is a positively homogeneous evenly quasiconvex function with $\mathbf{0} \in dom(f) \subseteq cl(L_S(f, 0))$ then f is a nonpositive l.s.c sublinear function.*

Proof. As in Lemma 125 the representation (93) holds and therefore we will analyze the function $c_{\mathbf{a}}$ for every $\mathbf{a}$. If $\mathbf{a} \notin (L_S(f, 0))^0$ there exists some $\mathbf{x}_0 \in L_S(f, 0)$ satisfying $r := \mathbf{a}^\top \mathbf{x}_0 > 0$. By Lemma 123 this yields for every $t > 0$ that $c_{\mathbf{a}}(t) = tr^{-1} c_{\mathbf{a}}(r) \leq tr^{-1} f(\mathbf{x}_0) < 0$ and so $c_{\mathbf{a}}(\infty) := \lim_{t\uparrow\infty} c_{\mathbf{a}}(t) = -\infty$. Since also by Lemma 123 the function $c_{\mathbf{a}}$ is nondecreasing this implies

$$\mathbf{a} \notin (L_S(f, 0))^0 \Rightarrow c_{\mathbf{a}} \equiv -\infty. \tag{95}$$

By assumption $f > -\infty$ and so it follows by relations (93) and (95) that

$$f(\mathbf{x}) = \sup\{c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) : \mathbf{a} \in (L_S(f,0))^0\}. \tag{96}$$

To analyze $c_{\mathbf{a}}(t)$ for $\mathbf{a} \in (L_S(f,0))^0$ and $t \leq 0$ we first assume that there exists some $\mathbf{x}_0 \in L_S(f,0)$ satisfying $\mathbf{a}^\top \mathbf{x}_0 = 0$. This shows for every $t \leq 0$ that $c_{\mathbf{a}}(t) \leq c_{\mathbf{a}}(0) \leq f(\mathbf{x}_0) < 0$ and applying Lemma 123 it follows that $c_{\mathbf{a}}(t) \leq c_{\mathbf{a}}(0) = -\infty$. Hence we have verified that

$$\mathbf{a} \in (L_S(f,0))^0 \text{ and } \exists_{\mathbf{x}_0 \in L_S(f,\,0)} \ \mathbf{a}^\top \mathbf{x}_0 = 0 \Rightarrow \forall_{t \leq\, 0}\ c_{\mathbf{a}}(t) = -\infty$$

and using again $f > -\infty$ and relation (96) this implies

$$f(\mathbf{x}) = \sup\{c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) : \mathbf{a} \in K_0\} \tag{97}$$

with $K_0 := \{\mathbf{a} \in (L_S(f,0))^0 : \mathbf{a}^\top \mathbf{y} < 0 \text{ for every } \mathbf{y} \in L_S(f,0)\}$. We will now analyze the behaviour of $\mathbf{x} \to c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x})$ for an arbitrary $\mathbf{a} \in K_0$. If $\mathbf{a}^\top \mathbf{x} > 0$ it follows for every $\mathbf{y}$ satisfying $\mathbf{a}^\top \mathbf{y} \geq \mathbf{a}^\top \mathbf{x} > 0$ that $\mathbf{y} \notin cl(L_S(f,0))$ and since $dom(f) \subseteq cl(L_S(f,0)$ this shows $f(\mathbf{y}) = \infty$ or equivalently $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = \infty$. Also, if $\mathbf{a}^\top \mathbf{x} = 0$ then for every $\mathbf{y}$ satisfying $\mathbf{a}^\top \mathbf{y} \geq \mathbf{a}^\top \mathbf{x} = 0$ it follows that $f(\mathbf{y}) \geq 0$ and since $\mathbf{0} \in dom(f)$ this implies $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = 0$. Finally, for $\mathbf{a} \in K_0$ and $\mathbf{a}^\top \mathbf{x} < 0$ it follows by Lemma 123 that $c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = \widetilde{r}_{\mathbf{a}} \mathbf{a}^\top \mathbf{x}$ with $\widetilde{r}_{\mathbf{a}} := -c_{\mathbf{a}}(-1)$ and since $L_S(f,0)$ is nonempty we obtain $0 < \widetilde{r}_{\mathbf{a}} \leq \infty$. Hence we have shown for every $\mathbf{a} \in K_0$ that $c_{\mathbf{a}}(0) = 0$ and

$$c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = \widetilde{r}_{\mathbf{a}} \mathbf{a}^\top \mathbf{x} \text{ if } \mathbf{a}^\top \mathbf{x} < 0 \text{ and } c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = \infty \text{ if } \mathbf{a}^\top \mathbf{x} > 0.\text{‘}$$

By relation (97) and $f > -\infty$ we obtain that the set $S := \{\mathbf{a} \in K_0 : 0 < \widetilde{r}_{\mathbf{a}} < \infty\}$ is nonempty and so

$$f(\mathbf{x}) = \sup\{c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) : \mathbf{a} \in S\}. \tag{98}$$

Since for $\mathbf{a} \in S$ it follows that $-\infty < c_{\mathbf{a}}(\mathbf{a}^\top \mathbf{x}) = \widetilde{r}_{\mathbf{a}} \mathbf{a}^\top \mathbf{x}$ for $\mathbf{a}^\top \mathbf{x} \leq 0$ and ∞ otherwise this is clearly a l.s.c sublinear function and by relation (98) the desired result follows. □

Using relation (92) and Lemma 124 up to 126 the following remarkable result follows immediately.

Theorem 127 *If $f : \mathbb{R}^n \to (-\infty, \infty]$ is a positively homogeneous evenly quasiconvex function and $\mathbf{0} \in dom(f)$ then f can be written as the minimum of a nonpositive l.s.c sublinear function and a nonnegative l.s.c sublinear function.*

Proof. If $L_S(f,0)$ is empty then $f \geq 0$ and the result follows by Lemma 125. Moreover, if $L_S(f,0)$ is nonempty then by relation (92) it follows that $f = \min(f_+, f_-)$ and applying Lemma 124 up to 126 yields the desired result. □

By Theorem 127 every positively homogeneous evenly quasiconvex function f satisfying $f > -\infty$ and $\mathbf{0} \in dom(f)$ must be the minimum of two l.s.c sublinear functions and so it is l.s.c. Observe by relation (64) these l.s.c sublinear functions can be written as support functions. This is a rather remarkable result which does not hold in general for evenly quasiconvex functions. As an example we mention the evenly quasiconvex function sign(x) given by

$$sign(x) = -1 \text{ if } x < 0,\ sign(0) = 0 \text{ and } sign(x) = 1 \text{ if } x > 0$$

which is neither upper or lower semicontinuous at 0. To conclude this subsection we observe that Theorem 127 is an extension of the main result in Crouzeix (cf.[18]). For related results see also [19], [17] and [16]. Introducing now the Dini upper directional derivative $\mathbf{d} \to f_+^D(\mathbf{x}, \mathbf{d})$ given by

$$f_+^D(\mathbf{x}, \mathbf{d}) := \limsup_{t \downarrow 0} t^{-1}(f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x}))$$

(cf.[8]) it is possible to use the above so-called Crouzeix representation theorem for positively homogeneous quasiconvex functions to analyze the global behaviour of the function $\mathbf{d} \to f_+^D(\mathbf{x}, \mathbf{d})$ for f quasiconvex (cf.[18], [45], [46], [29]). This concludes our discussion of positively homogeneous evenly quasiconvex functions and dual representations. In the next section we mention some mile stone papers and books within the long history of convex and quasiconvex analysis.

5 Some remarks on the history of convex and quasiconvex analysis.

In this section[2] we will discuss the origin of the important notions used in convex and quasiconvex analysis. It seems that the field of convex geometry and convex bodies in two and three dimensional space was first studied systematically by H.Brunn (cf.[9], [10]) and Minkowski (cf.[13]).

[2]The authors like to thank Prof.Kolumbán (Cluj) and Prof. Komlósi (Pecs) for pointing out some of the early developments.

Brunn (cf.[11]) and Minkowski (cf.[14]) also proved the existence of support hyperplanes. Also at the end of $19th$ and the beginning of the $20th$ century Farkas showed in a series of papers (cf.[46], [2]) the alternative theorem for linear inequality systems and this result became known as Farkas lemma within linear programming. Although this result was listed with an incorrect proof in some of his earlier papers a correct proof of this result appeared in [27]. More fundamental ideas about the related field of necessary optimality conditions for nonlinear optimization subject to inequality constraints can be found in papers by Fourier, Cournot, Gauss, Ostrogradsky, and Hamel (cf.[2]). On the other hand, more early references related to the study of convex sets are listed in the reprinted version of the 1934 book of Bonnesen and Fenchel (cf.[48]), Fenchel (cf.[54]), Valentine (cf.[5]) and Varberg (cf.[3]). Also at the beginning of the 20th century convex functions were introduced by Jenssen (cf.[30]) and more than forty years later a thorough study of conjugate functions in $\mathbb{R}^n$ was initiated by Fenchel (cf.[53]). Although Mandelbrojt (cf.[47]) already introduced the conjugate function in $\mathbb{R}^n$ for $n = 1$ (cf.[52]) it was Fenchel who first realized the importance of the conjugacy concept in convex analysis. Four years before the mile stone paper of Fenchel, also the first book on convex functions written in French by Popoviciu (cf.[50]) was published. In the English scientific community the unpublished lecture notes by Fenchel (cf.[54]) were a long time the main source of references. This book served as the main inspiration for the classical book of Rockafellar (cf.[44]) as noted in its preface. Also in this preface it is mentioned that Prof. Tucker suggested the name convex analysis and this became the standard word for this field. The introduction of quasiconvex functions started later. Although in most of the literature de Finetti ([7]) is mentioned as being the first author introducing quasiconvex functions these functions were already considered by von Neumann (cf.[38] and independently Popoviciu (cf.[49]). Actually von Neumann (cf.[38]) already proved in 1928 a minimax theorem on simplices for bifunctions which are quasiconcave in one variable and quasiconvex in the other variable and a slight generalization of this result was rediscovered by Sion (cf.[37]) 30 years later. For more details on the development of quasiconvex functions the reader is referred to [34]. To develop results for the surrogate dual concept developed by Glover (cf.[6]) an adhoc approach involving the $\overline{c_r}$-conjugate function was initiated by Greenberg and Pierskalla (cf.[12]). Their results were generalized and put into the proper framework of dual representations by Crouzeix in a series of mile stone papers (cf.[16], [17], [18], [19]). In these papers Crouzeix focussed his attention on the dual representation of the l.s.c hull of a quasiconvex function. Although Fenchel (cf.[55]) already introduced the concept of an evenly convex set the usefulness of this concept leading to a more symmetrical dual representation of an evenly quasiconvex function was discovered independently by Passy and Prisman (cf.[51]) and Martinez Legaz (cf.[33]). This concludes our short excursion, which is by no means complete, to the history of convex and quasiconvex analysis.

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