Transit Functions on Graphs (and Posets)

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Abstract

The notion of transit function is introduced to present a unifying approach for results and ideas on intervals, convexities and betweenness in graphs and posets. Prime examples of such transit functions are the interval function I and the induced path function J of a connected graph. Another transit function is the all-paths function. New transit functions are introduced, such as the cut-vertex transit function and the longest path function. The main idea of transit functions is that of 'transferring' problems and ideas of one transit function to the other. For instance, a result on the interval function I might suggest similar problems for the induced path function J. Examples are given of how fruitful this transfer can be. A list of Prototype Problems and Questions for this transferring process is given, which suggests many new questions and open problems.

Keywords: transit function, interval function, induced path, convexity, betweenness, path function, block graph.

MCS: 05C99, 05C12, 05C38, 05C75.

1 Introduction

The geodetic interval function I and the geodetic convexity of a connected graph are important tools in problems that involve distance and shortest paths (i.e. geodesics). But there are many other ways how one would like to move around in a graph. Another prime example is that where one uses induced paths instead of shortest paths to move from one vertex to another. This yields another type of convexity on a graph. As can be expected, proof techniques for the geodetic case and the induced path case are very different. On the other hand, problems and results that are obtained for the interval function I might suggest problems and questions for the induced path function J. An interesting example of this approach is the following. An interval $I(u, v)$ in a connected graph consists of all vertices that are in between u and v, that is, lie on some shortest u, v-path. A median graph is defined by the fact that $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$, for any three vertices u, v, w. These graphs allow a rich structure theory, and there are by now many applications available in such diverse areas as evolutionary biology, chemistry, location theory, consensus theory, and so forth, and so forth (see e.g. [19, 16, 14]). This condition on the intersection of three intervals and the theory that followed from it suggested to me already in 1991 to consider the question what one might say about graphs for which there are similar conditions on the intersection $J(u, v) \cap J(v, w) \cap J(w, u)$, for any three vertices u, v, w. Here $J(u, v)$ is the set of all vertices on induced u, v-paths. Because of the big difference between shortest path and induced path one could not expect right away that this idea would lead to anything. But interestingly enough, non-trivial results could be obtained, see [17], where results as well as proof techniques are quite different. In January 1998 at the Conference on "Graph Connections" at Cochi University of Science and Technology the idea came to me that this transferring of problems from one such function to another could be utilized to generate many problems and questions that might be worthwhile to study in their own right. This is the underlying motivation for the idea of transit function. A *transit function* is a very general notion that unifies the many ways how one could move around in a graph. But it can also be used to study other structural aspects of graphs, such as the structure of the blocks and cut-vertices, or convexity and betweenness. Moreover, other discrete structures than graphs, for instance partially ordered sets, or hypergraphs, can be considered as well. The general notion of transit function in itself might not be very interesting. But for this transferring process of ideas, problems and questions from one transit function to another it has already proven to be worthwhile, as can be concluded from the examples and references below.

A transit function $R : V \times V \to 2^V$ satisfies three simple axioms. The basic idea is: given a set V, a discrete structure σ on V and a transit function R on V, study the interrelations between σ and R. In our case, the structure σ is the set of edges E of a graph. The other pertinent structure from our point of view is that of a partial order $\leq$ on V. We will discuss this case only shortly. First the notion of transit function on a discrete structure (V, σ) is introduced in its most general form, with associated notions, amongst which 'interval', 'convexity', and 'betweenness'. In this framework, twenty prototype questions and problems are formulated. Most of these were suggested by known results on the interval function or the induced path function. Then we give examples, from the literature, but also some new ones to show the fruitfulness of this approach. Some ideas below are just first thoughts on the topic and may need adjustment or further formalization before they can be properly pursued.

2 Transit functions

□ A *discrete structure* (V, σ) consists of a finite set V and a 'structure' σ on V. Prime examples of discrete structures on V are a graph $G = (V, E)$, where $\sigma = E$ is the edge set of the graph, and a partially ordered set, poset for short, $(V, \leq)$, where $\sigma = \leq$ is a partial ordering on V. We denote the power set of V by 2^V.

□ A *transit function* on a discrete structure (V, σ) is a function $R : V \times V \to 2^V$ satisfying the three transit axioms

($t1$) $u \in R(u, v)$, for any u and v in V,

($t2$) $R(u, v) = R(v, u)$, for all u and v in V,

($t3$) $R(u, u) = \{u\}$, for all u in V.

Axioms as these, which are phrased in terms of the function only, will be called *transit axioms*. Axiom ($t3$) is to avoid degenerate cases. One might consider also transit functions that do not satisfy ($t3$). These could then be called *weak transit functions*. Note that the structure σ is not involved in the definition of a transit function. But, of course, the focus of our interest will be on the interplay between the function R and the structure σ. Usually, R will just be defined in terms of σ.

In this paper our main point of focus is transit functions on graphs. Then we are interested in the interrelations between the two structures E and R on V. Basically, a transit function on a graph G describes how we can get from vertex u to vertex v: via vertices in $R(u, v)$. A prime example is the *interval function* I defined by

$$I(u, v) = \{ \ w \mid d(u, w) + d(w, v) = d(u, v) \ \},$$

see [19] for an extensive study of the interval function. In the terminology of this paper one would call the interval function the *geodetic transit function*.

□ The *trivial transit function* $\mathbf{1}$ on (V, σ) is defined by $\mathbf{1}(u, u) = \{u\}$, for all u in V and $\mathbf{1}(u, v) = V$, for all distinct u, v in V. The *discrete transit function* $\mathbf{0}$ is defined by $\mathbf{0}(u, u) = \{u\}$, for all u in V, and $\mathbf{0}(u, v) = \{u, v\}$, for all distinct u, v in V.

□ Let R and S be two transit functions on (V, σ). The *join* of R and S is the transit function $R \vee S$ defined by $R \vee S(u, v) = R(u, v) \cup S(u, v)$. The *meet* of R and S is the function $R \wedge S$ defined by $R \wedge S(u, v) = R(u, v) \cap S(u, v)$. With this join and meet the family of transit functions on (V, σ) is a lattice with the discrete transit function $\mathbf{0}$ as universal lower bound and the trivial transit function $\mathbf{1}$ as universal upper bound.

□ The *underlying graph* G_R of transit function R is the graph with vertex set V, where distinct u and v in V are joined by an edge if and only if $|R(u, v)| = 2$.

□ The *transit hypergraph* of R has V as vertex set, and, for distinct u and v, a set $R(u, v)$ is an edge if and only if for each w in $R(u, v)$ distinct from u, v we have $R(u, v) = R(u, w)$. The *transit graph* of R is the underlying graph of the transit hypergraph, that is, u and v are adjacent if they are in the same edge of the transit hypergraph. Note that there may be other types of hypergraph to be associated with a specific transit function.

□ A set $W \subseteq V$ is an *R-convex set* in (V, σ) if $R(u, v) \subseteq W$, for all u, v in W. The family $\mathcal{C}_R$ of R-convex sets in (V, σ) is an abstract convexity: it is closed under intersections and both $\emptyset$ and V are R-convex sets. The family of R-convex sets is called the *R-convexity* on G.

Note that axiom $(t3)$ implies that the R-convexity satisfies the so-called separation axiom S_1, that is, singletons are R-convex. Note also that in the infinite case there is yet another condition to be satisfied by an abstract convexity: it must be closed under nested unions. Let us keep in mind that, in case we encounter a convexity that is not S_1, we may drop axiom $(t3)$ to obtain a weak transit function R to be associated with this convexity.

An R-convexity of a transit function R is also known as an *interval space*, cf. [34], and may be studied from the viewpoint of abstract convexity theory. Thus one might ask for the usual convexity parameters of a specific R-convexity. In the context of transit functions we will focus mainly on other types of questions: those related to the basic ideas of transit functions on discrete structures as exemplified below.

□ Let $\mathcal{C}$ be any convexity on the finite set V, that is, $\mathcal{C}$ is a family of subsets of V closed under intersection and containing $\emptyset$ and V. The sets in $\mathcal{C}$ are the convex sets of the convexity. The convex hull $co_{\mathcal{C}}(W)$ of a subset W of V with respect to the convexity $\mathcal{C}$ is the smallest convex set containing W, that is, the intersection of all convex sets containing W. We assume that the convexity $\mathcal{C}$ is S_1, so that all singletons are convex. The transit function $R_{\mathcal{C}}$ defined by $R_{\mathcal{C}}(u, v) = co_{\mathcal{C}}(u, v)$ is the *canonical transit function* of the convexity. Note that the convexity defined by $R_{\mathcal{C}}$ need not be $\mathcal{C}$ itself. For instance, take the convexity $\mathcal{C}$ consisting of V and all subsets of V of size at most 2. Then the convexity defined by $R_{\mathcal{C}}$ consists of all subsets of V.

□ Let R be a transit function on (V, σ). Let $\mathcal{C} = \mathcal{C}_R$ be the R-convexity, and let R^* be the canonical transit function of $\mathcal{C}$. Then, clearly, R and R^* define the same convexity $\mathcal{C}$ on V, even if R and R^* do not coincide. So, unless $R = R^*$, trivially there is always one other transit function defining the same convexity as R. If there is no transit function S on (V, σ) distinct from both R and R^* such that the R-convexity and the S-convexity coincide, then we say that the R-convexity is *uniquely determined by R*.

□ For a transit function R, we define

$$R(u, v, w) = R(u, v) \cap R(v, w) \cap R(w, u).$$

This intersection with respect to the interval function I of a connected graph plays a major role in the study of median graphs and modular graphs, cf. [19, 14], see also Section 4.1 below.

□ The discrete structure (V, σ) is said to be *R-monotone* if the sets $R(u, v)$ are R-convex, for all u, v in V, cf. the notion of interval monotone graph in [19].

□ For a transit function R, we define the transit function F_R by

$$F_R(u, v) = \{ \ w \mid R(u, w) \cap R(w, v) = \{w\} \ \}.$$

In case $R = I$, the transit function F_I played a role in the study of Hamming graphs, i.e. Cartesian products of complete graphs, see [19]. Of course, at that time it was not yet phrased in terms of transit functions.

□ Let R be a transit function on (V, σ), let W be a subset of V, and let z be a vertex of G outside W. A vertex x in W is an *R-gate* for z in W if x lies in $R(z, w)$, for any w in W. The set W is called an *R-gated set*, or an *R-prefiber*, if every vertex z outside W has a unique R-gate in W. We note here that the notions of R-gate and R-gated set may need some adjustments, or may give rise to some specializations. For instance, here we do not (yet?) require that each vertex w inside an R-gated set W has a unique R-gate in W. In the case of the interval function I, each vertex x in a set W trivially is its own gate in W.

The notion of gate was first introduced by Goldman and Witzgall [12] with respect to geodesics, that is, in our current terminology with respect to the interval function I. It played an essential role in the study of quasi-median graphs, which generalize median graphs to the non-bipartite case, see [19, 2]. Tardif [33] proposed the term prefiber for I-gated set, because these sets play an important role in the embedding of a graph into a Cartesian product of graphs. Loosely speaking, one may consider the Cartesian product to be a fabric of graphs woven with fibers. The fibers are the factors in the product. When one wants to embed a graph in a product one of the main problems is to find suitable factors, or fibers, for this product. Candidates are the I-gated sets, or *prefibers*.

□ For a transit function R, we define the transit function Q_R by

$$Q_R(u, u) = \{u\},$$
$$Q_R(u, v) = \{ \ w \in R(u, v) \mid v \notin R(u, w), u \notin R(v, w) \ \}.$$

This transit function highlights the elements w that might be "between" u and v. So it may play a role in the study of betweenness, see the next item.

□ A transit function R on (V, σ) is a *betweenness* if it satisfies the additional *betweenness axioms*:

($b1$) if $x \in R(u,v)$ and $x \neq v$, then $v \notin R(u,x)$, for u, v in V,

($b2$) if $x \in R(u,v)$, then $R(u,x) \subseteq R(u,v)$, for u, v in V.

Note that, as for transit functions, a betweenness can just be defined on the set V by ignoring the structure σ. Thus one can develop an abstract theory of betweenness as in [31]. The notion of betweenness in the above sense was first introduced in [17]. For a slightly different notion of betweenness, see [30].

□ In the rest of this section we restrict ourselves to graphs, that is, σ is the edge set of a graph $G = (V, E)$. Let Φ be a property of paths. A Φ-*path* is a path having property Φ. Formally, one takes Φ to be a subset of all paths in G, so that a Φ-path is a path in Φ. We will only consider *feasible path properties*, that is, path properties Φ such that for each pair of vertices in G there exists a Φ-path. A Φ-*path transit function*, or Φ-*path function* for short, is a transit function R on G such that

$$R(u,v) = \{\ x \mid x \text{ is on some } \Phi\text{-path between } u \text{ and } v\ \}.$$

A *path transit function*, or *path function* for short, is a Φ-path transit function for some feasible path property Φ. The family of path functions on a graph is a join semilattice. The all-paths function A is the universal upper bound of this semilattice, see Section 4.4. See [10] for a first systematic study of path functions.

□ Let $P = u_1 \to u_2 \to \ldots \to u_k$ be a path in G. Let z_i be a vertex not on P but adjacent to two consecutive vertices u_i, u_{i+1} of P. Then we say that the path $Q = u_1 \to u_2 \to \ldots \to u_i \to z_i \to u_{i+1} \to \ldots \to u_k$ is obtained from P by "replacing" the edge $u_i \to u_{i+1}$ by a triangle. A *triangular extension* of a path P is a path Q obtained from P by replacing some of the edges of P by triangles. We will call P a triangular extension of itself as well. Let Φ be a (feasible) path property on G. Then $\Phi^{\triangle}$ is the path property defined by

$$\Phi^{\triangle} = \{\ Q \mid Q \text{ is a triangular extension of some path in } \Phi\}.$$

Note that we have $\Phi \subseteq \Phi^{\triangle}$, with equality if and only if no path in Φ is involved in a triangle. In particular, we have equality in the case of a triangle-free graph. The path property $\Phi^{k\triangle}$ is defined recursively by $\Phi^{0\triangle} = \Phi$ and $\Phi^{k\triangle} = \left(\Phi^{(k-1)\triangle}\right)^{\triangle}$, for $k \geq 1$. And for $m \geq 1$ with $\Phi^{(m+1)\Delta} = \Phi^{m\Delta}$, we write $\Phi^{m\Delta} = \Phi^{\Delta^*}$. If R is the Φ-path transit function, then we write $R^{k\Delta}$ for the $\Phi^{k\Delta}$-path function, and R^{Δ^*} for the Φ^{Δ^*}-transit function.

□ A *rapid transit function* on (V, σ) is a function $r : V \times V \to 2^V$ satisfying the three rapid transit axioms

($rt1$) if $r(u,v) \neq \emptyset$, then $u \in r(u,v)$,

($rt2$) $r(u,v) = r(v,u)$, for all u and v in V,

($rt3$) for any two elements u and v in V, there exists a finite sequence of elements $u_1, u_2, \ldots, u_{k(u,v)}$ such that $u = u_1$ and $v = u_{k(u,v)}$ and $r(u_i, u_{i+1}) \neq \emptyset$, for $i = 1, \ldots, k(u,v)$.

A rapid transit function tells us what direct connections we have: if $r(u, v)$ is nonempty, then we have a direct connection, otherwise we have to change at intermediate elements given by $(rt3)$. The underlying graph G_r of the rapid transit function r is defined by: uv is an edge in G_r if and only if u and v are distinct vertices with $r(u, v) \neq \emptyset$. An example of a rapid transit function on a graph $G = (V, E)$ is: $r_k(u, v) = \{u, v\}$ if $d(u, v) \leq k$. The underlying graph of r_k is the k-th power of G. Hence r_1 just gives the edges of the graphs. The theme of rapid transit functions will be pursued elsewhere.

3 Prototype Problems and Questions

In this section we collect a number of prototype problems and questions. Some were inspired by known results on the interval function and are now presented for any transit function. We will give the references to the pertinent literature in the following sections, where we discuss specific examples of transit functions. Some of the problems might seem a little bit cryptic at first reading, but they will make sense in conjunction with the examples in the following sections.

We formulate all the problems and questions only for transit functions on graphs, but almost all of these can also be formulated for other discrete structures (V, σ).

1. Characterize transit function R solely in terms of transit axioms on R. This problem becomes relevant as soon as the transit function is defined in terms of the structure σ, for instance in the case of path functions.

2. Let $\mathcal{C}$ be a convexity on G, i.e. $\mathcal{C}$ is a family of subsets of V that is closed under intersections and both $\emptyset$ and V are in $\mathcal{C}$. Is there a transit function R on G such that $\mathcal{C}$ is precisely the R-convexity on G?

3. Let R be a transit function on G. Is the R-convexity uniquely determined by R? If not, determine all transit functions S such that the S-convexity and the R-convexity coincide.

4. Let R be a transit function on a graph G. Are G and G_R isomorphic? If not, characterize the graphs G for which G and G_R are isomorphic.

5. Characterize the R-monotone graphs, for transit function R.

6. Let R be a transit function. Study $R(u, v, w)$. For example, is it usually empty, or nonempty, or a large set? For which graphs is $|R(u, v, w)| \leq 1$, or $|R(u, v, w)| \geq 1$, for all triples of (distinct) vertices u, v, w. For which graphs is there always equality?

7. Let R be a transit function. Characterize the graphs for which R is a betweenness.

8. Let R be a transit function. Is there a feasible path property Φ such that R is a Φ-path transit function?

9. Let R be a Φ-path transit function. Study the above problems for R^{Δ}, and $R^{k\Delta}$. What is the smallest k such that $R^{k\Delta} = R^{\Delta^*}$ on G?

10. Let R be a transit function. Study parameters of the R-convexity, such as the Helly number, the Carathéodory number, the Radon number, the Tverberg number, and the exchange number.

11. Let R be a transit function. It is easy to see that, if $|R(u, v, w)| \geq 1$, for all triples of (distinct) vertices u, v, w, then the Helly number of the R-convexity is equal to 2. What can be said about the converse: if the Helly number of the R-convexity is 2, is then necessarily $|R(u, v, w)| \geq 1$, for all triples of (distinct) vertices u, v, w? If not, characterize the graphs with Helly number 2 for which we have $|R(u, v, w)| \geq 1$, for all triples of (distinct) vertices u, v, w.

12. Let R be a transit function on a connected graph G. For which graphs G are the R-convex sets connected in G? What properties should R have such that R-convex implies connected on any connected graph G?

13. Let R be a transit function on a connected graph G. Characterize the R-gated sets. Do the R-gated sets form a convexity on G? And if so, is there a transit function Q such that the R-gated sets form the Q-convexity?

14. Let Φ be a feasible path property on G, and let R be the Φ-path transit function on G. Let W be an R-gated set, let u be a vertex outside W with x as (unique) gate in W, and let w be an arbitrary vertex in W. By definition, we can find a Φ-path between u and w passing through x. What properties should Φ have so that we can find a Φ-path P between u and w such that the subpath of P between x and w is contained in W?

15. Let G be a connected graph. Study the join semilattice of paths functions on a graph G. For what graphs is this semilattice a lattice?

16. Let R be a transit function on G. Study the transit graph of R. For which graphs is the transit graph precisely G itself?

17. Let R be a transit function on G. Study the transit function F_R, for special instances of R.

18. Let R be a transit function on G. Study the transit function Q_R, for special instances of R.

19. Let R be a transit function. For which graphs do we have $R = F_R$, or $R = Q_R$?

20. Given a transit function R, does R have any interesting new properties that have not yet been studied for other transit functions? If so, then use these to formulate new Prototype Problems and Questions for transit functions.

4 Examples

In this section we present some well known examples of transit functions on graphs and propose a couple of possibly interesting new ones. The interval function I and the induced path function J are well studied. We just list some of the main results related to the prototype problems and questions above without going into detail. For the all-paths transit function a number of the prototype questions have been settled. In this case we include more details to give examples of the type of results that were inspired by the transit function approach. The prefiber transit function has been used already a couple of times in different guises in the literature. But, from the viewpoint of transit functions, it has only been studied recently, see [21]. The Δ-convexity played a role in the study of quasi-median graphs. Here a transit function for this convexity is given, and we prove some first simple results. Many of the prototype problems may be relatively easy ones for this transit function. But it is a transit function that has its uses and is not a path function. The cut-vertex transit function is another example of a non-trivial transit function that is not a path function. We include some first obvious results. The longest path function and the single paths functions are proposed as possibilities to extend the approach of transit functions. Except for the all-paths function and the longest path function, the idea of R^Δ and further triangular extensions make sense, but they are not listed separately here.

4.1 The geodetic transit function

The *interval function* I of a connected graph G is, of course, *the* prime example of a transit function. It is defined by

$$I(u,v) = \{\ x \mid x \text{ lies on a shortest } u,v\text{-path }\}.$$

Its origins probably belong to the folklore of graph theory. The first published use of the term might be in [19], which is certainly the first systematic study of the interval function. It was characterized in terms of transit axioms only by Nebeský, [25]. It is easily seen to be a betweenness on any graph, see [19]. The parameters of the I-convexity were determined by Duchet [11]. The I-monotone graphs were characterized in [19] by a forbidden minor. In general the set $I(u,v,w)$ may be empty. The graphs with $I(u,v,w)$ always nonempty are the *modular graphs*. The graphs, for which $|I(u,v,w)| = 1$ for all triples of vertices u,v,w, are known as *median graphs*. These were introduced independently by various authors: Avann [1], Nebeský [23], and Mulder and Schrijver [22]. By now, a rich structure theory is available for these graphs and related structures, see e.g. [19, 16, 14]. There are also many interesting generalizations of median graphs, to name but one: the quasi-median graphs (see [19, 2]). For the triangular extension I^Δ see for instance [10]. Prototype Question 3 above is still open.

Open problem (4.1.1) Is the geodesic convexity of a connected graph G uniquely determined by its interval function I?

4.2 The induced path transit function

The *induced path function*, or *minimal path function* on a graph G is the path transit function $J : V \times V \to 2^V$, where J is defined by

$$J(u, v) = \{\ x \mid x \text{ lies on an induced } u, v\text{-path }\}.$$

The J-convexity and its parameters were studied by Duchet [11]. The graphs, for which J is a betweenness, were characterized by forbidden subgraphs in [17]. Usually, $J(u, v, w)$ is a big chunk of vertices. The graphs with $|J(u, v, w)| = 0$ for all triples of distinct vertices u, v, w, and those with $|J(u, v, w)| = 1$ for all triples of distinct vertices u, v, w, were characterized in [17]. The *svelte graphs*, i.e. the graphs with $|J(u, v, w)| \leq 1$ for all triples of vertices u, v, w, were characterized by forbidden subgraphs in [17]. This paper was the first instance of fruitfully transferring questions or ideas from one transit function, viz. I, to another, viz. J, with unexpected new problems that were interesting in their own right. The J-monotone graphs are characterized by Changat and Mathews in [5], see also [7]. The theme of betweenness and monotonicity for J is pursued in [8, 7, 9]. Another question that arose because of the transit function approach was the problem whether the induced path function could be characterized by transit axioms only. Surprisingly enough this is not possible, as Nebeský [28] has shown using first order logic. Many problems are still open, for instance the following challenging problem.

Open problem (4.2.1) Is the minimal path convexity of a graph uniquely determined by the minimal path function J?

The $J^{|Delta}$-convexity has been studied by Changat and Mathews, see [4].

4.3 The Δ-convexity

Let $G = (V, E)$ be a connected graph. A subset W of V is Δ-*convex* if the following holds: if W contains an edge of a triangle, then it contains the whole triangle. Clearly, the family of Δ-convex sets is a convexity on G. In [19] a Δ-convex set was called Δ-closed. The Δ-closed sets played an essential role in the characterization of quasi-median graphs. It is easily seen that the following transit function defines the Δ-convexity:

$\delta(u, v) = \{u, v\}$ if u and v are not adjacent,

$\delta(u, v) = I^{\Delta}(u, v)$ if u and v are adjacent.

Thus, for adjacent vertices, $\delta(u, v)$ consists of all the cliques containing the edge uv. We restrict ourselves here to some first simple results. Clearly, δ is a path function if and only if G is complete. The following Lemma is obvious.

Lemma 1 *$\delta(u, v, w) \neq \emptyset$ if and only if u, v, w induce a triangle in G.*

Note that this means that Prototype Problem 6 reduces to a triviality.

Proposition 2 *A graph G is δ-monotone if and only if G does not contain $K_{1,1,2}$ as an induced subgraph.*

Proof. First assume that G contains an induced $K_{1,1,2}$ on the vertices u, v, x, y, where x, y are the nonadjacent vertices. Then all four vertices are in $\delta(u, v)$, whereas y is not in $\delta(u, x)$. So the latter set is not δ-convex.

Next assume that G does not contain an induced $K_{1,1,2}$. Pick any two distinct vertices u, v in G. If u, v are not adjacent, then $\delta(u, v) = \{u, v\}$, which is clearly δ-convex. So assume that u, v are adjacent. Since there is no induced $K_{1,1,2}$ in G, the set $\delta(u, v)$ is a clique K such that there is no triangle sharing exactly two vertices with K. Hence $\delta(u, v) = \delta(x, y)$, for any two distinct vertices x, y in $\delta(u, v)$. So $\delta(u, v)$ is δ-convex. □□□

4.4 The all-paths transit function

The results in this section can be found in [3]. Let G be a connected graph. The *all-paths transit function* A on G is the transit function defined by

$$A(u, v) = \{ \ w \mid w \text{ lies on some path between } u \text{ and } v \ \}.$$

Clearly, $A(u, u) = \{u\}$, for any u. Furthermore, for distinct u and v, it follows that $A(u, v)$ consists of all blocks in G intersecting $I(u, v)$ in at least two vertices. Here a *block* either is a maximal 2-connected subgraph or it consists of the two ends of a cut-edge. A set W in G is A-convex if and only if it induces a connected subgraph such that, if W contains two vertices of a block, then it contains the whole block. In particular, $A(u, v) = \{u, v\}$ if and only if it is a block, that is, uv is a cut-edge. Thus the all-paths transit function highlights the block structure of the graph. For instance, the edges in the transit hypergraph of A are precisely the blocks of G.

Theorem 3 *Let G be a connected graph. Then $G = G_A$ if and only if G is a tree.*

Theorem 4 *The A-prefibers are precisely the A-convex sets.*

Theorem 5 *Let G be a graph. Then G is a tree if and only if $|A(u, v, w)| = 1$, for every triple of vertices u, v, w of G.*

The block closure G^* of a connected graph G is the graph obtained from G by joining two vertices whenever they are in the same block of G. Thus G^* is the block graph with the same blocks as G. Recall that a *block graph* is a graph in which each block induces a complete subgraph, whence it is a tree-like structure built from complete graphs.

Theorem 6 *Let G be a connected graph. Then G^* is the transit graph of A.*

The function A is another example of a transit function that can be characterized by transit axioms only. This was the main result in [3].

Theorem 7 *Let $R : V \times V \to 2^V$ be a transit function satisfying the extra axioms*

$(a4)$ $w \in R(u,v) \Rightarrow R(w,v) \subseteq R(u,v)$;

$(a5)$ $R(u,x) \cap R(x,v) = \{x\} \Rightarrow R(u,x) \cup R(x,v) = R(u,v)$;

$(a6)$ $R(u,v) \not\subseteq R(u,w) \Rightarrow \exists x \in R(u,v)$ *with* $x \neq u$ *such that* $R(u,x) \cap R(x,w) = \{x\}$.

Then R is the all-paths transit function of the transit graph of R.

4.5 The cut-vertex transit function

Another way to analyze the block structure of a graph is to study the cut-vertices. For a connected graph $G = (V,E)$ we define the transit function C by

$$C(u,u) = \{u\},$$
$$C(u,v) = \{u,v\} \cup \{\ w \mid w \text{ is a cut-vertex between } u \text{ and } v\ \}.$$

Note that $C(u,v) = F_A(u,v)$, where A is the above all-paths function of G.

Clearly, $C(u,v) = \{u,v\}$ if and only if u and v are in the same block. The C-convexity is as follows. A set W is C-convex if and only if every cut-vertex between any two elements of W also lies in W. This is another example of a transit function R such that G and G_R do not coincide. We list some obvious first results on C.

Proposition 8 *Let G be a connected graph. Then C is a betweenness on G.*

Proposition 9 *Let G be a connected graph. Then G_C is the block closure of G.*

Proposition 10 *Every connected graph is C-monotone.*

Proposition 11 *Let G be a connected graph. Then $|C(u,v,w)| \leq 1$, for any three vertices u,v,w of G.*

Proof. If at least two of u,v,w lie in the same block, say u and v, then $C(u,v,w) \subseteq \{u,v\}$. Clearly, u and v can not simultaneously be cut-vertices between v,w and u,w respectively. Hence $C(u,v,w)$ is either empty or a singleton. So suppose that all three of u,v,w are in different blocks. First, one of the three, say v, might be in a block that is "in between" u and w. Now $C(u,v) \cap C(v,w) = \{v\}$, and we are done. Otherwise, there is either a unique cut-vertex x or a unique block B "in between" any pair of the three. In the latter case it is easy to see that the intersection $C(u,v,w)$ is empty. □□□

Proposition 12 *Let G be a connected graph. Then $|C(u, v, w)| = 1$ for any triple of vertices of G if and only if G is a tree.*

Proof. If G is a tree, then $C(u, v) = I(u, v)$, and we are done. If G is not a tree, then it contains a block with at least three distinct vertices, say u, v, w. Then we have $C(u, v, w) = \emptyset$. □□□

The transit function C is an instance of a transit function with $F_C = R_C = C$.

4.6 The prefiber transit function

For u and v in G, the prefiber interval between u and v is the set

$$F(u, v) = \{ \ w \mid I(u, w) \cap I(w, v) = \{w\} \ \}.$$

This notion of interval was introduced by Nebeský [24] to characterize the block graphs, in his words Husimi trees. Block graphs are special instances of quasi-median graphs. These graphs are isometric subgraphs of Hamming graphs. As observed above, F plays a role in one of the characterizations of Hamming graphs, see [19]. Many of the prototype problems and questions turn out to be quite interesting for the prefiber transit function, see [21].

4.7 The longest path transit function

Let G be a connected graph. The *longest-path transit function* L on G is the transit function defined by

$$L(u, v) = \{ \ w \mid w \text{lies on a longest path between } u \text{ and } v \ \}.$$

Clearly, $L(u, u) = \{u\}$, for any u. One may expect that this path transit function will have some very different and unexpected properties. Take, for example the graph consisting of two vertices u and v, three internally disjoint paths of length four between u and v, and a vertex x adjacent to u and v only. Then the L-convex sets are the trivial sets (i.e. the empty set, the singletons, and the whole vertex set) and the set of all vertices except x.

Let $\lambda(u, v)$ be the length of a longest u, v-path. Then it is easy to see that λ defines a metric on V. What properties does this metric have?

4.8 Single paths transit functions

Let G be a connected graph. Let Φ be a set of paths in G such that, for any two vertices u and v in G, there exits a unique path $P_{u,v}$ between u and v in Φ. We call such a path family a *single-path family*, and its Φ-path transit function R a single-path transit function.

If G is a geodetic graph, then, by definition, its interval function I is a single-path transit function. A characterization of geodetic graphs involving the interval function is given by Nebeský [27].

Another example is the following. Let G be a cycle, and let uv be a fixed edge of G. Let $P_{u,v}$ be the u, v-path containing the single edge uv. For any pair of vertices x, y distinct from u, v, let $P_{x,y}$ be the x, y-path in G not containing the edge uv. Then this set of paths is a single-path family, and its transit function R is a single-path transit function. Note that this transit function has some nice properties: the cycle is R-monotone, and R is a betweenness.

In addition to the prototype problems and questions that may be relevant, we mention just one other question.

Open problem (4.7.1) Characterize the graphs for which the longest path transit function is a single-path transit function.

5 Transit functions on posets

Many results on to the interval function I have their analogue in the area of partially ordered sets, semilattices and lattices. A prime example of this is the theory of median graphs. Let $G = (V, E)$ be a median graph, and let u be a vertex of G. If we define the ordering $\leq_u$ on V by $v \leq_u w$ if $v \in I(u, w)$, then the partially ordered set $(V, \leq_u)$ is a so-called median semilattice, see [19]. On the other hand, the Hasse graph of a median semilattice is a median graph. Here the Hasse graph of a poset is the Hasse diagram where the orientation is ignored. This relationship makes it possible to translate many results on the interval function of median graphs to their analogues of median semilattices. There are generalizations possible to partially ordered sets in general. It leads much to far to give e detailed presentation here. But it becomes quite clear that many of the above problems and ideas have their counterpart in the area of partially ordered sets. Not all of these are just a trivial transfer of a result on graphs into a result on posets. So, again, the transit function approach may provide us with many new ideas and problems to be pursued for their own sake. An early instance of relevant results, even on other structures than posets, is the work of Sholander, see [31, 32]. A first and interseting step in this area is represented by [15].

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