Duality and calculi without exceptions for convex objects

Jan Brinkhuis

Econometric Institute, Erasmus University Rotterdam

Econometric Institute Report EI 2008-07

March 2008

Abstract

The aim of this paper is to make a contribution to the investigation of the roots and essence of convex analysis, and to the development of the duality formulas of convex calculus. This is done by means of one single method: firstly *conify*, then work with the calculus of convex cones, which consists of three rules only, and finally *deconify*. This generates all definitions of convex objects, duality operators, binary operations and duality formulas, *all without the usual need to exclude degenerate situations*. The duality operator for convex function agrees with the usual one, the Legendre-Fenchel transform, only for proper functions. It has the advantage over the Legendre-Fenchel transform that the duality formula holds for improper convex functions as well. This solves a well-known problem, that has already been considered in Rockafellar's Convex Analysis [21]. The value of this result is that it leads to the general validity of the formulas of Convex Analysis that depend on the duality formula for convex functions. The approach leads to the systematic inclusion into convex sets of recession directions, and a similar extension for convex functions. The method to construct binary operations given in [21] is formalized, and this leads to some new duality formulas. An existence result for extended solutions of arbitrary convex optimization problems is given. The idea of a similar extension of the duality theory for optimization problems is given.

Keywords: convex sets, convex functions, duality, convex optimization.

1 Introduction

This paper is based on joint work with V.M. Tikhomirov, published in [7]. The aim of this work is a reduction to *calculus* of some main tasks of convex analysis. Ideally, this calculus should be as transparent and user-friendly as the celebrated differential calculus. For convex sets, this subject has its origins in the work of Minkowski [17], for convex functions in the work of Fenchel [8, 9]. The present paper aims to give a systematic and elegant development of the duality formulas of convex calculus and to avoid the exclusion of degenerate situations. In particular, the duality theorem for convex functions and all results depending on it, are valid for all convex functions if one uses the duality operator given in the present paper. This operator agrees for proper convex functions with the Legendre-Fenchel transform. We show that many constructions and results of convex analysis can be generated by one single method: firstly *conify*, then work with the calculus of convex cones, which consists of three rules only, and finally *deconify*.

Now we discuss what is the contribution that the present paper tries to make to the vast literature on this well-established subject. It lies in the systematic realization of the unified method. This leads to some new results, such as the result that all calculus formulas hold without exceptions. In order to show the efficiency of the method, we prove all results from first principles. All proofs can be based on simple figures, but for the sake of rigor, all proofs have been written down in a careful analytical style.

We begin by establishing, in section 2, the calculus for convex cones, which consists of three formulas only: the bipolar theorem and the duality formulas for images and inverse images under linear transformations. These three formulas are well-known, for example, they are given in Rockafellar's seminal book Convex Analysis [21]. In the present paper, an attempt has been made to give short self-contained proofs. This is done to emphasize the value of the conification method: for example, in [21], some properties on convex cones are derived from properties of convex functions.

In section 3, four fundamental types of convex objects are constructed by deconification of convex cones with suitable properties (of inclusion in some half space and/or containment of some ray): convex cones containing zero, convex sets including some recession directions as elements, a similarly extended concept of convex functions, and sublinear functions. Then, for each of these four types, the three duality formulas are derived by means of deconification. Of course, the connection between convex sets (resp. functions) with convex cones is well-known. It has been a main driving force in the works of Rockafellar [21, 22], Moreau [18] and Hörmander [14], and it has been exposed forcefully in Kutateladze-Rubinov [15]. Moreover, the importance of recession directions in the study of convex sets and functions is well-known; this is treated in detail in [21]. The contribution of the present paper is here the systematic way in which recession directions arise and are treated, as a consequence of deconification of special convex cones. A novelty of the present treatment appears to be that it does not require the exclusion of degeneracies, such as improper convex functions. For example, a formula for the duality operator for arbitrary (proper or improper) convex functions with recession is given in proposition 3.2. It appears to be novel and it involves the duality operators of all four fundamental types of convex objects: the Legendre-Fenchel operator, the subdifferential operator, the support function operator, and the polar operator. The value of this result is as follows. At first sight, the usual restriction to *proper* convex functions might seem appropriate, as *proper* convex functions are the real objects of study. However, working with proper convex can lead to improper functions. For this reason, Rockafellar recommends, in [22] and in [25], to include improper convex functions. However, the duality formula $f^{**} = \mathrm{cl} f$, does not hold for improper functions. Many formulas from Convex Analysis depend on the duality formula for convex functions. Therefore, the exclusion of improper convex functions spreads through the entire theory of Convex Analysis, and in the process the negative effect of this imperfection is strengthened. Having to take exceptions into account when working with calculus rules, is not very convenient. An attempt to address this problem is done in [22]: for improper convex functions, an ad-hoc closure operator is defined—not by means of the closure of the epigraph—an then the duality formula $f^{**} = \mathrm{cl} f$ holds always. However, this does not lead to completely satisfactory results—for example, then no improper convex function is closed apart from the two trivial functions that are identically $+\infty$ (resp. $-\infty$). In [25], this modified closure operation is not mentioned. Most modern accounts of Convex Analysis give the duality formula $f^{**} = \mathrm{cl} f$ for proper convex functions only. In fact, often convex functions are defined in such a way that they do not take the value $-\infty$. The present paper offers a solution for this problem that appears to leave nothing to be desired: we consider the duality operator on convex functions that is given by the conification method (conify, take the polar cone, deconify). This operator agrees with the usual one (called Legendre-Fenchel transform or conjugate convex function) for proper convex functions, *but not for improper ones.* This operator has the advantage over the Legendre-Fenchel transform that *the duality formula holds for all convex functions*, proper and improper ones, if we use this operator.

In section 4, we formalize the construction of binary operations on convex objects that is given in [21]. This formalization is based on a construction using special linear transformations, diagonal mappings and addition mappings. As a result, the duality formulas for binary operations are immediate consequences of the calculus of convex objects, established in the previous section. The device of using diagonal mappings in this context is known: for example, it is used in the textbook [11]. The contribution of the present paper is the consistent development, the inclusion of recession directions, and the fact that degenerate cases are not excluded. For example, convex functions are allowed to be improper, and sums of arbitrary convex functions are allowed, even if at a certain point one takes value $+\infty$ and the other value $-\infty$. Some of the duality formulas appear to have been derived for the first time by means of the present formalization, in [7].

In section 5, we consider convex optimization. By deconification of a result on convex cones, we get that every closed convex optimization problem has a solution in an extended sense. This suggests that it might be useful to include in algorithms for convex optimization the task to find extended solutions, along with ordinary solutions. Moreover, by deconification of this result on convex cones, one can derive the duality theory of convex optimization; the idea of this derivation is given.

Finally, we compare the present paper with [7] and [6]. The paper [7] intends to make a contribution to the development of the duality formulas of convex calculus presented in [16, 26, 27]. In the present paper, proofs are given in a rigorous analytical style, in [7] of all calculus formulas (sometimes for some representative examples). In the present paper, all formulas hold in general, and explicit conditions are given under which the closure operator in the formulas can be omitted, in [7] half of the formulas hold under conditions of general position, which are not made explicit. Moreover, improved conification and deconification procedures are given in the present paper, compared to [7]. Deconification is always possible. It turns out that it is preferable, to define the conification of a convex object in such a way that it is not necessarily unique. This improved procedures force the inclusion of 'recession elements' in convex sets and functions, and this in its turn allows the possibility to avoid exclusion of degenerate situations. This inclusion is equivalent to carrying out the analysis in cosmic space, rather than in ordinary n-dimensional space. Cosmic space contains a *horizon*, consisting of points at infinity. The properties of cosmic space are developed in [25]. The idea of including recession directions of convex sets as points of these sets is already developed in [21] and goes back to Steinitz. Finally, the exposition has been simplified. In [6], the fundamentals of Convex Analysis, but not its use in optimization, is considered; in that paper, the analysis is carried out systematically and explicitly in terms of cosmic space.

In the last fifteen years, there has been a renewal of interest in convexity, stimulated by progress in convex optimization algorithms by Nesterov and Nemirovski [19], and many books on convexity and in particular convex optimization have been published, for example [1–5, 10–13, 16, 19, 20, 25].

2 The three duality formulas for convex cones

In this section we recall the standard operations and facts of convex cones. We include self-contained proofs of this material. All results in the remainder of the paper will be based on this section. To be specific, the duality formulas for convex cones in finite dimensional vector spaces, always equipped with a *non-degenerate symmetric bilinear form* $\langle\cdot,\cdot\rangle$—called just *form*—are given. We will work with coordinates, considering vector spaces $\mathbb{R}^n$, together with a symmetric non-singular $n\times n$-matrix M, which determines the non-degenerate symmetric bilinear form $\langle u,v\rangle = \langle u,v\rangle_M = u^T M v$ for all $u,v\in\mathbb{R}^n$, where the superscript T denotes transposition. The matrix $M = I_n$ gives the *standard form* on $\mathbb{R}^n$, the Euclidean inner product $\langle u,v\rangle_{I_n} = \langle u,v\rangle_n = \sum_{k=1}^n u_k v_k$. A subset $C\subset\mathbb{R}^n$ is called a *convex cone* if it is closed under taking linear combinations with positive coefficients:

$\alpha_i > 0,\ x_i \in C,\ i = 1,2 \Rightarrow \alpha_1 x_1 + \alpha_2 x_2 \in C$. A convex cone is not required to contain the zero vector 0_n—sometimes denoted just 0—and it is allowed to be the empty set. For two convex cones $C_i \subset \mathbb{R}^n,\ i = 1,2$, the intersection $C_1 \cap C_2$ and the sum $C_1 + C_2 = \{x_1 + x_2 | x_i \in C_i,\ i = 1,2\}$ are convex cones in $\mathbb{R}^n$. The *polar cone* C° of a convex cone $C \subset \mathbb{R}^n$ is the closed convex cone $\{y \in \mathbb{R}^n | \langle y, x\rangle \leq 0\ \forall x \in C\}$. The closure $\mathrm{cl}C$ of a convex cone C is a convex cone, and it has the same polar cone as C. For two convex cones $C_i \subset \mathbb{R}^n,\ i = 1,2$, one has $(C_1 + C_2)^\circ = C_1^\circ \cap C_2^\circ$. The *bipolar cone* $C^{\circ\circ}$ of a convex cone $C \subset \mathbb{R}^n$ is the closed convex cone $(C^\circ)^\circ$. For a subspace $L \subset \mathbb{R}^n$, the polar L° with respect to the standard form equals the *orthogonal complement* $L^\perp = \{y \in \mathbb{R}^n | \langle y, x\rangle_n = 0\ \forall x \in L\}$. The *image* (respectively *inverse image*) of a convex cone $C \subset \mathbb{R}^n$ (respectively $\bar{C} \subset \mathbb{R}^m$) under a linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is the convex cone defined by $\Lambda C = \{\Lambda x | x \in C\}$ (respectively $\Lambda^{-1}\bar{C} = \{x \in \mathbb{R}^n | \Lambda x \in \bar{C}\}$). The *conjugate linear transformation* of a linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation $\Lambda' : \mathbb{R}^m \to \mathbb{R}^n$ given by $\langle \Lambda' y, x\rangle = \langle y, \Lambda x\rangle$ for all $x \in \mathbb{R}^n,\ y \in \mathbb{R}^m$ (in terms of matrices, this is the transposition operator). One has $\Lambda'' = \Lambda$. We will use the result from matrix theory that for a linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ the kernel of Λ, $\ker \Lambda = \Lambda^{-1}(0_m)$, and the image of the conjugate linear transformation with respect to the standard form, $\mathrm{Im}\ \Lambda' = \Lambda'(\mathbb{R}^m)$, are each others orthogonal complement. The *relative interior* $\mathrm{ri}C$ of a convex cone $C \in \mathbb{R}^n$ is the interior of C when C is regarded as a subset of $\mathrm{Span}C$, its linear span; $\mathrm{ri}C$ is a convex cone. We will use the following property: for each convex cone $C \subset \mathbb{R}^n$, the entire open interval connecting a point in $\mathrm{ri}C$ and a point in $\mathrm{cl}C$ is contained in $\mathrm{ri}C$. A convex cone in $\mathbb{R}^n$ is called a *polyhedral cone* if it is the solution set of a finite system of homogeneous linear nonstrict inequalities, $\sum_{j=1}^n \alpha_{ij} x_j \leq 0,\ 1 \leq i \leq m$.

Theorem 2.1 *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $C \subset \mathbb{R}^n$ and $\bar{C} \subset \mathbb{R}^m$ be convex cones. Then the following formulas hold true:*

1. $C^{\circ\circ} = \mathrm{cl}C$ *precisely if* $C \neq \emptyset$,
2. $(\Lambda C)^\circ = (\Lambda')^{-1}(C^\circ)$,
3. $(\Lambda^{-1}\mathrm{cl}\bar{C})^\circ = \mathrm{cl}(\Lambda'(\bar{C}^\circ))$ *precisely if either* $\bar{C} \neq \emptyset$ *or* $\ker \Lambda = 0$.

The closure operation can be omitted from formula 3. if one of the following two assumptions holds true:

a. $\mathrm{ri}\bar{C} \cap \mathrm{Im}\ \Lambda \neq \emptyset$*;*

b. $\bar{C}$ *is a polyhedral cone.*

For the proof, we need two consequences of the fact that the image of a *compact*—that is, closed and bounded—subset of $\mathbb{R}^n$ under a continuous mapping to $\mathbb{R}^m$ is compact. The first one is the extreme value theorem: a continuous function on a compact subset of $\mathbb{R}^n$ assumes its maximal and minimal value. The second one is the following lemma. Let S_{n-1} be the unit sphere in $\mathbb{R}^n$, that is, $S_{n-1} = \{x \in \mathbb{R}^n | \|x\| = 1\}$, where $\| \cdot \|$ is the euclidean norm given by $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. For each convex cone $C \subset \mathbb{R}^n$ that contains zero, $0_n \in C$, let $H(C)$ denote the intersection $H(C) = C \cap S_{n-1}$ of C with the unit sphere S_{n-1} in $\mathbb{R}^n$.

Lemma 2.2 *Let C be a convex cone containing zero in $\mathbb{R}^n$.*

1. *$H(C)$ is closed precisely if C is closed.*
2. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If C is closed and $C \cap \ker \Lambda = 0$, then ΛC is closed.*

Proof.

1. If C is closed, then $H(C)$ is closed, as it is the intersection of two closed sets, C and S_{n-1}. Conversely, if $H(C)$ is closed, then for each $N > 0$ the subset $\{x \in C | \|x\| \leq N\}$ of $\mathbb{R}^n$ is compact and so closed, as it is the image of the compact set $[0, N] \times H(C)$ under the continuous mapping $(\rho, x) \mapsto \rho x$. Therefore, C is closed.
2. $H(C)$ is compact by statement 1 of the lemma, as C is closed. Λ induces a surjective continuous mapping $\Lambda : H(C) \to H(\Lambda C) : c \mapsto \|\Lambda c\|^{-1} \Lambda c$; this is well-defined, as $C \cap \ker \Lambda = 0$. Therefore, $H(\Lambda C)$ is compact and so ΛC is closed, by statement 1 of the lemma.

Now we prove the theorem.

Proof.

We assume that the chosen form on $\mathbb{R}^n$ is the standard form; this does not restrict the generality of the argument by virtue of the following observation: if C, D are convex cones in $\mathbb{R}^n$, and D is the polar cone of C with respect to $\langle \cdot, \cdot \rangle_M$, then MD is the polar cone of C with respect to the standard form.

1. (a) Assume $C = \emptyset$. Applying the polar operator to C gives $\mathbb{R}^n$; applying the polar operator on this outcome gives 0. That is, $C^{\circ\circ} = 0$. However, $\text{cl}C = \emptyset$. Therefore, the formula does not hold.

   (b) Assume $C \neq \emptyset$.

   - The inclusion $C^{\circ\circ} \subset \text{cl}C$. For each $x \in \mathbb{R}^n \setminus \text{cl}\, C$, one takes the point y in $\text{cl}C$ that is closest to x; the existence of a closest point follows from the extreme value theorem, after addition of the constraint that y lies in some closed ball with center x and containing a point of $\mathrm{cl}\, C$ (in order to ensure boundedness and so compactness of the feasible set of the optimization problem); the uniqueness is not needed here, so its proof is not given. Now we check that $x - y \in C^\circ$. By the minimality property of y, we get that for all $z \in C$ the function $g : [0, +\infty) \to \mathbb{R} : t \mapsto \|(y + tz) - x\|^2$ has a local minimum at $t = 0$; therefore, $g'(0) \geq 0$, and this gives $\langle x - y, z\rangle \leq 0$. This shows that $x - y \in C^\circ$.

In order to prove that $x \notin C^{\circ\circ}$, it suffices to show that $\langle x - y, x\rangle > 0$. By the minimality property of y, we get that the function $h : [0, +\infty) \to \mathbb{R} : t \mapsto \|ty - x\|^2$ has a local minimum at $t = 1$; therefore, $h'(1) = 0$, and this gives $\langle x - y, y\rangle = 0$. Therefore, $\langle x - y, x\rangle = \langle x - y, x - y\rangle + \langle x - y, y\rangle = \|x - y\|^2$ and this is positive. Indeed, $x \notin \mathrm{cl}C$ and $y \in \mathrm{cl}C$, so $x - y \neq 0$ and so $\|x - y\| > 0$. It follows that $\langle x - y, x\rangle > 0$, as required.

- The inclusion $\mathrm{cl}C \subset C^{\circ\circ}$ follows immediately from the definitions.

2.
   - The inclusion $(\Lambda C)^\circ \subset (\Lambda')^{-1}(C^\circ)$. For each $y \in (\Lambda C)^\circ$ and each $x \in C^\circ$, one has $\langle \Lambda' y, x\rangle = \langle y, \Lambda x\rangle \leq 0$ and so $y \in (\Lambda')^{-1}(C^\circ)$.
   - The inclusion $(\Lambda')^{-1}(C^\circ) \subset (\Lambda C)^\circ$. For each $y \in (\Lambda')^{-1}(C^\circ)$, and each $x \in C$ one has $\langle y, \Lambda x\rangle = \langle \Lambda' y, x\rangle \leq 0$ and so $y \in (\Lambda C)^\circ$.

3. (a) Assume $\bar{C} = \emptyset$. Then the left hand side of the formula equals $\mathbb{R}^n$, and the right hand side equals $\mathrm{cl}(\Lambda'(\mathbb{R}^m))$, which is $\mathrm{Im}\ \Lambda' = (\ker \Lambda)^\perp$.

   Therefore, the third formula holds precisely if $(\ker \Lambda)^\perp = \mathbb{R}^n$, that is, if $\ker \Lambda = 0$.

   (b) Assume $\bar{C} \neq \emptyset$. Then the formula follows immediately from the first two formulas: $(\Lambda^{-1}\mathrm{cl}\bar{C})^\circ = (\Lambda^{-1}(\bar{C}^{\circ\circ}))^\circ$ by the first formula, and this equals $((\Lambda'\bar{C}^\circ)^\circ)^\circ$ by the second formula and by $\Lambda'' = \Lambda$; this equals $\mathrm{cl}(\Lambda'(\bar{C}^\circ))$ by the first formula.

It remains to prove the last statement of the theorem.

Case a. Assume that $\mathrm{ri}\bar{C} \cap \mathrm{Im}\ \Lambda \neq \emptyset$, say, $\tilde{d} \in \mathbb{R}^n$ with $\Lambda\tilde{d} \in \mathrm{ri}\bar{C}$.

1. We check the formula $\Lambda^{-1}\mathrm{cl}\bar{C} = \mathrm{cl}(\Lambda^{-1}\bar{C})$.

   - The inclusion $\mathrm{cl}(\Lambda^{-1}\bar{C}) \subset \Lambda^{-1}\mathrm{cl}\bar{C}$. As $\bar{C} \subset \mathrm{cl}\bar{C}$, one gets $\Lambda^{-1}\bar{C} \subset \Lambda^{-1}\mathrm{cl}\bar{C}$. As $\mathrm{cl}\bar{C}$ is closed, $\Lambda^{-1}\mathrm{cl}\bar{C}$ is closed. It follows that $\mathrm{cl}(\Lambda^{-1}\bar{C}) \subset \Lambda^{-1}(\mathrm{cl}\bar{C})$.
   - The inclusion $\Lambda^{-1}\mathrm{cl}\bar{C} \subset \mathrm{cl}(\Lambda^{-1}\bar{C})$. Choose an arbitrary $d \in \Lambda^{-1}\mathrm{cl}\bar{C}$, then $\Lambda d \in \mathrm{cl}\bar{C}$ and so $(1-\alpha)\Lambda\tilde{d} + \alpha\Lambda d \in \mathrm{ri}\bar{C} \subset \bar{C}$ for all $\alpha \in (0, 1)$, that is $(1-\alpha)\tilde{d} + \alpha d \in \Lambda^{-1}\bar{C}$ for all $\alpha \in (0, 1)$, and so taking the limit $\alpha \uparrow 1$ we get $d \in \mathrm{cl}(\Lambda^{-1}\bar{C})$, as required.

2. We check that the left hand side of formula 3. remains the same if the closure operator is omitted. By what has just been proved, the left hand side of the third formula equals $(\mathrm{cl}(\Lambda^{-1}\bar{C}))^{\circ}$, which equals $(\Lambda^{-1}\bar{C})^{\circ}$, as required.

3. We check that the right hand side of formula 3. remains the same if the closure operator is omitted. That is, we check that the convex cone $\Lambda'(\bar{C}^{\circ})$ is closed.To begin with, the convex cone $E = \bar{C} + \mathbb{R}_+(-\Lambda\tilde{d})$—where $\mathbb{R}_+ = [0, +\infty)$— is a subspace, by the following lemma, as the intersection $\mathrm{ri}E \cap (-E)$ contains the element $\Lambda\tilde{d}$, and so is nonempty.

   Lemma 2.3 *A convex cone C in $\mathbb{R}^n$ is a subspace precisely if* $\mathrm{ri}C \cap (-C) \neq \emptyset$.

   Proof. If C is a subspace, then $-C = C = \mathrm{ri}C \neq \emptyset$, and so $\mathrm{ri}C \cap (-C) = C \neq \emptyset$. Conversely, if $c \in \mathrm{ri}C \cap (-C)$, then we choose a neighborhood U in $\mathrm{Span}C$ of c contained in C. Then the set $-c+U$ in $\mathrm{Span}C$ is a neighborhood of zero in $\mathrm{Span}C$ that is contained in C, as $-c \in C$, $U \subset C$, and C is a convex cone; this implies, as C is a convex cone, that $C = \mathrm{Span}C$. Therefore, C is a subspace.

   We continue the proof of case *a.* The convex cone $\bar{C} + \mathrm{Im}\ \Lambda$ is a subspace, as $E = \bar{C} + \mathbb{R}_+(-\Lambda\tilde{d})$ is a subspace and $-\mathrm{Im}\ \Lambda = \mathrm{Im}\ \Lambda$. Taking the polar cone gives that $\bar{C}^{\circ} \cap \ker \Lambda'$ is a subspace. Choose a closed convex cone containing zero D in $\mathbb{R}^n$ such that $D + (\bar{C}^{\circ} \cap \ker \Lambda') = \bar{C}^{\circ}$—and so $\Lambda'(\bar{C}^{\circ}) = \Lambda'(D)$—and $D \cap (\bar{C}^{\circ} \cap \ker \Lambda') = 0$, that is, $D \cap \ker \Lambda' = 0$. Then $\Lambda'(D)$ is closed by lemma 2.2. Therefore, $\Lambda'(\bar{C}^{\circ})$ is closed, as required.

Case b. We observe that polyhedral cones in $\mathbb{R}^m$ are precisely the convex cones of the form $(\tilde{\Lambda}\mathbb{R}^p_+)^{\circ}$ for some natural number p and some linear transformation $\tilde{\Lambda} : \mathbb{R}^p \to \mathbb{R}^m$. Moreover, finitely generated convex cones containing zero in $\mathbb{R}^m$ are precisely the convex cones of the form $\tilde{\Lambda}(\mathbb{R}^p_+)$, for some natural number p and some linear transformation $\tilde{\Lambda} : \mathbb{R}^p \to \mathbb{R}^m$

We will use the second statement of the following lemma.

Lemma 2.4 *Let S be a subset of $\mathbb{R}^n$ and consider the convex cone containing zero C that is generated by S.*

1. *(Carathéodory's theorem). C consists of all conic combinations of linearly independent subsets of S; that is, C is the union of the convex cones containing zero that are generated by linearly independent subsets of S.*

2. *The convex cone C is closed if S is finite.*

Proof.

1. Choose $x \in C$. Let r be the minimal number for which x can be written as a conic combination of a finite subset T of S of r elements, $x = \sum_{t \in T} \alpha_t t,\ \alpha_t \geq 0 \forall t \in T$. Then T is linearly independent. Otherwise, one could choose a nontrivial linear relation $\sum_{t \in T} \beta_t t = 0$, and subtract a suitable multiple of it from the expression for x above, in order to write x as a conic combination of $T \setminus \{\bar{t}\}$ for some $\bar{t} \in T$; this would contradict the minimality property of r.

2. To prove the second statement, it suffices to combine the following three facts: 1) the first statement of the lemma, 2) the observation that the convex cone containing zero that is generated by a linearly independent subset of $\mathbb{R}^n$ is closed, 3) the union of a finite collection of closed sets is closed.

Thus prepared, we are ready to deal with case b. Assume that $\bar{C}$ is a polyhedral cone. Choose a natural number p and a linear transformation $\tilde{\Lambda} : \mathbb{R}^p \to \mathbb{R}^m$ for which $\bar{C} = (\tilde{\Lambda}(\mathbb{R}^p_+))^\circ$.

- We check that the left hand side of formula 3. remains the same if the closure operator is omitted. It suffices to note that $\bar{C}$ is closed, as it is the polar cone of some convex cone.

- We check that the right hand side of formula 3. remains the same if the closure operator is omitted. The convex cone $\Lambda'(\bar{C}^\circ)$ equals $\Lambda'((\tilde{\Lambda}(\mathbb{R}^p_+))^{\circ\circ})$ by the equality $\bar{C} = (\Lambda(\mathbb{R}^p_+))^\circ$. By formula 1 of theorem 2.1, this equals $\Lambda'(\mathrm{cl}(\tilde{\Lambda}(\mathbb{R}^p_+)))$. This equals $\Lambda'\tilde{\Lambda}(\mathbb{R}^p_+)$ by the second statement of lemma 2.4. It follows that the right hand side of formula 3. $\mathrm{cl}(\Lambda'(\bar{C}^\circ))$ equals $\mathrm{cl}(\Lambda'\tilde{\Lambda}(\mathbb{R}^p_+))$. This equals $\Lambda'\tilde{\Lambda}(\mathbb{R}^p_+)$ by the second statement of lemma 2.4. This equals $\Lambda'(\mathrm{cl}(\tilde{\Lambda}(\mathbb{R}^p_+))$ by the second statement of lemma 2.4. This in its turn equals $\Lambda'((\tilde{\Lambda}(\mathbb{R}^p_+))^{\circ\circ})$ by the first formula of theorem 2.1. Finally, this equals $\Lambda'(\bar{C}^\circ)$ by the equality $\bar{C} = (\Lambda(\mathbb{R}^p_+))^\circ$, as required.

3 The three duality formulas for convex objects

In this section, the four fundamental types of convex objects will be generated automatically from the consideration of suitable convex cones ('deconification'). Conversely, to each convex object of one of these four types, a convex cone will be associated ('conification'), not uniquely, but without loss of information. These constructions, given in proposition 3.1, are the base of the approach 'conify, work, deconify'. Thus the duality formulas for the four fundamental types of convex objects, given in proposition 3.2, are a corollary of the results in the previous section.

The *epigraph* $\mathrm{epi} f$ (resp. *strict epigraph* $\mathrm{sepi} f$) of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is the subset $\{(x, \rho) | \rho \geq f(x), x \in \mathbb{R}^n, \rho \in \mathbb{R}\}$ (resp. $\{(x, \rho) | \rho > f(x), x \in \mathbb{R}^n, \rho \in \mathbb{R}\}$). We will use that $\mathrm{sepi} f$ is contained in $\mathrm{epi} f$, and that they have the same closure. The notation $\mathbb{R}_+ = [0, +\infty),\ \mathbb{R}_{++} = (0, +\infty)$ will be used. We consider the following two spaces with form: $\mathbb{R}^n \times \mathbb{R}$ with form $\langle (x', \alpha'), (x, \alpha) \rangle =$

$\langle x', x\rangle_n - \alpha'\alpha$, and $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with form $\langle (x', \alpha', \beta'), (x, \alpha, \beta)\rangle = \langle x', x\rangle_n - \alpha\beta' - \alpha'\beta$. A set $A \subset \mathbb{R}^n$ is called a *convex set* if $\alpha_1 x_1 + \alpha_2 x_2 \in A$ for all $\alpha_i > 0,\ x_i \in A,\ i = 1, 2$ for which $\alpha_1 + \alpha_2 = 1$. The *recession cone* of a convex set $A \subset \mathbb{R}^n$ is the convex cone

$$0^+A = \{x \in \mathbb{R}^n | \mathbb{R}_{++}x + A \subset A\} = \{x \in \mathbb{R}^n | x + A \subset A\}.$$

A function $p : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is called a *sublinear function* if, in the first place, $p(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 p(x_1) + \alpha_2 p(x_2)$ for all $\alpha_i > 0, x_i \in \mathbb{R}^n,\ i = 1, 2$ for which $\{p(x_1), p(x_2)\} \neq \{+\infty, -\infty\}$, and, in the second place, $p(0) = 0$ or $p(0) = -\infty$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is called a *convex function* if $f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$ for all $\alpha_i > 0, x_i \in \mathbb{R}^n,\ i = 1, 2$ for which $\alpha_1 + \alpha_2 = 1$ and $\{f(x_1), f(x_2)\} \neq \{+\infty, -\infty\}$. The *recession function* of a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is the sublinear function $f0^+$ on $\mathbb{R}^n$ defined by $\mathrm{epi}(f0^+) = 0^+(\mathrm{epi} f)$.

Proposition 3.1 Characterization of convex objects in terms of convex cones. *We consider four examples of the following procedure.*

- Deconification. *For each nonnegative integer n, a set $\mathcal{C}_n$ of nonempty convex cones—called conifications—in a vector space C_n—called conification space—with form $\langle \cdot, \cdot \rangle$ is chosen. Then, for each convex cone $C \in \mathcal{C}_n$, an 'object' $O(C)$—called the deconification of C—is defined. Finally, an internal characterization of the set $\mathcal{O}_n$ of objects $O(C)$, where C runs over $\mathcal{C}_n$, is given.*
- Conification. *Conversely, for each convex object $O \in \mathcal{O}_n$, two convex cones $C_l(O)$ and $C_u(O)$ belonging to $\mathcal{C}_n$ are defined, having the following properties: $C_l(O)$ is contained in $C_u(O)$, they have the same closure, and for each $C \in \mathcal{C}_n$ one has $O(C) = O$—called C is a conification of O—precisely if $C_l(O) \subset C \subset C_u(O)$.*

1. Convex sets with recession.

  - Deconification. *To each convex cone $C \subset \mathbb{R}^n \times \mathbb{R}$ for which $0 \in C \subset \mathbb{R}^n \times \mathbb{R}_+$, the pair $(A(C), A_0(C)$ is associated, consisting of*

    $$A(C) = \{x \in \mathbb{R}^n | (x, 1) \in C\} \text{ and } A_0(C) = \{x \in \mathbb{R}^n | (x, 0) \in C\}.$$

    The pairs (A, A_0) that arise in this way—called convex sets in $\mathbb{R}^n$ with recession—are precisely the pairs consisting of a convex set $A \subset \mathbb{R}^n$ and a convex cone $A_0 \subset \mathbb{R}^n$ containing zero and contained in the recession cone 0^+A of A.
  - Conification. *For each convex set in $\mathbb{R}^n$ with recession (A, A_0), we define*

    $$C_l(A, A_0) = C_u(A, A_0) = C(A, A_0) = (\mathbb{R}_{++}(A \times 1)) \cup (A_0 \times 0).$$

2. Sublinear functions.

- Deconification. *To each convex cone* $C \subset \mathbb{R}^n \times \mathbb{R}$ *for which* $C \supset \mathbb{R}_{++}(0_n, 1)$*, the function* $p(C) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *is associated, where*

$$p(C)(x) = \inf\{\rho \in \mathbb{R} | (x, \rho) \in C\}$$

for all $x \in \mathbb{R}^n$*. The functions* $p : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *that arise in this way are precisely the sublinear functions on* $\mathbb{R}^n$

- Conification. *For each sublinear function* p*, we define*

$$C_l(p) = \text{sepi}p \text{ and } C_u(p) = \text{epi}p.$$

3. Convex functions with recession.

- Deconification. *To each convex cone* $C \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ *for which* $\mathbb{R}_{++}(0_n, 1, 0) \subset C \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$*, a pair* $(f(C), f_0(C))$ *of functions* $\mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *is associated, where*

$$f(C)(x) = \inf\{\rho \in \mathbb{R} | (x, \rho, 1) \in C\}$$

and

$$f_0(C)(x) = \inf\{\rho \in \mathbb{R} | (x, \rho, 0) \in C\}$$

for all $x \in \mathbb{R}^n$*. The pairs* (f, f_0) *that arise in this way—called convex functions with recession—are precisely the pairs for which* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *is a convex function and* $f_0 : \mathbb{R}^n \to \mathbb{R}$ *is a sublinear function for which* $f_0 \geq f0^+$*, where* $f0^+$ *is the recession function of* f*.*

- Conification. *For each convex function with recession* (f, f_0) *we define*

$$C_l(f, f_0) = C(\text{sepi}f, \text{sepi}f_0)) \text{ and } C_u(f, f_0) = C(\text{epi}f, \text{epi}f_0)).$$

4. Convex cones containing zero. *For convex cones in* $\mathbb{R}^n$ *containing zero, the deconification and conification operators are the identical operators.*

Remark 1. For each convex cone A_0 in $\mathbb{R}^n$ containing zero, the pair $(\emptyset, A_0)$ is a convex set with recession. Moreover, for each sublinear function f_0 on $\mathbb{R}^n$, the pair $(+\infty, f_0)$, where $+\infty$ denotes the function on $\mathbb{R}^n$ that is identically $+\infty$, is a convex function with recession.

Remark 2. A more intuitive description of the four types above can be given as follows. A convex set with recession, (A, A_0), can be viewed as one set, the elements of which are the points of A and the rays of A_0 (representing some of the recession directions of A). A convex function with recession, (f, f_0), can be viewed as one function: consider the set, the elements of which are the points of $\mathbb{R}^n$ and the rays of $\mathbb{R}^n$, and then consider the function on this set defined by $x \mapsto f(x)$ for all $x \in \mathbb{R}^n$ and $\mathbb{R}_{++}x \mapsto \|x\|^{-1} f_0(x)$ for all rays $\mathbb{R}_{++}x$. A convex cone containing zero, C, can be viewed as one set, the elements of which are the rays of C. A sublinear function p can be viewed as the function on the set of rays in $\mathbb{R}^n$, defined by $\mathbb{R}_{++}x \mapsto \|x\|^{-1} p(x)$ for all rays $\mathbb{R}_{++}x$.

Proof.

1. Convex sets with recession.
    - Deconification. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be a convex cone for which $0 \in C \subset \mathbb{R}^n \times \mathbb{R}_+$.
        - We check that $A(C)$ is a convex set. For all $\alpha_i > 0,\ x_i \in A(C),\ i = 1, 2$ for which $\alpha_1 + \alpha_2 = 1$, one has $(\alpha_1 x_1 + \alpha_2 x_2, 1) = \alpha_1(x_1, 1) + \alpha_2(x_2, 1) \in C$, as $(x_i, 1) \in A(C) \times 1 \subset C,\ i = 1, 2$ and C is a convex cone. Therefore, $(\alpha_1 x_1 + \alpha_2 x_2, 1) \in C \cap (\mathbb{R}^n \times 1) = A(C) \times 1$. That is, $\alpha_1 x_1 + \alpha_2 x_2 \in A(C)$.
        - We check that the two given definitions of 0^+A are equivalent. For this, it suffices to prove that $\alpha x + a \in A$ for each choice of an element $x \in \mathbb{R}^n$ for which $x + A \subset A$, an element $a \in A$ and a number $\alpha > 0$. Let r be the nonnegative integer for which $r \leq \alpha \leq r+1$. By repeated application of $x + A \subset A$, it follows that $rx + a, (r+1)x + a \in A$. The element $\alpha x + a$ is a convex combination of $rx + a$ and $(r+1)x + a$: indeed, $\alpha x + a = (r + 1 - \alpha)(rx + a) + (\alpha - r)((r + 1)x + a)$, and so, as A is a convex set, we obtain $\alpha x + a \in A$.
        - We check that 0^+A is a convex cone containing zero. One has $0 + A \subset A$, and for all $\alpha_i > 0, x_i \in 0^+A,\ i = 1, 2$ one has $\alpha_1 x_1 + \alpha_2 x_2 + A \subset A$, that is, $\alpha_1 x_1 + \alpha_2 x_2 \in 0^+A$.
        - We check the stated properties of $A_0(C)$. $A_0(C)$ is a convex cone containing zero as $A_0(C) \times 0$ is the intersection of two convex cones containing zero, C and $\mathbb{R}^n \times 0$. The inclusion $A_0(C) \subset 0^+A(C)$ holds as for each $x \in A_0(C)$ and each $y \in A(C)$, one has $(x, 0) \in C$ and $(y, 1) \in C$ and so $(x + y, 1) = (x, 1) + (y, 0) \in C$ as C is a convex cone. Therefore, $(x + y, 1) \in C \cap (\mathbb{R}^n \times 1) = A(C) \times 1$. That is, $x + y \in A(C)$.
    - Conification. Let (A, A_0) be a convex set in $\mathbb{R}^n$ with recession. By definition, $0 \in C(A, A_0) \subset \mathbb{R}^n \times \mathbb{R}_+$.
        - We check that $C(A, A_0)$ belongs to $\mathcal{C}_n$. By definition, $C(A, A_0) \subset \mathbb{R}^n \times \mathbb{R}_+$. It remains to check that $C(A, A_0)$ is a convex cone. For this, it suffices, by the definition of $C(A, A_0)$, to check that $\alpha_1 v_1 + \alpha_2 v_2 \in C(A, A_0)$ for all $\alpha_i > 0,\ i = 1, 2$ and each pair of elements $v_1, v_2 \in C(A, A_0)$ of one of the following three types:
            (a) $v_i = (x_i, 1),\ i = 1, 2$, so $x_i \in A,\ i = 1, 2$; then $\alpha_1 v_1 + \alpha_2 v_2 = (\alpha_1 + \alpha_2)(\alpha_1(\alpha_1 + \alpha_2)^{-1} x_1 + \alpha_2(\alpha_1 + \alpha_2)^{-1} x_2, 1)$, and so, as A is a convex set, $\alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}_{++}(A \times 1)$. Therefore, $\alpha_1 v_1 + \alpha_2 v_2 \in C(A, A_0)$.

(b) $v_i = (x_i, 0),\ i = 1,2$, so $x_i \in A_0,\ i = 1,2$; then $\alpha_1 v_1 + \alpha_2 v_2 = (\alpha_1 x_1 + \alpha_2 x_2, 0)$ and so, as A_0 is a convex cone, $\alpha_1 v_1 + \alpha_2 v_2 \in A_0 \times 0$. Therefore, $\alpha_1 v_1 + \alpha_2 v_2 \in C(A, A_0)$.

(c) $v_1 = (x_1, 1), v_2 = (x_2, 0)$, so $x_1 \in A, x_2 \in A_0$; then $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(x_1 + \frac{\alpha_2}{\alpha_1} x_2, 1)$; and so, as $A_0 \subset 0^+ A$, one has $\alpha_1 v_1 + \alpha_2 v_2 = \in \mathbb{R}_{++}(A \times 1)$. Therefore, $\alpha_1 v_1 + \alpha_2 v_2 \in C(A, A_0)$.

- We have that $C(A, A_0) \in \mathcal{C}_n$ and, by definition, $C(A, A_0) \cap (\mathbb{R}^n \times 1) = A \times 1$ and $C(A, A_0) \cap (\mathbb{R}^n \times 0) = A_0 \times 0$. For a convex cone $C \in \mathcal{C}_n$, the condition $O(C) = (A, A_0)$ means that $C \cap (\mathbb{R}^n \times 1) = A \times 1$ and $C \cap (\mathbb{R}^n \times 0) = A_0 \times 0$. Now we use that convex cones in $\mathcal{C}_n$ are determined by their intersections with the hyperplanes $\mathbb{R}^n \times 1$ and $\mathbb{R}^n \times 0$. It follows that for convex cones $C \in \mathcal{C}_n$, one has $O(C) = (A, A_0)$ precisely if $C = C(A, A_0)$.

2. Sublinear functions.

- Deconification. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be a nonempty convex cone for which $C \supset \mathbb{R}_{++}(0_n, 1)$.
  - We check that $p(C)$ satisfies the defining inequality of sublinear functions. Take arbitrary $\alpha_i > 0, x_i \in \mathbb{R}^n,\ i = 1,2$ for which $\{p(C)(x_1), p(C)(x_2)\} \neq \{+\infty, -\infty\}$. We want to verify the inequality $p(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 p(C)(x_1) + \alpha_2 p(C)(x_2)$. We assume that $p(C)(x_i) < +\infty, i = 1,2$ as we may: otherwise the inequality holds as the right hand side is $+\infty$. For all $(x_i, \rho_i) \in \text{sepi}p(C) \subset C,\ i = 1,2$, one has, as C is a convex cone, that $\alpha_1(x_1, \rho_1) + \alpha_2(x_2, \rho_2) \in C \subset \text{epi}p(C)$. That is, $p(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 \rho_1 + \alpha_2 \rho_2$. Taking the infimum over all $\rho_i,\ i = 1,2$ for which $(x_i, \rho_i) \in \text{sepi}p(C)$ $i = 1,2$, we obtain the inequality $p(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 p(C)(x_1) + \alpha_2 p(C)(x_2)$.
  - We check the stated property of $p(C)(0)$. By the inequality that is just proved, with $x_1 = x_2 = 0$, one gets that $p(C)(0)$ equals $0, -\infty$ or $+\infty$. As $(0_n, 1) \in C \subset \text{epi}p(C)$, we get that $1 \geq p(C)(0)$, so $p(C)(0) \neq +\infty$. Therefore, $p(C)(0)$ equals 0 or $-\infty$.
- Conification. Let $p : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a sublinear function.
  - We use that sepi$p$ is contained in epi$p$, and that they have the same closure. It follows, by definition of $C_l(p)$ and $C_u(p)$, that $C_l(p)$ is contained in $C_u(p)$, and that they have the same closure.
  - We check that $C_l(p)$ is a convex cone. Choose arbitrary $\alpha_i > 0, (x_i, \rho_i) \in C_l(p),\ i = 1,2$. Then $p(x_i) > -\infty,\ i = 1,2$, so $\{p_1(x_1), p(x_2)\} \neq \{+\infty, -\infty\}$, so $p(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 p(x_1) + \alpha_2 p(x_2)$, as p is a sublinear function. Therefore, $\alpha_1 \rho_1 + \alpha_2 \rho_2 > \alpha_1 p(x_1) + \alpha_2 p(x_2) \geq p(\alpha_1 x_1 + \alpha_2 x_2)$, that is, $(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 \rho_1 + \alpha_2 \rho_2) \in \text{sepi}p = C_l(p)$. It follows that $\alpha_1(x_1, \rho_1) + \alpha_2(x_2, \rho_2) = (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 \rho_1 + \alpha_2 \rho_2)$ is contained in $C_l(p)$.
  - We check that $C_l(p)$ contains the convex cone $\mathbb{R}_{++}(0_n, 1)$. This follows from $C_l(p) = \text{sepi}p$ and from $p(0) = 0$ or $p(0) = -\infty$, which implies $p(0) \leq 0$, that is, sepi$p$ contains the convex cone $\mathbb{R}_{++}(0_n, 1)$.

– The verifications that $C_u(p)$ is a convex cone containing $\mathbb{R}_{++}(0_n,1)$ are similar.

– It follows that $C_l(p) = \text{sepi}p$ and $C_u(p) = \text{epi}p$ belong to $\mathcal{C}_n$. For a convex cone $C \in \mathcal{C}_n$, the condition $O(C) = p$ means that $\text{sepi}p \subset C \subset \text{epi}p$, that is, $C_l(p) \subset C \subset C_u(p)$.

3. Convex functions with recession.

- Deconification. Let $C \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ be a convex cone for which $\mathbb{R}_{++}(0_n,1,0) \subset C \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$.

  – We check that $f(C)$ is a convex function. Choose arbitrary $\alpha_i > 0, x_i \in \mathbb{R}^n,\ i = 1,2$ for which $\alpha_1 + \alpha_2 = 1$ and $\{f(C)(x_1), f(C)(x_2)\} \neq \{+\infty, -\infty\}$. We want to verify the inequality $f(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(C)(x_1) + \alpha_2 f(C)(x_2)$. We assume that $f(C)(x_i) < +\infty,\ i = 1,2$, as we may without restriction of the generality; otherwise, the inequality holds, as the right hand side is $+\infty$. For all $\rho_i \in \mathbb{R},\ i = 1,2$ for which $(x_i, \rho_i) \in \text{sepi}f(C),\ i = 1,2$, one has $(x_i, \rho_i, 1) \in \text{sepi}f(C) \times 1 \subset C,\ i = 1,2$, and so, one has, as C is a convex cone, that $\alpha_1(x_1, \rho_1, 1) + \alpha_2(x_2, \rho_2, 1) \in C \cap (\mathbb{R}^n \times \mathbb{R} \times 1) \subset (\text{epi}f(C)) \times 1$. That is, $f(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 \rho_1 + \alpha_2 \rho_2$. Taking the infimum over all $\rho_i,\ i = 1,2$ for which $(x_i, \rho_i) \in \text{sepi}f(C),\ i = 1,2$, we obtain the inequality $f(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(C)(x_1) + \alpha_2 f(C)(x_2)$.

  – We check that $f_0(C)$ satisfies the defining inequality of sublinear functions. We choose arbitrary $\alpha_i > 0, x_i \in \mathbb{R}^n,\ i = 1,2$ for which $\{f_0(C)(x_1), f_0(C)(x_2)\} \neq \{+\infty, -\infty\}$. We want to verify the inequality $f_0(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f_0(C)(x_1) + \alpha_2 f_0(C)(x_2)$. We assume that $f_0(C)(x_i) < \infty,\ i = 1,2$, as we may without restriction of the generality; otherwise, the inequality holds, as the right hand side is $+\infty$. For all $\rho_i \in \mathbb{R},\ i = 1,2$ for which $(x_i, \rho_i) \in \text{sepi}f_0(C),\ i = 1,2$, one has $(x_i, \rho_i, 0) \in \text{sepi}f_0(C) \times 0 \subset C \cap (\mathbb{R}^n \times 0),\ i = 1,2$, and so, one has, as C is a convex cone, that $\alpha_1(x_1, \rho_1, 0) + \alpha_2(x_2, \rho_2, 0) \in C \cap (\mathbb{R}^n \times \mathbb{R} \times 0) \subset \text{epi}f_0(C) \times 0$. That is, $f_0(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 \rho_1 + \alpha_2 \rho_2$. Taking the infimum over all $\rho_i,\ i = 1,2$ for which $(x_i, \rho_i) \in \text{sepi}f_0(C),\ i = 1,2$, we obtain the inequality $f_0(C)(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f_0(C)(x_1) + \alpha_2 f_0(C)(x_2)$.

  – We check that $f_0(C)(0) = 0$ or $f_0(C)(0) = -\infty$. By the inequality that is just proved, with $x_1 = x_2 = 0$, one gets that $f_0(C)(0)$ equals $0, -\infty$ or $+\infty$. As $(0_n, 1, 0)) \in C \cap (\mathbb{R}^n \times \mathbb{R} \times 0) \subset \text{epi}f_0(C)$, we get $1 \geq f_0(C)(0)$, so $f_0(C)(0) \neq +\infty$. Therefore, $f_0(C)(0)$ equals 0 or $-\infty$.

- Conification. Let (f, f_0) be a convex function on $\mathbb{R}^n$ with recession.

  – We use that $\text{sepi}f$ (resp. $\text{sepi}f_0$) is contained in $\text{epi}f$ (resp. $\text{epi}f_0$) and that they have the same closure. It follows, by the definitions of $C_l(f, f_0)$ and $C_u(f, f_0)$, that $C_l(f, f_0)$ is contained in $C_u(f, f_0)$, and that they have the same closure.

– We check that $C_l(f, f_0)$ (resp. $C_u(f, f_0)$) belongs to $\mathcal{C}_n$. This verification can be reduced to what has been proved already for convex sets with recession by means of the equality $C_l(f, f_0) = C(\text{sepi}f, \text{sepi}f_0)$ (resp. $C_u(f, f_0) = C(\text{epi}f, \text{epi}f_0)$).

– We have that $C_l(f, f_0)$ and $C_u(f, f_0)$ belong to $\mathcal{C}_n$ and, by definition, $C_l(f, f_0) \cap (\mathbb{R}^n \times \mathbb{R} \times 1) = \text{sepi}f$, $C_l(f, f_0) \cap (\mathbb{R}^n \times \mathbb{R} \times 0) = \text{sepi}f_0$, $C_u(f, f_0) \cap (\mathbb{R}^n \times \mathbb{R} \times 1) = \text{epi}f$, and $C_u(f, f_0) \cap (\mathbb{R}^n \times \mathbb{R} \times 0) = \text{epi}f_0$. For convex cones $C \in \mathcal{C}_n$, one has $O(C) = (f, f_0)$ precisely if $\text{sepi}f \times 1 \subset C \cap (\mathbb{R}^n \times \mathbb{R} \times 1) \subset \text{epi}f \times 1$ and $\text{sepi}f_0 \times 0 \subset C \cap (\mathbb{R}^n \times \mathbb{R} \times 0) \subset \text{epi}f_0 \times 0$. Now we use that convex cones that belong to $\mathcal{C}_n$ are determined by their intersections with the hyperplanes $\mathbb{R}^n \times \mathbb{R} \times 1$ and $\mathbb{R}^n \times \mathbb{R} \times 0$. It follows that for convex cones $C \in \mathcal{C}_n$, one has $O(C) = (f, f_0)$ precisely if $C_l(f, f_0) \subset C \subset C(f, f_0)$.

The following proposition gives the result of transferring some concepts from convex cones to convex objects. We need some definitions. Sometimes we indicate the dependence of $\mathcal{C}_n$ and C_n on the type $\mathcal{O}$ by writing $\mathcal{C}_n(\mathcal{O})$ and $\mathrm{C}_n(\mathcal{O})$ instead. For each function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and each subset $S \subset \mathbb{R}^n$, the *restriction of f to S* is denoted by $f|_S$. The *effective domain* $\text{dom}f$ of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is the set $\{x \in \mathbb{R}^n | f(x) < +\infty\}$. This is a convex set (resp. a convex cone) if f is a convex function (resp. a sublinear function). We let $\text{cl}A$ denote the topological closure in $\mathbb{R}^n$ of a convex set $A \subset \mathbb{R}^n$ and we let $\text{cl}f$ be the function $\mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ determined by $\text{epi}(\text{cl}f) = \text{cl}(\text{epi}f)$ for a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. The linear transformation $\mathbb{R}^{n+s} \to \mathbb{R}^{m+s}$, $s = 0, 1, 2$ defined by operating with Λ on the first n coordinates and by acting by the identity operator on the other s coordinates, is again denoted by Λ.

Proposition 3.2 *1. For each type of convex objects $\mathcal{O}$, there is a unique type $\mathcal{O}^D$—called the dual type of $\mathcal{O}$—for which $C^\circ \in \mathcal{C}_n(\mathcal{O}^D)$ for all $C \in \mathcal{C}_n(\mathcal{O})$. Explicitly, 'convex cones' and 'convex functions with recession' are self dual, 'sublinear functions' and 'convex sets with recession' are each others dual type.*

2. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, $\mathcal{O}$ one of the four fundamental classes and O (resp. $\bar{O}$) a convex object of type $\mathcal{O}$ over $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Choose a convex cone C in $\mathcal{C}_n$ (resp. $\bar{C}$ in $\mathcal{C}_m$) for which $O(C) = O$ (resp. $O(\bar{C}) = \bar{O}$). The following concepts are independent of the choice of C (resp. $\bar{C}$):

(a) Duality operator. *The object O^D of type $\mathcal{O}^D$ over $\mathbb{R}^n$—called the dual object of O—that is defined by $O^D = O(C^\circ)$. Explicitly:*

i. convex cones containing zero C: $C^D = C^\circ$, where the convex cone C°—the polar cone—is defined by $\{y \in \mathbb{R}^n | \langle y, x\rangle \leq 0 \ \forall x \in C\}$;

ii. convex sets with recession (A, A_0): $(A, A_0)^D = sA|_{A_0^\circ}$, where the sublinear function sA—the support function of A—is defined by $(sA)(y) = \sup\{\langle y, x\rangle | x \in A\}$ for all $y \in \mathbb{R}^n$;

iii. sublinear functions p: $p^D = (\partial p, (\mathrm{dom}p)^\circ)$, where the convex set ∂p—the subdifferential of p—is defined by $\{y \in \mathbb{R}^n | \langle y, x\rangle \le p(x)\ \forall x \in \mathbb{R}^n\}$;

iv. convex functions with recession (f, f_0): $(f, f_0)^D = (f^*|_{\partial f_0}, s(\mathrm{dom}f)|_{(\mathrm{dom}f_0)^\circ})$, where the convex function f^*—the conjugate function (or Legendre-Fenchel transform)—is defined by $f^*(y) = \sup\{\langle y, x\rangle - f(x) | x \in \mathbb{R}^n\}$ for all $y \in \mathbb{R}^n$.

(b) Closure. *The object $\mathrm{cl}O$ of type $\mathcal{O}$ over $\mathbb{R}^n$—called the closure of O—that is defined by $\mathrm{cl}O = O(\mathrm{cl}C)$. Explicitly:*

i. convex sets with recession (A, A_0): $\mathrm{cl}(A, A_0)$ equals $(\mathrm{cl}A, 0^+\mathrm{cl}A)$ if $A \ne \emptyset$ and it equals $(\emptyset, \mathrm{cl}A_0)$ if $A = \emptyset$;

ii. convex functions with recession (f, f_0): $\mathrm{cl}(f, f_0) = (\mathrm{cl}f, (\mathrm{cl}f)0^+)$ if $f \not\equiv +\infty$ and it equals $(+\infty, \mathrm{cl}f_0)$ if $f \equiv +\infty$;

iii. for cones containing zero and sublinear functions, the unified definition of closure agrees with the usual one;

(c) Image. *The object ΛO of type $\mathcal{O}$ over $\mathbb{R}^m$—called the image of O under Λ—that is defined by $\Lambda O = O(\Lambda C)$.*

(d) Inverse image. *The object $\Lambda^{-1}\bar{O}$ of type $\mathcal{O}$ over $\mathbb{R}^n$—called the inverse image of $\bar{O}$ under Λ—that is defined by $\Lambda^{-1}O = O(\Lambda^{-1}\bar{C})$.*

Remark. We emphasize a special case of interest. Statement 2.$(a)iv$ shows that the duality operator on convex functions defined by the conification method—conify, take the polar cone, deconify—is different from the usual duality operator, the Legendre-Fenchel transform for improper convex functions (other than the trivial ones: the function that is identically $+\infty$ and the function that is identically $-\infty$). Moreover, this statement shows that the two duality operators agree for proper convex functions. We will see later that the duality operator defined by conification has advantages over the usual one (see the remark following theorem 3.4).

The explicit versions for images and inverse images are the obvious ones—for example $\Lambda(A, A_0) = (\Lambda A, \Lambda A_0)$ for convex sets with recession—and have not been displayed in the result above.

Proof.

1. The property of convex cones in $\mathbb{R}^n$ to contain zero is preserved by taking polar cones. The polar cone operator interchanges the properties of nonempty convex cones in $\mathbb{R}^n \times \mathbb{R}$ to be contained in $\mathbb{R}^n \times \mathbb{R}_+$ and to contain $\mathbb{R}_{++}(0_n, 1)$. The property of convex cones in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ to be contained in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$ and to contain $\mathbb{R}_{++}$ is preserved by the polar cone operator.

2. (a) Duality operator. The equality $O(C) = O$ gives, by proposition 3.1, the inclusions $C_l(O) \subset C \subset C_u(O)$. Taking closures and using $\mathrm{cl}C_l(O) = \mathrm{cl}C_u(O)$ gives $\mathrm{cl}C_l(O) = \mathrm{cl}C = \mathrm{cl}C_u(O)$.

Taking polar cones, and using that a convex cone has the same polar cone as its closure, that $C_l(O)^\circ = C^\circ = C_u(O)^\circ$. Therefore, $O^D = O(C^\circ)$ is independent of the choice of C.

i. Convex cones C. There is nothing to check here.

ii. Convex sets with recession (A, A_0). It suffices to check the equality $C(A, A_0)^\circ = C(sA|_{A_0})$. For an element $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ to belong to $C(A, A_0)^\circ$ means that $\langle (y, \beta), (x, \alpha) \rangle \le 0$ for all $(x, \alpha) \in C(A, A_0)$. By the definition of $C(A, A_0)$, this amounts to the conditions $\langle (y, \beta), (x, 1) \rangle = \langle y, x \rangle_n - \beta \le 0$ for all $x \in A$, and $\langle (y, \beta), (x, 0) \rangle = \langle y, x \rangle_n \le 0$ for all $x \in A_0$. This can be formulated as follows: $(y, \beta) \in \mathrm{epi}(sA)$ and $y \in A_0^\circ$. That is, (y, β) is an element of $\mathrm{epi}(sA|_{A_0^\circ})$, that is, of $C(sA|_{A_0})$.

iii. Sublinear functions p. It suffices to check the equality $C_u(p)^\circ = C(\partial p, (\mathrm{dom} p)^\circ)$. For an element $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ to belong to $C_u(p)^\circ$ means that $\langle (y, \beta), (x, \alpha) \rangle \le 0$ for all $(x, \alpha) \in C_u(p)$. By the definition of $C_u(p)$, this amounts to the condition $\langle (y, \beta), (x, \alpha) \rangle = \langle x, y \rangle_n - \beta\alpha \le 0$ for all $(x, \alpha) \in \mathrm{epi} p$. For $\beta = 1$ this reduces to $\langle y, x \rangle_n \le \alpha$ for all $(x, \alpha) \in \mathrm{epi} p$, that is, $y \in \partial p$. For $\beta = 0$ this reduces to $\langle y, x \rangle_n \le 0$ for all $(x, \alpha) \in \mathrm{epi} p$, that is, $y \in (\mathrm{dom} p)^\circ$. That is, (y, β) is an element of $C(\partial p, (\mathrm{dom} p)^\circ)$.

iv. Convex functions with recession (f, f_0). It suffices to check the equality $C_u(f, f_0)^\circ = C(f^*|_{\partial f_0}, s(\mathrm{dom} f)|_{\mathrm{dom} f_0})$. For an element $(y, \beta, \delta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ to belong to $C_u(f, f_0)^\circ$ means that $\langle (y, \beta, \delta), (x, \alpha, \gamma) \rangle \le 0$ for all $(x, \alpha, \gamma) \in C_u(f, f_0)$. By the definition of $C(f, f_0)$, this amounts to the conditions $\langle (y, \beta, \delta), (x, \alpha, 1) \rangle = \langle y, x \rangle_n - \beta - \delta\alpha \le 0$ for all $(x, \alpha) \in \mathrm{epi} f$, and $\langle (y, \beta, \delta), (x, \alpha, 0) = \langle y, x \rangle_n - \delta\alpha \le 0$ for all $(x, \alpha) \in \mathrm{epi} f_0$. For $\delta = 1$ this reduces to $\beta \ge \langle y, x \rangle_n - \alpha$ for all $(x, \alpha) \in \mathrm{epi} f$, that is, $(y, \beta) \in \mathrm{epi} f^*$, and $\alpha \ge \langle y, x \rangle_n$ for all $(x, \alpha) \in \mathrm{epi} f_0$, that is, $y \in \partial f_0$; these conditions can be written as $(y, \beta) \in \mathrm{epi}(f^*|_{\partial f_0})$. For $\delta = 0$ this reduces to $\beta \ge \langle y, x \rangle_n$ for all $(x, \alpha) \in \mathrm{epi} f$, that is, $(y, \beta) \in \mathrm{epi} s(\mathrm{dom} f)$, and $\langle y, x \rangle_n \le 0$ for all $(x, \alpha) \in \mathrm{epi} f_0$, that is, $y \in (\mathrm{dom} f_0)^\circ$; these conditions can be written as $(y, \beta) \in \mathrm{epi}(s(\mathrm{dom} f)|_{(\mathrm{dom} f_0)^\circ})$. In all, we obtain that $(y, \beta, \delta) \in C(f^*|_{\partial f_0}, s(\mathrm{dom} f)|_{(\mathrm{dom} f_0)^\circ})$.

(b) Closure. The equality $O(C) = O$ gives, again, $\mathrm{cl} C_l(O) = C = C_u(O)$. Therefore, $\mathrm{cl} O = O(\mathrm{cl} C)$ is independent of the choice of C.

i. Convex sets with recession (A, A_0). It suffices to check that $\mathrm{cl} C(A, A_0)$ is equal to $C(\mathrm{cl} A, 0^+\mathrm{cl} A)$ if $A \ne \emptyset$ and that it equals $C(\emptyset, \mathrm{cl} A_0)$ if $A = \emptyset$.

- The case $A = \emptyset$. Then $\mathrm{cl} C(A, A_0) = \mathrm{cl}(A_0 \times 0) = (\mathrm{cl} A_0) \times 0 = C(\emptyset, \mathrm{cl} A_0)$, as required.
- The case $A \ne \emptyset$.
  - We check the inclusion $\mathrm{cl} C(A, A_0) \subset C(\mathrm{cl} A, 0^+\mathrm{cl} A)$. We take an arbitrary element $(x, \rho) \in \mathrm{cl} C(A, A_0)$. We choose an infinite sequence $(x_k, \rho_k)_k$ in $C(A, A_0)$

that converges to (x,ρ). If $\rho>0$, then it suffices to show that $\rho^{-1}x\in \text{cl}A$, as this implies $(x,\rho)=\rho(\rho^{-1}x,1)\in\mathbb{R}_{++}(\text{cl}A\times 1)\subset C(\text{cl}A,0^+\text{cl}A)$. We assume that $\rho_k>0$ for all k as we may by going over to a suitable subsequence. Then, for each k we have $(x_k,\rho_k)\in\mathbb{R}_{++}(A\times 1)$, and so $\rho_k^{-1}x_k\in A$. Taking the limit $k\to+\infty$ gives $\rho^{-1}x\in\text{cl}A$, as required. If $\rho=0$, then it suffices to show that $x\in 0^+\text{cl}A$, as this implies $(x,\rho)=(x,0)\in 0^+\text{cl}A\times 1\subset C(\text{cl}A,0^+\text{cl}A)$. We distinguish two cases:

- $*$ $\rho_k=0$ for at most finitely many k; we assume that $\rho_k>0$ for all k, as we may by going over to a suitable subsequence. Then for all k one has $(\rho_k^{-1}x_k,1)\in C(A,A_0)\cap((\mathbb{R}^n\times 1)=A\times 1$, that is $\rho_k^{-1}x_k\in A$. We take an arbitrary $a\in A$, and choose an infinite sequence $(a_k)_k$ in A that converges to a. Then $a+x=\lim_{k\to+\infty}(\frac{1}{1+\rho_k}a_k+\frac{\rho_k}{1+\rho_k}(\rho_k^{-1}x_k))\in\text{cl}A$. Therefore, $x\in 0^+A$, as required.
- $*$ $\rho_k=0$ for infnitely many k; we assume that $\rho_k=0$ for all k, as we may by going over to a suitable subsequence. Then for all k one has $x_k\in A_0\subset 0^+A$. We take an arbitrary $a\in\text{cl}A$ and choose an infinite sequence $(a_k)_k$ in A that converges to a. Then $a+x=\lim_{k\to+\infty}a_k+x_k$. One has for each k that $a_k+x_k\in A$ as $a_k\in A$ and $x_k\in 0^+A$. Therefore, $a+x\in\text{cl}A$.

– We check the inclusion $C(\text{cl}A,0^+\text{cl}A)\subset\text{cl}C(A,A_0)$. We take an arbitrary element $(x,\rho)\in C(\text{cl}A,0^+\text{cl}A)$. If $\rho>0$, then $(\rho^{-1}x,1)\in(\mathbb{R}^n\times 1)\cap C(\text{cl}A,0^+\text{cl}A)=\text{cl}A\times 1$, and so $\rho^{-1}x\in\text{cl}A$. We choose an infinite sequence $(y_k)_k$ in A that converges to $\rho^{-1}x$. Then $(x,\rho)=\lim_{k\to+\infty}(\rho y_k,\rho)$ and so $(x,\rho)\in\text{cl}C(A,A_0)$. If $\rho=0$, then we choose $a\in A$. We have $(x,\rho)=(x,0)=\lim_{k\to+\infty}(k^{-1}a+x,k^{-1})=\lim_{k\to+\infty}k^{-1}(a+kx,1)$. As $a\in A$ and $x\in 0^+\text{cl}A$, we have for all k that $a+kx\in\text{cl}\mathbb{R}_{++}(\text{cl}A\times 1)\subset\text{cl}(\mathbb{R}_{++}(A\times 1)\subset\text{cl}C(A,A_0)$. It follows that $(x,\rho)\in\text{cl}C(A,A_0)$.

ii. Convex functions with recession (f,f_0). It suffices to check that $\text{cl}C(f,f_0)$ equals $C(\text{cl}f,(\text{cl}f)0^+)$ if $f\not\equiv+\infty$ and that it equals $C(+\infty,\text{cl}f_0)$ if $f\equiv+\infty$. These statements follow immediately from the explicit formulas for $\text{cl}(A,A_0)$ by virtue of the defining formula $C_u(f,f_0)=C(\text{epi}f,\text{epi}f_0)$. definition of $C(f,f_0)$.

iii. Convex cones containing zero and sublinear functions. These statements follow immediately from the definitions.

(c) Image. For convex sets with recession and convex cones, this follows immediately from the definitions and for convex functions with recession and sublinear functions, this follows from the inclusions $\text{sepi}\Lambda f\subset\Lambda(\text{sepi}f)\subset\Lambda(\text{epi}f)\subset\text{epi}\Lambda f$.

(d) Inverse image. For convex sets with recession and convex cones, this follows immediately from the definitions and for convex functions with recession and sublinear functions, this follows from the inclusions $\text{sepi}\Lambda^{-1}\bar{f} \subset \Lambda^{-1}(\text{sepi}\bar{f}) \subset \Lambda^{-1}(\text{epi}\bar{f}) \subset \text{epi}\Lambda^{-1}\bar{f}$.

A convex object is called closed if it is equal to its closure.

Proposition 3.3 *Each closed convex object has a unique closed conification. Explicitly, the unique closed convex cones corresponding to closed convex objects are as follows:*

1. *for a closed convex set with recession (A, A_0), it is $C(A, A_0)$;*
2. *for a closed sublinear function p, it is $C_u(p)$;*
3. *for a closed convex function with recession (f, f_0), it is $C_u(f, f_0)$.*

Proof. This is a consequence of the propositions 3.1. and 3.2.

Now we are ready to give the main result on convex objects. The *relative interior* $\text{ri}A$ of a convex set is the interior of A, when A is regarded as a subset of its affine hull; this is a convex set. The usual definitions of polyhedrality of convex objects are that a convex set A is called *polyhedral* if it is the intersection of a finite collection of closed half spaces; a convex function f is called *polyhedral* if its epigraph is a polyhedral convex set. The unified definition is that a closed convex object is called a *polyhedral convex object* if its closed conification is a polyhedral cone. Explicitly, a closed convex set with recession (A, A_0) is polyhedral if A and A_0 are polyhedral, a closed convex function with recession (f, f_0) is polyhedral if f and f_0 are polyhedral. For convex cones and sublinear functions, the unified definition of polyhedrality agrees with the usual one.

Theorem 3.4 *Let $\mathcal{O}$ be one of the four fundamental classes of convex objects. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, let O (resp. $\bar{O}$) be a convex object of type $\mathcal{O}$ over $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Then the following formulas hold true :*

1. $O^{DD} = \text{cl}O$,
2. $(\Lambda O)^D = (\Lambda')^{-1}(O^D)$,
3. $(\Lambda^{-1}\text{cl}\bar{O})^D = \text{cl}(\Lambda'(\bar{O}^D))$.

Both sides of formula 3. remain the same if the closure operator is omitted, provided one of the following two assumptions holds true:

a. In the case of convex sets (and so of convex cones): if $\bar{A} \neq \emptyset$, then $\text{ri}\bar{A} \cap \text{Im }\Lambda \neq \emptyset$, if $\bar{A} = \emptyset$, then $\text{ri}\bar{A}_0 \cap \text{Im }\Lambda \neq \emptyset$; in the case of convex functions and (and so of sublinear functions): if $\bar{f} \not\equiv +\infty$, then $\text{ridom}\bar{f} \cap \text{Im }\Lambda \neq \emptyset$, if $\bar{f} \equiv +\infty$, then $\text{ridom}\bar{f}_0 \cap \text{Im }\Lambda \neq \emptyset$

b. $\bar{O}$ is a polyhedral object.

Remark. A special case of interest of this result concerns convex functions. The first statement of this result means that the duality formula for convex functions holds for all convex functions, proper as well as improper, if we define the duality operator on convex functions by means of the conification method: conify, take the polar cone, deconify. From proposition 3.2 2. $(a)iv$, it follows that this duality operator equals the usual duality operator, the Legendre-Fenchel transform, for proper convex functions, but not for improper convex functions (except the convex functions that are identically $+\infty$ or $-\infty$. As many results from Convex Analysis depend on the duality formula for convex functions, it follows that in this way one gets satisfactory versions of these results, without the need to exclude certain special cases.

Proof.

1. Choose a convex cone $C \in \mathcal{C}_n$ for which $O(C) = O$. By theorem 2.1, one has $C^{\circ\circ} = \text{cl}C$. Moreover, one has $O(C^{\circ\circ} = O(C^\circ)^D = O(C)^{DD} = O^{DD}$ and $O(\text{cl}C) = \text{cl}O(C) = \text{cl}O$. It follows that $O^{DD} = \text{cl}O$.

2. Choose a convex cone $C \in \mathcal{C}_n$ for which $O(C) = O$. By theorem 2.1, one has $(\Lambda O)^D = (\Lambda')^{-1}O^D$. Moreover, one has $O((\Lambda C)^\circ) = O(\Lambda C)^D = (\Lambda(O(C)))^D = (\Lambda O)^D$ and $O((\Lambda')^{-1}(C^\circ) = (\Lambda')^{-1}O(C^\circ) = (\Lambda')^{-1}O(C)^D = (\Lambda')^{-1}(O^D)$. It follows that $(\Lambda O)^D = (\Lambda')^{-1}(O^D)$.

3. Choose a convex cone $\bar{C} \in \mathcal{C}_m$ for which $O(\bar{C}) = \bar{O}$. By theorem 2.1 one has $(\Lambda^{-1}\text{cl}\bar{C})^\circ = \text{cl}(\text{cl})((\Lambda')\bar{C}^D)$. Moreover, one has

$$O(\Lambda^{-1}\text{cl}\bar{C})^\circ = (O(\Lambda^{-1}\text{cl}\bar{C}))^D = (\Lambda^{-1}O(\text{cl}\bar{C})) = (\Lambda^{-1}\text{cl}O(\bar{C}))^D = (\Lambda^{-1}\text{cl}\bar{O})^D$$

   and $O\text{cl}(\Lambda'(\bar{C}^\circ)) = \text{cl}O((\Lambda'(\bar{C}^\circ)) = \text{cl}\Lambda' O(\bar{C}^\circ) = \text{cl}\Lambda'(O(\bar{C})^D) = \text{cl}(\Lambda'\bar{O}^D)$.

It remains to translate the conditions in theorem 2.1 that imply that the closure operation may be omitted in the third formula from convex cones to convex objects. For the second condition this is clear. Now we do the first condition. Let $\bar{C} \in \mathcal{C}_m$ be a convex cone for which $O(\bar{C}) = \bar{O}$. In the case of convex sets with recession, $\bar{O} = (\bar{A}, \bar{A}_0)$, we have $\bar{C} = C(\bar{A}, \bar{A}_0)$. Therefore, the condition is $\text{Im}\ \Lambda \cap \text{ri}C(\bar{A}, \bar{A}_0) \neq \emptyset$. The relative interior of $C(A, A_0)$ equals $\mathbb{R}_{++}(\text{ri}A \times 1)$ if $\bar{A} \neq \emptyset$ and it equals $\text{ri}A_0 \times 0$ if $A = \emptyset$. Translation of this condition gives : if $A \neq \emptyset$, then $\text{ri}\bar{A} \cap \text{Im}\ \Lambda \neq \emptyset$, if $A = \emptyset$, then $\text{ri}\bar{A}_0 \cap \text{Im}\ \Lambda \neq \emptyset$. In the case of convex functions with recession, $\bar{O} = (\bar{f}, \bar{f}_0)$, we can take $\bar{C} = C_u(\bar{f}, \bar{f}_0) = C(\text{epi}f, \text{epi}f_0)$. Therefore, the condition is $\text{Im}\ \Lambda \cap \text{ri}C_u(f, f_0) \neq \emptyset$. The relative interior of $C_u(f, f_0)$ equals the relative interior of $C(\text{sepi}(f|_{\text{ridom}}), \text{sepi}(f_0|_{\text{ridom}}))$; here we use the easy fact that a convex function is continuous on the relative interior of its domain. Using what we have proved above for convex sets with recession, we get that the condition translates into: if $f \not\equiv +\infty$, then $\text{ridom}\bar{f} \cap \text{Im}\ \Lambda \neq \emptyset$, if $\bar{f} \equiv +\emptyset$, then $\text{ridom}\bar{f}_0 \cap \text{Im}\ \Lambda \neq \emptyset$

Remark The following subclasses of convex objects are of special interest.

1. *Self dual subclasses.* The classes 'convex sets with recession' and 'sublinear functions' are each others dual. The following subclass of the convex sets with recession (resp. sublinear functions) is dual to itself: *convex sets containing zero with recession* (B, B_0) and *nonnegative sublinear functions* (or *gauges* or *generalized norms*) n. Indeed, for both subclasses, the corresponding convex cones are the convex cones $C \subset \mathbb{R}^n \times \mathbb{R}$ that are contained in the closed halfspace $\mathbb{R}^n \times \mathbb{R}_+$ and that contain the convex cone generated by $(0_n, 1)$; the polar operator acts on this collection of convex cones. The explicit expressions for the polar operators is as follows: $(B, B_0)^D$ is equal to $(B^\circ, (0^+B)^\circ)$, where the *polar set* B° is defined by $\{y \in \mathbb{R}^n | \langle x, y\rangle \leq 1 \forall x \in B\}$ and where B_0° is the polar cone of B_0; $n^D = n^*$, where the *dual gauge* n^* is defined by $n^*(y)$ is the smallest element in $[0, +\infty]$ for which $\langle y, x\rangle \leq n^*(y)n(x)$ for all $x \in \mathbb{R}^n$.

2. *Linear subspaces.* For the subclass of convex cones, consisting of the subspaces, the duality operator reduces to the *orthogonal complement operator*, and the closure operator in the formulas of theorem 3.6 can be omitted as all subspaces of $\mathbb{R}^n$ are closed.

3. *Polyhedral convex objects.* It can be shown that the property 'polyhedrality' of a convex object is preserved under the duality operators and under taking images and inverse images of linear transformations.

4 Duality of binary operations on convex objects

The aim of this section is to formalize the procedure to construct the binary operations on convex objects given in [R], but to avoid the exclusion of exceptional cases; moreover, a simple unified proof of the duality formulas for binary operations, including some new formulas, is given.

Consider the special linear transformations $+_n : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (x_1, x_2) \mapsto x_1 + x_2$ and $\Delta_n : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n : x \mapsto (x, x)$.

Lemma 4.1 *The linear operators* $+_n$ *and* Δ_n *are dual to each other:* $(+_n)' = \Delta_n$ *and* $(\Delta_n)' = +_n$.

Proof. To prove this lemma, it suffices to show that (x_1, x_2) has the same inner product with $(+_n)'y$ as with $\Delta_n y$: for all $x_1, x_2, y \in \mathbb{R}^n$: indeed, $\langle (x_1, x_2), (+_n)'y\rangle$ equals $\langle \Delta_n(x_1, x_2), y\rangle$, by definition of the dual linear transformation; this equals $\langle x_1 + x_2, y\rangle$; moreover, $\langle (x_1, x_2), \Delta_n y\rangle = \langle (x_1, x_2), (y, y)\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$.

The two well-known binary operations on convex cones in $\mathbb{R}^n$ containing zero can be expressed in terms of these linear transformations as follows: $C_1 + C_2 = +_n(C_1 \times C_2)$ and $C_1 \cap C_2 = \Delta_n^{-1}(C_1 \times C_2)$.

If a factorization of the given vector space as a product of r factors is chosen, say, $\mathbb{R}^n = \prod_{i=1}^r \mathbb{R}^{n_i}$ with $n = \sum_{i=1}^r n_i$, then this gives a factorization $\mathbb{R}^n \times \mathbb{R}^n = \prod_{i=1}^r \mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$. Then one gets 2^r binary operations $\odot_\omega$ on the convex cones in $\mathbb{R}^n$ by considering one of the 2^r sequences ω of r minus $(-)$ and plus $(+)$ symbols, and by acting on the i-th factor $\mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$ of $\mathbb{R}^n \times \mathbb{R}^n$ $(1 \le i \le r)$ by $+_{n_i}$ if $\omega_i = +$ and by $\Delta_{n_i}^{-1}$ if $\omega_i = -$. The definition of the convex cone containing zero $C_1 \odot_\omega C_2 \subset \mathbb{R}^n$—for all convex cones containing zero $C_j \subset \mathbb{R}^n,\ j = 1, 2$ and each sequence ω of r minus $(-)$ and plus $(+)$ symbols—can be given in the following explicit form. We denote for each $x \in \mathbb{R}^n$ its projection on the i-th factor of $\mathbb{R}^n$ by $x^{(i)} \in \mathbb{R}^{n_i}$ $(1 \le i \le r)$. For all elements $x(j) \in C_j, j = 1, 2$ for which $x(1)^{(i)} = x(2)^{(i)}$ for all $i \in \{1, \ldots, r\}$ for which $\omega_i = -$, we define the element $x(1) \odot_\omega x(2) \in \mathbb{R}^n$ by $(x(1) \odot_\omega x(2))^{(i)} = x(1)^{(i)} + x(2)^{(i)}$ if $\omega_i = +$ and $(x(1) \odot_\omega x(2))^{(i)} = x(1)^{(i)} = x(2)^{(i)}$ if $\omega_i = -$. Then $C_1 \odot_\omega C_2 \subset \mathbb{R}^n$ is defined to consist of all elements $x(1) \odot_\omega x(2) \in \mathbb{R}^n$ that arise in this way. These binary operations $\odot_\omega$ on convex cones in $\mathbb{R}^n$ are commutative and associative. For a type of convex objects $\mathcal{O}$ and a natural number n, the *standard factorization*—into $s + 1$ factors—of the conification space $\mathrm{C}_n(\mathcal{O}) = \mathbb{R}^{n+s}$ is $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ if $s = 2$, it is $\mathbb{R}^n \times \mathbb{R}$ if $s = 1$ and it is $\mathbb{R}^n$ if $s = 0$. For later use, the binary operation resulting from replacing in ω each choice of $+$ by $-$, and vice versa, is denoted by ω^d. Note that $\omega^{dd} = \omega$. The following binary operations will arise automatically by means of this construction: 1) on convex sets: intersection, sum, convex hull of the union, and Kelley sum (defined by $A_1 \boxplus A_2 = \cup_{0 \le \alpha \le 1}(\alpha A_1 \cap (1 - \alpha)A_2)$); 2) on convex functions: maximum, sum (with the convention $\infty + (-\infty) = -\infty$), convex hull of the minimum, convolution (defined by $f_1 \oplus f_2(x) = \inf\{f_1(x_1) + f_1(x_2) | x_j \in \mathbb{R}^n, j = 1, 2,\ x = x_1 + x_2\}$).

To construct binary operations on convex objects, we apply again the procedure of proposition 3.2.

Proposition 4.2 *Let $\mathcal{O}$ be one of the four fundamental types of convex objects, let n be a natural number, let the conification space $\mathrm{C}_n(\mathcal{O})$ be $\mathbb{R}^{n+s}$, and let ω be a sequence of $s + 1$ minus and plus symbols. Let $O_i,\ i = 1, 2$ be convex objects of type $\mathcal{O}$ over $\mathbb{R}^n$. Choose convex cones $C_i \in \mathcal{C}_n(\mathcal{O}),\ i = 1, 2$ for which $O(C_i) = O_i,\ i = 1, 2$. Then the following statements hold true.*

1. $C_1 \odot_\omega C_2 \in \mathcal{C}_n(\mathcal{O})$.
2. *$O(C_1 \odot_\omega C_2)$ is a convex object of type $\mathcal{O}$ over $\mathbb{R}^n$ that is independent of the choices of $C_i,\ i = 1, 2$.*
3. *The binary operation $(O_1, O_2) \mapsto O_1 \odot_\omega O_2 = O(C_1 \odot_\omega C_2)$ on the convex objects of type $\mathcal{O}$ over $\mathbb{R}^n$ is commutative and associative.*
4. *Explicit formulas for the binary operations.*

   (a) Convex sets with recession. $(A(1), A_0(1)) \odot_\omega (A(2), A_0(2))$ *equals*

i. $(A(1) \cap A(2), A_0(1) \cap A_0(2))$ *if* $\omega = (--)$;

ii. $(A(1) + A(2), A_0(1) + A_0(2))$ *if* $\omega = (+-)$;

iii. $((A(1)\text{co} \cup A(2)) \cup (A(1) + A_0(2)) \cup (A_0(1) + A(2)), A_0(1) + A_0(2))$ *if* $\omega = (++)$;

iv. $((A(1) \boxplus A(2)) \cup (A(1) \cap A_0(2)) \cup (A_0(1) \cap A(2)), A_0(1) \cap A_0(2))$ *if* $\omega = (-+)$.

(b) Convex functions with recession. $(f(1), f_0(1)) \odot_\omega (f(2), f_0(2))$ *equals*

i. $(\max(f(1), f(2)), \max(f_0(1), f_0(2)))$ *if* $\omega = (---)$;

ii. $(f(1) + f(2), f_0(1) + f_0(2))$ *if* $\omega = (-+-)$;

iii. $(f(1) \oplus f(2), f_0(1) \oplus f_0(2))$ *if* $\omega = (++-)$;

iv. $((f(1)\text{co} \wedge f(2)) \vee (f(1) \oplus f_0(2)) \vee (f_0(1) \oplus f(2)), f_0(1) \oplus f_0(2))$ *if* $\omega = (+++)$.

(c) Convex cones and sublinear functions. *For convex cones one has the sum and the intersection; for sublinear functions, one has the maximum, the sum, the convolution, and the convex hull of the minimum.*

Proof.

1. To verify the inclusion $C_1 \odot_\omega C_2 \in \mathcal{C}_n(\mathcal{O})$, we consider each of the four types of convex objects $\mathcal{O}$ separately.

   (a) Convex cones contain zero. Then $s = 0$ and so $r = s + 1 = 1$. Choose $x(j) = 0$, $j = 1, 2$. Then $x(j) \in C_j$, $j = 1, 2$ as $C_j \in \mathcal{C}_n(\mathcal{O})$, $j = 1, 2$, and $x(1) = x(2)$; moreover, $x(1) \odot_\omega x(2) = 0$. Therefore, $0 \in C_1 \odot_\omega C_2$. That is, $C_1 \odot_\omega C_2 \in \mathcal{C}_n(\mathcal{O})$.

   (b) Convex sets with recession. Then $s = 1$ and so $r = s + 1 = 2$. Choose arbitrary $x(j) \in C_j$, $j = 1, 2$ for which $x(1)^{(i)} = x(2)^{(i)}$ if $i = -$. Then $x(j)^{(2)} \geq 0$, $j = 1, 2$ and so $(x(1) \odot_\omega x(2))^{(2)} \geq 0$. Therefore, $C_1 \odot_\omega C_2 \in \mathcal{C}_n(\mathcal{O})$.

   (c) Sublinear functions. Then $s = 1$ and so $r = s + 1 = 2$. Choose $x(j) = (0_n, 1)$, $j = 1, 2$. Then $x(j) \in C_j$, $j = 1, 2$ as $C_j \in \mathcal{C}_n(\mathcal{O})$, and $x(1) = x(2)$. Moreover, $x(1) \odot_\omega x(2)$ is a positive scalar multiple of $(0_n, 1)$ (the scalar is 1 if $\omega_2 = -$ and it is 2 if $\omega_2 = +$). Therefore, $(0_n, 1) \in C_1 \odot_\omega C_2$, and so $C_1 \odot_\omega C_2 \in \mathcal{C}_n(\mathcal{O})$.

   (d) Convex functions with recession. Then $s = 2$ and so $r = s + 1 = 3$. To settle this case, one has to establish the inclusions $(0_n, 1) \in C_1 \odot_\omega C_2$ and $C_1 \odot_\omega C_2 \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$. The first (resp. second) one of these inclusions can be verified in the same way as in the case of sublinear functions (resp. convex sets with recessions).

2. The independence of $O(C_1 \odot_\omega C_2)$ of the choices of C_j, $j = 1, 2$ follow from proposition 3.2 (c) and (d).

3. The commutativity and associativity of the binary operation $\odot_\omega$ on convex objects of type $\mathcal{O}$ is an immediate consequence of the same properties of the binary operation $\odot_\omega$ on convex cones.

4. Now we display the calculations, without comment, for the verification of half of the formulas; the verification of the other half is similar.

   (a) Convex sets with recession. We recall that $C(A, A_0) = \mathbb{R}_{++}(A\times 1)\cup(A_0\times 0) = \{\rho(x,1)|\rho > 0, x \in A\} \cup \{(x,0)|x \in A_0\}$. For each ω, we want to compute $u_1 \odot_\omega u_2$ for all $u_i \in C(A(i), A_0(i)),\ i = 1,2$. We have to distinguish four cases:

      i. $u_1 = \rho_1(x_1, 1)$ and $u_2 = \rho_2(x_2, 1)$;
      ii. $u_1 = \rho_1(x_1, 1)$ and $u_2 = (x_2, 0)$;
      iii. $u_1 = (x_1, 0)$ and $u_2 = \rho_2(x_2, 1)$;
      iv. $u_1 = (x_1, 0)$ and $u_2 = (x_2, 0)$.

      - $\omega = (--)$.
         i. $\rho_1 x_1 = \rho_2 x_2$ and $\rho_1 = \rho_2$. This gives $\{\rho(x,1)|x \in A(1) \cap A(2)\}$
         ii. $\rho_1 x_1 = x_2$ and $\rho_1 = 0$. This leads to contradiction as $\rho_1 > 0$.
         iii. $x_1 = \rho_2 x_2$ and $0 = \rho_2$. This leads to contradiction as well.
         iv. $x_1 = x_2$ and $0 = 0$. This gives $\{(x,0)|x \in A_0(1) \cap A_0(2)\}$.
      - $\omega = (++)$.
         i. $\rho_1 x_1 + \rho_2 x_2$ and $\rho_1 + \rho_2$. This gives $\{\rho((1-\alpha)x_1 + \alpha x_2, 1)|\rho > 0, 0\alpha < 1, x_i \in A(i), i = 1,2\}$.
         ii. $\rho_1 x_1 + x_2$ and ρ_1. This gives, after replacing x_2 by $\rho^{-1}x_2$, $\{\rho(x_1+x_2, 1)|\rho > 0, x_1 \in A(1), x_2 \in A_0(2)\}$.
         iii. The same as the previous case with the roles of 1 and 2 interchanged.
         iv. $x_1 + x_2$ and 0. This gives $\{(x,0)|x \in A_0(1) + A_0(2)\}$.

   (b) Convex functions with recession. We recall that $C_u(f, f_0) = (\mathbb{R}_{++}(\mathrm{epi}f\times 1))\cup(\mathrm{epi}f_0\times 0) = \{\rho(x,\alpha,1)|\rho > 0, \alpha \geq f(x)\} \cup \{(x,\alpha,0)|\alpha \geq f_0(x)\}$. For each ω, we want to compute $u_1 \odot_\omega u_2$ for all $u_i \in C_u(f(i), f_0(i)),\ i = 1,2$. We have to distinguish four cases:

      i. $u_1 = \rho_1(x_1, \alpha_1, 1)$ and $u_2 = \rho_2(x_2, \alpha_2, 1)$;
      ii. $u_1 = \rho_1(x_1, \alpha_1, 1)$ and $u_2 = (x_2, \alpha_2, 0)$;
      iii. $u_1 = (x_1, \alpha_1, 0)$ and $u_2 = \rho_2(x_2, \alpha_2, 1)$;
      iv. $u_1 = (x_1, \alpha_1, 0)$ and $u_2 = (x_2, \alpha_2, 0)$.

      - $(---)$.
         i. $\rho_1 x_1 = \rho_2 x_2$ and $\rho_1\alpha_1 = \rho_2\alpha_2$ and $\rho_1 = \rho_2$. This gives

$$\{\rho(x,\alpha,1)|\alpha \geq \max(f(1), f(2))(x)\}.$$

ii. $\rho_1 x_1 = x_2$ and $\rho_1\alpha_1 = \alpha_2$ and $\rho_1 = 0$. This leads to contradiction as $\rho_1 > 0$.

iii. $x_1 = \rho_2 x_2$ and $\alpha_1 = \rho_2\alpha_2$ and $0 = \rho_2$. This leads again to contradiction.

iv. $x_1 = x_2$ and $\alpha_1 = \alpha_2$ and $0 = 0$. This gives $\{\rho(x, \alpha, 0)|\alpha \geq \max(f_0(1), f_0(2))\}$.

- $(+++)$.

i. $\rho_1 x_1 + \rho_2 x_2$ and $\rho_1\alpha_1 + \rho_2\alpha_2$ and $\rho_1 + \rho_2$. This gives

$$\{(\rho_1+\rho_2)(\frac{\rho_1}{\rho_1+\rho_2}x_1+\frac{\rho_2}{\rho_1+\rho_2}x_2, \frac{\rho_1}{\rho_1+\rho_2}\alpha_1+\frac{\rho_2}{\rho_1+\rho_2}\alpha_2, 1)|\rho_i > 0, \alpha_i \geq f(x_i),\ i = 1, 2\}.$$

ii. $\rho_1 x_1 + x_2$ and $\rho_1\alpha_1 + \alpha_2$ and ρ_1. This gives

$$\{\rho(x_1 + x_2, \alpha_1 + \alpha_2, 1)|\rho > 0,\ \alpha_1 \geq f(1)(x_1),\ \alpha_2 \geq f_0(1)(x_2)\}.$$

iii. Similar to the previous case, with the roles of 1 and 2 interchanged.

iv. $x_1 + x_2$ and $\alpha_1 + \alpha_2$ and 0. This gives $(\mathrm{epi} f_0(1) + \mathrm{epi} f_0(2)) \times 0$.

Remark. The procedure above leads to all binary operations on convex objects given in [R], but with the difference that here we allow recession directions as elements of our convex objects, and that here the binary operations are defined for all pairs of convex objects. This gives two binary operations on convex cones containing zero, four on sublinear functions, four on on convex sets with recession, and eight on convex functions with recession. In the proposition above, we have displayed from the eight binary operations on convex functions with recession, only the four best known ones. Now we will describe the other four—nameless—binary operations. These have been described in theorem 5.8 of [R] ('... One is led similarly to *eight* "natural" commutative associative binary operations in the collection of all convex functions on $\mathbb{R}^n$, ...' p.39 of [R]). We define for convex functions $f_i,\ i = 1, 2$ on $\mathbb{R}^n$ the following four convex functions on $\mathbb{R}^n$ (again observing the convention $\infty + (-\infty) = \infty$):

$$f_1 \boxplus f_2(x) = \inf\{\max(f_1(x_1), f_2(x_2))|x_j \in \mathbb{R}^n,\ j = 1, 2,\ x = x_1 + x_2\},$$

$$f_1 \heartsuit f_2(x) = \inf_{0<\alpha<1}\{\max(\alpha f_1(x_1), (1-\alpha)f_2(x_2))|x_j \in \mathbb{R}^n,\ j = 1, 2,\ x = \alpha x_1 + (1-\alpha)x_2\},$$

$$f_1 \diamondsuit f_2(x) = \inf_{0<\alpha<1}\{\alpha f_1(x_1) + (1-\alpha)f_2(x_2)|x_j \in \mathbb{R}^n,\ j = 1, 2,\ x = \alpha x_1 = (1-\alpha)x_2\},$$

$$f_1 \spadesuit f_2(x) = \inf_{0<\alpha<1}\{\alpha \max(f_1(x_1), (1-\alpha)f_2(x_2))|x_j \in \mathbb{R}^n,\ j = 1, 2,\ x = \alpha x_1 = (1-\alpha)x_2\}.$$

The promised remaining four binary operations $\odot_\omega$ on convex functions with recession on $\mathbb{R}^n$ are as follows:

$(f(1) \boxplus f(2), f_0(1) \boxplus f_0(2))$ if $\omega = (+--)$;

$((f(1)\heartsuit f(2)) \vee (f(1) \boxplus f_0(2)) \vee (f_0(1) \boxplus f(2)), f_0(1) \boxplus f(2))$ if $\omega = (+-+)$;

$((f(1)\diamondsuit f(2)) \vee (f(1) + f_0(2)) \vee (f_0(1) + f(2)), f_0(1) + f_0(2))$ if $\omega = (-++)$;

$((f(1)\spadesuit f(2)) \vee (\max(f(1), f_0(2))) \vee (\max(f_0(1), f(2))), \max(f_0(1), f_0(2)))$ if $\omega = (-++)$.

Now we come to the main result of this section.

Theorem 4.3 *Consider convex objects $O_i,\ i=1,2$ of type $\mathcal{O}$ over $\mathbb{R}^n$. Then the following formula holds true:*

$$(\mathrm{cl}O_1 \odot_\omega \mathrm{cl}O_2)^D = \mathrm{cl}(O_1^D \odot_{\omega^d} O_2^D).$$

Both sides of this formula remain the same if the closure operator is omitted, provided one of the following three assumptions holds true:

a the first sign occurring in the sequence ω is $+$;

b $O_i, i=1,2$ is a polyhedral convex object;

c in the case of convex sets with recession $O_i=(A(i),A_0(i)),\ i=1,2,$ (and so of convex cones containing zero): $\mathrm{ri}\bar{A}(1)\cap\mathrm{ri}\bar{A}(2)\neq\emptyset$; in the case of convex functions with recession $O_i=(f(i),f_0(i))$ $i=1,2$ (and so of sublinear functions): $\mathrm{ridom}f(1)\cap\mathrm{ridom}f(2)\neq\emptyset$.

Proof. The statement of the theorem follows from the construction of the binary operation together with theorem 3.4.

This gives four formulas for convex sets, four for sublinear functions, eight for convex functions, two for convex cones, four for gauges, four for convex sets containing zero. To illustrate that some of these formulas have a complicated appearance, we display two of these, for the binary operations 'intersection' and 'convex hull of the union' of convex sets:

$$s(\mathrm{cl}A(1)\cap\mathrm{cl}A(2))|_{(D(A(1),A_0(1))\cap D(A(2),A_0(2)))^\circ} = \mathrm{cl}(s(A(1))|_{A_0(1)^\circ}\mathrm{co}\wedge s(A(2))|_{A_0(2)^\circ}),$$

$$s(\max(A(1)\mathrm{co}\cup A(2)),((A(1)+A_0(2)),(A_0(1)+A(2)))|_{A_0(1)+A_0(2))^\circ} =$$
$$\max(s(A(1))|_{A_0(1)^\circ}\mathrm{co}\wedge s(A(2))|_{A_0(2)^\circ}).$$

Remark. Four of the formulas obtained by applying this result are novel: the following formulas for two pairs of dual binary operations for convex functions appear not to be in the literature (for the other two pairs the formulas are essentially the formulas of Rockafellar-Moreau and Dubovitsky-Milyutin). For simplicity, we display these formulas only for the special case of proper convex functions $f_i:\mathbb{R}^n\to\mathbb{R}\cup\{\pm\infty\}$ (proper means 'not taking the value $-\infty$, and taking a finite value in at least one point'):

$$(f_1\spadesuit f_2)^*=\mathrm{cl}(f_1^*\heartsuit f_2^*);\quad (f_1\heartsuit f_2)^*=f_1^*\spadesuit f_2^*;\quad (f_1\diamondsuit f_2)^*=\mathrm{cl}(f_1^*\boxplus f_2^*);\quad (f_1\boxplus f_2)^*=f_1^*\diamondsuit f_2^*.$$

5 On convex optimization

The aim of this section is to give some essential aspects of convex optimization theory from the point of view of reduction to convex cones.

5.1 On supporting hyperplanes to convex cones

In this section we give the main properties of the convex cone problem to find, for a given convex cone C, elements $x \in C$ and $y \in C^{\circ}$ for which $\langle x, y \rangle = 0$.

A *ray* in $\mathbb{R}^n$ is a set of the form $R_x = \{\rho x | \rho > 0\}$ for some nonzero $x \in \mathbb{R}^n$. For each convex cone $C \subset \mathbb{R}^n$ for which $C \neq \mathbb{R}^n$, one can consider the *convex cone problem* P_C to find a pair of rays (R_x, R_y) where $x \in C \setminus \{0_n\}$ and $y \in C^{\circ} \setminus \{0_n\}$ such that $\langle x, y \rangle = 0$. The geometric interpretation is that the hyperplane $\{v \in \mathbb{R}^n | \langle y, v \rangle = 0$ supports the convex cone C at the point x: that is, this hyperplane contains the point $x \in C$ and it has the entire convex cone C lying entirely on one of its two sides. The *boundary* ∂S of a set $S \subset \mathbb{R}^n$ is the set of points in $\mathbb{R}^n$ that are neither in the interior of S nor in the interior of its complement $\mathbb{R}^n \setminus S$. In particular, ∂S is a closed set in $\mathbb{R}^n$.

Proposition 5.1 *Let C be a closed convex cone in $\mathbb{R}^n$ for which $C \neq (0_n), \mathbb{R}^n$.*

1. *For each solution (R_x, R_y) of the problem P_C, one has $x \in \partial C \setminus \{0_n\}$ and $y \in \partial(C^{\circ}) \setminus \{0_n\}$.*
2. *The sets $\partial C \setminus \{0_n\}$ and $\partial(C^{\circ}) \setminus \{0_n\}$ are nonempty and their intersections with the unit sphere are closed (and so compact).*
3. *For each $x \in \partial C \setminus \{0_n\}$ there exists a solution (R_x, R_y) of the problem P_C, and for each $y \in \partial(C^{\circ}) \setminus \{0_n\}$ there exists a solution (R_x, R_y) of the problem P_C.*

Proof.

1. Assume that (R_x, R_y) is a solution of P_C. To prove the inclusion $x \in \partial C$, we argue by contradiction. Assume $x \notin \partial C$. Then $x \in \text{int}C$, and so one has $\langle \bar{x}, y \rangle = \langle x + \bar{x}, y \rangle \leq 0$ for all sufficiently small $\bar{x} \in \mathbb{R}^n$, and therefore $y = 0_n$. This contradicts the requirement $y \neq 0_n$. In the same way, one proves $y \in \partial C^{\circ}$.
2. As $C \neq (0_n), \mathbb{R}^n$, one can choose nonzero elements $x_i \in \mathbb{R}^n,\ i = 1, 2$ such that $x_1 \in C,\ x_2 \notin C$, and x_1, x_2 are linearly independent: indeed, if C is contained in some line through the origin, then we can choose $x_1 \in C, x_1 \neq 0_n$ and $x_2 \in \mathbb{R}^n \setminus \mathbb{R}x_1$; otherwise, we can choose $x_2 \in \mathbb{R}^n \setminus C$ and then we can choose $x_1 \in C$ such that $x_1 \notin \mathbb{R}x_2$. The intersection of the closed interval with

endpoints $x_i,\ i = 1, 2$ with C is the closed interval with endpoints x_1 and d for some $d \in \mathbb{R}^n$. Then d is an element of $\partial C \setminus \{0_n\}$. Moreover, the intersection of $\partial C \setminus \{0_n\}$ with the unit sphere equals the intersection of ∂C with the unit sphere, and so it is closed (and so compact).

As $C \neq (0_n), \mathbb{R}^n$, it follows that $C^\circ \neq (0_n), \mathbb{R}^n$. Therefore, it can be proved in the same way that $\partial C \setminus \{0_n\}$ is nonempty and that its intersection with the unit sphere is closed (and so compact).

3. Let $x \in \partial C \setminus \{0_n\}$. Then $C + \mathbb{R}_{++}(-x)$ is not equal to $\mathbb{R}^n$. To prove this, it suffices to demonstrate the existence of an element $z \in \mathbb{R}^n \setminus (C + \mathbb{R}_{++}(-x))$. To this end we observe that one can choose an infinite sequence of nonzero vectors $(x_i)_i$ converging to 0 for which $x + x_i \notin C$, and then one can take a limit point z of the infinite sequence $(\|x_i\|^{-1}x_i)_i$. Then $z \in \mathbb{R}^n \setminus \{0_n\}$ and $x + tz \notin C$ for all $t > 0$. To prove that $z \notin C + \mathbb{R}_{++}(-x)$, we argue by contradiction. Assume that $z \in C + \mathbb{R}_{++}(-x)$, say, $z = c + \alpha(-x)$ with $c \in C$ and $\alpha > 0$. Then $x + \alpha^{-1}z = \alpha^{-1}c \in C$, which contradicts the property of y above.

   It follows from $C + \mathbb{R}_{++}(-x) \neq \mathbb{R}^n$, that the polar cone $(C + \mathbb{R}_{++}(-x))^\circ$ contains a nonzero element y. Then (R_x, R_y) is a solution of the problem P_C.

5.2 Convex optimization problems: existence of solutions

The aim of this section is to give the connection between convex optimization problems and convex cone problems P_C. This leads to an extended solution concept for convex optimization problems. For each *closed* convex optimization problem, an extended solution exists.

A continuous parametrization, $(r_\rho)_{\rho\in[-\infty,+\infty]}$, of the upper unit circle $\{x \in \mathbb{R} \times \mathbb{R}^+ | \|x\| = 1\}$ is defined by $r_\rho = \|(-\rho, 1)\|^{-1}(-\rho, 1)$ for $\rho \neq \pm\infty$, $r_{-\infty} = (+1, 0)$ and $r_{+\infty} = (-1, 0)$. Then a parametrization, $(R_\rho)_{\rho\in[-\infty,+\infty]}$, of the rays of the convex cone $\mathbb{R} \times \mathbb{R}_+$ is defined by $R_\rho = \mathbb{R}_{++}r_\rho$ for all $\rho \in [-\infty, +\infty]$.

Proposition 5.2 *Let (f, f_0) be a convex function with recession over $\mathbb{R}^n$.*

1. *Assume that (f, f_0) is closed. Let $\rho \in [-\infty, +\infty]$. Then the following conditions are equivalent:*
   (a) *there exists a ray R_v in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ for which the pair of rays $(R_v, R_{(0_n, r_\rho)})$ is a solution of the problem $P_{C_u(f,f_0)}$;*
   (b) *$\rho = \inf f$ or $\rho = -\infty$.*
2. *If $\inf f \neq \pm\infty$, then for each ray R_v in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ the following conditions are equivalent:*
   (a) *the pair of rays $(R_v, R_{(0_n, r_{\inf f})})$ is a solution of the problem $P_{C_u(f,f_0)}$;*

(*b*) *v is a positive scalar multiple of $(\widehat{x}, f(\widehat{x}), 1)$, where $\widehat{x}$ is a solution of the convex optimization problem $f(x) \to \min,\ x \in \mathbb{R}^n$, or $v = (\bar{x}, 0, 0)$ for some $\bar{x} \in \mathbb{R}^n$ for which $f_0(\bar{x}) = 0$.*

Proof.

1. We will use the property that $(0_n \times \mathbb{R} \times \mathbb{R}) \cap \partial((C_u(f, f_0))^\circ) \setminus \{0_n\}$ consists of the rays $R_{(0_n,\rho)}$ with $\rho = \inf f$ and $\rho = -\inf f$. Here, we only display the verification that $(0_n, \inf f) \in \partial((C_u(f, f_0))^\circ)$; the remaining verifications are similar.

   - The case $\inf f \neq \pm\infty$. Then one has $(C_u(f, f_0))^\circ = (C(\text{epi}f, 0))^\circ$ and
   $$r_{\inf f} = \|(-\inf f, 1)\|^{-1}(-\inf f, 1).$$
   One has $\langle(0_n, r_{\inf f}), (x, \rho, 1)\rangle = \|(-\inf f, 1)\|^{-1}(\inf f - \rho) \leq 0$ for all $(x, \rho) \in \text{epi}f$. This proves the inclusion $(0_n, r_{\inf f}) \in (C_u(f, f_0))^\circ$. Moreover, for each $\varepsilon > 0$, one can take $(x, \rho) \in \text{epi}f$ with $\rho < \inf f + \varepsilon$, and then one has $\langle(0_n, r_{\inf f+\varepsilon}), (x, \rho, 1)\rangle = \|(-\inf f - \varepsilon, 1)\|^{-1}(-\rho + \inf f + \varepsilon) > 0$. This proves that $(0_n, r_{\inf f+\varepsilon}) \notin (C_u(f, f_0))^\circ$. It follows that $(0_n, r_{\inf f}) \in \partial((C_u(f, f_0))^\circ)$, as required.
   - The case $\inf f = +\infty$. Then one has $C_u(f, f_0) = C(\emptyset, \text{epi}f_0)$ and $r_{\inf f} = (-1, 0)$. One has $\langle(0_n, r_{\inf f}), (x, \rho, 0)\rangle = 0$ for all $(x, \rho) \in \text{epi}f_0$. This proves the inclusion $(0_n, r_{\inf f}) \in (C_u(f, f_0))^\circ$. Moreover, for each $\varepsilon > 0$, one has $(0_n, 1) \in \text{epi}f_0$ and
   $$\langle(0_n, r_{\inf f} + (0, -\varepsilon)), (0_n, 1, 0)\rangle = \varepsilon > 0.$$
   This proves that $(0_n, r_{\inf f} + (0, -\varepsilon)) \notin (C_u(f, f_0))^\circ$. It follows that $(0_n, r_{\inf f}) \in \partial((C_u(f, f_0))^\circ)$ as required.
   - The case $\inf f = -\infty$. Then one has $C_u(f, f_0) = C(\mathbb{R}^n \times \mathbb{R}, 0)$ and $r_{\inf f} = (1, 0)$. One has $\langle(0_n, r_{\inf f}), (x, \rho, 1)\rangle = -1 \leq 0$ for all $x \in \mathbb{R}^n, \rho \in \mathbb{R}$. This proves the inclusion $(0_n, r_{\inf f}) \in (C_u(f, f_0))^\circ$. Moreover, for each $\varepsilon > 0$, one has $(0_n, -2\varepsilon^{-1}) \in \mathbb{R}^n \times \mathbb{R}$, and $\langle(0_n, r_{\inf f} + (0, \varepsilon)), (0_n, -2\varepsilon^{-1}, 1)\rangle = -1 + 2 = 1 > 0$. This proves that $(0_n, r_{\inf f} + (0, \varepsilon)) \notin (C_u(f, f_0))^\circ$. It follows that $(0_n, r_{\inf f}) \in \partial((C_u(f, f_0))^\circ)$ as required.

2. Now we prove the first statement of the proposition. Assume that (f, f_0) is closed and let $\rho \in [-\infty, +\infty]$.

   - $(a) \Rightarrow (b)$. Assume (a) holds. Then, by statement 1 of proposition 5.1, $(0_n, r_\rho)$ is contained in $\partial((C_u(f, f_0))^\circ)$, and so $\rho = \inf f$ or $\rho = -\infty$. That is, statement (b) holds true.
   - $(b) \Rightarrow (a)$. Assume (b) holds. Then $(0_n, r_\rho) \in \partial((C_u(f, f_0))^\circ \setminus \{0_n\}$, and so, by statement 3 of proposition 5.1, statement (a) holds true.

3. Now we prove the second statement of the theorem. Assume that $\inf f \neq \pm\infty$ and let R_v be a ray in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$.

   - $(a) \Rightarrow (b)$. Assume that condition (a) holds true. Then
   $$(0_n, r_{\inf f}) = \|(-\inf f, 1)\|^{-1}(0_n, -\inf f, 1).$$
   We distinguish two cases:
     - v is a positive scalar multiple of $(x, \rho, 1)$ where $\rho \geq f(x)$: then we have by condition (a) that $\langle (x, \rho, 1), (0_n, -\inf f, 1)\rangle = 0$. That is, $-\rho + \inf f = 0$. It follows that x is a solution of the convex optimization problem $f(x) \to \min, x \in \mathbb{R}^n$ and that v is a positive scalar multiple of $(x, f(x), 1)$.
     - $v = (x, \rho, 0)$ where $\rho \geq f_0(x)$: then we have by condition (a) that
     $$\langle (x, \rho, 0), (0_n, -\inf f, 1)\rangle = 0.$$
     That is, $-\rho = 0$ and $\rho \inf f_0 \geq 0$. Therefore, $f_0(x) \leq \rho = 0$. As $f_0 \geq 0^+ f$ and $\inf f \neq \pm\infty$, it follows that $f_0(x) \geq 0$. It follows that $v = (x, 0, 0)$ and $f_0(x) = 0$.
   - $(b) \Rightarrow (a)$. This implication is proved in the same way, by reversing the arguments.

The second statement of this proposition suggests to define the concept extended solution of the convex optimization problem determined by the convex function with recession (f, f_0) to be a ray R_v in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ for which the pair $(R_v, R_{(0_n, r_{\inf f})})$ is a solution of the problem $P_{C_u(f,f_0)}$.

Theorem 5.3 *Let (f, f_0) be an arbitrary closed convex function with recession over $\mathbb{R}^n$. Then there exists an extended solution of the convex optimization problem determined by (f, f_0). If $\inf f \neq \pm\infty$, then an extended solution is either an element $\widehat{x} \in \mathbb{R}^n$ for which $f(x) \geq f(\widehat{x})$ for all $x \in \mathbb{R}^n$ or a ray $R_{\bar{x}}$ in $\mathbb{R}^n$ for which $f0^+(\bar{x}) = 0$; if $\inf f = -\infty$, then it is a ray in the epigraph of $f0^+$; if $\inf f = +\infty$, then it is a ray in the epigraph of f_0.*

Proof. The existence of an extended solution follows from the first statement of proposition 5.2. The given concrete interpretation of extended solutions follows in the case $\inf f \neq \pm \inf$ from the second statement of proposition 5.2. We do not display the verifications in the cases $\inf f = +\infty$ and $\inf f = -\infty$.

Remark. If $\inf f = \pm\infty$, then the rays R_v in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ for which the pair of rays $(R_v, R_{(0_n, r_\rho)})$ is a solution of $P_{C_u(f,f_0)}$ correspond to the rays $\mathbb{R}_{++}(x, \alpha)$ of $\mathrm{epi} f_0$. Therefore, it would be natural for $\inf f = \pm\infty$ to consider a stricter solution concept, requiring for example that the angle between the ray $\mathbb{R}_{++}(x, \alpha)$ in $\mathrm{epi} f_0$ and the ray $\mathbb{R}_{++}(0_n, -1)$ is minimal. We note that there exists an extended solution in this strict sense for the convex optimization problem determined by an arbitrary closed convex function (f, f_0).

5.3 Duality of convex optimization problems

One can proceed in a similar way to derive an extended version of the duality theorem for convex optimization problems. Here we will only give the main idea. One should consider for a convex function with recession (F, F_0) on $\mathbb{R}^n \times \mathbb{R}^m$, the problem $P_{C_u(F,F_0)}$ with the additional condition that one looks for solutions (R_v, R_w) where w is of the form $(0_n, y, r_\rho)$ for suitable $y \in \mathbb{R}^m$ and $\rho \in [-\infty, +\infty]$.

5.4 Conditions of optimality and subdifferential calculus.

The *subdifferential* of a convex function f on $\mathbb{R}^n$ at a feasible point $\widehat{x}$ is defined to be $\partial f(\widehat{x}) = \{x' \in \mathbb{R}^n | \; f(\widehat{x}) + \langle x', h\rangle \leq f(\widehat{x}+h) \; \forall h \in \mathbb{R}^n\}$. Applications of the subdifferential in optimization are based on the criterion $0 \in \partial f(\widehat{x})$ for solutions $\widehat{x}$ of convex optimization problems $f(x) \to \min$, $x \in \mathbb{R}^n$. Part of the convex calculus for subdifferentials of convex functions is contained in the following result (we do not give the chain rule for convex functions and not the list of subdifferentials of basic convex functions). For simplicity, we have omitted in the notation the point at which the subdifferential is taken.

Theorem 5.4 *1.* $\partial(\Lambda f) = (\Lambda')^{-1}\partial f$,

2. $\partial(\Lambda^{-1}\mathrm{cl} f) = \mathrm{cl}\Lambda'(\partial f)$,

3. $\partial(\mathrm{cl} f_1 \vee \mathrm{cl} f_2) = \mathrm{cl}(\partial f_1 \mathrm{co} \cup \partial f_2)$,

4. $\partial(f_1 \boxplus f_2) = \partial f_1 \boxplus \partial f_2$,

5. $\partial(\mathrm{cl} f_1 + \mathrm{cl} f_2) = \mathrm{cl}(\partial f_1 \oplus \partial f_2)$,

6. $\partial(f_1 \oplus f_2) = \partial f_1 \cap \partial f_2$,

7. $\partial(\mathrm{cl} f_1 \spadesuit \mathrm{cl} f_2) = \mathrm{cl}(\partial f_1 \mathrm{co} \cup \partial f_2)$,

8. $\partial(f_1 \heartsuit f_2) = \partial f_1 \boxplus \partial f_2$,

9. $\partial(\mathrm{cl} f_1 \diamondsuit \mathrm{cl} f_2) = \mathrm{cl}(\partial f_1 \oplus \partial f_2)$,

10. $\partial(f_1 \mathrm{co} \wedge f_2) = \partial f_1 \cap \partial f_2$,

Proof. We give a brief sketch of the proof. The subdifferential of a convex function f on $\mathbb{R}^n$ at the point $\widehat{x}$ equals the subdifferential of the sublinear function function $p_{\widehat{\mathrm{x}}}$ on X such that $\mathrm{strictepi}(p_{\widehat{\mathrm{x}}}) = \mathrm{cone}(\mathrm{strictepi} f - (\widehat{x}, f(\widehat{x})))$ under a suitable assumption of general position, for example that $\widehat{x}$ is an internal point of dom$f$. Therefore, applying the calculus rules for subdifferentials of convex functions, all required formulas are obtained.

Remark. By applying the conditions from theorems 3.4 and 4.3 under which closure operations can be omitted, one gets conditions under which closure operations can be omitted in the formulas of theorem 5.5.

References

[1] A. Barvinok. *A Course in Convexity.* Graduate Studies in Mathematics, Volume 54, American Mathematical Society, 2002.

[2] A. Ben Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, SIAM, Philadelphia, 2001.*

[3] D. Bertsekas, A. Nedić and A.E. Ozdaglar. *Convex Analysis and Optimization.* Athena Scientific, Belmont, Massachusetts, 2003.

[4] J. Borwein and A.S. Lewis. *Convex Analysis and Nonlinear Optimization.* Springer Verlag, New York, 2000.

[5] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2003.

[6] J. Brinkhuis. On a Conic Approach to Convex Analysis, Econometric Institute Report EI 2008-05.

[7] J. Brinkhuis and V.M. Tikhomirov. Duality and calculus of convex objects, Sbornik: Mathematics 198:2 171-206, 2007.

[8] W. Fenchel. On Conjugate Convex Functions. *Canad. J. of Math.* Vol. 1, pp. 73-77, 1949.

[9] W. Fenchel. *Convex Cones, Sets, and Functions.* Mimeographed Notes, Princeton Univ. 1951.

[10] P.M. Gruber. *Convex and Discrete Geometry.* Grundlehren der mathematischen Wissenschaften 336, Springer Verlag, Heidelberg, 2007.

[11] J.-P. Hiriart-Urrutty and C. Lemaréchal. *Fundamentals of Convex Analysis.* Grundlehren Text Editions, Springer Verlag, Heidelberg, 2004.

[12] J.-B. Hiriart-Urrutty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms.* Springer Verlag, Heidelberg, 1993. Two volumes.

[13] L. Hörmander. *Notions of Convexity.* Modern Birkhäuser Classics, 2007.

[14] L. Hörmander. Sur la fonction d'appui des ensembles convexes dans un espace localement convexe. *Ark. Mat.*, 3(12):181-186, 1954.

[15] S.S. Kutateladze, A.M. Rubinov. *Minkowski duality and its applications.* Nauka, 1976.

[16] G.G. Magaril-Il'yaev and V.M. Tikhomirov. *Convex Analysis, Theory and Applications.* Translations of Mathematical Monographs, Volume 222, AMS, Providence, 2003.

[17] H. Minkowski. *Theorie der Konvexen Körper, Insbesondere Begründung Ihres Ober Flachenbegriffs.*, Gesammelte Abhandlungen, II, Teubner, Leipzig, 1911.

[18] J.-J. Moreau. *Fonctionelles convexes.* Séminaire "Équations aux dérivées partielles". Collège de France, Paris, 1966.

[19] Yu.E. Nesterov and A.S. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM Studies in Applied Mathematics 13. SIAM, Philadelphia, 1994.

[20] Yu. Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course.* Kluwer Academic Publishers, Dordrecht, 2004.

[21] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, 1970.

[22] R.T. Rockafellar. *Conjugate Duality and Optimization.* Regional Conference Series in Applied Mathematics 16. SIAM Publications, 1974.

[23] R.T. Rockafellar and R.J.-B. Wets. *Variational Analysis.* Springer Verlag, Heidelberg, 1998.

[24] E. Steinitz. Bedingt konvergente Reihen und konvexe Systeme, I, II, III, Journal der Reine und Angewandte Mathematik, 143-144-146, 128-175;1-40;1-52, 1913-1916.

[25] R.T. Rockafellar and R.J.-B. Wets. *Variational Analysis.* Springer Verlag, Heidelberg, 1998.

[26] V.M. Tikhomirov. *Fundamental Principles of the Theory of Extremal Problems.* John Wiley and Sons, 1986.

[27] V.M. Tikhomirov. Convex Analysis. In R.V. Gamkrelidze (Ed.), *Analysis II.* Encyclopaedia of Mathematical Sciences, Volume 14, Springer-Verlag, Heidelberg, 1980.