Public Choice **81**: 363–380, 1994.
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The solution to the Tullock rent-seeking game when R > 2: Mixed-strategy equilibria and mean dissipation rates*

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Accepted 6 December 1993

Abstract. In Tullock's rent-seeking model, the probability a player wins the game depends on expenditures raised to the power R. We show that a symmetric mixed-strategy Nash equilibrium exists when $R > 2$, and that overdissipation of rents does not arise in any Nash equilibrium. We derive a tight lower bound on the level of rent dissipation that arises in a symmetric equilibrium when the strategy space is discrete, and show that full rent dissipation occurs when the strategy space is continuous. Our results are shown to be consistent with recent experimental evidence on the dissipation of rents.

1. Introduction

In Tullock (1980) the following interesting rent-seeking game is described. Consider two players who bid for a political favor commonly known to be worth Q dollars ($Q > 0$ and finite). Their bids influence the probability of

* An earlier version of this paper circulated under the title, "No, Virginia, There is No Overdissipation of Rents." We are grateful to Dave Furth and Frans van Winden for stimulating conversations, and for comments provided by workshop participants from the CORE-ULB-KUL IUAP project, Purdue University, Pennsylvania State University, Rijksuniversiteit Limburg, and Washington State University. We also thank Max van de Sande Bakhuyzen and Ben Heijdra for useful discussions, and Geert Gielens for computational assistance. An earlier version of the paper was presented at the ESEM 1992 in Brussels and the Mid-West Mathematical Economics Conference in Pittsburgh. All three authors would like to thank CentER for its hospitality during the formative stages of the paper. The second author has also benefited from the financial support of the Katholieke Universiteit Leuven and the Jay N. Ross Young Faculty Scholar Award at Purdue University. The third author benefitted from visiting IGIER where part of the paper was written. The third author also benefitted from grant IUAP 26 of the Belgian Government.

receiving the favor. Let x and y denote the bids of agents 1 and 2 respectively, and let $\pi(x,y)$ denote the probability the first agent is awarded the political favor. The payoff to agent 1 from bidding x when the other agent bids y is

$$U_1(x|y) = \pi(x,y)Q - x, \tag{1}$$

while that of player two is symmetrically defined:

$$U_2(y|x) = [1 - \pi(x,y)]Q - y$$

Because the politician awarding the prize may have other considerations, or because he can only imperfectly discriminate between the bids (if bids are not made in the money metric), the high bidder is not guaranteed the prize. This is a common assumption in (1) the principal-agent literature (Lazear and Rosen, 1981; Nalebuff and Stiglitz, 1983; Bull, Schotter and Weigelt, 1987), (2) the political campaign expenditure literature (Snyder, 1990); and (3) the literature on rationing by waiting in line (Holt and Sherman, 1982). Presumably, given y, the probability of winning is an increasing function of x. Tullock suggested the specification

$$\pi(x,y) = \begin{cases} \dfrac{1}{2} & \text{if } x = y = 0 \\[2ex] \dfrac{x^R}{x^R + y^R} & \text{otherwise } (x \geq 0,\ y \geq 0), \end{cases} \tag{2}$$

where $R > 0$. This specification has become standard in the rent-seeking literature and other fields, see, e.g., Snyder (1990).[1] The case where $R = 1$ is studied most (Ellingsen, 1991; Nitzan, 1991a; Paul and Wilhite, 1991), but it is of interest to consider other values of R, as in Applebaum and Katz (1986) and Millner and Pratt (1989). Loosely speaking, the case $0 < R < 1$ represents decreasing returns, while $R > 1$ represents increasing returns to aggressive bidding. While the two agent pure strategy symmetric Nash equilibrium is straightforward to calculate from the first-order conditions when $0 < R \leq 2$, this is not the case when $R > 2$. Consequently Tullock (1980) devoted a large part of his discussion to these latter cases.

To date, there are only conjectures concerning the existence of a Nash equilibrium for $R > 2$ but finite. Rowley (1991), in his review of Tullock's work, lists this as one of the three important theoretical problems for a research program in the area of rent-seeking. The problem is not so much that the first-order condition for a maximum cannot be calculated; the problem is that the symmetric $(x = y)$ solution to the two players' first-order conditions does not

Table 1. Millner and Pratt (1989) hypotheses and experimental results

Exponent	H_0	Experiment R = 1	H_0	Experiment R = 3
Mean individual	2	2.24	6	3.34
Expenditures		(2.42)		(−24.28)
Mean dissipation	50%	56%	150%	84%
Rates		(2.37)		(−13.37)
Number of observations		146		100

necessarily yield a global maximum (if R > 2 the symmetric solution to the first-order conditions implies a negative expected payoff, which is dominated by a zero bid). In such a case the sum of the solutions to the first-order conditions exceeds the value of the prize Q; there is the *false appearance of an over dissipation of rents*. Tullock (1980, 1984, 1985, 1987, 1989) devoted considerable attention to the case of over dissipation because of the induced excess social waste; see Dougan (1991) for a critical comment, and Laband and Sophocleus (1992) for estimates of the resource expenditures. In Tullock (1984) it was acknowledged that over dissipation may be due to a failure of the second-order conditions.[2] In the vernacular of the game theory, over dissipation is not part of a Nash equilibrium. This notwithstanding, the possibility of over dissipation is a recurrent theme in the rent-seeking literature.

In particular, Millner and Pratt (1989) examined the rent-seeking model experimentally for the cases where R = 1 and R = 3. Due to the use of laboratory dollars, the strategy space used in their experiment is discrete. For a prize worth 8 U.S. dollars they formulate two hypotheses concerning the mean of the individual expenditures and the mean dissipation rates. These hypotheses are stated in Table 1, together with their experimental results.[3] Both hypotheses are rejected for either value of R, but at markedly different p-values. The p-value for the R = 1 case is at least .015, while the p-value for R = 3 is at the most 10^{-40}. Thus, H_0 is only rejected marginally for the case R = 1, while H_0 is strongly rejected for the case R = 3. Shogren and Baik (1991) point out, however, that the null hypothesis for the case R = 3 is not the correct one. The problem, however, is that the equilibrium to the game is not known when R > 2. Our paper resolves this issue.

More specifically, for R = 1, the symmetric Nash equilibrium is known, and the associated expenditure and dissipation rates are readily verified to correspond with the hypothesized values in Table 1. This is further corroborated by a recent experiment by Millner and Pratt (1991) which shows that risk aversion can explain the discrepancies between the hypothesized and realized values in Table 1 for the case when R = 1. A major benefit of the results presented below is that we will be able to explain the discrepancy between the hypothesized values and experimental results for the case when R = 3. The punchline is that the formula based on the first-order equations (which yields a rent dissipation of 150%) is incorrect. In fact, there is not a symmetric pure-strategy equilibrium when R = 3. We characterize the "correct" Nash equilibrium, and show that the results of the Millner-Pratt experiments are in line with the theoretically correct Nash equilibrium mixed strategies. To this end we mainly focus on the two agent case in discrete strategy space. In the last section we consider a continuous strategy space by taking limits of the finite game.

Before we embark on this, we briefly review the approaches others have used to deal with the R > 2 case. The approach in the existing literature is to *modify the original game* to remove the apparent over dissipation of rents. In his original contribution Tullock (1980) suggested three modifications. The first is to let R be infinite, which turns the game into an all-pay auction. Within the rent-seeking literature this version has been studied by Hillman and Samet (1987). The complete characterization of all equilibrium strategies has been obtained by Baye, Kovenock, and de Vries (1990), and the equilibrium level of rent dissipation is derived in Baye, Kovenock and de Vries (1993). The second type of modification is to change the one shot game into a dynamic game. Tullock (1980) discusses the case of alternating bids, and this has been formalized recently by Leininger (1990) and Leininger and Yang (1990). In Corcoran (1984), Corcoran and Karels (1985), and Higgins, Shughart, and Tollison (1987) the game is changed into a two-stage game. In the first stage the number of participants is selected such that, when the rent-seeking game is played in stage two, the number of participants is consistent with (almost) complete rent dissipation. Similarly, Michaels (1988) devises a setting within which the politician has the incentive to adjust the exponent such that the first- and second-order conditions are met. The third modification deals with asymmetries between the players. This was briefly dealt with in Tullock (1980) and has been further investigated by Allard (1988). Finally Nitzan (1991b) introduces coalition behavior on the part of the contestants. *None of these contributions, though, offers a solution to the original simultaneous move rent-seeking game when R > 2.* The next section provides this solution and relates it to the experimental and theoretical literatures.

2. Solving the rent-seeking game

Consider the two-agent rent-seeking game with conditional payoffs and winning probabilities as given in equations (1) and (2). The exponent satisfies R > 0. Suppose a pure strategy equilibrium exists. Given y > 0, the first- and second-order conditions for an unconstrained (local) maximum of $U_1(x|y)$ are readily calculated as

$$Q\frac{Ry^R x^{R-1}}{(x^R + y^R)^2} - 1 = 0, \tag{3}$$

and

$$Q\frac{Ry^R x^{R-2}}{(x^R + y^R)^3} [(R-1)(x^R + y^R) - 2Rx^R] < 0. \tag{4}$$

Assuming a symmetric solution, condition (3) yields $x = y = QR/4$, for which condition (4) is readily seen to hold locally for any $R > 0$. Substituting back into equation (1) yields

$$U_i(x = y = \frac{QR}{4}) = \frac{Q}{2} (1 - \frac{R}{2}); i = 1, 2. \tag{5}$$

Note that in this case $U_i(.|.)$ is non-negative as long as $R \leq 2$. Moreover, for any $x, y > 0$ the factor $(R-1)(x^R + y^R) - 2R x^R$ in the second-order condition (4) is unambiguously negative if $R \leq 1$, while it is positive over some interval to right of $x = 0$ if $R > 1$ and becomes negative thereafter. In particular, (4) is satisfied when $x = y$. Thus for $R \leq 2$, the symmetric solution $x = y = QR/4$ constitutes a Nash Equilibrium. For $R > 2$, $U(QR/4|QR/4)$ in (5) becomes negative and hence the first-order conditions do not yield a symmetric Nash equilibrium point (because one can choose $x = 0$ given that $y = QR/4$; and earn a higher payoff. But if $x = 0$ is chosen, player two has an incentive to lower y to small $\varepsilon > 0$). Generally, the first- and second-order conditions (3) and (4) fail to characterize the global maximum when $R > 2$.[4]

In order to find a solution for the case $R > 2$, we focus on the game with a discrete strategy space. This yields a version of the game similar to that used in the laboratory experiments by Millner and Pratt (1989), 1991).[5] Due to the use of laboratory dollars, the bids are necessarily discrete, and thus the game is a so-called finite game.[6] Nash's (1951) theorem guarantees that every finite game has a mixed-strategy equilibrium.[7] It follows immediately that the Tullock rent-seeking game in discrete strategy space has a Nash equilibrium, possibly in non-degenerate mixed strategies, for any $R > 2$. While it is in general difficult to characterize the equilibria, we may be more specific in this case. Note that for any strategy pair (x,y), the payoff to the second agent is the same as the payoff to the first agent if the strategies played by the two agents are interchanged; the game is symmetric. Recalling that an equilibrium is defined to be a symmetric equilibrium if all players choose the same strategy, we may apply Dasgupta and Maskin's (1986) Lemma 6; a finite symmetric game has a symmetric mixed-strategy equilibrium.

In summary, the Tullock rent-seeking game with a discrete strategy space certainly has a symmetric Nash equilibrium, even when $R > 2$. These results immediately raise the following questions: (i) Can we characterize the equilibria for $R > 2$, even though previous authors have been unable to do so? In particular, is it possible to provide an explicit solution for the symmetric equilibria that arise for different values of R? (ii) Can the equilibria of the finite game be used to shed light on infinite game (continuous strategy space) equilibria? A derivative question is: (iii) How do the answers to these questions relate to the experimental work reported by Millner and Pratt for the case $R = 3$?

We answer question (i) by employing a device which was first used by Shilony (1985). The payoffs to the game will be written in matrix format. We then show this yields a matrix equation which can be manipulated to yield the symmetric mixed strategy solution. Some numerical examples and a special case of this procedure are provided. To answer the derivative question (iii) we manipulate the matrix equation to obtain tight bounds on the equilibrium dissipation rate. Question (ii) is answered by letting the mesh of the strategy space become small relative to the value of the prize.

Recall equation (1) which gives the conditional payoffs for agent 1. To obtain the unconditional or expected payoffs from playing x, $EU_1(x)$, the conditional payoffs are premultiplied by the (mixed-strategy) probability p_y that a particular y value is being played by player one's opponent, and subsequently these are summed over y. Thus

$$EU_1(x) = \sum_{y=0}^{Q} p_y \, \pi(x,y) \, Q - x. \tag{6}$$

Denote the expected payoffs to agents 1 and 2 in an arbitrary Nash equilibrium by v_1 and v_2 respectively. In the case of a symmetric Nash equilibrium note that the players' expected payoffs are identical, $v_1 = v_2 = v$ (however, v need not be unique). The manipulations below make repeated use of the following general result.

Theorem 1. In any equilibrium: (i) $EU_1(x) \leq v_1$, (ii) $EU_1(x) = v_1$ when $p_x > 0$, while (iii) $p_x = 0$ if $EU_1(x) < v_1$. Similar results hold for player 2.

A proof of this theorem can be found in Vorob'ev (1977, sec. 3.2.2., 3.4.2. and 3.4.3.). For a symmetric equilibrium – which we know exists by Lemma 6 in Dasgupta and Maskin (1986) – we can use equations (6) and (2) to restate the condition $EU_1(x) \leq v$ as

$$\sum_{y=0}^{Q} p_y \frac{x^R}{x^R + y^R} \leq \frac{v+x}{Q}. \tag{7}$$

Conditions (ii) and (iii) in Theorem 1 imply a complementary slackness-type condition for a symmetric equilibrium of the form

$$\forall\, x\colon p_x \left[\sum_{y=0}^{Q} p_y \frac{x^R}{x^R + y^R} - \frac{v + x}{Q} \right] = 0. \tag{7'}$$

Now note that $EU_1(x = Q) \leq 0$, and in fact $EU_1(x = Q) < 0$ if $p_{y=0} < 1$ (and R is finite). Thus in a symmetric equilibrium no mass will be placed at Q, i.e. $p_{x=q} = p_{y=q} = 0$. Suppose (without loss of generality but for ease of notation) that $Q \varepsilon N$, and that x and y can only take on the integer values, 0, 1, ..., Q. Note that there are exactly Q conditions (7) for $x = 0, 1, \ldots, Q-1$. These can be conveniently expressed in matrix format:

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & \frac{1}{1+2^R} & & \frac{1}{1+(Q-1)^R} \\ 1 & \frac{2^R}{2^R+1} & \frac{1}{2} & \cdots & \frac{2^R}{2^R+(Q-1)^R} \\ \vdots & & \vdots & & \vdots \\ 1 & \frac{(Q-1)^R}{(Q-1)^R+1} & \frac{(Q-1)^R}{(Q-1)^R+2^R} & \cdots & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{Q-1} \end{bmatrix} \leq \begin{bmatrix} \frac{v}{Q} \\ \frac{v+1}{Q} \\ \frac{v+2}{Q} \\ \vdots \\ \frac{v+Q-1}{Q} \end{bmatrix} \tag{8}$$

In addition to this Q x Q matrix condition, the following constraints must be imposed:

$$\sum_{y=0}^{Q-1} p_y = 1;\ p_y \geq 0,\ y = 0, 1, \ldots, Q. \tag{9}$$

Condition (8), together with the constraints (9) and the complementary slackness condition (7′) provide a complete, but implicit characterization of the symmetric equilibrium, which we know exists by Dasgupta and Maskin's Lemma 6. These conditions form a linear programming problem which, at least in principle, can be solved for $(p_0, \ldots, p_{Q-1}, v)$. We have thus proved

Theorem 2. Suppose the strategy space is discrete. Then for any $R > 2$, the Tullock rent-seeking game has a symmetric mixed-strategy Nash equilibrium, defined implicitly by the solution to conditions (7′), (8) and (9).

In order to illustrate the practical utility of Theorem 2, we will investigate two special cases: $R = \infty$ and $R = 3$. The latter case is that examined in Millner and Pratt's experiments, while the former is the discrete strategy space version of the all pay auction examined in Baye, Kovenock, and de Vries (1990; 1993).

We begin with the case when the exponent $R = \infty$ and assume $Q > 1$ for simplicity. In this case the matrix expression in (8) becomes

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{Q-1} \end{bmatrix} \leq \begin{bmatrix} \frac{v}{Q} \\ \frac{v+1}{Q} \\ \frac{v+2}{Q} \\ \vdots \\ \frac{v+Q-1}{Q} \end{bmatrix} \qquad (10)$$

It is straightforward to find symmetric equilibria if it is assumed that all $p_i > 0$. In this case the matrix inequality (10) becomes an equality by Theorem 1. The lower triangular matrix equation can then be solved through recursive substitution. This yields $p_0 = p_2 = p_4 = \ldots = 2v/Q$ and $p_1 = p_3 = p_5 = \ldots = 2(1-v)/Q$. In addition to (8), conditions (9) and (7′) have to hold. For even values of Q this restricts $v \in [0, 1]$, while for odd values of Q, we necessarily have $v = 1/2$ (see Bouckaert, Degrijse, and de Vries, 1992, for a proof of this claim).

Note that we may make the grid in the formulation of the game (7) finer and finer and normalize the value of the prize to be one by dividing all dollar units by Q and letting Q tend to infinity. The equilibrium distributions in this discrete game with $R = \infty$ then converge uniformly to the continuous uniform distribution, and the expected payoff v/Q converges to zero; there is full rent dissipation. Also note that equations (1) and (2) can be expressed as

$$U_1(x|y) = \begin{cases} Q - x & \text{if } x > y \\ \frac{1}{2}\,Q - x & \text{if } x = y \\ -\,x & \text{if } x < y \end{cases} \qquad (11)$$

which is precisely the definition of the all-pay auction (cf. Baye, Kovenock, and de Vries, 1993). It follows that the symmetric equilibria of the discrete all-pay auction converge to the unique (see Baye et al., 1990) equilibrium of the continuous strategy space *two player* all-pay auction.

Next, consider the case of finite exponents. When $0 < R \leq 2$, the game has a symmetric pure strategy equilibrium ($x = y = QR/4$) as discussed earlier. Because $R = 3$ is used in Millner and Pratt's experimental work on the game, and as pointed out by Shogren and Baik (1991) the "solution" examined by Millner and Pratt is not really a Nash equilibrium, we will focus on this case.[8] For $R > 2$ and finite, the solutions to the game cannot be given in the same compact form as the solution for $R = \infty$, although conditions (7′), (8) and (9) still provide a complete but implicit description of the game and its solution. For any specific values of R and Q, it can be solved explicitly through linear programming. We list some examples.

(i) $R = 3, Q = 1$. There is one pure strategy solution: both agents bid zero and receive $v = 1/2$. Inter alia, this result holds for any finite value of R.

(ii) $R = 3, Q = 2$. There exist multiple pure strategy solutions: (1) both bid zero and receive $v = 1$, (2) one agent bids zero and the other bids one with respective payoffs $v_1 = 0$ and $v_2 = 1$, and (3) both agents bid one and receive $v = 0$. Mixed strategies whereby agents randomize over (some) of the pure strategy solutions exist as well.

(iii) $R = 3, Q = 3$. This case is still solvable by hand. In particular, condition (8) becomes

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{9} \\ 1 & \frac{8}{9} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \leq \begin{bmatrix} \frac{v}{3} \\ \frac{v+1}{3} \\ \frac{v+2}{3} \end{bmatrix} \tag{12}$$

It is readily verified that $(p_0, p_1, p_2) = (\frac{1}{7}, \frac{3}{7}, \frac{3}{7})$ and $v = \frac{3}{14}$ satisfy condition (12) and the other conditions of Theorem 2, and hence constitute an equilibrium to the game.

(iv) $R = 3, Q = 4$. This case is already too cumbersome to solve by hand, so we relied on the analytical computer program "Derive" to solve this game. It can be checked that there are two symmetric solutions: (i) $(p_0, p_1, p_2, p_3) =$

$\left(\frac{5}{14}, 0, \frac{9}{14}, 0\right)$ with $v = \frac{5}{7}$, and (ii) $(p_0, p_1, p_2, p_3) = \left(\frac{3}{38}, 0, \frac{35}{38}, 0\right)$ with $v = \frac{3}{19}$.

For R = 3 and Q > 4, one generally finds that all probability mass is loaded on the first few probabilities p_y, with most mass loaded on the higher p_y's, and $0 < v < 1$. For Q > 15 the computational burden increases rapidly and exact solutions take an excessive amount of computer time. This is a bit unfortunate because the experiment conducted by Millner and Pratt (1989) used R = 3 and a grid of Q = 80 (at the end of the experiment the laboratory dollars were converted into U.S. dollars at an exchange rate of 10. But subject payments were also rounded to the nearest 25 cents, generating a grid of 32 with unequal grid sizes). Their hypotheses and tests, however, all concern mean individual expenditures and mean dissipation rates. The question therefore is whether we have something to offer concerning these quantities, without explicitly calculating the solutions.[9]

The expected individual expenditures and the expected dissipation rates can be calculated from equation (6). Note that premultiplication of $EU_1(x)$ by p_x and summation over x gives the expected equilibrium payoff to player 1 in a symmetric equilibrium:

$$EU_1 = \sum_{x=0}^{Q} p_x EU_1(x) = \sum_{x=0}^{Q} p_x \left(\sum_{y=0}^{Q} p_y \pi(x,y)Q - x \right) = \sum_{x=0}^{Q} p_x v = v, \tag{13}$$

because player one only loads mass on those x's which generate the same (highest) expected payoff equal to v (see Theorem 1 above).

In order to dispel the claim that over dissipation of rents is expected when R > 2, first note that if agent 1 chooses x = 0 with probability 1, then

$$EU_1 = \sum_{y=0}^{Q} p_y \pi(0,y)Q = p_0 \frac{1}{2} Q \geq 0. \tag{14}$$

Hence each player can guarantee a non-negative expected payoff. Secondly, the expected dissipation rate is easily calculated from $EU_1 + EU_2$. Note that in any equilibrium,

$$v_1 = EU_1 = \text{Prob}\{\text{agent 1 wins}\}\ Q - \bar{x},$$

where $\bar{x} = \Sigma p_x x$ is the average individual expenditure. Adding up yields

$$v_1 + v_2 = [\text{Prob}\{\text{agent 1 wins}\} + \text{Prob}\{\text{agent 2 wins}\}]\, Q - \bar{x} - \bar{y}.$$

But since the prize is always awarded, there is always a winning agent and hence by (14)

$$0 \leq v_1 + v_2 = Q - \bar{x} - \bar{y}, \tag{15}$$

so that $\bar{x} + \bar{y} \leq Q$. The expected rate of rent dissipation, D, is defined as $D = (\bar{x} + \bar{y})/Q$. Thus

$$D = 1 - \frac{v_1 + v_2}{Q} \leq 1. \tag{16}$$

We have thus proved:

Theorem 3. The two player finite rent-seeking game devised by Tullock never involves over dissipation in any (possibly mixed-strategy) Nash equilibrium for any $R > 0$. That is, $D \leq 1$ always.

The dissipation rate is also bounded from below. But in contrast with the upper bound, the lower bound depends on the value of the exponent R. This can be easily seen by investigating the two limiting cases $R = 0$ and $R = \infty$. In the former case there is no dissipation, while in the latter case dissipation can be complete. Therefore, we will investigate specific values of R. To explain the Millner-Pratt experimental results for the case $R = 3$, one requires precise information about the size of D, and hence the tighter the lower bound on D the better. It is not too difficult to show for $Q > 2$, $R > 2$, that in any equilibrium the dissipation rate is at least 50%. With more effort, for $Q > 3$ a sharper lower bound for the symmetric equilibria is obtained in Theorem 4.

Theorem 4. In any symmetric Nash equilibrium of the two player Tullock rent-seeking game with $\infty > R > 2$ and $\infty > Q > 2$, the dissipation rate is bounded from below by $1 - \dfrac{2}{Q}$.

Proof. The proof comes in two parts. In Part 1 we assume that $p_0 > 0$, and show that this implies $v \leq 1$. Hence $D \geq 1 - 2/Q$. In Part 2 we show that $p_0 = 0$ implies $v < 1$. Some of the computations from Part 2 are relegated to the Appendix.

Part 1. Suppose that $p_0 > 0$. Then (by Theorem 1) for $x = 0$ condition (7) necessarily becomes an equality: $p_0 = 2\,v/Q$, so that $v = Qp_0/2$. Because p_0 is bounded above by 1, v is bounded above by Q/2. This implies $D \geq 0$. To improve the upper bound on v, i.e., to lower it from Q/2 to 1, we continue the presumption $p_0 > 0$. From condition (7), for $x = 1$ we have

$$p_0 + \alpha \leq \frac{v+1}{Q}\,;\ 0 \leq \alpha < \frac{1}{2}\,.$$

To see this note that all the probabilities $\pi(1,y)$ except the first in the second row of matrix condition (8) are less than or equal to 1/2. Combine the presumption $p_0 = 2v/Q$ with the above inequality to get

$$\alpha \leq \frac{1}{Q} - \frac{v}{Q}\,. \tag{17}$$

Hence $1 - v \geq \alpha Q \geq 0$. Therefore $1 \geq v$.

Part 2. We now show that $p_0 = 0$ implies $v < 1$. Let x be the first row for which $p_x > 0$, $x \neq 0$, i.e., $p_0 = \ldots = p_{x-1} = 0$. Then condition (7) holds as an equality for this row, i.e.,

$$\frac{1}{2}\,p_x + \frac{x^R}{x^R + (x+1)^R}\,p_{x+1} + \ldots + \frac{x^R}{x^R + (Q-1)^R}\,p_{Q-1} = \frac{v+x}{Q}\,, \tag{18}$$

We will show that $v \geq 1$ and $p_0 = 0$ are incompatible. For $x + 1$, condition (7) reads as follows:

$$\frac{(x+1)^R}{(x+1)^R + x^R}\,p_x + \frac{1}{2}\,p_{x+1} + \ldots + \frac{(x+1)^R}{(x+1)^R + (Q-1)^R}\,p_{Q-1} \leq \frac{v+1+x}{Q}\,. \tag{19}$$

Compute p_x from the equality (18), and substitute this into the weak inequality (19). This yields the following weak inequality:

$$\left[\frac{1}{2} - 2\,\frac{(x+1)^R}{(x+1)^R + x^R}\,\frac{x^R}{x^R + (x+1)^R}\right] p_{x+1} + \ldots +$$

$$\left[\frac{(x+1)^R}{(x+1)^R + (Q-1)^R} - 2\,\frac{(x+1)^R}{(x+1)^R + x^R}\,\frac{x^R}{x^R + (Q-1)^R}\right] p_{Q-1}$$

$$\leq \frac{1}{Q} \{v+1+x - 2(v+x) \frac{(x+1)^R}{(x+1)^R + x^R} \}. \tag{20}$$

In the Appendix we manipulate the two sides of inequality (20) to show that if $v \geq 1$ the left-hand side is non-negative while the right-hand side is strictly negative. (Note that the proof would be particularly simple if $R = \infty$, since then (20) reduces to $0 \leq p_{x+1} \leq 2(1-v-x)/Q$.) This yields a contradiction so that the supposition $p_0 = 0$ and $v \geq 1$ are incompatible. QED

3. Millner and Pratt revisited

How do the above theoretical results compare with the experimental evidence reported by Millner and Pratt (1989)? Note that for Q large Theorems 3 and 4 provide tight bounds. In particular, given the values of $R = 3$ and $Q = 80$ used in the Millner and Pratt experiments, the symmetric (mixed-strategy) equilibrium expected outlays are $\bar{x} = \bar{y} = 3.9$ (after conversion to U.S. dollars) and the corresponding interval for the expected rent dissipation is $D \in [97.5\%, 100\%]$ – it is not the 150 percent dissipation rate used as the null hypothesis by Millner and Pratt. Using the experimental evidence reported by Millner and Pratt, we find the following t-statistics for the null hypotheses: -5.11 and -2.73 respectively.[10] Compare these to the values reported by Millner and Pratt and reproduced in Table 1 above. (If the rounding to the nearest 25 cents in the actual payout is taken into account, the mean dissipation rate is reduced to approximately 93.75, which does not differ significantly from the experimental result at the 5% level). Note that these t-statistics are of the same order of magnitude as those for the case $R = 1$. Also recall the experimental work by Millner and Pratt (1991) which relates the relatively small discrepancy for the case $R = 1$ to the existence of risk aversion.[11] Our conjecture is that the remaining discrepancy for the case $R = 3$ can be explained in a similar way. Importantly, though, the above shows that when the correct symmetric (mixed-strategy) Nash equilibrium is used as the theoretical benchmark to form the null hypothesis, Millner and Pratt's empirical results for the case $R = 3$ and $Q = 80$ accord well with state-of-the art rent-seeking theory. Individuals seem to behave quite efficiently after all.

4. Summary and results for the continuous strategy space case

In this paper we have solved the original rent-seeking game devised by Tullock for the case where the rent-seeking exponent (R) exceeds two. A constructive method was used to find the explicit solution for the finite game (i.e., the

Tullock game in discrete strategy space). Our theoretical results, which establish that rents are under dissipated when $R > 2$, accord well with the existing experimental evidence. We also provide tight bounds on the rate of dissipation as the mesh of the strategy space decreases.

Up to this point we have not addressed the solution to the infinite rent-seeking game, i.e., when the strategy space is continous and $R > 2$. It turns out the payoff functions in equation (1) satisfy the conditions of Theorem 6 in Dasgupta and Maskin (1986), guaranteeing the existence of a symmetric mixed strategy equilibrium for the rent-seeking game with a continuous strategy space. The proof of their theorem relies on finite approximation of the game and then letting the grid size become finer and finer, as we did in our example with an infinite R. Thus the construction of the equilibrium to the finite game in the previous section is driven to the limit. Under sufficient regularity conditions this method indeed yields a solution to the infinite game.

The application of Dasgupta and Maskin's Theorem 6 requires four conditions, each of which is satisfied for the Tullock game with a continuous strategy space. In particular, this theorem requires: (i) The sum of the payoffs must be upper semi-continous. From equations (1) and (2) we easily see that $U_1(x|y) + U_2(x|y) = Q - x - y$, which is continuous and therefore upper semi-continuous as well. (ii) The subset of discontinuities in the payoffs must be of a dimension lower than 2, and one must be able to express the elements of this subset as functions which relate the strategy of one player to the strategy of the other. For the Tullock game with $R < \infty$, this condition is simple to check, as $x = y = 0$ constitutes the only point of discontinuity. The condition guarantees that the discontinuities are relatively unimportant (have measure zero). (iii) The payoff $U_1(x|y)$ must be bounded. This holds evidently as $-Q \leq U_1(x|y) \leq Q$ on $[0,Q]$. (iv) Finally, $U_1(x|y)$ must be weakly lower semi-continuous. The only point where there could arise a problem is at the point of discontinuity, but as $U_1(x|y = 0)$ is lower semi-continuous, it is certainly weakly lower semi-continuous. This last condition guarantees that, loosely speaking, a player does not want to put weight on the discontinuity point even if the other player does, because payoffs may jump down but do not jump up.

Thus we conclude that a symmetric mixed strategy equilibrium exists for the continuous strategy space rent-seeking game for all $R > 2$ as well. An explicit closed form solution remains for future investigation. For the special case $R = \infty$, a full characterization of all the equilibria is available even when there are more than two players; see Baye, Kovenock, and de Vries (1990, 1993). Other interesting questions include the explicit solution to asymmetric versions of the game, as well as further experimental work along the lines suggested above. These remain the focus of our future research.

Notes

1. While our focus is on the Tullock specification in (2), several of our results are valid for other functional forms of π. In particular, so long as $0 \leq \pi(x,y) \leq 1$ and $\pi(x,y) + \pi(y,x) = 1$, any Nash equilibrium satisfies the properties stated in Theorem 1 below, and our Theorem 3 on the impossibility of overdissipation carries through. Only the tight lower bound on the dissipation rate given in Theorem 4 is dependent on the specific functional form in (2). We thank a referee for encouraging us to point this out.
2. Briefly considering the n-player variant, $n \geq 2$, the second order conditions fail if $R > n/(n-2)$, cf. Tullock (1984) (where the reverse condition is reported erroneously). Note that for the case $n = 2$ the second order conditions are always satisfied. But it is easily checked that for $R > 2$ the symmetric solution to the first-order conditions yields $U_1(.|.) < 0$, and hence is not a global maximum. Thus the two agent case is the most interesting case to consider, because with $n > 2$ the posited solutions obviously do not make sense if $R > n/(n-2)$.
3. The null hypotheses should be interpreted with caution because the experimental setup of Millner and Pratt (1989) is not entirely congruent with the simultaneous move requirement (neither does it fit the alternating move version studied in Leininger, 1990; Leininger and Yang (1990).
4. Baye, Tian, and Zhou (1993) show that one cannot generally blame the non-existence of a pure-strategy equilibrium on the failure of payoff functions to be quasi-concave or upper semi-continuous.
5. Although Millner and Pratt claim to be testing the Tullock model, the experiment actually allows the rent-seekers to expend resources continuously over a small time interval. Hence, the experiment does not formally test the original one-shot simultaneous-move Tullock game. This problem is corrected in the experiments of Shogren and Baik (1991), who do not reject the theoretical prediction when $R = 1$.
6. The continuous strategy space (infinite game) is dealt with below.
7. The mixed strategies may be degenerate, i.e., in the case of a pure strategy equilibrium.
8. Shogren and Baik (1991) state that the behavioral inconsistency reported in Millner and Pratt "... is due to the nonexistence of a Nash equilibrium. In this case there is no predictable behavioral benchmark to measure the experimental evidence against." Our Theorem 2, however, provides such a benchmark. Shogren and Baik are referring to the non-existence of a symmetric pure strategy Nash equilibrium.
9. In future work it may be of interest to repeat the experiment for $R = 3$ and Q small such that all the properties of the symmetric equilibrium can be evaluated, i.e., the values of the p_y's.
10. Calculations are based on $(3.34 - 3.9)/s_1 = -5.11$ and $(84 - 97.5)/s_2 = -2.73$, where s_1 and s_2 were calculated from Millner and Pratt (1989) using $(3.34 - 6)/s_1 = -24.28$ and $(84 - 150)/s_2 = -13.37$.
11. See also Shogren and Baik, who run a related experiment for $R = 1$ and find that the Nash equilibrium dissipation hypothesis cannot be rejected at the 90 percent level.

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Appendix

In this Appendix we show that, for $v \geq 1$, the left-hand side of inequality (20) is non-negative, while the right-hand side is strictly negative.

Manipulate the right-hand side as follows:

$$v+1 + x - 2(v+x)\frac{(x+1)^R}{(x+1)^R + x^R} \lesseqgtr 0$$

$$\Leftrightarrow$$

$$(v+1+x)x^R \lesseqgtr (v-1+x)(x+1)^R$$

$$\Leftrightarrow$$

$$1 + \frac{2}{v+x-1} \lesseqgtr (1 + \frac{1}{x})^R.$$

Note that the left-hand side of this last inequality is decreasing in v. Hence, to show that the right-hand side of (20) is negative, it is sufficient to show that such is the case for $v = 1$. Assuming that $v = 1$, we can further manipulate the last inequality:

$$1 + \frac{1}{1+x} \lesseqgtr (1 + \frac{1}{x})^{R-1}$$

$$\Leftrightarrow$$

$$1 + \frac{1}{1+x} \lesseqgtr (1 + \frac{1}{x})(1 + \frac{1}{x})^{R-2}.$$

Evidently, for any $x > 0$

$$1 + \frac{1}{1+x} < 1 + \frac{1}{x}\cdot$$

Thus for any $R \geq 2$ and $x \geq 1$ the right-hand side of (20) is strictly negative for any $v \geq 1$.

To obtain the left-hand side result we need to show that for any t such that

$$Q - 1 \geq t \geq x + 1,$$

$$\frac{(x+1)^R}{(x+1)^R + t^R} \geq 2 \frac{(x+1)^R}{(x+1)^R + x^R} \frac{x^R}{x^R + t^R} .$$

Manipulation yields

$$[(x+1)^R + x^R][x^R + t^R] \lesseqgtr 2x^R[(x+1)^R + t^R]$$

$$\Leftrightarrow$$

$$x^R(x+1)^R + t^R(x+1)^R + x^{2R} + x^R t^R \lesseqgtr 2x^R (x+1)^R + 2x^R t^R$$

$$\Leftrightarrow$$

$$[(x+1)^R - x^R][t^R - x^R] \lesseqgtr 0.$$

Because $t \geq x+1 > x$, the left-hand side of this last inequality is unequivocally positive, and hence the left-hand side of (20) is non-negative.