Decision support for inventory systems with complete backlogging

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Beslissingsondersteuning voor voorraadsystemen
met volledige nalevering

PROEFSCHRIFT

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Emőke Bázsa-Oldenkamp
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Chapter 1

Introduction

In the past couple of years it is an increasingly prevalent practice that stores do not keep stocks, especially in the crowded cities where lack of space is an important issue. For quite some time, the new trend in managing supply chains has been the so called just-in-time (JIT) approach: inventory arrives on the factory floor only as and when it is needed. Reducing lead times, production setup times and costs, have indeed a huge impact on the functionality of a supply chain. In particular in the telecoms and computer businesses, over-stocking has been a costly misjudgement. Cisco Systems, for example, had to write down its inventory by $2.25 billion at the end of 2000 (The Economist). "Inventory is evil" sounds a JIT slogan (Zipkin (2000)). Does this mean then, that all the different inventory models developed since 1913 (Harris (1913)) have become redundant?

After the September 11th events, with security tightened at the ports, airports and land borders, goods were taking much longer than expected to reach their destinations. In its September 22nd number, The Economist notes that "any perception of an increase in risk tilts the balance away from JIT and in favor of JIC, the just-in-case strategy of holding inventory against the risk of an unexpected disturbance in the supply chain. This balance has been tilted further by sharp falls in interest rates, which have cut the holding costs by more or less half, and thus reduced the need for JIT systems" (The Economist). The truth in fact is, there is no universally, in- any -case -applicable inventory model; there is more to inventory management than just inventory control. Surely, *if* leadtime and setup costs are negligible relative to holding costs and other factors *then* JIT can be applied (Zipkin (2000)). This point of consideration is even more important than the risk factor. While JIT proved to work fine in the telecoms and computer businesses for example, other types of products/environment may require a completely different approach. So, the "old"

models are still of great value.

In principle, in managing inventories it is not enough to know what *in general* the possibilities for improvement may be, since some of the alternatives might be very costly for the actual situation. Therefore, one needs to know what the anticipated benefits for *that particular* situation are (Zipkin (2000)). Hence, it is also essential to know what kind of tools we have at our disposal. Often, a supply chain can be optimized by reducing the problem to several single item single location problems (e.g. see Chapter 5). However, the hidden or obvious relations between the various components of the supply chain induce a dependency structure on these 'simple' single location problems, triggering the need for more sophisticated solution techniques. That is, the rougher the approximate solution of such a component, the weaker the overall solution for the whole supply chain is. Or conversely, an exact solution for the single location problems can assure a good global solution for the supply chain. In this spirit, the aim of this thesis is to revisit the basic components of a supply chain, starting at the simplest models, that is, with exogenous demand, without intricate dependency structures, then moving on to models with various endo- and exogenous dependencies. Finally, it is demonstrated for a specific supply chain how the newly developed techniques can be applied to obtain an exact solution. It is not only the explosive improvement of computer capacities and capabilities which motivate the need for exact solution. Den Iseger (2000) developed a technique for inverting Laplace transform, obtaining results which are exact almost up to machine precision. This enables us to calculate the closed form expressions, obtained for the performance measures, with the same precision.

Chapter 2 opens with an overview of the simplest single item single location models, with stochastic but exogenous demand. The literature on this topic may well be the most extensive, due to the fact that researchers focused in their analysis on a single class of models at a time. However, in spite of the overflow of research results on this topic, there is enough room for improvement. The models can be divided in classes, according to the most important characteristics: the type of policy employed (e.g. base stock, (s, S), (s, Q) policies; continuous or periodic review), the nature of the lead time (constant or stochastic), the type of the demand (unit or random, discrete or continuous individual demand; Poisson arrivals or more general). In 1986, Zipkin (1986) was the first to perform an analysis for one model, which captured all these characteristics together, assuming relatively[1] weak regularity conditions. His contribution is especially remarkable for its asymptotic analysis under a stochastic leadtimes assumption. The new achievements presented in this chapter, yield an

[1]His model captured all policies and demand processes dealt with in such models up to that time.

analysis which allows more general demand processes and policies than those considered by Zipkin. A significant example is the time- nonhomogeneous compound Poisson demand process, which does not satisfy the regularity conditions imposed by Zipkin, but does fit into our model. Although this example is already important in itself, it also lights a new perspective on stochastic leadtimes, while yielding an interesting equivalence between Zipkin's orders' arrival mechanism with stochastic leadtimes, and a model with constant leadtime but nonstationary demand. In fact, our improvement on the 'simple' models is a forerunner of the model of Chapter 4, which extends 'simple' models to incorporate intricate dependency structures.

The next chapter deals with the optimization of the policies controlling a single item single location inventory model. In particular, the optimization of the so-called (s, S) policies has attracted the attention of the researchers for a long time. In 1991 Zheng and Federgruen have published an optimization algorithm for (s, S) policies with discrete order sizes, which represented a breakthrough in this line of research. However, their algorithm does not work for continuous order size models, hence the literature still lacked an efficient algorithm for the continuous case. Chapter 3 fills this gap, while giving an overview and comparison with already known algorithms. The optimization of (s, S) policies with continuous action space is considered to be a difficult problem, since a Markov Decision Processes technique requires clever modifications (Federgruen and Zipkin (1985)). However, the algorithm presented in this chapter is very simple, in the sense that every step is motivated by purely geometrical considerations. Moreover, the algorithm is always converging monotonically, and every cycle results in a policy improvement. The resulting policy is ε–optimal for an arbitrary $\varepsilon > 0$.

Chapter 4 moves on to the analysis of single item single location models with endo- and/or exogenous dependencies. An example of such a model with exogenous dependency structure is where demand has Markov–modulated arrivals. Song and Zipkin (1993) call such a demand process *world–dependent* demand: the rate of the Poisson arrivals changes with the state of an exogenous, independent (discrete-state) stochastic process, called the *world*. The analysis becomes considerably more intricate with endogenous dependencies. However, the new results derived in this chapter for such models are surprisingly simple, as a consequence of the modeling idea: we assume that there exists a Harris chain (intuitively: a positive recurrent Markov chain with continuous state space), which captures all the relevant information about the past. In practice, such a process is fairly easy to identify. Consider for example the case when the leadtime demand depends on the past through the residual life process. Such a problem occurs in case of periodic review policies in combination with compound renewal demand (thus *not* Poisson), or the example of Chapter 5.

It is known (Sigman (1990)) that the residual life process is a Harris chain, hence it perfectly suits our needs: we identified an embedded Harris chain, which carries all the relevant information about the past. The resulting average cost expressions are in terms of the arrival cycles, that is, the simplest possible form, while the optimization can be done on the transitions of the inventory position and the embedded Harris chain. The latter means that nonstationary optimal policies can be identified.

Finally, the new methods and techniques developed for single item single location problems are applied in order to solve a two–level decentralized distribution system. The system consists of one warehouse and several non-identical retailers. External demand at the retailers is compound renewal. Employing a decentralized policy means that one solves the problem for each component separately: each retailer sees only its own demands, and the warehouse sees only the incoming order streams. However, the components will not belong to the class of 'simple' single item single location models, because the hidden interrelation among the components induces a dependency structure on each component. Therefore, so far for such decentralized systems only approximate performance measure expressions were derived. Yet, applying the theory of Chapter 4 helps us come around the difficulties: Chapter 5 derives exact average cost and service measure (waiting time) expressions.

A guide to notation

The characteristics of stochastic single item single location inventory models are described by stochastic processes; these characteristics are standard and we wish to establish a common notation for them throughout the chapters of this thesis. Inventory models assuming complete backlogging will yield a radically different behaviour than models assuming lost sales, since the characteristics are different for the two cases. As stated earlier, here we only focus on models with complete backlogging. This guide is only meant to serve as a quick reference for notation used throughout the following chapters, it does not fully describe the models analyzed further on.

- $\mathbf{IN}(t)$ is the inventory level or net inventory at time t = the stock on hand minus backorders
- $\mathbf{IP}(t)$ is the inventory position at time t is the net inventory plus outstanding replenishment orders
- $\mathbf{D}(t)$ is the cumulative demand up to time t, a point process
- $\{\mathbf{t}_n, n \in I\!N \cup \{0\}\}$ denotes the arrival process of customers
- $\mathbf{T}_n := \mathbf{t}_n - \mathbf{t}_{n-1}, n \in I\!N$ are the interarrival times
- $\mathbf{N}(t)$ is the stochastic counting process related to the arrival process of customers,

$$\mathbf{N}(t) := \sum_{n=1}^{\infty} 1_{\{\mathbf{t}_n \leq t\}}.$$

- $\mathbf{N}_\Delta(t)$ denotes the 'increment process' of the stochastic counting process, that is, $\mathbf{N}_\Delta(t) := \mathbf{N}(t+\Delta) - \mathbf{N}(t)$, $t \geq 0$.
- $\mathbf{Y}_n$ is the individual demand at the nth demand epoch $\mathbf{t}_n$

- L is the leadtime (for stochastic leadtimes we like to use $\mathbf{L}$) = the time elapsed from the moment of placing an order until it arrives at the facility.

With the notation above the cumulative demand process $\mathbf{D}(t)$ is given by the expression

$$\mathbf{D}(t) := \sum_{n=1}^{\mathbf{N}(t)} \mathbf{Y}_n. \tag{1.0.1.1}$$

The costs we consider, related to an inventory system, have the following components:

- $K \geq 0$ is a fixed setup cost, incurred every time an order is placed
- $h > 0$ is the cost rate of holding a unit item in inventory, referred to as holding cost
- $p > 0$ is the cost rate of backlogging a unit item, referred to as penalty cost
- f is the cost rate function $f(x) = h(x^+) - p(x^-)$, $x \in I\!R$.

Besides the characteristics and cost components of a model, we use the following notation.

- boldface characters always denote random variables or stochastic processes, for example $\mathbf{X} := \{\mathbf{X}(t) : t \geq 0\}$ is a general stochastic process
- $\mathbf{X}(a, b] = \mathbf{X}(b) - \mathbf{X}(a)$, with $0 \leq a < b$
- ϕ_s, $s \geq 0$ is a shift operator, that is, $\phi_s(\mathbf{X})(t) := \mathbf{X}(t + s)$ for every $t \geq 0$
- F_X stands for the cumulative distribution function of the random variable $\mathbf{X}$
- LS_{F_X} and L_{F_X} denote the Laplace- Stieltjes transform and the Laplace transform of F_X, respectively
- $F * G$ denotes the convolution of the two cumulative distribution functions F and G, defined as

$$(F * G)(x) = \int_0^x F(x - s)G(ds)$$

- "i.i.d." is the abbreviation for "independent and identically distributed"
- " $\stackrel{d}{=}$ " means "distributed as" or "distributionally equals"
- U.I. stands for "uniformly integrable"
- $f \in \mathcal{B}^+$ means that f is a positive Borel measurable function

Chapter 2

Asymptotic behaviour of stochastic models

2.1 Introduction

The mathematical modeling of stochastic single item single location inventory systems has already attracted the attention of researchers since the 50's. More precisely, one is interested in the asymptotic behaviour of such a system: calculating performance measures and optimizing decision variables when the system has reached a steady state. The performance measures associated with an inventory model are usually calculated for (or related to) one of the characteristics of such a model: the inventory level. These characteristics are inter-related through the so called *flow-conservation relation*: all and only those orders placed by time t will arrive by $t+L$, that is,

$$\mathbf{IN}(t+L) = \mathbf{IP}(t) - \mathbf{D}(t, t+L],$$

where $\mathbf{D}(t, t+L]$ is called the leadtime demand, that is, demand during leadtime. In order to obtain the asymptotic performance measures, which are related to the net inventory process, one needs to analyze the asymptotic behaviour of the characteristic stochastic processes. It was shown for certain demand processes and order policies that the limiting variables exist, and

$$\mathbf{IN}_\infty = \mathbf{IP}_\infty - \mathbf{D}_\infty(L). \tag{2.1.2.1}$$

Here, the random variables $\mathbf{IN}_\infty, \mathbf{IP}_\infty$ and $\mathbf{D}_\infty(L)$ are distributed with the limiting distributions of the corresponding processes. Relation (2.1.2.1) in its own would

not be a fundamental result, but the following two properties turn it into a strong statement:

$$\mathbf{IP}(t) \text{ and } \mathbf{D}(t, t+L) \text{ are asymptoticly independent,} \tag{2.1.2.2}$$

$$\mathbf{IP}_\infty \text{ has the limiting distribution of the Markov chain } \mathbf{IP}(\mathbf{t}_n). \tag{2.1.2.3}$$

The last relation substantially simplifies the calculation of the limiting distribution of the inventory position process. The embedded Markov chain is often formed by observing $\mathbf{IP}(t)$ just after orders are placed (see Sahin (1979) for example); needless to point out that the latter approach results in cycle lengths which could be cumbersome to calculate. For this reason observing **IP** just after arrival epochs proves to be a better approach (Zipkin (1986), Bázsa and Den Iseger (2001)). Properties (2.1.2.1) and (2.1.2.2) significantly simplify the calculation of the limiting distribution of $\mathbf{IN}(t)$, resulting in the ability to directly compute performance measures. This chapter therefore focuses on this key result: under which condition do properties (2.1.2.1) - (2.1.2.3) hold, and how can one calculate the primary performance measures efficiently?

Let us first focus on the constant leadtime case. For the basic (s, Q) model, where a fixed quantity Q is ordered whenever $\mathbf{IP}(t)$ reaches the reorder point s, this key result was proven by Galliher, Morse and Simond (1959) for the case when demand is generated by a simple Poisson process (thus demand is discrete unit). It is easy to see that in this case the limiting distribution of the embedded Markov chain (2.1.2.3) is the uniform distribution on the integers from $s+1$ to $s+Q$. The independency property (2.1.2.2) obviously follows, since demand has independent increments. Obviously, for the unit demand case the (s, S) policy (an order is placed as soon as the inventory position drops to or below the reorder level s, and the size of the order is such that **IP** is raised to the so called order-up-to level S) coincides with the simple (s, Q) policy, through $S - s = Q$. This is not the case for compound demand, that is, when the individual demand size can exceed the unit size, resulting in a so called *undershoot* just before placing an order. Sivazlian (Sivazlian (1974)) extends the result to renewal-process unit demand. Hadley and Whitin (1963) report the same result for compound-Poisson demand and an (s, nQ) policy (the order quantity is always an integer multiple of Q, chosen in such a way, that the inventory position is raised to a level between s and $s+Q$). They show that the transition matrix of the embedded Markov chain is double stochastic, thus the uniform distribution represents the unique solution. Richards (1975) extends their results to compound-renewal demand processes, provided that a customer is allowed to ask for a single unit. The embedded Markov chain he considers is in the moments just after ordering. More

recently, Zheng and Chen (1992) derived simple performance measures, together with an optimization algorithm and sensitivity analysis, with compound renewal demand, allowing the demand size to depend on the time since the last demand.

The key result was shown to hold for (s, S) policies with compound Poisson demand (Galliher, Morse and Simond (1959)). In this case the limiting distribution of the embedded Markov chain is not uniform. Stidham (1974) obtains similar results for independent-increments demand processes. Sahin (1979) treats the case where demand is generated by a compound renewal demand. He obtains as first the limiting distribution of the embedded Markov chain, in the points of customer arrivals. He shows that the inventory position and the leadtime demand form regenerative processes with the times of order placements, hence, through the flow conservation relation, the net inventory process forms a regenerative process too. This approach is used by Bázsa, Frenk and Den Iseger (2001), with a summary of basic policies and their properties. This line of research, the regenerative processes approach, constituting a vast part of the literature, is evaluated critically in Section 2.2. Richards (1975) also proves that the limiting distribution of the inventory position related to an (s, S) process is uniform if and only if the individual demand size is strictly unit. Federgruen and Schrechner (1983) assume compound renewal demand, allowing the demand size to depend on the time since the last demand, and derive discounted costs. The most significant contribution to this line of research is the work of Zipkin (Zipkin (1986)), which comprises and generalizes the earlier research. He assumes demand generated by a compound counting process, with i.i.d. demand sizes, independent of the interarrival times, demand epochs forming an asymptoticly stationary counting process, while the ordering policy is general, only assuming that it is based on the observation of the inventory position. He distinguishes between the two cases when the embedded Markov chain is assumed to be aperiodic and when periodicity is allowed. In the latter case his analysis too builds on the regenerative processes technique. His approach can also handle stochastic leadtimes. We will discuss this model in detail in Section 2.3. Recently, Bázsa and Den Iseger (2001) extended these results further, by relaxing Zipkin's condition on the stochastic counting process related to the arrival process of customers, hence extending the class of allowable demand processes to nonstationary point processes. Furthermore, under Zipkin's condition for the counting process, the model can deal with unit demand processes. The main assumption used by Bázsa and Den Iseger (2001), which relaxes Zipkin's conditions, is that the *limiting distribution in the Cesaro sense* of the joint process $(\mathbf{IP}, \mathbf{N}_\Delta$ exists. Since the Cesaro sense limiting distribution is a long run (time- or event-) average (Sigman (1995)), it is clearly less restrictive than the weak limit assumption (Zipkin (1986)). Section 2.4 is devoted to the discussion of their approach.

The model with stochastic leadtimes has also been analyzed extensively. There was a so called standard approach (Hadley and Whitin (1963)): $\mathbf{D}_\infty$ denotes the marginal leadtime demand, that is, its distribution is obtained by mixing those of $\mathbf{D}_\infty(L)$ over L. Then the asymptotic flow conservation law (2.1.2.1) is used with $\mathbf{D}_\infty$, treating $\mathbf{IP}_\infty$ and $\mathbf{D}_\infty$ again independent. The question which arises in this case is, which conditions justify the modified relation (2.1.2.1)? Hadley and Whitin (1963) consider stochastic leadtimes with a simple Poisson demand process assuming two mutually contradictory conditions: orders should never cross in time, and still, leadtimes should be independent. Kaplan (1970) considers a finite planning horizon and discounted costs; he also assumes that orders never cross in time, and that the probability of arrival of an outstanding order is independent of the number and size of outstanding orders, it only depends on the time since the order was placed. He builds on the probabilities p_k that in any period, all orders which have been placed for k or more periods arrive in the current period. Kaplan also proves that the ordering policy only depends on the inventory position, hence the optimal policy is easily computed and characterized. Nahmias (1979) goes further with the model of Kaplan, interpreting the model as having a sequence of independent identically distributed random variables $\mathbf{A}_0, \mathbf{A}_1, \ldots$ assuming values on $0, 1, \ldots, L$ such that $p_k = I\!P\{\mathbf{A}_n = k\}$ for all $n \geq 0$. $\mathbf{A}_n$ is interpreted as the age of the oldest order arriving in period n. Ehrhardt (1981) adopts the leadtime model related to the i.i.d. sequence $\mathbf{A}_0, \mathbf{A}_1, \ldots$ of Nahmias, deducing that the event $\{\mathbf{L} = i\}$ is equivalent to the event $\{\mathbf{A}_n > 0, \mathbf{A}_{n+1} > 1, \ldots, \mathbf{A}_{n+i-1} > i-1, \mathbf{A}_{n+i} \leq i\}$. The distribution of the leadtime is derived from this equivalence relation.

Zipkin's stochastic leadtime model (Zipkin (1986)) is the most intuitive of all, being at the same time more realistic as well. He assumes that the order arrival mechanism is driven by a stationary, ergodic stochastic process $\mathbf{U}$, which operates independently of the demand. Then, if an order is placed at time u, it arrives at time $v = \min\{t : t - \mathbf{U}(t) \geq u\}$, which assures the no order crossing condition. He also establishes some kind of equivalence with the Kaplan- Nahmias- Ehrhardt model, by assuming an i.i.d. sequence $\{\mathbf{U}'(t)\}$ with $p_j = I\!P\{\mathbf{U}'(t) = j\}$, and setting $\mathbf{U}(t+1) = \min\{\mathbf{U}'(t+1), \mathbf{U}(t) + 1\}$. Then $\{\mathbf{U}(t)\}$ is a Markov chain satisfying $\mathbf{U}(t+1) \leq \mathbf{U}(t) + 1$ and $I\!P\{\mathbf{U}(t+1) = j | \mathbf{U}(t) = i\} = p_j, 0 \leq j \leq i$, independently of i. This construction yields that if an order is placed at time u, it arrives at time $v = \min\{t : t - \mathbf{U}(t) \geq u\}$. Zipkin also points out that only a limited class of marginal leadtime distributions can be realized with the model of Kaplan, Nahmias and Ehrhardt (for instance, the geometric distribution is ruled out). Zipkin's model originates from relaxing the rather restrictive assumption of $I\!P\{\mathbf{U}(t+1) = j | \mathbf{U}(t) = i\} = p_j, 0 \leq j \leq i$, independently of i. He also mentions that although a policy

solely based on the inventory position might not be optimal, unlike for the Kaplan model, it is reasonable to adopt such a policy for the sake of simplicity. Zipkin's mechanism of order arrivals with stochastic leadtimes is discussed in Section 2.3. As it was mentioned previously, a recent article of Bázsa and Den Iseger (2001) generalizes the results of Zipkin (1986). Moreover, they show that, in fact, one does not need to distinguish between the fixed and stochastic leadtimes cases, as long as no order crossing is assumed. They also deduce a surprising equivalence between fixed leadtime - models with time- nonhomogeneous compound Poisson demand, and stochastic leadtime - models with a compound point process demand which yields to Zipkin's condition, as depicted in Section 2.6.

Very often, inventory models are categorized according to the review policy, that is, continuous or periodic review models. We show in Section 2.8 that under certain regularity conditions for the underlying stochastic counting process, periodic review models have the same asymptotic behaviour as continuous review models.

We provide a unified treatment of average costs and service measures, by exploiting the asymptotic independence result (2.1.2.2). The end result is a general cost expression which yields any desired cost or service measure by solely substituting the appropriate cost-rate function. Furthermore, the convolution structure of these performance measure expressions, enables us to obtain a closed form expression for their Laplace transformations, without a significant effort (see Section 2.7). We then make use of a recently developed Laplace transform inversion technique (Den Iseger(2000)), which facilitates us to invert these Laplace transforms in any point. The obtained results are exact almost up to machine precision.

2.2 The Regenerative Processes Perspective with Constant Leadtimes

This section presents the asymptotic analysis of inventory systems with renewal theoretic tools. The policies considered in this section, which we will always refer to as basic inventory policies, are solely dependent on the inventory position process, and the initial inventory position is assumed to be independent of the demand process. Assuming also that demand forms a compound renewal process yields that the inventory position process form a regenerative process. This implies through the flow conservation relation that the net inventory process is also regenerative. Having obtained that the net inventory process is regenerative, one can apply the renewal reward theorem to obtain a simple closed form expression for the long run average

cost. Before explaining in more detail how this analysis has really been done in the literature, we recall the definition of regenerative processes (Asmussen (1987)).

Definition 2.2.1 *The stochastic process* $\mathbf{X}$ *is called a pure regenerative process if there exists an increasing sequence* σ_n, $n \in I\!N \bigcup \{0\}$ *with* $\sigma_0 := 0$ *of random points, satisfying*

1. *The random variables* $\sigma_{n+1} - \sigma_n, n \in I\!N \bigcup \{0\}$, *are independent and identically distributed with right continuous cumulative distribution function* F_σ *satisfying* $F_\sigma(0) = 0$ *and* $F_\sigma(\infty) = 1$.

2. *For each* $n \in I\!N \bigcup \{0\}$ *the post-*σ_n *process* $\{\mathbf{X}_{\sigma_n}(t) \ : \ t \in T\}$ *is independent of* $\sigma_0, \ldots, \sigma_n$.

3. *The distribution of* $\{\mathbf{X}_{\sigma_n}(t) \ : \ t \in T\}$ *is independent of* n.

It is also known (Asmussen (1987)), that the regenerative property is preserved under measurable functions. This implies that if one can show that the inventory position process is regenerative with a subset of arrival points, through the flow conservation relation it follows that **IN** is also regenerative.

The most prevalent method of modeling with regenerative processes in the literature is to consider the moments just after placing an order as regeneration times for the inventory position process (Sahin (1979), Sahin (1990), Bázsa, Frenk and Den Iseger (2001)). In 1979 Sahin proved this result for the case of an (s, S) policy, together with an asymptotic analysis. He deduces the limiting distributions (through all $t \geq 0$) of the characteristic stochastic processes (inventory position, leadtime demand and net inventory), concluding that the inventory position and the leadtime demand are asymptoticly independent. He also summarized these results in his book in 1990 (Sahin (1990)). However, there is a different approach which considers **IP** as a delayed regenerative process with regeneration times being the moments of customer arrivals (the finite delay cycle being the period of time until **IP** reaches steady state (Zipkin (1986), Bázsa and Den Iseger (2001)). On the other hand, the advantage of considering "order -point -cycles" is that no assumption needs to be made about the embedded Markov chain $\mathbf{IP}_n$, since at every order moment the inventory position is precisely S. Bázsa, Frenk and Den Iseger (2001), before discovering the results of Zipkin, drawing on the work of Sahin (1990), developed a regenerative processes framework for simple single item inventory models. Later, Bázsa and Den Iseger (1999) refined this work, realizing that the long run average cost of an (s, S) policy as well of an (s, nQ) model can be expressed in terms of the cost of a base stock policy, since the inventory position process in all cases forms a delayed regen-

erative process with the moments of customers' arrival. We briefly present the latter approach. Recall, we assume demand is a compound-renewal process.

2.2.1 The base stock policy

Under this inventory control rule a replenishment order is placed as soon as a demand occurs, such that the inventory position is raised again to a predefined level S. We also refer to this policy as an $(S-1,S)$ policy, although demand is not necessarily of unit size. In the general case, the inventory position after customer arrival and just before ordering will drop to $S-\mathbf{Y}_n$, $n \in I\!N$. By this control rule it results that the regeneration points of the stochastic process $\{\mathbf{IN}(t+L) : t \geq 0\}$ will be given by the moments of customers' arrival, $\mathbf{t}_n,\ n \in I\!N \cup \{0\}$. The proof of this statement can be found in many textbooks on inventory models (e.g. Silver, Pyke and Peterson (1998)). The average cost related to this model is obtained by means of the renewal reward theorem (see for instance Ross (1970)), and it is given by the expression

$$\mathbf{C}(S) = \frac{K + I\!E\mathbf{t}_1 I\!E f(S - \mathbf{Y}_1 - \mathbf{D}(L))}{I\!E\mathbf{t}_1}. \tag{2.2.2.1}$$

In the next sections we are going to show that the cost calculations for an (s,S) model and an (s,nQ) model reduce to the cost calculation of this simple model, setting $\Delta = S - s$ and $\Delta = Q$, respectively.

2.2.2 The (s,S) model

Under this rule an order is triggered in the moment the level of the inventory position drops below the reorder level $s < S$ and the size of the order is such that the level of the inventory position process is raised to order-up-to level S. Similarly as for the $(S-1,S)$ model, one can identify the regeneration points of the inventory position process at the moments of customers' arrival $\mathbf{t}_n,\ n \in I\!N \cup \{0\}$. In order to prove that these are indeed regeneration points, we proceed as follows. Let the random variable $\mathbf{V}_n$ denote the difference between the order-up-to level S and the inventory position at time of arrival of the n^{th} customer, i.e. $\mathbf{V}_n := S - \mathbf{IP}(\mathbf{t}_n),\ n \in I\!N$. By the definition of the policy it immediately follows that

$$\mathbf{V}_{n+1} = (\mathbf{V}_n + \mathbf{Y}_{n+1})\, 1_{\{\mathbf{V}_n + \mathbf{Y}_{n+1} \leq S-s\}}, \tag{2.2.2.2}$$

hence $\mathbf{V}_{n+1}$ is completely determined by $\mathbf{V}_n$ and $\mathbf{Y}_{n+1}$, the individual demand of the $(n+1)^{\text{th}}$ customer. Hence, $\{\mathbf{V}_n,\ n \in I\!N \cup \{0\}\}$ is a Markov chain. By the theory of Markov chains (Feller (1968), Ross (1970)) we know that if $\{\mathbf{V}_n,\ n \in I\!N \cup \{0\}\}$ is positive recurrent and aperiodic then it has an invariant distribution. We aim to

show now that this invariant distribution given by $\frac{U_0(x)}{U_0(S-s)}$, where U_0 denotes the renewal function related to the renewal sequence $\{\mathbf{Y}_0, \mathbf{Y}_0 + \mathbf{Y}_1, \dots\}$ and it is given by

$$U_0(x) := \sum_{k=0}^{\infty} F_Y^{k\star}(x).$$

By relation (2.2.2.2) the following equation holds for every $0 \leq x \leq S - s$

$$F_V(x) = h + (F_V \star F_Y)(x), \tag{2.2.2.3}$$

where $h := 1-(F_V \star F_Y)(S-s)$ is a constant. By repeated substitution and induction in relation (2.2.2.3) one can determine that its solution is given by

$$F_V(x) = hU_0(x). \tag{2.2.2.4}$$

The constant h can be determined by the condition $F_V(S-s) = 1$, therefore we obtain that the invariant distribution of the Markov chain $\mathbf{V}_n$ is given by

$$F_V(x) = \frac{U_0(x)}{U_0(S-s)}. \tag{2.2.2.5}$$

Thus we have that $\mathbf{V}_{n+1}$ only depends on $\mathbf{V}_n$ and $\mathbf{Y}_{n+1}$ and the Markov chain $\{\mathbf{V}_n, n \in I\!N\}$ has an invariant distribution, and the demand process is a compound renewal process. Therefore, since every aperiodic and positive recurrent discrete time Markov chain admits a successful coupling (cf. Theorem 3.2. of Thorisson (2000)) for any initial distribution, we can deduce that that the joint stochastic process $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ is delayed regenerative with the sequence of regeneration points given by the arrival moments of customers $\mathbf{t}_n, n \in I\!N \cup \{0\}$. Hence, the netstock process $\{\mathbf{IN}(t+L) : t \geq 0\}$ is also delayed regenerative. Furthermore, the expected cost of the delay cycle is finite (as it is proven in Appendix 4.7 for a more general case with embedded Harris chains).

Although Sahin determined the limiting distribution of the inventory position process, his approach was different: first he determined the distribution of $\mathbf{IP}(t)$ for every $t \geq 0$ and then he took the limit with respect to t. His approach can be improved by realizing that, since the inventory position is constant between arrivals of customers, it is sufficient to determine the stationary distribution of the embedded Markov chain, which is more intuitive and consequently less tedious. Although the cost expression deduced with the arrival-epochs-cycle approach coincides with the one deduced with the order -placement- cycles approach, the latter is considerably murkier. Now the drawback of our approach for an (s, S) policy is clearly that one needs to assume that the embedded Markov chain $\mathbf{IP}_n$ is aperiodic. To assume

positive recurrence and irreducibility alone would not be very restrictive (this is also an assumption in Zipkin (1986)), but assuming aperiodicity rules out a number of demand processes, such as unit demand. On the other hand, assuming that the embedded Markov chain starts in the invariant distribution yields that the $\mathbf{IP}(t)$ is a pure regenerative process, implying further that $\mathbf{IN}(t+L)$ is a pure regenerative process too. There is a trade–off, and given a specific situation one must decide which approach is more suitable for the given data.

It remains now to determine the long run average cost. Note first that the condition that an order was triggered is given by the expression

$$I\!E(\mathbf{V}+\mathbf{Y}>S-s)=1-I\!EF_Y(S-s-\mathbf{V}).$$

Again using the renewal reward theorem yields that $\mathbf{C}(s,S-s)$ equals

$$\frac{K(1-I\!EF_Y(S-s-\mathbf{V}))+\int_0^{S-s} I\!E\left(\int_L^{\mathbf{t}_1+L} f(S-x-\mathbf{D}(t))dt\right)F_{V_n}(dx)}{I\!E\mathbf{t}_1},$$

where $F_{V_n}=F_V$, the invariant distribution of the Markov chain. Observe now that the integral in the formula can be expressed in terms of the cost expression of the $(S-1,S)$ model with order-up-to level $S:=S-x$

$$I\!E\left(\int_L^{\mathbf{t}_1+L} f(S-x-\mathbf{D}(t))dt\right)=\mathbf{C}(S-x)I\!E\mathbf{t}_1-K.$$

Thus we can express the average cost of an (s,S) model in terms of the cost expression of an $(S-1,S)$ model. In conclusion, the average cost of the (s,S) model is given by the formula

$$\mathbf{C}(s,S-s)=\int_0^{S-s}\mathbf{C}(S-x)\frac{U_0(dx)}{U_0(S-s)}-K\frac{I\!EF_Y(S-s-\mathbf{V})}{I\!E\mathbf{t}_1}. \qquad (2.2.2.6)$$

This form simplifies the evaluation of the cost expression.

2.2.3 The (s,nQ) model

According to this inventory rule an order is triggered at the moment the inventory position (after demand and just before placing an order) drops below or equals the reorder level s. The order size is chosen to be an integer multiple of Q, such that after ordering the inventory position process will be between s and $s+Q$. This model is also discussed by Richards (1975) and Chen and Zheng (1992). The invariant distribution of the Markov chain $\{\mathbf{IP}(\mathbf{t}_n): n\in I\!N\cup\{0\}\}$ is given by the uniform distribution on $\{s+1,\ldots,s+Q\}$, in case of discrete order size and on $(s,s+Q]$ in case of continuous

order size. The definition of an (s, nQ) policy yields the following inequality for the inventory position process:

$$s \leq \mathbf{IP}(t) \leq s + Q, \text{ for every } t \geq 0. \tag{2.2.2.7}$$

Suppose that $\mathbf{IP}(\mathbf{t}_n) = s + Q\mathbf{U}$, $n \in I\!N \cup \{0\}$, with $\mathbf{U}$ a uniformly distributed random variable on $[0, 1]$, independent of the demand process. Again by the definition of an (s, nQ) policy we obtain that

$$\mathbf{IP}(\mathbf{t}_{n+1}) = s + Q\mathbf{U} - \mathbf{Y}_n + kQ, \; k \in I\!N \cup \{0\}$$

and k is chosen such that $\mathbf{IP}(\mathbf{t}_{n+1})$ satisfies relation (2.2.2.7), i.e.

$$0 \leq Q\mathbf{U} - \mathbf{Y}_n + kQ \leq Q.$$

We are interested now in the distribution of the inventory position process in the point $\mathbf{t}_{n+1}$:

$$I\!P\{0 \leq \mathbf{IP}(\mathbf{t}_{n+1}) < x\} = \sum_{k=0}^{\infty} I\!P\{kQ - x < \mathbf{Y}_1 - Q\mathbf{U} \leq kQ\}.$$

Conditioning on the uniformly distributed random variable $\mathbf{U}$ and taking the right hand derivative of the expression (which exists since F_Y is right continuous) we obtain that the probability distribution of $\mathbf{IP}(\mathbf{t}_{n+1})$ equals

$$\frac{1}{Q} \sum_{k=0}^{\infty} \int_{kQ-x}^{(k+1)Q-x} F_Y(dy) = \frac{1}{Q}.$$

From this result it follows immediately that there are no jumps, i.e. $\mathbf{IP}(\mathbf{t}_{n+1}) \stackrel{d}{=} s + Q\mathbf{U}$, therefore, reasoning in the same manner as for the (s, S) model, the Markov chain $\mathbf{IP}(\mathbf{t}_n)$ has an invariant distribution which equals the invariant distribution. After obtaining this result it is immediately clear that the stochastic process $\{\mathbf{IN}(t + L) : t \geq 0\}$ is regenerative. As for the long run average cost one has

$$\mathbf{C}(s, Q) = \frac{K(1 - I\!E(F_Y(Q\mathbf{U}))) + I\!E\left(\int_0^{\mathbf{t}_1} f(s + Q\mathbf{U} - \mathbf{D}(y + L))dy\right)}{I\!E\mathbf{t}_1}.$$

One can immediately observe that the expression in the numerator

$$I\!E\left(\int_0^{\mathbf{t}_1} f(s + Qu - \mathbf{D}(y + L))dy\right)$$

can be expressed again in terms of the cost function of an $(S - 1, S)$ model with $S := s + Qu$, hence the previous expression equals

$$\mathbf{C}(s + Qu)I\!E\mathbf{t}_1 - K,$$

thus the average cost generated by applying an (s, nQ) policy equals

$$\mathbf{C}(s,Q) = \frac{K\mathbb{E}(F_Y(Q\mathbf{U}))}{\mathbb{E}\mathbf{t}_1} + \int_0^1 \mathbf{C}(s+Qu)du. \qquad (2.2.2.8)$$

Thus, the real difference between this and Sahin's approach is the most apparent for the (s, S) policy. However, the regenerative processes technique still lacks the flexibility of Zipkin's model (Zipkin (1986)) and that of Bázsa and Den Iseger (2001).

2.3 Zipkin's Model: Constant and Stochastic Leadtimes

As the title suggests (Zipkin (1986)), originally this article's only intention was to develop a model, a mechanism of order arrival, which assures the validity of the asymptotic results (2.1.2.1) – (2.1.2.3) under a stochastic leadtimes assumption. However, the "preliminary" results for the constant leadtimes case are at least as important, providing an entirely new perspective for the analysis of inventory models. Let us therefore discuss the findings of Zipkin separately for the constant and stochastic leadtimes cases. The preliminary assumptions are that

(a) individual demands $\mathbf{Y}_n, n \in \mathbb{N}$ are nonnegative, i.i.d. and independent of the stochastic counting process $\mathbf{N}$

(b) the initial inventory position $\mathbf{IP}(0)$ is chosen independently of the demand process; orders are placed according to a stationary policy Γ, depending only on the inventory position

(c) the chain $\mathbf{IP}_n$ ($\mathbf{IP}_n := \mathbf{IP}(\mathbf{t}_n)$) is irreducible and positive recurrent,

therefore it has a unique stationary density π.

2.3.1 Constant Leadtimes

It is first assumed that $\mathbf{IP}(0) \stackrel{d}{=} \pi$ and that the stochastic counting process $\mathbf{N}_\Delta(t)$ has stationary increments. This coincides with the case discussed in the previous section, except that here the policy is not specified exactly. Instead it covers any policy agreeing with condition (b). As for the calculation of performance measures (or performance criteria, such as expected average inventory) the employment of the weak ergodic theorem for stationary processes is suggested (Heyman and Sobel (1982)). In this case, in our opinion, the approach of the previous section works better.

The second level is to assume that although $\mathbf{N}_\Delta$ is not stationary, it is converging to one in limit, that is, for every $t_2 > t_1 \geq 0$,

(d) $$\lim_{t\uparrow\infty} I\!P\{\mathbf{N}(t_2+t)-\mathbf{N}(t_1+t) = k\} = I\!P\{\mathbf{N}_e(t_2-t_1) = k\},$$

where $\mathbf{N}_e$ is stationary. At this point one needs to distinguish between periodic and aperiodic chains $\{\mathbf{IP}_n\}$. In case of aperiodicity the proof of the stationary results and asymptotic independency of the inventory position and leadtime demand is relatively simple, it is a direct estimation of sup norms. In case of periodic chains $\{\mathbf{IP}_n\}$ assumption (d) is not sufficient anymore. Therefore, it is replaced by a kind of 'indirect' regeneration property, namely

(d') there exists a (possibly delayed) regenerative process $\mathbf{A}$, such that $\{\mathbf{N}(t+s) - N(t)\}$ depend on the past only through $\mathbf{A}(t)$ for all $s \geq 0$, and the conditional increments given $\mathbf{A}(t)$ are stationary.

However, this condition is still not sufficient to deduce the asymptotic results. Zipkin remarks that a regeneration point of $\mathbf{A}$ need not occur at a point of $\mathbf{A}$! This could be a serious drawback of this model, since the task of calculating costs is made considerably harder by a different cycle length than those defined by arrivals. It is actually shown in Chapter 4 that for most cases one *can* find a process $\mathbf{A}$, satisfying condition (d'), for which the regeneration points do coincide with the points of $\mathbf{N}$, hence costs can be expressed in terms of arrival- cycles. Zipkin proceeds in the following way: Denoting now with σ_m the regeneration points of $\mathbf{A}$, it follows that $\mathbf{IP}'_m := \mathbf{IP}(\sigma_m)$ is a positive recurrent Markov chain (because $\mathbf{IP}_n$ is positive recurrent), but it need not be irreducible. This yields that the additional assumption

(e) the Markov chain $\{\mathbf{IP}'_m\}$ is irreducible

is needed. This condition is equivalent to the condition that the joint chain $\{\mathbf{IP}_n, \mathbf{A}_n\}$ is irreducible, with $\mathbf{A}_n = \mathbf{A}(\mathbf{t}_n)$, as observed by Zipkin. He also deduces that condition (e) will hold if one assumes that the semi Markov process $\mathbf{A}(t)$ is aperiodic, which in turn holds for a wide class of demand processes. Under these conditions the desired asymptotic results hold.

Condition (d) can be further relaxed by taking a Cesaro limit instead of the weak limit, while maintaining the validity of the asymptotic results and extending the class of plausible demand processes, as it is shown in the next section. Before introducing the order-arrival mechanism of Zipkin (1986), we ought to mention that there is an interesting remark made at the end of this section:

" One more result may be worth nothing: Suppose $\mathbf{N}(t)$ is the superposition of several independent processes satisfying (d), and at least one of them satisfies (d').

Then, it is not hard to show that the asymptotic results are valid. (Such processes arise in multi echelon models, as in Deuermeyer and Schwarz (1981). This result was pointed out to me by Ward Whitt.)"

Just how valuable this result really is, is shown in Chapter 5 of this thesis.

2.3.2 Stochastic leadtimes

When dealing with stochastic leadtimes Zipkin (Zipkin (1986)) follows the approach of Kaplan (1970) by focusing on the arrival mechanism of the orders rather than on a direct specification of how leadtimes are generated when orders are placed. Although Zipkin establishes a certain equivalence between his own model and that of Kaplan (later improved by Nahmias (1979) and Ehrhardt (1981)), it would be farfetched to claim that either of them had envisioned the same model as Zipkin. Zipkin's arrival mechanism of orders is based on a real-valued, stationary, ergodic stochastic process $\{\mathbf{U}(t) : t \in I\!R\}$ (see Chapter 4 for the precise definitions) satisfying the assumptions

(i) $\mathbf{U}(t) \geq 0$ and $I\!E\mathbf{U}(t) < \infty$;

(ii) $t - \mathbf{U}(t)$ is nondecreasing, such that $t - \mathbf{U}(t) \to \infty$ a.s. as $t \to \infty$;

(iii) sample paths of $\mathbf{U}(t)$ are continuous to the right;

(iv) $\mathbf{U}$ is independent of the placement and size of orders and of the demand process.

Now, if an order is placed at time u, then it arrives at time $v = \min\{t : t - \mathbf{U}(t) \geq u\}$. Intuitively, "...whenever $t - \mathbf{U}(t)$ equals (or jumps over) the value u, a time when an order was placed, the process forces the order to arrive at time t. In particular, any order placed at or before time $t - \mathbf{U}(t)$ arrives by t, while all placed after $t - \mathbf{U}(t)$ arrive after t, by (ii)" (Zipkin (1986)). Thus, the no order crossing condition is satisfied: if $u_1 \leq u_2$ then $t - \mathbf{U}(t) \geq u_2$ implies $t - \mathbf{U}(t) \geq u_1$. Moreover, the marginal distribution of the leadtime is given by the distribution of $\mathbf{U}$.

In order to prove the asymptotic results, the stationary and limiting cases are handled separately again. As in the previous section let us first assume that $\mathbf{IP}(0) \stackrel{d}{=} \pi$ and $\mathbf{N}_\Delta$ is stationary, and assume also that $\{\mathbf{IP}(t), \mathbf{D}(t, t+L]\}$ can be extended to be a stationary process over t, for all $L \geq 0$. The idea is to write the flow conservation relation in terms of $\mathbf{U}$, that is, $\mathbf{IN}(t) = \mathbf{IP}(t - \mathbf{U}(t)) - \mathbf{D}(t - \mathbf{U}(t), \mathbf{U}(t)]$. Conditioning consequently on $\mathbf{U}$, and taking the expectation with respect to $\mathbf{U}$, the desired asymptotic results follow by the independency and stationary assumptions. In the case when (d) or (d') and (e) are assumed the following equalities hold (Zipkin

(1986)):

$$\begin{aligned}
\lim_{t\uparrow\infty} \quad & I\!P\{\mathbf{IP}(t-\mathbf{U}(t)) \le x,\ \mathbf{D}(t-\mathbf{U}(t),\mathbf{U}(t)] \le y]\} \\
= \quad & \lim_{t\uparrow\infty} \int_0^\infty I\!P\{\mathbf{IP}(t-u) \le x,\ \mathbf{D}(t-u,u] \le y\} F_U(du) \quad \text{(by (iv))} \\
= \quad & \int_0^\infty \lim_{t\uparrow\infty} I\!P\{\mathbf{IP}(t-u) \le x,\ \mathbf{D}(t-u,u] \le y\} F_U(du) \\
= \quad & \int_0^\infty I\!P\{\mathbf{IP}_\infty \le x,\ \mathbf{D}_\infty(u) \le y\} F_U(du) = I\!P\{\mathbf{IP}_\infty \le x\} \int_0^\infty \mathbf{D}_\infty(u) F_U(du),
\end{aligned}$$

$\mathbf{D}_\infty$ has the distribution obtain by mixing those of the $\mathbf{D}_\infty(\mathbf{U})$ over $\mathbf{U}$.

The next section shows that relaxing condition (d) leads to a flexible model which can handle stochastic and constant leadtimes jointly *as long as* no order crossing in time is assumed. Moreover, this order arrival mechanism of Zipkin can be deduced with a time transformation from the nonstationary demand case (see Section 2.6).

2.4 A Time- and Event- Averages Approach

A recent work (Bázsa and Den Iseger (2001)) proves that statements (2.1.2.1) – (2.1.2.3) are valid in case of more general demand processes or policies. This extension can be made more apparent by concentrating on the differences in the assumptions needed for the main results, compared to those made by Zipkin (1986). That is, instead of requiring the more restrictive condition (d), we assume the following: the *limiting distribution in the Cesaro sense* of the joint process $(\mathbf{IP}, \mathbf{N}_\Delta)$ exists. Since the Cesaro sense limiting distribution is a long run average, it is clearly less restrictive than the limit assumption. Furthermore, since we also prove asymptotic independence of the processes $\mathbf{IP}$ and $\mathbf{D}_L$, or equivalently, $\mathbf{IP}$ and $\mathbf{N}_L$, in order to be able to 'take apart' the joint limiting distribution in the Cesaro sense of $\mathbf{IP}$ and $\mathbf{N}_\Delta$, we will consider the following cases:
(i) the limiting distribution in the Cesaro sense of $\mathbf{N}_\Delta(t)$ AND the limiting distribution of $\mathbf{IP}(t)$ exist, or
(ii) the limiting distribution of $\mathbf{N}_\Delta(t)$ AND the limiting distribution in the Cesaro sense of $\mathbf{IP}(t)$ exists.

Under case (i) we can list models such as demand modeled by a nonhomogeneous compound Poisson process, i.e., the limit for the corresponding counting process $\mathbf{N}_\Delta$ does not exist, while the Cesaro limit does. Under assumption (ii) models such as an (s, S) policy with unit demand can be analyzed: the weak limit of the $\mathbf{IP}$ process does not exist while the limiting distribution in the Cesaro sense does. Furthermore, the asymptotic independence of the inventory position process and the leadtime demand

is exploited to provide a unified treatment for service measures. All the results of the present paper hold true in the case of stochastic leadtimes, provided that orders do not cross in time, as it is explained in detail in Section 2.6.1.

Let us thus assume in the rest of the chapter conditions (a) – (c) and the Cesaro variant of condition (d).

2.4.1 Cesaro sense limiting distribution

For the stochastic counting process $\mathbf{N}_{\Delta}(t)$ or the inventory position process **IP**, instead of the existence of a limiting distribution we only assume the existence of the *time* or *event stationary distributions*. These distributions are defined as follows (Sigman (1995)).

Definition 2.4.1 *Consider a compound point process* $\mathbf{X} = \{\mathbf{X}(t) : t \geq 0\}$ *and the sequence of events* $\{\mathbf{s}_n, n \in I\!N\}$ *related to* $\mathbf{X}$. *The distribution*

$$F^c_{\infty}(x) = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t F_{\mathbf{X}(\mathbf{s})}(x)ds \tag{2.4.2.1}$$

is called the time stationary distribution for $\mathbf{X}$ *(Sigman (1995)). The distribution given by*

$$F^e_{\infty}(x) = \lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=1}^{n} F_{\mathbf{X}(\mathbf{s}_k)}(x) \tag{2.4.2.2}$$

is defined as the event stationary distribution for $\mathbf{X}$ *(Sigman (1995)).*

Since the time and event stationary distributions are defined by Cesaro limits (time and resp. event averages!), their existence is a much less restrictive condition than that of a limiting distribution. This is the reason for referring to these distributions suggestively, under a common phrase, as limiting distributions in Cesaro sense. A good example for the generality of these distributions is the time-nonhomogeneous compound Poisson process: its limit in distribution does not exist, while both the time and event stationary distributions exist. Furthermore, related to the Cesaro sense limiting distributions, we define a concept called "asymptotic independence in the Cesaro sense", that is,

Definition 2.4.2 *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are two stochastic processes such that* $\mathbf{X}$ *has a limiting distribution and the time- stationary distribution for* $\mathbf{Y}$ *exists, then they are asymptoticly independent if and only if*

$$\lim_{T\uparrow\infty} \frac{1}{T} \int_0^T I\!P\{\mathbf{X}(t) \leq x, \mathbf{Y}(t) \leq y\}dt = I\!P\{\mathbf{X}_{\infty} \leq x\} I\!P\{\mathbf{Y}^c_{\infty} \leq y\}, \tag{2.4.2.3}$$

where $\mathbf{X}_\infty$ *is a random variable distributed with the limiting distribution of* $\mathbf{X}$*, and* $\mathbf{Y}^c_\infty$ *is a random variable distributed with the time- stationary distribution of* $\mathbf{Y}$*.*

Observe that this definition can be easily adjusted in case of discrete stochastic processes and an event- stationary distribution.

2.4.2 The inventory position process

The definitions of this general control system imply for the inventory position process in the epochs of customers' arrival that $\mathbf{IP}(\mathbf{t}_n)$ only depends on the previous state $\mathbf{IP}(\mathbf{t}_{n-1})$, the individual demand of the nth customer $\mathbf{Y}_n$, and the magnitude of the replenishment order $\mathbf{Z}_n$, if there was any order placed at $\mathbf{t}_n$. Since the individual demands $\mathbf{Y}_n$ are independent and identically distributed, and $\mathbf{Z}_n$ only depends on $\mathbf{IP}(\mathbf{t}_{n-1})$ and $\mathbf{Y}_n$, $\{\mathbf{IP}(\mathbf{t}_n) : n \in I\!N\}$ is a Markov chain. If the chain $\{\mathbf{IP}(\mathbf{t}_n) : n \in I\!N \cup \{0\}\}$ has a unique limiting distribution (see Ross(1970), Thorisson (2000)), then it is given by

$$\pi := \lim_{n\uparrow\infty} I\!P\{\mathbf{IP}(\mathbf{t}_n) \le x\} = I\!P\{\mathbf{IP}_\infty \le x\}, \tag{2.4.2.4}$$

where $\mathbf{IP}_\infty$ is a random variable distributed with the limiting distribution of the Markov chain $\{\mathbf{IP}(\mathbf{t}_n),\ n \in I\!N\}$. Otherwise, if only the limiting distribution in the Cesaro sense exists, that is, the event stationary distribution for $\mathbf{IP}_n$, then it is denoted by

$$\pi^c := \lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=1}^{n} I\!P\{\mathbf{IP}(\mathbf{t}_k) \le x\} = I\!P\{\mathbf{IP}^c_\infty \le x\}. \tag{2.4.2.5}$$

The step function structure of the sample paths of the inventory position implies that

$$\mathbf{IP}(t) = \mathbf{IP}(\mathbf{t}_{\mathbf{N(t)}}), \text{ for all } t \ge 0. \tag{2.4.2.6}$$

The asymptotic independence of $\mathbf{IP}$ and $\mathbf{N}_L$ is justified by the following theorem, for any inventory system where the control rule solely depends on the inventory position:

Theorem 2.4.3 *Assuming that the Markov chain* $\{\mathbf{IP}(\mathbf{t}_n) : n \in I\!N\}$ *is ergodic, and the time- stationary distribution for the stochastic counting process* $\mathbf{N}_L$ *exists, while* $\mathbf{N}_L(t) \longrightarrow \infty$ *a.s. as* $t \longrightarrow \infty$*, the inventory position process* $\mathbf{IP}(t)$ *and the leadtime demand* $\mathbf{D}(t, t+L]$ *are asymptoticly independent. Moreover,*

$$\lim_{t\uparrow\infty} I\!P\{\mathbf{IP}(t) \le x\} = I\!P\{\mathbf{IP}_\infty \le x\} = \pi, \tag{2.4.2.7}$$

for all $x \in \mathbb{R}$, where $\mathbf{IP}_\infty$ and π were defined by relation (2.4.2.4). Conversely, if the event- stationary distribution π^c for $\{\mathbf{IP}_n\}$ exists (defined by relation (2.4.2.5)), together with a limiting distribution of $\mathbf{N}_L$, with $\mathbf{N}(t) \longrightarrow \infty$ a.s. as $t \longrightarrow \infty$, the inventory position process $\mathbf{IP}(t)$ and the leadtime demand $\mathbf{D}(t, t+L]$ are asymptoticly independent.

Proof Let us introduce the notation

$$P_n(t) := I\!P\{\phi_t \mathbf{N}(0, L] = n\},$$

$n \in \mathbb{N}$, for the time- stationary distribution of $\phi_t \mathbf{N}(0, L]$

$$P_n(\infty) := I\!P\{\mathbf{N}^c_\infty(0, L] = n\},$$

and the joint event

$$\mathbf{A}_{n,n_0}(t) := (\phi_t \mathbf{N}(0, L] = n, \mathbf{N}(t) \geq n_0).$$

Proving the identity

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T I\!P\{\mathbf{IP}(t) \leq x, \phi_t \mathbf{N}(0, L] = n\} dt = \pi P_n(\infty) \tag{2.4.2.8}$$

is adequate to conclude all the statements of the theorem. The fact that **IP** has a limiting distribution means that for all $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$

$$| I\!P\{\mathbf{IP}(\mathbf{t}_n) \leq x\} - \pi(x)| < \varepsilon, \tag{2.4.2.9}$$

for all $x \in \mathbb{R}$. Let us now start with the expression under the limit in (2.4.2.8), that is,

$$I\!P\{\mathbf{IP}(t) \leq x, \phi_t \mathbf{N}(0, L] = n\}$$

equals the sum of probabilities

$$I\!P\{\mathbf{IP}(t) \leq x, \phi_t \mathbf{N}(0, L] = n, \mathbf{N}(t) < n_0\} + I\!P\{\mathbf{IP}(t) \leq x, \phi_t \mathbf{N}(0, L] = n, \mathbf{N}(t) \geq n_0\}.$$

Knowing that $\mathbf{N}(t) \longrightarrow \infty$ a.s. as $t \longrightarrow \infty$ yields

$$I\!P\{\mathbf{IP}(t) \leq x, \phi_t \mathbf{N}(0, L] = n, \mathbf{N}(t) < n_0\} \leq I\!P\{\mathbf{N}(t) < n_0\} \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

Relation (2.4.2.6) states that $\mathbf{IP}(t) = \mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})})$ for all $t \geq 0$. Suppose that $I\!P\{\phi_t \mathbf{N}(0, L] = n\} > 0$ (otherwise the result is obviously 0), hence

$$I\!P\{\mathbf{IP}(t) \leq x, \mathbf{A}_{n,n_0}(t)\}$$

equals

$$I\!P\{\mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})}) \leq x \mid \mathbf{A}_{n,n_0}(t)\} I\!P\{\mathbf{A}_{n,n_0}(t)\}.$$

Since in the above expression $\mathbf{N}(t) \geq n_0$, we know by (2.4.2.9) that

$$\mid I\!P\{\mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})}) \leq x\} - \pi(x)| < \varepsilon,$$

independently of $\mathbf{N}(t)$ and $\phi_t \mathbf{N}(0, L]$. That is, since $\lim_{t\uparrow\infty} I\!P\{\mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})}) \leq x\} = \lim_{k\uparrow\infty} I\!P\{\mathbf{IP}_k \leq x\}$, and the random variable $\mathbf{IP}_k$ is only determined by the random variables $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ and the size of the replenishment orders place at or before the kth arrival, which are in turn independent of $\mathbf{N}$. These arguments imply that

$$\mid I\!P\{\mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})}) \leq x \mid \mathbf{A}_{n,n_0}(t)\} - \pi(x)| < \varepsilon. \tag{2.4.2.10}$$

The assumption that $\mathbf{N}(t) \longrightarrow \infty$ a.s. as $t \longrightarrow \infty$ also implies that

$$\lim_{T\uparrow\infty} \frac{1}{T} \int_0^T I\!P\{\mathbf{A}_{n,n_0}(t)\} dt = P_n(\infty). \tag{2.4.2.11}$$

Finally we obtain that

$$\mid I\!P\{\mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})}) \leq x \mid \mathbf{A}_{n,n_0}(t)\} I\!P\{\mathbf{A}_{n,n_0}(t)\} - \pi(x) P_n(\infty)|$$

is less or equal than

$$\mid I\!P\{\mathbf{IP}(\mathbf{t}_{\mathbf{N}(\mathbf{t})}) \leq x \mid \mathbf{A}_{n,n_0}(t)\} - \pi(x)| \cdot I\!P\{\mathbf{A}_{n,n_0}(t)\} + (I\!P\{\mathbf{A}_{n,n_0}(t)\} - P_n(\infty)) \mid \pi(x)|.$$

Taking the Cesaro limit with respect to t of the above expression, and using intermediate the results (2.4.2.10) and (2.4.2.11), yields that the expression is bounded by ε. This completes the proof.

Remark 2.4.4 *In the discrete case the conclusions of the theorem remain valid, assuming that the event- stationary distribution for the stochastic counting process* $\mathbf{N}$ *exists.*

As an example we will give the (s, nQ) policy: the limiting distribution of $\mathbf{IP}$ was already deduced in the previous section, however here we give a different approach.

The (s, nQ) policy

In Section 2.2.3 it was deduced that limiting distribution of the Markov chain $\{\mathbf{IP}(\mathbf{t}_n) : n \in I\!N\}$ is given by the uniform distribution on $(s, s + Q]$, that is

$$\lim_{n\uparrow\infty} \mathbf{IP}(\mathbf{t}_n) = s + Q\mathbf{U}, \quad n \in I\!N, \tag{2.4.2.12}$$

with $\mathbf{U}$ a uniformly distributed random variable on $(0,1]$. It is possible though to generalize this case even further. Assume that the inventory position $\mathbf{IP}_n$ has the steady state distribution $s+Q\mathbf{U}_n$, with $\mathbf{U}_n$ uniformly distributed as before, and assume that the individual demand $\mathbf{Y}_i$ are not identically distributed anymore, perhaps not even independent. It follows now that $\mathbf{IP}_{n+1} \stackrel{d}{=} [Q\mathbf{U}_n + \mathbf{Y}_{n+1}] \bmod Q \stackrel{d}{=} Q\mathbf{U}_{n+1}$, and $\mathbf{IP}_{n+2} \stackrel{d}{=} Q\mathbf{U}_{n+2}$, hence the distribution of the inventory position process remains uniformly distributed on $(s, s+Q]$. Furthermore, $\mathbf{IP}_n$ is independent of $\{\mathbf{Y}_k : k = 1, \ldots, n\}$.

Furthermore, the time- and event- averages approach also extends the combination of policy/demand processes: for instance it can handle an (s,S) policy with unit demand, such that the stochastic counting process $\mathbf{N}_\Delta$ converges in limit to a stationary one. This case is the combination of a periodic chain $\mathbf{IP}_n$ and condition (d) from Section 2.3. Zipkin's model can only handle the combination of periodic chain and a demand process with a regenerative structure, that is, under assumptions (d) and (e).

The asymptotic independence results obtained can be best exploited by deriving efficient performance measures. The aim is to obtain any significant performance measure without using much algebra. Such a unified treatment is explained in the next section.

2.5 Performance measures

2.5.1 The cost structure

In general, the cost of an inventory control system (most commonly: long run average cost) is associated with the net inventory process. It is well-known that the sample paths of the net inventory process form a step function, with two types of jumps: (downwards) jumps occurring due to the arrival of customers, called type I jumps, and (upwards) jumps caused by the arrival of a replenishment order, called type II jumps. We associate three kinds of costs with the netstock process. The first type of cost is incurred between events, that is, between jumps. The second and third kinds of costs are associated with the type I and type II jumps, respectively, as follows:

first When $\mathbf{IN}(t) = \mathbf{IN}(\mathbf{J}_n) = x$ a.s. for $\mathbf{J}_n \leq t < \mathbf{J}_{n+1}$, where $x \in I\!R$ is a constant and $\mathbf{J}_n, n \in I\!N$ are the points of time when a jump occurs, then one can introduce a cost-rate function $f(x)$ related to this event. This cost will give us a very important characteristic, the average holding cost (and penalty cost), therefore we refer to this type of cost in the remainder of the chapter as the *average holding cost*.

second Similarly, we introduce a cost-rate function g_1 related to the type I jumps of the sample paths of the netstock process, that is, the cost of the jump in time point $\mathbf{t}_n$ is given by $g_1(\mathbf{IN}(\mathbf{t}_n^-), \mathbf{Y}_n)$. This type of "cost" usually provides us with service measures, since it is related to the arrival of customers. Therefore we refer to the cost of the type I jumps as *service measures*. Observe, that by altering the cost-rate function g_1, we obtain any specific service measure one needs. Later we also show that this cost-rate function is most of the time given by a simple algebraic expression.

third Introduce also a function G_2, related to the type II jumps, that is, the *cost of the control policy*: for a replenishment order placed at time point $\mathbf{t}_n$ it is given by $G_2(\mathbf{Z}_n)$. By the definition of $\mathbf{Z}_n$, $\mathbf{Z}_n = h(\mathbf{IP}(\mathbf{t}_n^-) - \mathbf{Y}_n)$, where h is a function dependent on the control rule, the cost of the control rule is given by $g_2(\mathbf{IP}(\mathbf{t}_n^-) - \mathbf{Y}_n)$, with $g_2 = G_2 \circ h$.

Before starting with the actual computation of these costs, we discuss some properties related to the expected long run average cost associated with a stochastic process. The average cost associated with a positive function l (or if l is a function with bounded variation) and a stochastic process $\mathbf{X}$ is given by

$$\lim_{t\uparrow\infty} I\!E\left(\frac{1}{t}\int_0^t l(\mathbf{X}(s))ds\right). \tag{2.5.2.1}$$

Assume now that the time-stationary distribution for the stochastic process $\mathbf{X}$ exists, and $\mathbf{X}_\infty^c$ denotes a random variable distributed with this time- stationary distribution of the process $\mathbf{X}$. Let

$$\mathbf{C}_t := \frac{1}{t}\int_0^t l(\mathbf{X}(s))ds,$$

then $\mathbf{C}_t \Rightarrow \mathbf{C}_\infty$, that is, the corresponding probability measures $\mu_t \Rightarrow \mu$. Applying theorem 25.7 of Billingsley yields that for any measurable mapping $l : I\!R^1 \to I\!R^1$ (with measurable set of discontinuities) $\mu_t l^{-1} \Rightarrow \mu l^{-1}$. Now, by Skorohod's theorem (Theorem 25.7 of Billingsley), there exist random variables $\mathbf{Z}_t$ and $\mathbf{Z}$ on a common probability space, such that $\mathbf{Z}_t$ has distribution μ_t and $\mathbf{Z}$ has distribution μ, and $\mathbf{Z}_t \to \mathbf{Z}$ a.s. Furthermore, since $\mathbf{Z}_t$ and $\mathbf{C}_t$ have the same distribution, it follows that if $\mathbf{C}_t$ is uniformly integrable then so is $\mathbf{Z}_t$ and vice versa. Now, in order to obtain L^1 convergence is to use Theorem 13.7 of Williams (1991), and Scheffé's Lemma (Williams (1991)) establishing a sufficient and necessary condition for L^1

convergence:

$$\begin{gathered}\lim_{t\uparrow\infty} I\!E\left(\frac{1}{t}\int_0^t l(\mathbf{X}(s))ds\right) = I\!E(l(\mathbf{X}^c_\infty)) \\ \text{if and only if} \\ \left\{\frac{1}{t}\int_0^t l(\mathbf{X}(s))ds : t \geq> 0\right\} \text{ is uniform integrable.}\end{gathered} \tag{2.5.2.2}$$

One can also define costs as event-averages, in the following way. The costs related to the jumps (type I or type II) are associated with events, hence we define an event-average cost in the following manner. The event-average cost related to the series of events $\{\mathbf{s}_n : n \in I\!N \cup \{0\}\}$ associated with a stochastic process $\mathbf{X}$ and a positive cost-rate function l is given by

$$\lim_{n\uparrow\infty} I\!E\left(\frac{1}{n}\sum_{j=1}^n l(\mathbf{X}(\mathbf{s}_j))\right). \tag{2.5.2.3}$$

Similarly as relation (2.5.2.2) for the continuous case, we obtain that

$$\begin{gathered}\lim_{n\uparrow\infty} I\!E\left(\frac{1}{n}\sum_{j=1}^n l(\mathbf{X}(\mathbf{s}_j))\right) = I\!E\left(l(\mathbf{X}^e_\infty)\right) \\ \text{if and only if} \\ \left\{\frac{1}{n}\sum_{j=1}^n l(\mathbf{X}(\mathbf{s}_j)) : n \in I\!N\right\} \text{ is uniform integrable,}\end{gathered} \tag{2.5.2.4}$$

where $\mathbf{X}^e_\infty$ is a random variable distributed with the event stationary distribution F^e_∞ defined by (2.4.2.2). Let us summarize relations (2.5.2.1) - (2.5.2.4) in the following theorem.

Theorem 2.5.1 *Assuming that the time and event stationary distributions, F^c_∞ resp. F^e_∞, for the stochastic process $\mathbf{X}$ exist, then*

$$\lim_{t\uparrow\infty} I\!E\left(\frac{1}{t}\int_0^t l(\mathbf{X}(s))\right) ds = I\!E\left(l(\mathbf{X}^c_\infty)\right), \tag{2.5.2.5}$$

if and only if $\left\{(1/t)\int_0^t l(\mathbf{X}(s))ds : t > 0\right\}$ is uniform integrable; and

$$\lim_{n\uparrow\infty} I\!E\left(\frac{1}{n}\sum_{j=1}^n l(\mathbf{X}(\mathbf{s}_j))\right) = I\!E\left(l(\mathbf{X}^e_\infty)\right), \tag{2.5.2.6}$$

if and only if $\left\{(1/n)\sum_{j=1}^n l(\mathbf{X}(\mathbf{s}_j)) : n \in I\!N\right\}$ is uniform integrable. Expressions (2.5.2.5) and (2.5.2.6) are the **time-, respectively event-average costs** *related to the process $\mathbf{X}$ and the cost-rate function l. Moreover, if $\mathbf{N}(t)/t \longrightarrow \lambda$ a.s. as $t \longrightarrow \infty$ then*

$$\lim_{t\uparrow\infty} I\!E\left(\frac{1}{t}\sum_{j=1}^{\mathbf{N}(t)} l(\mathbf{X}(\mathbf{s}_j))\right) = \lambda I\!E\left(l(\mathbf{X}^e_\infty)\right), \tag{2.5.2.7}$$

and $\left\{(1/t)\sum_{j=1}^{\mathbf{N}(t)} l(\mathbf{X}(\mathbf{s}_j)) : t>0\right\}$ *uniform integrable.*

Obviously, if the limiting distribution of the stochastic process $\mathbf{X}$ exists then it coincides with the distributions defined by relations (2.4.2.1) and (2.4.2.2). The right hand side of relation (2.5.2.7) is the time-average version of the cost defined on a set of events.

Assumption 2.5.2 *For the case of the inventory systems considered, we assume for the rest of the paper that* $\left\{(1/t)\int_0^t l(\phi_L \mathbf{IN}(s))ds : t>0\right\}$ *is uniform integrable and / or* $\left\{(1/n)\sum_{j=1}^{n} l(\phi_L \mathbf{IN}(\mathbf{s}_j)) : n\in I\!N\right\}$ *is uniform integrable.*

2.5.2 Average holding cost

Since in this case we are interested in long run time-average costs we aim to compute the expression

$$\lim_{t\uparrow\infty}\frac{1}{t}\int_0^t I\!Ef(\phi_L \mathbf{IN}(s))ds. \tag{2.5.2.8}$$

The flow conservation relation gives us a powerful tool to compute the average cost. By the definition of the demand process (1.0.1.1) the average cost equals

$$\lim_{t\uparrow\infty}\frac{1}{t}\int_0^t I\!Ef\left(\mathbf{IP}(s)-\sum_{k=1}^{\phi_s \mathbf{N}(0,L]}\mathbf{Y}_k\right)ds. \tag{2.5.2.9}$$

Proposition 2.5.3 *The average holding cost defined by relation (2.5.2.9) equals*

$$I\!E_{\mathbf{IP}_\infty}\left(\left(f * F_{D_\infty(0,L]}\right)(\mathbf{IP}_\infty)\right), \tag{2.5.2.10}$$

where $\mathbf{D}_\infty(0,L] := \sum_{k=1}^{\mathbf{N}^c_\infty(0,L]}\mathbf{Y}_k$.

Proof As deduced in Theorem 2.4.3, $\mathbf{IP}(t)$ and $\mathbf{D}(t,t+L]$ are asymptoticly independent. Since $\mathbf{IP}_\infty, \mathbf{N}^c_\infty(0,L]$ and $\mathbf{Y}_k$ are pair by pair independent, the statement of the proposition follows immediately.

Observe that the expression for the demand process can be written in the form

$$I\!P\{\sum_{k=1}^{\mathbf{N}^c_\infty(0,L]}\mathbf{Y}_k\le x\} = \sum_{k=0}^{\infty} I\!P\{\mathbf{N}^c_\infty(0,L]=k\}F_Y^{k*}(x),$$

and taking the Laplace- Stieltjes transform of this we obtain

$$LS_{F_{D_\infty}}(\alpha) = \sum_{k=0}^{\infty} I\!P\{\mathbf{N}^c_\infty(0,L]=k\}LS^k_{F_Y}(\alpha) = P_{\mathbf{N}^c_\infty(0,\mathrm{L}]}(LS_{F_Y}(\alpha)),$$

where $P_{\mathbf{N}^c_\infty(0,\mathtt{L}]}(\cdot)$ denotes the z-transform of $\mathbf{N}^c_\infty(0,L]$. In conclusion, if we can determine $P_{\mathbf{N}^c_\infty(0,\mathtt{L}]}$ then with the previously mentioned Laplace transform inversion algorithm we obtain a piece-wise polynomial approximation for $f * F_{D_\infty(0,L]}$, say $P_{f*F_{D_\infty(0,L]}}$. We are now able to approximate equation (2.5.2.10) by

$$I\!E_{\mathbf{IP}_\infty}\left(P_{f*F_{D_\infty(0,L]}}(\mathbf{IP}_\infty)\right), \tag{2.5.2.11}$$

obtaining a result which is almost up to machine precision.

2.5.3 Service measures

The long run event-average cost, as given by relation (2.5.2.3), of the (type I) jumps associated with the cost-rate function g_1 is of the form

$$\lim_{n\uparrow\infty}\frac{1}{n}\sum_{j=1}^{n} I\!E\left(g_1(\mathbf{IN}(\mathbf{t}_j^-),\mathbf{Y}_j)\right). \tag{2.5.2.12}$$

Furthermore, by the definition of the demand process it is obvious that $\mathbf{IN}(\mathbf{t}_n^-)$ and $\mathbf{Y}_n$ are independent for any $n \in I\!N \cup \{0\}$, and $\mathbf{Y}_n, n \in I\!N \cup \{0\}$ are identically distributed. Using now the flow conservation relation, the above average cost expression equals

$$\lim_{n\uparrow\infty}\frac{1}{n}\sum_{j=1}^{n} I\!E g_1\left(\mathbf{IP}(\mathbf{t}_j^- - L) - \sum_{k=1}^{\phi_{\mathbf{t}_j}\mathbf{N}(-L,0]}\mathbf{Y}_k, \mathbf{Y}_\infty\right),$$

where the notation $\mathbf{Y}_\infty$ stands for a random variable distributed as $\mathbf{Y}_1$. As deduced in section 2.4.2, $\mathbf{IP}(t)$ has a limit in distribution, and by the assumptions the event stationary distribution for $\phi_t\mathbf{N}(0,L]$ exists. We obtain thus by relation (2.5.2.4) the following proposition.

Proposition 2.5.4 *The event-average cost of the type I jumps equals*

$$SM_{event} := I\!E g_1\left(\mathbf{IP}_\infty - \sum_{k=1}^{\mathbf{N}^e_\infty(0,L]}\mathbf{Y}_k, \mathbf{Y}_\infty\right) = I\!E_{\mathtt{IP},\mathtt{Y}_\infty}\left(g_1(\cdot,\mathbf{Y}_\infty)\star F_{\mathtt{D^e}(0,\mathtt{L}]}(\mathbf{IP}_\infty)\right), \tag{2.5.2.13}$$

where $\mathbf{N}^e_\infty(0,L]$ is a random variable distributed with the event stationary distribution for $\mathbf{N}$, given by

$$\lim_{n\uparrow\infty}\frac{1}{n}\sum_{j=1}^{n} I\!P\{\phi_{\mathbf{t}_j}\mathbf{N}(-L,0] = k\}. \tag{2.5.2.14}$$

Proof: The statement is a direct consequence of Theorem 2.4.3 and Remark 2.4.4.

Observe that by Theorem 2.5.1 the event-average cost (2.5.2.13) is easily convertible to a time-average cost expression. That is, if

$$\lim_{t\uparrow\infty} \frac{\mathbf{N}(t)}{t} = \lambda \text{ a.s.},$$

then the long run time-average cost of the jumps, SM_{time}, is given as in relation (2.5.2.7) by

$$SM_{\text{time}} = \lambda \cdot SM_{\text{event}}. \tag{2.5.2.15}$$

An intuitive example for the cost of the type I jumps would be **the expected number of items short up to time** t, which is one of the most frequently used service measures in the literature. In this case the function g_1 related to the jumps is given by

$$g_1(X,Y) := (Y-X)^+ - (-X)^+, \tag{2.5.2.16}$$

where X is the level from where the jump occurs and Y is the size of the jump. Obviously, $X := \mathbf{IN}(\mathbf{t}_k^-)$ and $Y := \mathbf{Y}_k$. **Figure** 2.1 provides some intuition for the definition of the function g_1 in this case.

2.5.4 The cost of the control rule

As we discussed at the beginning of Section 2.5, the type II jumps are related to the inventory position process. These jumps in the sample paths of the inventory position process occur due to placement of replenishment orders. This implies a suggestive name for this type of cost: *the cost of the control rule*. Thus, as in section 2.5.3 we can derive the following proposition.

Proposition 2.5.5 *The time-average cost of the type II jumps equals*

$$\lambda I\!\!E g_2(\mathbf{IP}_\infty - \mathbf{Y}_\infty). \tag{2.5.2.17}$$

The most obvious example for the cost of the control rule is the setup cost. In this case the cost-rate function is given by

$$g_2(A) = K1_{\{A\le s\}},$$

where K and s are given parameters.

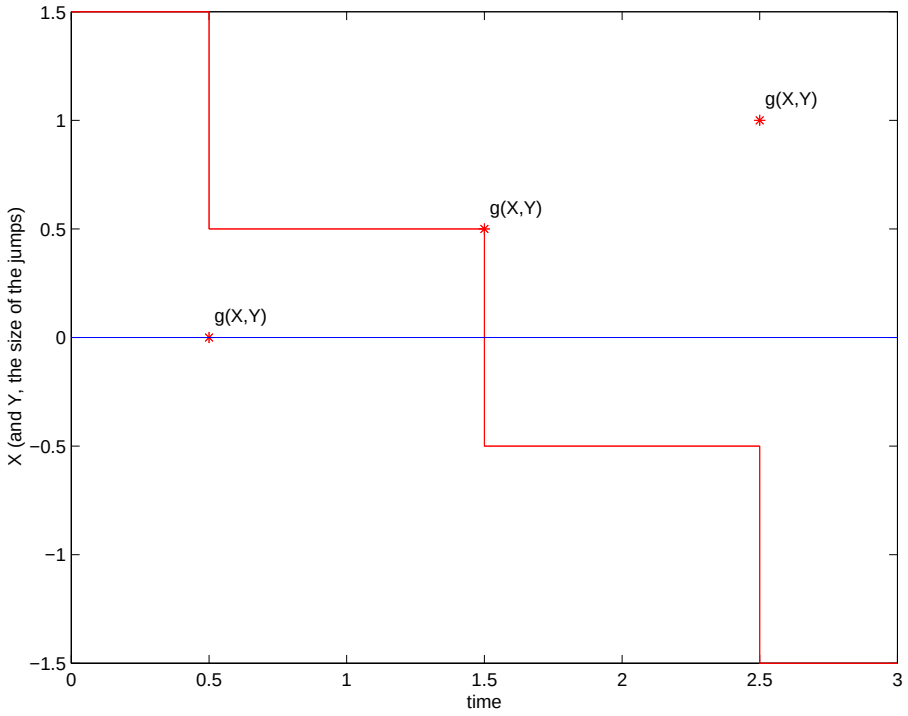

Figure 2.1: The g_1 cost-rate function, related to the type I jumps

2.6 Stochastic Leadtimes, Nonstationary Demand, and Limiting Distributions in the Cesaro Sense

2.6.1 Establishing equivalent models

Let us assume for the moment that the leadtime is fixed $L > 0$, and the arrival rate of the demand process is nonhomogeneous, but known, given by the function $\lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. As it was described earlier, we are interested in the limiting distribution of the leadtime demand, that is,

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t I\!P\{\mathbf{D}(s, s+L] \leq x\} ds,$$

which can be interpreted as $\phi_{\mathbf{U}t} I\!P\{\mathbf{D}(s, s+L] \leq x\}$, with $\mathbf{U}$ a uniformly distributed random variable on $[0, t]$. This, according to relation (2.5.2.5) is in fact

the time stationary version of the leadtime demand, which can be further written as $I\!P\{\mathbf{D}(\mathbf{U}T, \mathbf{U}T + L] \leq x\}$, $T > 0$. The Laplace transform of the latter is given by

$$\exp\left(-L\left(\frac{1}{L}\int_{\mathbf{U}T-L}^{\mathbf{U}T}\lambda(v)dv\right)(1 - LS_{F_Y}(\alpha))\right).$$

Hence we define a new demand rate

$$\widehat{\lambda} := \frac{1}{L}\int_{\mathbf{U}T-L}^{\mathbf{U}T}\lambda(v)dv,$$

the stationary version of the nonhomogeneous demand rate. Clearly, the new rate is now homogeneous but stochastic!

The message is the following. In the expression of the Laplace transform of the leadtime demand $\exp(-\Lambda(\mathbf{U})(1 - LS_{F_Y}(\alpha)))$, with $\Lambda(\mathbf{U}) = \int_{\mathbf{U}T-L(\mathbf{U}T)}^{\mathbf{U}T}\lambda(v)dv$, it is only $\Lambda(\mathbf{U})$ which is changing or uncertain. This yields that if the distribution of $\Lambda(\mathbf{U})$ stays the same, the distribution of L and λ (or the functions $L(u)$ and $\lambda(u)$) can change, leading to the same stationary version of the leadtime demand, thus to the same average cost. Suppose for instance that the rate of the leadtime demand process is nonhomogeneous, with $\Lambda(\mathbf{U}) = \int_{\mathbf{U}T-L_0}^{\mathbf{U}T}\lambda(v)dv$, and fixed leadtime L_0. By keeping the distribution of $\Lambda(\mathbf{U})$ fixed, we can always transform the model into an equivalent stochastic leadtime and constant demand demand rate (say, λ_0) model. In order to achieve this, set $L(\mathbf{U}T) = (1/\lambda_0)\int_{\mathbf{U}T-L_0}^{\mathbf{U}T}\lambda(v)dv$, yielding $\Lambda(\mathbf{U}) = \lambda_0 L(\mathbf{U}T)$, that is, stochastic leadtime, constant demand rate. The next subsection expresses the same idea with a more intuitive construction.

There is one more interesting observation to make. The limiting distribution in the Cesaro sense of the leadtime demand, $\lim_{t\uparrow\infty}(1/t)\int_0^t I\!P\{\mathbf{D}(u, u + L] \leq x\}du$ is almost surely equal to $\lim_{t\uparrow\infty}(1/t)\int_0^t 1_{\{\mathbf{D}(u,u+L]\leq x\}}du$, which is just the mathematical justification of the approximations so often used in practice. In conclusion, this means that one doesn't need to know the distribution of the whole demand process, not even that of the leadtime demand, only the fraction of time that $\mathbf{D}(L) \leq x$.

2.6.2 Stochastic leadtimes vs. nonhomogeneous demand processes

Although the title of this subsection might be surprising, there is indeed an interesting relation between inventory models with stochastic leadtimes and a compound renewal demand process and inventory models with fixed leadtimes and a time-nonhomogeneous compound Poisson demand process. Consider therefore a model with fixed leadtime $L > 0$, and time-nonhomogeneous compound Poisson demand

with rate $\lambda(t)$. A standard result in mathematical statistics yields that one can define a kind of 'aggregate rate'

$$\Lambda(t) := \int_0^t \lambda(s)ds,$$

such that the process will be 'homogeneous' Poisson with the rate $\Lambda(t)$ (e.g. Bain and Engelhardt (1987)), that is, its Laplace transform on the interval T is given by $\exp(-\Lambda(T)(1-z))$. The idea is now to perform a time transformation[1] $s := \Lambda^{-1}(t)$. Since $\Lambda(t)$ is a strictly increasing function its inverse is well defined. Intuitively, imagine that the original time axis would consist of a *nonhomogeneous* rubber material, which we can stretch out until the arrival moments will get into balance, such that they will correspond to a now homogeneous Poisson arrival with rate 1 on this 'new' transformed (stretched) axis. Indeed, $\Lambda(s) = \Lambda(\Lambda^{-1}(t)) = t$. While the behaviour of the demand process is cured in this way, the distances such as the leadtime L are not the same as in the time before transformation: they became 'nonhomogeneous', that is, stochastic! This construction of stochastic leadtimes has a very nice property: orders do not cross in time. In this way we arrived at the stochastic leadtimes model described in Zipkin (1986). The long run expected average cost of the initial system with the nonhomogeneous arrivals is

$$\lim_{t\uparrow\infty} I\!E\left(\frac{1}{t}\int_0^t f(\mathbf{IP}(u) - \mathbf{D}(u, u+L])du\right).$$

Using the change of variables $u := \Lambda^{-1}(z)$ (the transformed time system, where the arrivals are homogeneous), and the previous relation transforms into

$$\lim_{t\uparrow\infty} I\!E\left(\frac{\Lambda(t)}{t}\frac{1}{\Lambda(t)}\int_0^{\Lambda(t)} f(\mathbf{IP}(\Lambda^{-1}(z)) - \mathbf{D}(\Lambda^{-1}(z), \Lambda^{-1}(z)+L])d\Lambda^{-1}(z)\right).$$

Assuming that $\Lambda(t)/t$ converges a.s. as t goes to infinity to $\lambda > 0$, it follows that the average cost expression becomes

$$\lambda\left(\lim_{s\uparrow\infty}\frac{1}{s}\int_0^s f(\mathbf{IP}(u) - \tilde{\mathbf{D}}(u, u+\mathbf{L}])\frac{1}{\lambda(u)}du\right).$$

This relation, practically speaking, is the same kind of transformation as the one given in relation (2.5.2.15) between time and event averages. Besides, this relation can be interpreted as the long run average cost expression in homogeneous time, having stochastic leadtimes with probability density $1/\lambda(u)$, less a normalization factor.

[1] J.B.G. Frenk, Erasmus University, Private communication

The lead time $\mathbf{L}$ is now determined from the relation $\Lambda^{-1}(u)+L = \Lambda^{-1}(u+\mathbf{L}(u))$ (Λ can also be considered stochastic) , that is, $\mathbf{L}(u) = \Lambda(\Lambda^{-1}(u) + L) - u$. With this specific $\mathbf{L}$ we can determine the ergodic stochastic process $\mathbf{U}(t)$, which drives the leadtime mechanism of Zipkin (1986), obtaining thus an equivalence between the two models. Denoting with u the moment a replenishment order was placed, its arrival time v is obtained in Zipkin (1986), as $v = \min\{t : t - \mathbf{U}(t) \geq u\}$ (having $t-\mathbf{U}(t)$ nondecreasing). For our model this means $v-u = \Lambda(\Lambda^{-1}(u)+L)-u$, yielding $v = \Lambda(\Lambda^{-1}(u)+L)$. Substituting this specific v into the expression $v-\mathbf{U}(v) = u$, and letting $t := \Lambda(\Lambda^{-1}(u) + L)$, yields $\mathbf{U}(t) = t - \Lambda(\Lambda^{-1}(t) + L)$. Having $\Lambda(\Lambda^{-1}(t) + L)$ increasing in t, $\mathbf{U}$ satisfies all the conditions of Zipkin (1986).

To conclude this section, there is an interesting observation to make. One often encounters in the literature a random leadtime models controlled by stationary policies. Since any nonhomogeneous demand and fixed leadtime model can be transformed into a random leadtime model, one can say that a stationary control policy can also be applied for the nonhomogeneous demand case with an equivalent reliability.

2.7 Numerical Examples

2.7.1 Computational issues: Laplace (– Stieltjes) transforms and inversion techniques

Although the expressions obtained for the performance measures might seem complicated for the first sight, their convolution structure enables us to obtain a closed form expression for their Laplace (or Laplace-Stieltjes) transformations, without a significant effort. We then make use of a recently developed Laplace transform inversion technique of Den Iseger (2002), which facilitates inversion of these Laplace transforms in any point. The obtained result are exact almost up to machine precision. A short outline of this inversion algorithm is given after a short introduction on Laplace and Laplace-Stieltjes transforms. For discrete problems the most well known and widely used inversion algorithm is the Fourier transform inversion method of Abate and Whitt (Abate and Whitt (1992b), Abate and Whitt (1992)). To introduce some short hand notation we observe for $h : [0, \infty) \to I\!R$, a function of bounded variation, that its Laplace transform is denoted by

$$L_h(\alpha) := \int_0^\infty \exp(-\alpha x)h(x)dx.$$

Moreover, its Laplace-Stieltjes transform is given by

$$LS_h(\alpha) = \int_{0-}^{\infty} \exp(-\alpha x) h(dx).$$

Clearly the parameter α is chosen in such a way that the above integrals are well defined. Also, for any function $f : I\!R \to I\!R$, vanishing on $(-\infty, 0]$, and a cumulative distribution function F on $[0, \infty)$ we introduce the convolution $f \star F : [0, \infty) \to I\!R$ given by

$$(f \star F)(x) := \int_{0-}^{x} f(x-y) F(dy) \tag{2.7.2.1}$$

Again we assume that f is chosen in such a way that $f \star F$ is well defined. It is now well known that

$$L_{\mathtt{f}\star\mathtt{F}}(\alpha) = L_f(\alpha) LS_F(\alpha) \tag{2.7.2.2}$$

For example, the Laplace-Stieltjes transform of the cumulative distribution function of a compound Poisson process on a fixed interval L can be found in any standard textbook on Laplace transforms (Tijms (1994)); it is given by

$$LS_{F_{D(L)}}(\alpha) = \exp\left(-\frac{L}{I\!E\mathbf{t}_1} (1 - LS_{F_Y}(\alpha)) \right). \tag{2.7.2.3}$$

The Laplace transform of the cumulative distribution function of a nonhomogeneous compound Poisson process is more complicated. Since we are interested in asymptotic results we aim to determine the Laplace transform of the limiting distribution. We proceed as follows: First, the z-transform of the stochastic counting process of a non-homogeneous Poisson process in the interval $(s - L, s)$ can be written formally as

$$P_{(\mathtt{s}-\mathtt{L},\mathtt{s})}(z) = \sum_{k=1}^{\infty} z^k I\!P\{\mathbf{N}(s) - \mathbf{N}(s-L) = k\}. \tag{2.7.2.4}$$

Let I be an arbitrary interval and $\Delta > 0$ defined by $\Delta := \frac{I}{n+1}$ for $n \in I\!N$ fixed. Then

$$\mathbf{N}(I) = \sum_{k=0}^{n} \mathbf{N}(k\Delta, (k+1)\Delta), \tag{2.7.2.5}$$

and observe for $\Delta \downarrow 0$ that the process $\mathbf{N}$ on any interval of length Δ, $\mathbf{N}(k\Delta, (k+1)\Delta)$, is a homogeneous Poisson process with rate $\lambda_k = \lambda(k\Delta)$ and its z-transform $P_{(\mathtt{k}\Delta,(\mathtt{k}+1)\Delta)}(z) = \exp(-\lambda_k \Delta(1-z))$, a well-known result for Poisson processes. From relations (2.7.2.4) and (2.7.2.5) follows that

$$P_{(\mathtt{s}-\mathtt{L},\mathtt{s})}(z) = \prod_{k=0}^{n} P_{(\mathtt{k}\Delta,(\mathtt{k}+1)\Delta)}(z), \tag{2.7.2.6}$$

that is $P_{(\mathbf{s}-\mathbf{L},\mathbf{s})}(z)$ is given by the following expression

$$\prod_{k=0}^{n} \exp(-\lambda_k \Delta(1-z)) = \exp(-\sum_{k=0}^{n} \lambda_k \Delta(1-z)) = \exp(-\int_{s-L}^{s} \lambda(t)dt(1-z)).$$

Finally, the Laplace transform of the limiting distribution of the non-homogeneous compound Poisson demand process on the interval $(s-L, s)$ is obtained from

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-L_{F_Y}(\alpha)) \int_{s-L}^{s} \Lambda(z)dz\right) d\left(\sum_{k=1}^{\infty} I\!P\{\mathbf{t}_k \le s\}\right) =$$

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-L_{F_Y}(\alpha)) \int_{s-L}^{s} \Lambda(z)dz\right) d\left(\int_0^s \lambda(u)du\right) =$$

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-L_{F_Y}(\alpha)) \int_{s-L}^{s} \Lambda(z)dz\right) \lambda(s)ds.$$

In most cases such an expression for a Laplace (– Stieltjes) transform cannot be inverted analytically, therefore one must rely on numerical inversions, although these are not always numerically stable. Den Iseger (2000) developed a new method for numerically inverting Laplace transforms, which is robust, accurate and numerically stable.

The starting-point of his method is the Poisson Summation Formula (PSF), which enables the reduction of the problem of Laplace transform inversion to that of Fourier series (z-transform) inversion. Now the inversion of z-transforms is relatively easy and it can be done with machine precision. The right hand-side of the PSF can be represented as an operator function of a compact skew self-adjoint integral operator, say K. Employing for this operator function the Riesz functional calculus, it can be represented as a contour integral. Consequently, the PSF can be extended to a large class of functions, including polynomials. In fact, the induced operator turns out to be invariant and invertible on the space of nth degree polynomials (for all $n = 1, 2, \ldots$).

Furthermore, the operator K is similar to the multiplication operator. Identifying a set of orthogonal polynomials for the inner product induced by the multiplication operator leads finally to finite rank approximations. Since these polynomials are similar to the Legendre polynomials, the technique is related to the framework of Gaussian quadrature integration. The developed technique enables approximation of operator functions with accuracy of almost machine precision, which, in conclusion yields that Laplace transforms can be inverted with the same precision. Moreover,

this inversion method is norm preserving, that is, that approximated function has the same L^2 norm as the original function. The method is also numerically stable; in fact, the difference between the approximations of the inverse of two different Laplace transforms L_f and L_g equals to the difference of the the difference of the Laplace transforms themselves.

Based on this theory, a number of type of Laplace transform inversion algorithms were developed: pointwise inversion algorithms, piece wise polynomial inversion algorithms, and multi dimensional inversion algorithms. The running time of the algorithm is a fraction of a second.

With the help of these tools, the following subsections demonstrate the procedure and the results of calculating cost functions and service measures for several inventory systems with different demand processes.

2.7.2 Time-nonhomogeneous compound Poisson demand

Average holding cost

In case of **non-homogeneous compound Poisson demand** with arrival rate given by $\Lambda(t)$, $t \geq 0$, we obtain that the z-transform of the time stationary distribution for the stochastic counting process is given by

$$P_{\mathbf{N}^c_\infty(0,L]}(z) = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-z)\int_{s-L}^{s} \lambda(z)dz\right) ds. \tag{2.7.2.7}$$

Therefore the average cost can again easily be computed as it was described earlier. In **Figure** 2.2 we plotted the values of the average cost of an (s, S) policy with variable s and $S - s$ values in case when demand is given by a non-homogeneous compound Poisson process. The demand rate function varies every (unit) interval, such that if $t \in [2k, 2k+1)$ then $\Lambda(t) = \lambda_1$ and if $t \in [2k+1, 2k+2)$ then $\Lambda(t) = \lambda_2$. The individual demands follow a Gamma distribution with shape parameter 2.5 and scale parameter 2.5 (Tijms (1994)). Furthermore we considered a piecewise linear cost-rate function given by

$$f(x) = \begin{cases} -px & \text{if } x < 0 \\ h_1 x & \text{if } 0 \leq x \leq q \\ h_1 q + h_2(x-q) & \text{if } x \geq q \end{cases} \tag{2.7.2.8}$$

where q denotes a critical level of inventory, from which the inventory holding cost increases to h_2 per unit ($h_2 > h_1 > 0$). In the costs we also included a fixed ordering cost $K > 0$ (see section 2.5.4) for every placement of a replenishment order. The

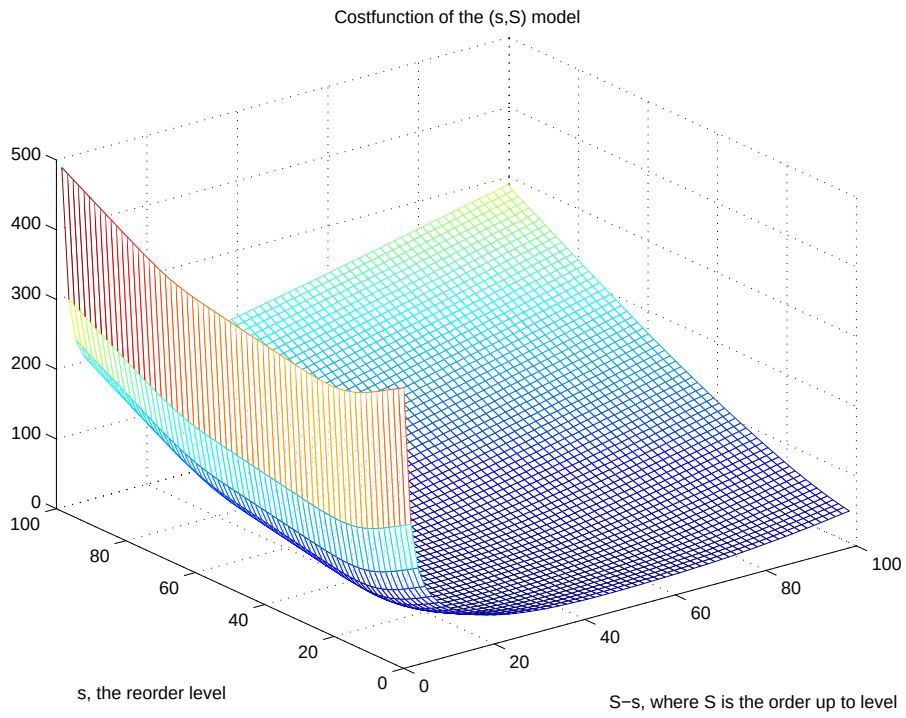

Figure 2.2: Average holding cost in case of an (s, S) policy with non-homogeneous compound Poisson demand; parameters are $K = 20$, $L = 1$, $\lambda_1 = 25/2$, $\lambda_2 = 45/2$, $q = 50$, $p = 3$, $h_1 = 1$, $h_2 = 3$)

expression (2.7.2.7) is easy to calculate, because one only needs the fraction of time that the demand has a certain rate, obtaining $1/2 \exp(-(1-z)\lambda_1 L) + 1/2 \exp(-(1-z)\lambda_2 L)$.

Service measures

In case of non-homogeneous compound Poisson demand with rate $\Lambda(t)$ we obtain that the z-transform of $\mathbf{N}^e_\infty(0, L]$ is given by

$$P_{\mathbf{N}^e_\infty(0,L]}(z) = \lim_{n\uparrow\infty} \frac{1}{n} I\!E \sum_{j=1}^{n} \exp\left(-(1-z) \int_{\mathbf{t}_j - L}^{\mathbf{t}_j} \Lambda(z) dz \right).$$

Conditioning on $\mathbf{t}_j$ we obtain that the above equals

$$P_{\mathbf{N}^e_\infty(0,\mathrm{L}]}(z) = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-z)\int_{s-L}^{s} \Lambda(z)dz\right) \frac{\Lambda(s)}{\lambda} ds, \tag{2.7.2.9}$$

where the normalization factor λ is given by

$$\lambda = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \Lambda(s)ds,$$

which is actually the rate $\lim_{t\uparrow\infty}(I\!E(\mathbf{N}(t)/t)$. This implies that relation (2.7.2.9) is in fact the time stationary transformation of the event stationary version; the transformation formula was given by relation (2.5.2.15). Both of the cases can be computed with the help of the Laplace transform inversion algorithm (Den Iseger (2002)).

2.7.3 Compound renewal demand

Average holding cost

In case of compound renewal demand, we obtain for the stochastic counting process that

$$\lim_{t\uparrow\infty} \phi_t \mathbf{N}(0,L] = \lim_{t\uparrow\infty} (\mathbf{N}(t+L) - \mathbf{N}(t)) \stackrel{d}{=} \mathbf{N}_0(L - \mathbf{A}), \tag{2.7.2.10}$$

where $\mathbf{A}$ is a random variable distributed with the limiting distribution of the residual life process (Tijms (1994)) and $\mathbf{N}_0$ denotes the arrival process with a renewal in time point 0. Let us use the notation

$$\Psi_k(t) := I\!P\{\mathbf{N}_0(t) = k\},$$

then the probability distribution of (2.7.2.10) equals $(\Psi_k \star F_A)(L)$. Straightforwardly

$$\Psi_k = F_X^{(k-1)*} - F_X^{k*},\ k \geq 1 \text{ and } \Psi_0(t) = 1_{\{t\geq 0\}}$$

and the Laplace-Stieltjes transform of F_A is given by

$$LS_{F_A}(\beta) = \frac{1 - LS_{F_X}(\beta)}{\beta I\!E X_1}.$$

It follows that the two dimensional Laplace-Stieltjes transform of $\mathbf{D}(0,L]$ is given by

$$\frac{(1 - LS_{F_X}(\beta))^2 LS_{F_Y}(\alpha)}{\beta I\!E\mathbf{X}(1 - LS_{F_Y}(\alpha)LS_{F_X}(\beta))} + \frac{(1 - LS_{F_X}(\beta))}{\beta I\!E\mathbf{X}}.$$

Thus β is the argument of the Laplace -Stieltjes transform taken with respect to the leadtime L, while α is the argument of the Laplace -Stieltjes transform taken with respect to the individual demand $\mathbf{Y}$. With this construction we are able to calculate the long run average cost with the help of the two dimensional inversion algorithm (Den Iseger (2002)).

Service measures

In case of compound renewal demand by a reversed time argument we obtain that

$$\lim_{j\uparrow\infty} I\!P\{\mathbf{N}(\mathbf{t}_j^-) - \mathbf{N}(\mathbf{t}_j^- - L) = k\} = I\!P\{\mathbf{N}(L) = k\}, \tag{2.7.2.11}$$

that is $\mathbf{N}_\infty^e(0, L] \stackrel{d}{=} \mathbf{N}(L)$. Hence relation (2.5.2.12) equals

$$I\!Eg_1(\mathbf{IP}_\infty - \sum_{k=1}^{\mathbf{N}(L)} \mathbf{Y}_k, \mathbf{Y}_\infty). \tag{2.7.2.12}$$

A special case of a general compound renewal demand process with Gamma dis-

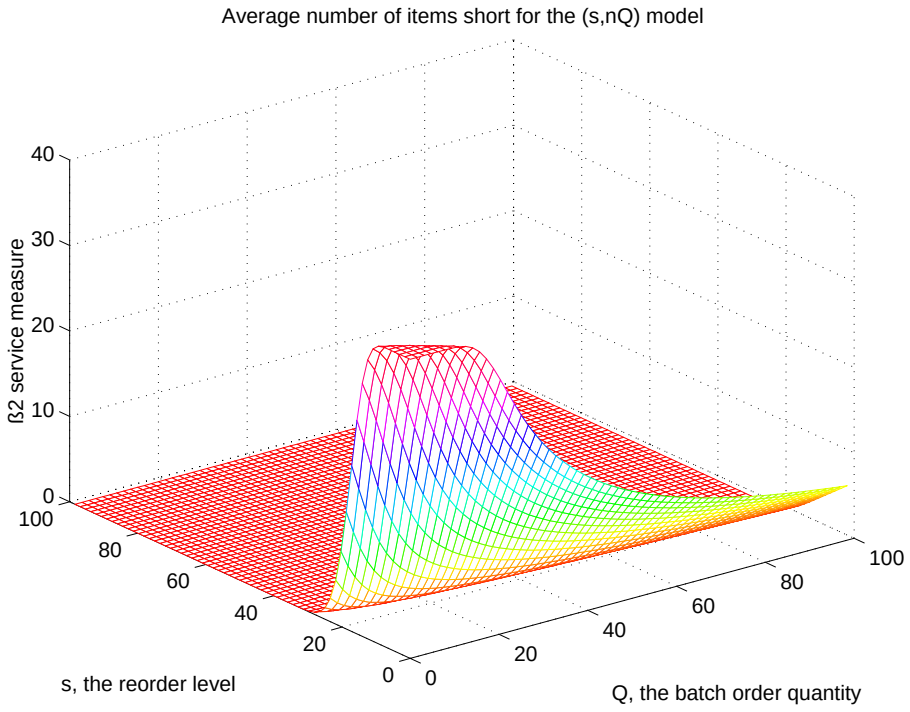

Figure 2.3: Average number β_2 of items short in case of an (s, Q) policy $(L = 0.5)$

tributed arrival process (shape=5/2,scale= 1/14) and i.i.d. Gamma distributed individual demands with shape resp. scale parameters $\alpha = \beta = 2.5$ are considered in case of an (s, Q) control rule. The expected (time-) average number of items short,

given by relation (2.7.2.12) with g_1 given by (2.5.2.16), is plotted in **Figure** 2.3, with respect to the decision variables s and Q.

2.8 A Few Words about Periodic Review Policies with compound renewal Demand

Periodic review models are most of the time perceived as discrete time models, that is, one assumes unit length periods and considers demand aggregated during one period as if all demand arrived at the end (or beginning) of the period. Therefore, if for example one considers compound renewal demand with periodic review models, it is assumed that the demand of the individual periods is independent and identically distributed. In this sense, the equivalence between periodic review and continuous review models is just the difference between discrete time and continuous time models (see for instance Chapter 1 of Sahin (1990)). However, for the compound renewal demand process case there is a different way of modeling periodic review systems: consider a continuous time inventory model with a fixed period of length $R > 0$ such that the interarrival times of customers are distinct from R. The idea is the same argument which which was discussed under Section 2.7.3 for the leadtime demand generated by a compound renewal processes, see relation (2.7.2.10). When observing an arbitrary period, this might have begun during an inter-demand epoch. Since the arrival process in this case is not memoryless $\mathbf{N}((k+1)R) - \mathbf{N}(kR)$ is *not* distributed as $\mathbf{N}(R)$. Yet, there is a different solution: we can shift time, that is, shift the beginning of the period until it coincides with the first arrival after the beginning of the period. The shifted time lag is just the residual life $\mathbf{A}$ (Tijms (1994)), and now starting in a renewal, the stochastic counting process $\mathbf{N}(t+R) - \mathbf{N}(t) \stackrel{d}{=} \mathbf{N}_0(R - \mathbf{A}(t))$, where $\mathbf{N}_0$ denotes the arrival process $\mathbf{N}$ with a renewal in time point 0. Since the limiting distribution of the residual life process exists, and it is given by

$$\lim_{t\uparrow\infty} I\!P\{\mathbf{A}(t) \leq x\} = I\!P\{\mathbf{A}_\infty \leq x\} = \frac{1}{I\!E\mathbf{t}_1} \int_0^x (1 - F_{\mathbf{t}_1}(y))dy, \; x \geq 0,$$

it follows that the limiting distribution of $\mathbf{N}(t+R) - \mathbf{N}(t)$ exists, and it is given by

$$\lim_{t\uparrow\infty} (\mathbf{N}(t+R) - \mathbf{N}(t)) \stackrel{d}{=} \mathbf{N}_0(R - \mathbf{A}_\infty).$$

Thus, this is actually just condition (d) of Zipkin's approach (Zipkin (1986)), see Section 2.3. This results suggests that with this construction it is possible to view

periodic review models as discrete time continuous review models with R as unit time, and $\mathbf{D}_0(R)$ is the 'individual demand' in this discretized time point. Unfortunately, with this construction not all periodic review analogies of the basic continuous review models allow the same straightforward asymptotic analysis as discussed in Section 2.2. The difficulty arises at the (s,S) model, namely, the inventory position will depend on the residual life process. Thus, the asymptotic independence of the embedded Markov chain $\mathbf{IP}_n$ and the leadtime demand fails to hold. This prevents us from applying the same asymptotic results as for a continuous review (s,S) policy. Yet, there is an other way to perform the asymptotic analysis, but first let us examine the asymptotic behaviour of the periodic versions of the base stock and (s,nQ) policies.

In case an (R,s,nQ) policy is employed, the embedded Markov chain $\mathbf{IP}_n := \mathbf{IP}(nR)$ remains uniformly distributed for every $n \in I\!N \cup \{0\}$ and a compound renewal demand process (analogously to Section 2.4.2), defining the individual demands $\mathbf{Y}_n := \mathbf{D}_0(nR) - \mathbf{D}_0((n-1)R)$. This implies that $\mathbf{IP}_n$ is independent of the residual life process. The same holds for the periodic review base stock policy, the so called (R,S) policy: the inventory position is S at the beginning of every period R, whether there was any demand during the previous period or not. This result implies that the (R,S) and (R,s,nQ) periodic review models can be analyzed in the same manner as the continuous review models of sections 2.2.1 and 2.2.3.

That is, *now* it is possible to view these periodic review models as discrete time models, with R as unit time, and $\mathbf{D}_0(R)$ is the 'individual demand' in the discretized R unit time points. This approach is the reverse of the idea which was presented for instance in Chapter 1 of the book of Sahin (1990), who suggests that continuous review models can be analyzed analogously to periodic review models (by periodic review he actually means the unit size period, that is, discrete time models). This equivalence also implies that the long run average cost expression of an (R,s,nQ) model can be reduced to the cost of a simple (R,S) policy, just as it was done in Section 2.2.3.

The average cost of an (R,S) policy, the periodic review analogue of the $(S-1,S)$ policy, is given by the expression

$$\mathbf{C}(R,S) = \frac{1}{R}\left(K I\!E\left(1_{\{A_\infty \le R\}}\right) + \int_L^{R+L} I\!E f(S - \mathbf{D}_0(t))dt \right), \qquad (2.8.2.1)$$

since $\mathbf{D}_0$ denotes a demand process with a renewal in 0.

The average cost of an (R,s,nQ) policy is given by the expression

$$\mathbf{C}(R,s,Q) = \frac{1}{R}\left(K(1 - I\!E(F_{D_0(R)}(Q\mathbf{U}))) + I\!E\left(\int_L^{R+L} f(s + Q\mathbf{U} - \mathbf{D}_0(y))dy \right) \right).$$

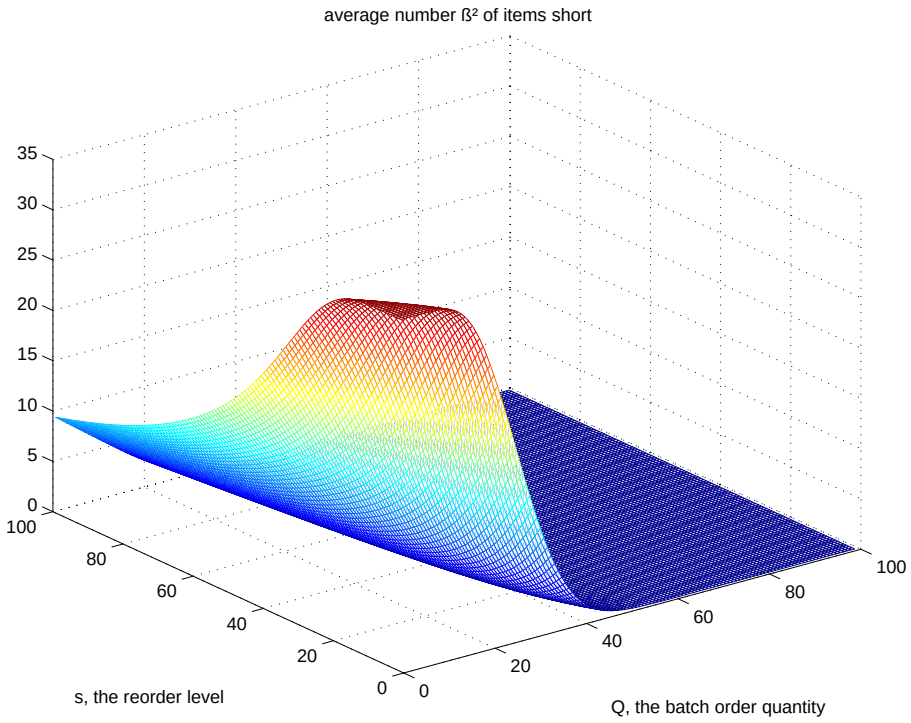

Figure 2.4: Average number β_2 of items short with parameters $R = 1/2, L = 1/2, \lambda = 35$ and shape resp. scale parameters $\alpha = \beta = 2.5$

One can immediately observe that the expression in the numerator

$$I\!E\left(\int_L^{R+L} f(s+Qu-\mathbf{D}_0(y))dy\right)$$

can be expressed again in terms of the cost function of an (R,S) model with order-up-to level $S := s + Qu$, hence the previous expression equals

$$\mathbf{C}(R, s+Qu)R - KI\!E\left(1_{\{A_\infty \le R\}}\right),$$

thus the long run average cost generated by applying an (R, s, nQ) policy equals

$$\mathbf{C}(R,s,Q) = \frac{K(I\!E\left(1_{\{A_\infty \le R\}}\right) - I\!E(F_Y(Q\mathbf{U})))}{R} + \int_0^1 \mathbf{C}(R, s+Qu)du. \quad (2.8.2.2)$$

One can deduce service measures just as easily as long run average costs, in the same way as for the continuous review analogies. In Section 2.7.3 a plot of the the β_2

service measure of a continuous review (s, nQ) policy was given, as a function of the decision variables s and Q. To illustrate also visually just how similar the periodic and continuous review analogue policies are, we plotted in **Figure** 2.4 the same β_2 service measure of a periodic review (R, s, nQ) policy, as a function of the decision variables s and Q, while holding R fixed $(R = 1/2)$.

For the (R, s, S) policy we can proceed in the following manner: the dependency of the inventory position process and the demand process is only through the residual life process $\mathbf{A}$. Hence, conditioning on $\mathbf{A}$ solves the problem. Yet, this method triggers a computational difficulty: one needs to deal with the joint distribution of the inventory position and residual life processes. Solving this problem differently, without causing computational difficulties is the topic of a future paper. Here we are only going to consider the compound Poisson demand case. The memoryless property of the Poisson arrivals imply that $\mathbf{D}(nR) - \mathbf{D}((n-1)R) \stackrel{d}{=} \mathbf{D}(R)$. Similarly as for the (s, S) policy, the condition that an order is being triggered is given by

$$I\!E(\mathbf{V} + \mathbf{D}(R) > S - s) = 1 - I\!E F_{D(R)}(S - s - \mathbf{V}).$$

Hence, the average cost is given by the expression

$$\mathbf{C}(R, s, S - s) = \frac{K(1 - I\!E F_{D(R)}(S - s - \mathbf{V})) + I\!E\left(\int_L^{R+L} f(\mathbf{IN}(t))dt\right)}{R}$$

with

$$\mathbf{IN}(t) = \mathbf{IP}(nR) - \mathbf{D}(t) = S - \mathbf{V}_n - \mathbf{D}(t),$$

for every $nR + L \leq t < (n+1)R + L,\ n \in I\!N \cup \{0\}$. The analogue of $\mathbf{V}_n$ from Section 2.2.2, for an (R, s, S) policy is defined as the difference between the order-up-to level, S and the inventory position at time of the n^{th} review nR, i.e. $\mathbf{V}_n := S - \mathbf{IP}(nR),\ n \in I\!N$. By the control policy it follows that

$$\mathbf{V}_{n+1} = (\mathbf{V}_n + \mathbf{D}((n+1)R))\, 1_{\{\mathbf{V}_n + \mathbf{D}((n+1)R) \leq S - s\}}$$

The invariant distribution of the Markov chain $\{\mathbf{V}_n : n \in I\!N\}$ is given by

$$I\!P\{\mathbf{V} \leq x\} = \frac{U_0(x)}{U_0(S - s)}, \tag{2.8.2.3}$$

where U_0 denotes the renewal function related to the renewal sequence $\{\mathbf{D}(R), \mathbf{D}(2R), \dots\}$. The long run average cost $\mathbf{C}(R, s, S - s)$ is given by the expression

$$\frac{1}{R}\left(K(1 - I\!E F_{D(R)}(S - s - \mathbf{V})) + \int_0^{S-s} I\!E\left(\int_L^{R+L} f(S - x - \mathbf{D}(t))dt\right) F_V(dx)\right).$$

This can be expressed in terms of the average cost of the (R, S) model, by observing that

$$I\!E\left(\int_L^{R+L} f(S-x-\mathbf{D}(t))dt\right) = \mathbf{C}(R, S-x)R - KI\!E\left(1_{\{A_\infty \leq R\}}\right),$$

with $\mathbf{C}(R, S-x)$ denoting the average cost of an (R, S) model with order-up-to level $S := S - x$. This yields for the final expression

$$\frac{K(I\!E\left(1_{\{A_\infty \leq R\}}\right) - I\!EF_{D(R)}(S-s-\mathbf{V}))}{R} + \int_0^{S-s} \mathbf{C}(R, S-x)\frac{U_0(dx)}{U_0(S-s)}. \quad (2.8.2.4)$$

The analysis of periodic review policies with demand generated by general point processes is considerably murkier, it needs more sophisticated techniques. The problem here is that the residual life process associated with a general point process might not have a limiting distribution, and then condition (d) fails to hold. Depending on the nature of the point process in consideration, one might try to find or construct a process $\mathbf{A}$ such that condition (d') of Section 2.3 would be satisfied. A somewhat similar but more general approach is presented in Chapter 4, which extends the analysis to demand driven by general stochastic processes, and extends the class of inventory policies considered so far.

2.9 Discussion

The mainstream research on stochastic single item inventory models can be divided into a renewal theoretic and a Markovian approach. Under the 'renewal theoretic' stream we understand the approach applied by Sahin (1979) and others (Bázsa, Frenk and Den Iseger (2001), Tijms (1994), and so on): separately for the different models they identify the regeneration points of the inventory position process at the time points of order placement. Their asymptotic analysis is a tedious evaluation of algebraic expressions. The accent is rather on developing efficient approximations, which is motivated by the objectives and computational capacities of computers at that time. Zipkin (1986) developed a new perspective of stochastic single item inventory models. He is the first to observe that all the basic policies have the same properties and one can perform the analysis on a single general policy which comprises all of these: a policy which only depends on the inventory position process. However, this is not the most innovative feature of the paper; its contribution is the proof of the asymptotic results (2.1.2.1) – (2.1.2.3) for constant and stochastic leadtimes. With

this approach performance measures as long run average costs and service measures can be deduced without much algebra. More importantly, it provides insight into the behaviour of such systems. The stochastic leadtime model is based on an arrival mechanism of the orders, similarly to the model of Kaplan (Kaplan (1970), Nahmias (1979), Ehrhardt (1981)), yet more general and much simpler. Zipkin's stochastic leadtimes model was such a success that it made scholars overlook the results for the constant leadtimes case (such as Sahin (1990), Bázsa, Frenk and Den Iseger (2001)). His results can be extended further by relaxing his assumption on the existence of the limiting distribution of the stochastic counting process $\mathbf{N}$ (Bázsa and Den Iseger (2001)). The advantage of the latter approach lies on one hand in the approach of treating constant and stochastic leadtimes jointly (but under the no order crossing condition), on the other hand, extending the class of demand processes to nonstationary point processes. In fact, this approach also extends the combination of policy/demand processes: for instance it can handle an (s, S) policy with unit demand, such that the stochastic counting process $\mathbf{N}_\Delta$ converges in the limit to a stationary one. Zipkin's model can only handle the combination of periodic chain/regenerative demand process. Furthermore Bázsa and Den Iseger (2001) deduce a surprising equivalence between the stochastic leadtimes model of Zipkin and a model with constant leadtimes but nonstationary demand, concluding: if in certain cases it is reasonable to use stationary policies in combination with stochastic leadtimes, then it is just as reasonable to use stationary policies in combination with nonstationary demand.

Chapter 3

Optimal continuous order quantity (s, S) models

3.1 Introduction

Research on optimization of stationary (s, S) policies has a long history, in fact, it might even be surprising that there is still anything new to tell about it. As the title already clarifies, the main algorithm we would like to discuss in this chapter is for the case when demand is continuous not only in time but also as quantity, that is, the individual demand sizes $\mathbf{Y}_n$ are non-discrete. The latest and until now best result on this line of research was presented by Federgruen and Zipkin (1985). The strategy employed by their algorithm is policy iteration, as a direct extension of the algorithm of Federgruen and Zipkin from 1984 for discrete order size (s, S) models. For discrete order quantity (s, S) policies there is a very efficient algorithm, that of Zheng and Federgruen (1991), which in its original form, is not applicable to the continuous order quantity case. In 2001 Bázsa and Den Iseger (2001b) developed an efficient and intuitive algorithm for continuous order quantity (s, S) models. It can be shown that their continuous–case algorithm reduces to the Zheng-Federgruen algorithm in the discrete case. Moreover, every step taken can be followed on a drawing, while for the earlier algorithms (except maybe the algorithm of Zheng and Federgruen (1991) it is difficult to find the precise geometrical interpretation. This gave us the idea to present this chapter somewhat unconventionally: first we present our algorithm (thus the newest) and discuss only afterwards the two most relevant algorithms from the literature: the discrete case Zheng–Federgruen algorithm and the continuous case Federgruen–Zipkin algorithm. This will enable us to point out the analog steps of

our algorithm, providing in this way a geometrical interpretation and motivation of their constructions – when possible.

In fact, the algorithm of Bázsa and Den Iseger exploits solely the geometrical properties of the cost function. In order to minimize a function with two variables, one can write down two optimality equations (the derivatives w.r.t. these variables equal 0). Based on these optimality relations we construct two aid functions, $s = \phi(S)$ and $s = \psi(S)$. It turns out that these auxiliary functions have two simple but crucial properties: (1) both of them increase slower than 45 degrees; and (2) ψ always intersects ϕ in a maximum or minimum of ϕ, and these are the only stationary points for ϕ.

Having obtained these results, now the algorithm builds on two simple observations:

(a) For a fixed order-up-to level S_0 one can always find a unique $\bar{s}$, which minimizes $C(s, S_0)$. Hence, $\bar{s}$ can be determined with no effort.

(b) The local minimum points $(\bar{s}, \bar{S})$ of $C(s, S)$ coincide with the local maximum points of ϕ, such that $\bar{s} = \phi(\bar{S})$. Moreover, the global minimum of C coincides with the global maximum of ϕ.

Thus the problem reduces to finding the global maximum of ϕ. First we construct a local search, $LM(S_0)$, which finds the closest maximum of ϕ, starting at S_0. The subroutine LM converges monotonically to this maximum point, such that between the starting point S_0, and the found local maximum there will be no other stationary points for ϕ (thus also not for C). LM is only based on properties (1), (2), and observations (a) and (b). This maximum point of ϕ defines the "relevance level" $\hat{s}_k$ (k is the actual number of iterations). Now solely using property (1) we construct an other subroutine, which finds the first point where ϕ increases above the relevance level $\hat{s}_k$. At this point we restart the local search LM, finding the next maximum, which determines the next relevance level, $\hat{s}_{k+1}$. Obviously, due to this construction, $\hat{s}_k < \hat{s}_{k+1}$, that is, $C(\hat{s}_k, S_k) > C(\hat{s}_{k+1}, S_{k+1})$, thus the algorithm converges monotonically to the global optimum. All the subroutines converge linearly. Moreover, since every iteration finds a local minimum of the expected average cost, the number of iterations is at most N, where $N < \infty$ represents the number of local minimums.

All of these algorithms mentioned above use the optimality of (s, S) policies of several standard infinite horizon single item inventory models. As first, in 1963 Iglehart (1963) proved the existence of stationary (s, S) policies which minimize the total discounted cost and average cost over an infinite horizon and for all initial inventory positions. His proofs for the discounted cost case start from the n–period model, establishing the existence of a non–stationary sequence (s_n, S_n), the optimal parameters for the n–stage problem. He goes on with establishing a limit point

for the sequence (s_n, S_n) (when $n \to \infty$), (s_0, S_0), while the sequence of n–period total cost functions $f_n(x)$ converge to $f(x)$. He verifies then that the limit function f satisfies the optimality equation (for its definition see for instance Ross (1983)). Finally it is verified that the limit point (s_0, S_0) achieves the minimum to the right of the optimality equation, while the solution of the optimality equation is unique. Hence $f(x)$ represents the optimal discounted cost, with the optimal policy (s_0, S_0). Similarly, the average cost case is also deduced from the finite horizon case. Later Veinott (1966) extended the results of Iglehart to more general models (quasiconvex instead of convex expected holding and shortage cost function), while Veinott and Wagner (1965) summarized the results. Veinott and Wagner also point out some of the optimal policies found with the discounted cost model might not be optimal for certain initial inventory positions. Bell (1970) identified optimal inventory policies via an optimal stopping rule for all initial inventory positions. Johnson (1968) uses a policy improvement method as Federgruen and Zipkin (1984a, 1985), yet he restricts the problem to finite state Markov decision processes. The most elegant proof for the optimality of (s, S) policies both for the discounted cost and average cost models is given by Zheng (1991). He realizes that indeed, the analysis of stationary infinite horizon models should be simpler than their finite horizon counterparts, since the latter encounters the well-known problem of end-of-horizon effect. The method is also built on Markov decision processes theory. The simple proof for the average cost case uses a relaxation of the model which is necessary to overcome the problems that arise under unbounded one-stage expected holding and shortage cost functions. Since the proof is indeed simple and short, we start the next section with its demonstration.

3.2 Zheng's simple optimality proof

Although the proofs are performed for discrete time periodic review models, a simple transformation (see Section 2.8) yields that they also apply to continuous time continuous review models. It is assumed that demands in consecutive periods are i.i.d. Denote by $\mathbf{D}$ the generic demand random variable, with $p_j = I\!P\{\mathbf{D} = j\}$. Zero leadtime is assumed for the sake of simplicity, but the proofs work for fixed leadtimes as well. The one period expected holding and shortage cost function is denoted by $G(y)$, as a function of the inventory level at the beginning of the period. It is assumed further that $-G$ is unimodal and $G(y) \to \infty$ as $\|y\| \to \infty$. $C(s, S)$ stands for the long run average cost as function of the decision variables, while $(s^\star, S^\star)$ denotes the optimal policy which minimizes the long run average cost, yielding C^*. $I(s, i)$ de-

notes the expected inventory holding and shortage costs incurred until the inventory level drops to or below the reorder level s, when the initial inventory level is i, $i > s$. $M(j)$ is the expected time (number of periods) to deplete j units of inventory. With these definitions Zheng deduces (see relation (8) of Zheng (1991)) that

$$I(s,i) = \sum_{j=0}^{i-s-1} G(i-j)m(j), \text{ for } i > s, \text{ and } M(i) = \sum_{j=0}^{i-1} m(j), \text{ for } i > 0, \quad (3.2.3.1)$$

where m is the renewal density. An analogue of Lemma 1 of Zheng and Federgruen (1991 is proven: if $G(y)$ attains its minimum in y^0, then there exists an $s^\star$ and $S^\star$ that satisfy:

(i) $C^* = C(s^\star, S^\star)$;

(ii) $s^\star < y^0 \leq S^\star$;

(iii) $G(s^\star) \geq C^* > G(s^\star + 1)$;

(iv) $G(S^\star) \leq C^*$.

The standard approach for the optimality proof is to show that the (optimal) policy $(s^\star, S^\star)$ minimizes the average cost optimality equation (cf. Theorem 2.1 of Ross (1983)):

$$h(i) = \inf_{j \geq i} \{K\delta(j-i) + G(i) - C^* + \sum_{l=0}^{\infty} p_l h(j-l)\}, \text{ for all integer } i, \quad (3.2.3.2)$$

where h is a bounded function. As Zheng notes (Zheng (1991)), while it is easy to construct a solution h of (3.2.3.2), such a solution is unbounded and for these (3.2.3.2) fails to imply the optimality of $(, s^\star, S^\star)$. Zheng comes around this difficulty by considering a relaxation of the original model, which does have a bounded solution, hence (3.2.3.2) implies the existence of an optimal stationary policy $(s^\star, S^\star)$. This solution is also optimal for the original model.

The relaxation is applied to the nonnegativity constraint for order sizes. Thus, the optimality equation remains identical to the one of the original model, except that the constraint $j \geq i$ in (3.2.3.2) is omitted. According to this construction, the function h is defined in the following way:

$$h(i) = \begin{cases} K, & i \leq s^\star \\ K + I(s^\star, i) - C^* M(i - s^\star), & s^\star < i \leq S^\star \\ \min\{K, G(i) - C^* + \sum_{l=0}^{\infty} p_l h(i-l)\}, & i > S^\star. \end{cases} \quad (3.2.3.3)$$

This definition assures that h is bounded and that it solves the following equations:

$$h(i) = K + h(S^\star), \qquad i \leq s^\star \tag{3.2.3.4}$$

$$h(i) = G(i) - C^* + \sum_{l=0}^{\infty} p_l h(i-l), \qquad s^\star < i \leq S^\star \tag{3.2.3.5}$$

$$h(i) = \min\{K, G(i) - C^* + \sum_{l=0}^{\infty} p_l h(i-l)\}, \qquad i > S^\star. \tag{3.2.3.6}$$

If h now satisfies the optimality equation, since it is also bounded it yields an optimal stationary policy $(s^\star, S^\star)$ for which we have (Zheng (1991)): when the inventory level i is below or equal $S^\star$, use the $(s^\star, S^\star)$ policy; when the inventory level i is above $S^\star$, reduce it to $S^\star$ by placing a negative order if $h(i) = K$ and do not order otherwise. This policy only differs from the 'normal' $(s^\star, S^\star)$ policy in *at most one* period: a negative order (disposal) is placed if the initial inventory level is too high, but once it gets at or below level $S^\star$ it never exceeds $S^\star$ again. Since the long run average cost is not affected by the cost of any finite period of time, this policy must have the same optimal cost C^* as the original model.

To show that h satisfies the optimality equation is not a difficult task. Since $h(S^\star) = 0$, the optimality equation is satisfied for $i > S^\star$. For $i \leq S^\star$ one needs to verify the following equations:

$$h(i) \leq G(i) - C^* + h(S^\star), \qquad s^\star < i \leq S^\star \tag{3.2.3.7}$$

$$h(i) \leq K + h(j)\}, \qquad i \leq S^\star. \tag{3.2.3.8}$$

In view of the definition of h, relation (3.2.3.8) is equivalent to $0 \leq G(i) - C^*$, $i \leq s^\star$, which, in turn, follows from (ii) and (iii) above. On the other hand, (i) implies that $h(j) \geq 0$, hence for relation (3.2.3.8) it suffices to show that $h(i) \leq K$. This follows again from the definition of the function h (we omit this one technical detail, since the precise proof can be found in Zheng (1991)).

The optimization algorithms listed further all use the result of the optimality of the (s, S) policies. The next sections present the recently developed algorithm of Bázsa and Den Iseger (2001b), followed by a discussion of the Zheng–Federgruen (Zheng and Federgruen (1991) and Federgruen–Zipkin (Federgruen and Zipkin (1985)) algorithms.

3.3 The model and the marginal cost

We assume that demand is generated by a compound renewal process $\mathbf{D}(t)$, $U(x)$ denotes the renewal function related to the sequence of individual demands with a renewal in 0, while $m(x)$ denotes its density. K is the fixed cost to place an order. The long run expected average cost of a system associated with an (s, S) policy is given by Section 2.2.2, that is,

$$C(s,S) = \frac{K/I\!E\mathbf{t} + \int_0^{S-s} I\!Ef(S-t-\mathbf{D}_\infty(0,L])U(dt)}{U(S-s)}, \tag{3.3.3.1}$$

where f is a given cost-rate function (its form is irrelevant for the further analysis), $\mathbf{D}_\infty(0, L]$ is the limiting distribution of the lead time demand, and $I\!E\mathbf{t}$ is the expected interarrival time. If we denote with $c(s)$ the long run expected average cost of an $(s-1, s)$ model (cf. Section 2.2.2) but without ordering costs, knowing that this is given by $I\!Ef(s - \mathbf{D}(0, L])$, the average cost of an (s, S) policy $C(s, S)$ can be written in terms of $c(s)$:

$$C(s,S) = \frac{K + \int_0^{S-s} c(S-t)U(dt)}{U(S-s)}, \tag{3.3.3.2}$$

where K is normalized as $K := K/I\!E\mathbf{t}$. If there is no ordering cost, that is, $K = 0$ the optimal policy satisfies $S = s$ and $C(s, S) = c(s)$. Let $s^\star$ be the optimal order-up-to level for an $(s-1, s)$ policy without ordering cost $(K = 0)$, that is, $s^\star = \arg\min c(s)$. From these observations it also follows for any given s and S, that

$$c(s^\star) < C(s,S). \tag{3.3.3.3}$$

Let us assume that $-c(s)$ is unimodal such that

$$c_s(s) < 0 \text{ for all } s < s^\star, \text{ and } \lim_{s \downarrow -\infty} c(s) = +\infty. \tag{3.3.3.4}$$

The following lemma gives a marginal – cost condition for the optimality of the reorder level for a given order-up-to level (see also **Figure** 3.1). This lemma can be interpreted as the continuous version of Lemma 1 of Zheng and Federgruen (1991).

Lemma 3.3.1 *For any fixed order-up-to level S_0, the cost function $-C(s, S_0)$ is unimodal in $s \in (-\infty, s^\star)$ and reaches its minimum in $\bar{s}$. Moreover, $\bar{s}$ is the unique solution of the equality*

$$C(s, S_0) = c(s), \tag{3.3.3.5}$$

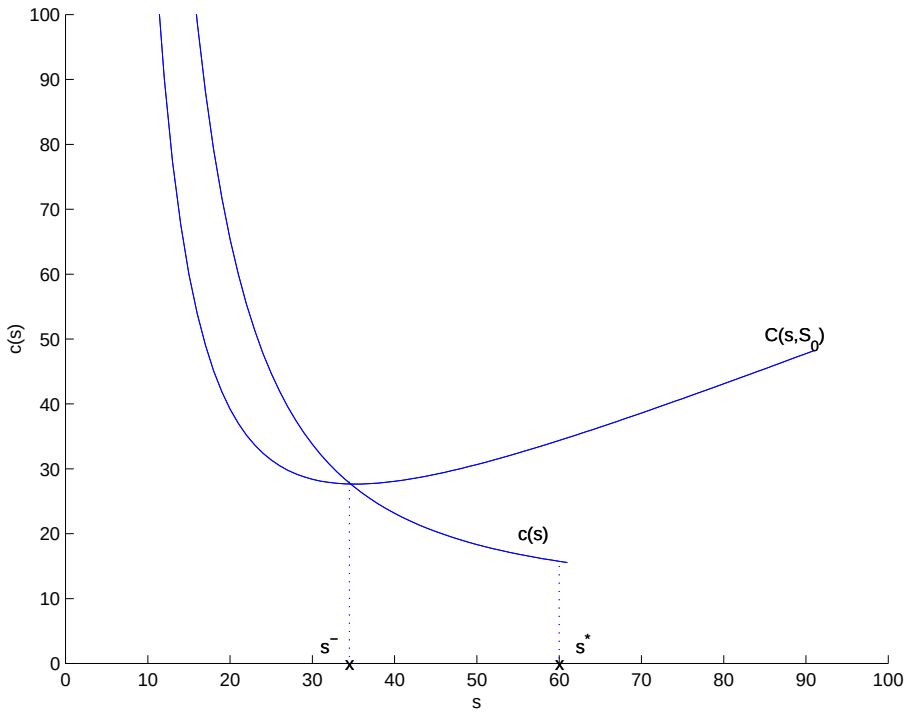

Figure 3.1: The marginal cost relation for fixed S_0

and the following inequalities hold

$$C(s,S_0) \;<\; c(s) \quad \textit{if and only if} \quad s<\bar{s}, \tag{3.3.3.6}$$

$$C(s,S_0) \;>\; c(s) \quad \textit{if and only if} \quad \bar{s}<s\leq s^\star. \tag{3.3.3.7}$$

Proof Let us start from relation (3.3.3.2), and take the derivative of $C(s,S_0)$ w.r.t. s. This yields

$$C_s(s,S_0)=(C(s,S_0)-c(s))\frac{m(S_0-s)}{U(S_0-s)}. \tag{3.3.3.8}$$

Having observed relation (3.3.3.3) it follows that

$$C_s(s^\star,S_0)>0 \quad \text{for all } S_0. \tag{3.3.3.9}$$

Let us suppose now that there exists a stationary point $s_0<s^\star$ of $C(s,S_0)$ (that is, $C_s(s_0,S_0)=0$). The second order derivative of $C(s,S_0)$ in this point is given by

$$C_{ss}(s_0,S_0)=-c_s(s_0). \tag{3.3.3.10}$$

Since $s_0 < s^\star$, it is clear through relation (3.3.3.4) that $C_{ss}(s_0, S_0) > 0$, which implies that s_0 is a local minimum for $C(s, S_0)$. This means, that any stationary point $s_0 < s^\star$ must be a local minimum, which is impossible. We can conclude therefore, that there is only one minimum: $\bar{s} < s^\star$. Furthermore, $\bar{s}$ is a minimum for $C(s, S_0)$ if and only if $C_s(\bar{s}, S_0) = 0$, that is, if and only if $C(\bar{s}, S_0) = c(\bar{s})$, proving thus (3.3.3.5).

Furthermore, if $\bar{s}$ is a global minimum, and $-C(s, S_0)$ is unimodal on $(-\infty, \bar{s}]$, then for $s < \bar{s}$ $C_s(s, S_0) < 0$. This implies directly that $C(s, S_0) > c(s)$ for all $s < \bar{s}$ (see also **Figure** 3.1). On the other hand, if $\bar{s} < s < s^\star$ then $C_s(s, S_0) > 0$, which means that $C(s, S_0) < c(s)$ for all $\bar{s} < s < s^\star$. It only remains to prove that there exists a stationary point for $C(s, S_0)$, S_0 fixed. Splitting the expression for C with respect to $s^\star$ yields:

$$C(s, S_0) \leq c(s)\frac{U(S_0 - s) - U(S_0 - s^\star)}{U(S_0 - s)} + \int_0^{S_0 - s^\star} c(S_0 - t)\frac{U(dt)}{U(S_0 - s)}.$$

Taking $s \longrightarrow -\infty$ yields $C_s(-\infty, S_0) < 0$, on the other hand $C_s(s^\star, S) > 0$, which implies that a stationary point for C does exist. This completes the proof.

Applying similar ideas to a maintenance model, Dekker and Plasmeijer (2001) use marginal costs to determine optimal parameters and to carry out a simple and efficient sensitivity analysis. They have found that when one of the parameters changes, then it is not necessary to solve the optimization anew, it is possible to asses the optimal parameter values by solving single root equations. For further details the reader is referred to their article.

3.4 Bounds for the optimal reorder and order-up-to levels

The optimal order-up-to level $s^\star$ of the $(s-1, s)$ policy (with $K = 0$) represents an upper bound for the optimal reorder level s, and a lower bound for the optimal order-up-to level S of an (s, S) policy. These bounds were first discovered by Veinott and Wagner (1965).

Lemma 3.4.1 *If $(\bar{s}, \bar{S})$ is an optimal policy, that is, $(\bar{s}, \bar{S})$ is a global minimum for $C(s, S)$, then*

$$\bar{s} < s^\star \qquad \textit{and} \qquad s^\star < \bar{S}. \tag{3.4.3.1}$$

Moreover, the following assertions hold:

$$\text{if} \quad \min_{S>s^\star} C(s,S) \;>\; c(s) \quad \text{then} \quad \bar{s} < s; \tag{3.4.3.2}$$

$$\text{if} \quad \min_{S>s^\star} C(s,S) \;<\; c(s) \quad \text{then} \quad s < \bar{s}. \tag{3.4.3.3}$$

Proof Suppose that the contrary of (3.4.3.1) is true, that is $\bar{S} \leq s^\star$. This means that there exists a $\delta > 0$ such that $\bar{S} + \delta = s^\star$. The cost in these points is given by

$$C(\bar{s}+\delta, \bar{S}+\delta) = \frac{K + c(\bar{S}+\delta) + \int_0^{\bar{S}-\bar{s}} c(\bar{S}+\delta-t)U(dt)}{U(\bar{S}-\bar{s})}.$$

Since $c(\bar{S}+\delta) = c(s^\star) = \min_S c(S)$, and c decreases on $(-\infty, s^\star]$, it follows that the former expression is smaller than

$$\frac{K + c(\bar{S}) + \int_0^{\bar{S}-\bar{s}} c(\bar{S}-t)U(dt)}{U(\bar{S}-\bar{s})} = C(\bar{s}, \bar{S}),$$

in conclusion, $C(\bar{s}+\delta, \bar{S}+\delta) < C(\bar{s}, \bar{S})$, which is a contradiction with the optimality of the policy $(\bar{s}, \bar{S})$, proving that $\bar{S} > s^\star$. Suppose now that $\bar{s} > s^\star$, then there exists a $\delta > 0$ such that $\bar{s} - \delta = s^\star$. By a similar argument as before we obtain that $C(\bar{s}-\delta, \bar{S}-\delta) < C(\bar{s}, \bar{S})$, which is a contradiction, concluding thus $s^\star > \bar{s}$.

If $(\bar{s}, \bar{S})$ is an optimal policy and the inequality in relation (3.4.3.3) holds, then

$$c(s) \;<\; \min_{S>s^\star} C(s,S) \leq C(s, \bar{S}),$$

and this implies by Lemma 3.3.1, (3.3.3.7), that $\bar{s} < s < s^\star$. Since c is non increasing, we obtain that relation (3.4.3.3), i.e.,

$$c(\bar{s}) = C(\bar{s}, \bar{S}) \leq \min_{S>s^\star} C(s,S) < c(s)$$

implies $s < \bar{s}$. This completes the proof.

One can also derive bounds for the optimal cost, which will generate an upper bound for the optimal order-up-to level. This upper bound is tighter than the one presented by Zheng and Federgruen (1991) (Lemma 2).

Lemma 3.4.2 *Let C^* denote the optimal cost achieved with the optimal policy $(\bar{s}, \bar{S})$, that is,*

$$C^* = C(\bar{s}, \bar{S}) = \min_{s,S} C(s,S).$$

The following inequality holds:

$$C^* \geq K(1 - F_Y(\bar{S} - \bar{s})) + c(\bar{S}). \tag{3.4.3.4}$$

This generates an upper bound

$$S^u := \sup\{S > s^\star : C^* \geq K(1 - F_Y(S - \bar{s})) + c(S)\} \tag{3.4.3.5}$$

for the optimal order-up-to level.

Proof For any function f, define the shift operator $\phi_s,\ s \in I\!R$ as $\phi_s f(x) := f(s+x)$ for all x, and define $\bar{C}$ as

$$\bar{C}(x) := K + (\phi_s c * U)(x).$$

Straightforward calculation of $\bar{C} * F_Y$ yields

$$\bar{C}(x) = \phi_s c(x) + K(1 - F_Y(x)) + (\bar{C} * F_Y)(x),$$

hence the cost function $C(s, S)$ can be written in terms of $\bar{C}$, and

$$C(s,S) = \frac{\bar{C}(S-s)}{U(S-s)} = \frac{c(S) + K(1 - F_Y(S-s)) + (\bar{C} * F_Y)(S-s)}{U(S-s)}.$$

Since C^* is the optimal cost, $C^* \leq C(s, S)$ for all s, S, thus $C^*U(S - t - s) \leq C(s, S - t)U(S - t - s) = \bar{C}(S - t - s)$, for all $0 \leq t \leq S - s$. This implies that

$$C(s,S) \geq \frac{c(S) + K(1 - F_Y(S-s)) + ((C^*U) * F_Y)(S-s)}{U(S-s)},$$

and the last term is equal to

$$C^* + \frac{c(S) + K(1 - F_Y(S-s)) - C^*}{U(S-s)}.$$

This yields in particular that

$$0 \geq \frac{c(\bar{S}) + K(1 - F_Y(\bar{S} - \bar{s})) - C^*}{U(\bar{S} - \bar{s})},$$

which implies obviously the conclusion of the lemma. This completes the proof.

Remark 3.4.3 *Suppose that $(\bar{s}, \bar{S})$ is the optimal policy. $\bar{s}$ is optimal if and only if $C(\bar{s}, \bar{S}) = c(\bar{s})$ (cf. Lemma 3.3.1). Substituting this into relation (3.4.3.4) (in Lemma 3.4.2) we obtain:*

$$c(\bar{s}) \geq K(1 - F_Y(\bar{S} - \bar{s})) + c(\bar{S}).$$

3.5 Derivatives and their properties

The standard method for locating extreme values of a function is to determine the roots of its derivative. How difficult it is to determine such roots, is a different question. Let us first start with this standard procedure. The derivative of the cost function $C(s,S)$ with respect to S is given by

$$C_S(s,S) = h(s,S) - C_s(s,S), \tag{3.5.3.1}$$

where $C_s(s,S)$ is given by relation (3.3.3.8), and $h(s,S)$ is given by

$$h(s,S) := \frac{\int_0^{S-s} c_S(S-t)U(dt)}{U(S-s)}. \tag{3.5.3.2}$$

The higher order derivatives of h yield the higher order derivatives of the cost function C, and the former are given by

$$h_s(s,S) = (h(s,S) - c_s(s))\frac{m(S-s)}{U(S-s)}, \tag{3.5.3.3}$$

$$h_S(s,S) = -h_s(s,S) + \Omega(s,S), \tag{3.5.3.4}$$

where Ω is given by

$$\Omega(s,S) := \frac{\int_0^{S-s} c_{SS}(S-t)U(dt)}{U(S-s)} + (c_S(s^{\star+}) - c_S(s^{\star-}))\frac{m(S-s^\star)}{U(S-s)}. \tag{3.5.3.5}$$

Assumption 3.5.1 *We assume for the rest of the chapter that c is convex, such that relation (3.3.3.4) holds and $c_s(s) \geq 0$ for all $s > s^\star$.*

Note, that c is not necessarily strictly convex.

Lemma 3.5.2 *The derivative of the function $h(s,S)$ w.r.t. S is given by relation (3.5.3.4). Moreover, the function $\Omega(s,S)$, defined by relation (3.5.3.5), is positive for every $s \leq s^\star \leq S$.*

Proof Consider now the decomposition of $h(s,S)$:

$$h(s,S) = \int_{0-}^{(S-s^\star)^-} c_S(S-t)\frac{U(dt)}{U(S-s)} + \int_{(S-s^\star)^+}^{S-s} c_S(S-t)\frac{U(dt)}{U(S-s)},$$

and take the derivative of the two terms with respect to S. The expression for h_S results immediately. We consider now two cases: when c_s is continuous in $s^\star$ and when c_s is not continuous in $s^\star$.

If c_s is continuous in $s^\star$ then, since $c_s(s) < 0$ for $s < s^\star$ and $c_s(s) > 0$ for $s > s^\star$, it is not possible that $c_{ss}(s) = 0$ in a neighborhood of $s^\star$. This yields that $\Omega(s, S) > 0$ and the term

$$(c_S(s^{\star+}) - c_S(s^{\star-}))\frac{m(S - s^\star)}{U(S - s)} = 0.$$

If, on the other hand, c_s is not continuous in $s^\star$, then we obtain

$$(c_S(s^{\star+}) - c_S(s^{\star-}))\frac{m(S - s^\star)}{U(S - s)} > 0,$$

since $s^{\star-} < s^\star$ and $c_s(s) < 0$ for all $s < s^\star$, while $s^{\star+} > s^\star$ and $c_s(s) \geq 0$ for all $s < s^\star$. This yields again that $\Omega(s, S) > 0$. This completes the proof.

Lemma 3.5.3 *The function $h(s, S)$ is increasing in s.*

Proof Let's decompose the expression (3.5.3.2) in the following way:

$$h(s, S) = \int_{0-}^{(S-s^\star)^-} c_S(S - t)\frac{U(dt)}{U(S - s)} + \int_{(S-s^\star)^+}^{S-s} c_S(S - t)\frac{U(dt)}{U(S - s)}. \quad (3.5.3.6)$$

Since c_S is a non decreasing function (c is convex), the previous term is greater or equal than

$$c_S(s^\star)\frac{U(S - s^\star)}{U(S - s)} + c_S(s)\frac{U(S - s) - U(S - s^\star)}{U(S - s)} \; > \; c_S(s),$$

having $c_S(s) < 0$ and $c_S(s^\star) \geq 0$. Summarizing these relations yields that for all $s < s^\star$ and all S

$$h(s, S) > c_s(s). \quad (3.5.3.7)$$

This together with (3.5.3.3) implies immediately that

$$h_s(s, S) > 0, \quad (3.5.3.8)$$

that is, h is increasing with respect to s. This completes the proof.

While the marginal cost relation provides the iterations for the optimal s, finding each time a stationary point for a fixed S, we need to solve now $C_S(s, S) = 0$. Considering the form of C_S (see relation (3.5.3.1)), this does not promise an efficient search; instead we can make use of the following remark.

Remark 3.5.4 *If for a fixed order-up-to level S_0 the reorder level s_0 represents a local minimum for $C(s, S)$, then $h(s_0, S_0) = 0$ if and only if S_0 is a stationary point for $C(s, S)$.*

3.6 Looking for the global minimum

3.6.1 The functions ϕ and ψ

Consider the following optimality and pseudo-optimality equations (see relation (3.3.3.5) and Remark 3.5.4)

$$C(s,S) = c(s) \quad \text{and} \tag{3.6.3.1}$$

$$h(s,S) = 0, \tag{3.6.3.2}$$

and define $\phi(S)$ and $\psi(S)$ respectively:

$$\phi(S) := \{\phi \in I\!R \ : \ C(\phi,S) = c(\phi)\}, \tag{3.6.3.3}$$

$$\psi(S) := \{\psi \in I\!R \ : \ h(\psi,S) = 0\}. \tag{3.6.3.4}$$

Since (3.6.3.1) has a unique solution for every S (see Lemma 3.3.1) ϕ is a well defined function. Consider now an arbitrarily fixed S_0. Since $h(s^\star,S_0) > 0$, while $h(-\infty,S_0) < 0$, the equation $h(s,S_0) = 0$ certainly has a solution in $(-\infty,s^\star]$. Knowing that $h_s(s,S_0) > 0$ (cf. Lemma 3.5.3), we can conclude that this solution is unique. Hence, ψ is also a well defined function. Now, if equations (3.6.3.1) and (3.6.3.2) are simultaneously satisfied for a pair (s_0,S_0), then this point is a stationary point for C. By the definition of ϕ and ψ, for this stationary point $\phi(S_0) = \psi(S_0)$, thus S_0 is an intersection point of the two functions. But can we possibly find every intersection point of ϕ and ψ, and at what cost (complexity)? Which of these intersection points represent a local minimum for the total cost, and how can we filter them? This section deals with these questions.

Taking derivatives in relations (3.6.3.1) and (3.6.3.2) with respect to S, yields the first order derivatives of ϕ and ψ:

$$\phi_S(S) = \frac{C_S(\phi(S),S)}{c_\phi(\phi(S))}.$$

Relations (3.5.3.1), (3.6.3.1) and Lemma 3.3.1 imply together that $C_S(\phi(S),S) = h(\phi(S),S)$, yielding

$$\phi_S(S) = \frac{h(\phi(S),S)}{c_\phi(\phi(S))}. \tag{3.6.3.5}$$

Similarly,

$$\psi_S(S) = \frac{-h_S(\psi(S),S)}{h_\psi(\psi(S),S)}.$$

By relation (3.5.3.4) this becomes

$$\psi_S(S) = 1 - \frac{\Omega(\psi(S), S)}{h_\psi(\psi(S), S)}. \tag{3.6.3.6}$$

Remark 3.6.1 *The function ϕ has a stationary point in the intersection points with ψ, that is, if $\phi(S_0) = \psi(S_0)$ then $\phi_S(S_0) = 0$, and these are the only stationary points for ϕ.*

What do these first order derivatives tell us about ϕ and ψ? The answer is summarized in the following lemma.

Lemma 3.6.2

$$\phi_S(S) < 1 \quad \textit{and} \quad \psi_S(S) < 1. \tag{3.6.3.7}$$

Moreover, $\phi(S) < s^\star$, $\phi_S(s^\star) > 0$, $\psi(s^\star) = s^\star$.

Proof

Relation (3.5.3.7) and $c_s(s) < 0$ for all $s < s^\star$ imply together that $h(\phi(S), S)/c_\phi(\phi(S)) < 1$, that is $\phi_S(S) < 1$ (cf. (3.6.3.5)). The strict positivity of $\Omega(s, S)$ (cf. Lemma 3.5.2) and $h_s(s, S)$ (cf. Lemma 3.5.3) yield that $\psi_S(S) < 1$. The last statement of the Lemma can be verified by direct computations. This completes the proof.

The second order derivative $\phi_{SS}(S)$ of ϕ is given by

$$\frac{h_\phi(\phi(S), S)\phi_S(S) + h_S(\phi(S), S)}{c_\phi(\phi(S))} - \frac{h(\phi(S), S)c_{\phi\phi}(\phi(S))\phi_S(S)}{c_\phi^2(\phi(S))}.$$

We already know by Remark 3.6.1 that ϕ has a local minimum or maximum in the intersection points with ψ. The second order derivative gives more information, namely, if $\phi(S_0) = \psi(S_0)$, then

$$\phi_{SS}(S_0) = \frac{h_S(\psi(S_0), S_0)}{c_\phi(\psi(S_0))},$$

since $\phi_S(S_0) = 0$. Multiplying by $h_\psi(\psi(S_0), S_0)/h_\psi(\psi(S_0), S_0)$ and using expression (3.5.3.3) yields

$$\phi_{SS}(S_0) = \psi_S(S_0)\frac{m(S_0 - \psi(S_0))}{U(S_0 - \psi(S_0))}. \tag{3.6.3.8}$$

In conclusion, we have two types of intersection points: the first is such that ψ is decreasing and it intersects ϕ in a local maximum, the second type is when ψ is increasing and it meets ϕ in a local minimum (see Figure 3.2). This gives us a lot of information about the behaviour of the two functions. Before the first type of intersection points the function ψ is decreasing, and since the intersection point itself is a maximum point for ϕ, it will increase until the intersection and it decreases afterwards. Let us summarize this in the following proposition.

Proposition 3.6.3 *$\phi(S) > \psi(S)$ if and only if $\phi_S(S) < 0$, while $\phi(S) < \psi(S)$ if and only if $\phi_S(S) > 0$. Moreover, if $\psi_S(S_0) < 0$ and $\phi(S_0) = \psi(S_0)$ then $\phi_{SS}(S_0) < 0$ and if $\psi_S(S_0) > 0$ and $\phi(S_0) = \psi(S_0)$ then $\phi_{SS}(S_0) > 0$.*

Proof Suppose that $\phi(S) > \psi(S)$. Then, knowing that $h(s, S)$ is increasing in s (see Lemma 3.5.3) $h(\phi(S), S) > h(\psi(S), S) = 0$. Furthermore, since $c_s(s) < 0$ $(s < s^\star)$, we obtain

$$\phi_S(S) = \frac{h(\phi(S), S)}{c_\phi(\phi(S))} < 0.$$

If, in turn, we suppose that $\phi_S(S) < 0$, this implies immediately by (3.6.3.5) that $h(\phi(S), S) > 0$. By the definition of ψ, $h(\psi(S), S) = 0$, and knowing that h is increasing in its first variable, these statements imply together that $\phi(S) > \psi(S)$.

When $\phi(S) < \psi(S)$, using the same reasoning as before, we obtain $\phi_S(S) > 0$. The remainder of the Proposition was proven by relation (3.6.3.8). This completes the proof.

It only remains to check now which of these intersection points of ϕ and ψ represent a local minimum for the total cost function C. Suppose that (s_0, S_0) is a stationary point for $C(s, S)$ (obviously $s_0 = \phi(S_0) = \psi(S_0)$), then the Hessian of C in (s_0, S_0) is

$$H(s_0, S_0) = \begin{bmatrix} -c_s(s_0)\frac{m(S_0 - s_0)}{U(S_0 - s_0)} & 0 \\ 0 & c_s(s_0)\frac{m(S_0 - s_0)}{U(S_0 - s_0)} + \Omega(s_0, S_0) \end{bmatrix}, \tag{3.6.3.9}$$

where Ω was defined by relation (3.5.3.5). In view of the definitions of $\psi_S(S)$ (see relation (3.6.3.6)) and Ω, the Hessian can be written in the form

$$H(s_0, S_0) = -c_s(s_0)\frac{m(S_0 - s_0)}{U(S_0 - s_0)} \begin{bmatrix} 1 & 0 \\ 0 & -\psi_S(S_0) \end{bmatrix}. \tag{3.6.3.10}$$

Indeed, this form yields us immediately the answer for the question: which intersection points of ϕ and ψ represent a local minimum for C?

Proposition 3.6.4 *If (s_0, S_0) is a stationary point for $C(s, S)$ such that $\psi_S(S_0) < 0$ and $\phi_S(S_0) = 0$, that is, S_0 is a local maximum for ϕ, then (s_0, S_0) is a local minimum for $C(s, S)$.*

3.6.2 Sorting out the stationary points

What is left to do now is to define a search routine which finds all the intersection points of ϕ and ψ. For this purpose Lemma 3.6.2 and Proposition 3.6.3 will be

of great help. Indeed, Lemma 3.6.2 asserts that neither ϕ nor ψ increases steeper than 45 degrees. Assume now, that having taken an arbitrary S_0, $\phi(S_0) < \psi(S_0)$. We know then from Proposition 3.6.3 that $\phi(S)$ is increasing. Let us take in this point a secant of 45 degrees. $\psi_S(S) < 1$ guarantees that this line will intersect ψ in strictly one point, say S_1. Furthermore $\phi_S(S) < 1$ guarantees that the 45 degree secant will not intersect ϕ in any other point than S_0. Repeat now the previous step for $\phi(S_1)$, obtaining S_2, and so on. We want to prove that $\{S_k : k \geq 1\}$ is converging monotonically to S^*, where $S^* := \inf\{S > S_0 : \phi(S) = \psi(S)\}$. When $\phi(S_0) > \psi(S_0)$ we proceed in exactly the same way and then the algorithm will converge monotonically to the left from the starting point.

Lemma 3.6.5 *Consider s and S fixed. The function $\Gamma(t) := C(s+t, S+t)$ is strictly convex in t. Moreover, t_0 minimizes Γ if and only if $h(s+t_0, S+t_0) = 0$.*

Proof The optimality condition is that the derivative of Γ w.r.t. t has to be 0; that is, $C_s(s+t_0, S+t_0) + C_S(s+t_0, S+t_0) = h(s+t_0, S+t_0) = 0$. It only remains to check if the second derivative is positive:

$$\Gamma_{tt}(t) = h_s(s+t, S+t) + h_S(s+t, S+t) = \Omega(s+t, S+t) > 0,$$

where Ω is given by (3.5.3.5). This completes the proof.

Let us summarize the results we found so far in a subroutine (which finds a local minimum for C) and prove convergence formally.

Subroutine: LM(S_0)

1. input S_0, $k := 0$;
2. calculate s_0 such that $C(s_0, S_0) = c(s_0)$;

repeat

3. $t_k := \arg\min_t C(s_k + t, S_k + t)$;
4. $S_{k+1} := S_k + t_k$;
5. calculate s_{k+1} such that $C(s_{k+1}, S_{k+1}) = c(s_{k+1})$;
6. $k := k + 1$

until $t_k = 0$.

7. $LM := S_k$;

Theorem 3.6.6 *If S_0 is the starting point and the previously described subroutine converges monotonically to S^*, then the cost reaches a (local) minimum in S^*. More-*

over, the subroutine always finds the closest[1] *minimum point to* S_0*, such that, if* $S^* < S_0$ *then there is no other stationary point in* $[S^*, S_0]$ *and if* $S^* > S_0$ *then there is no other stationary point in* $[S_0, S^*]$.

Proof Suppose that $\psi(S_k) > \phi(S_k)$. Since $\psi_S(S) < 1$ the 45 degree secant $(\Phi(S_k) + t, S_k + t),\ t \geq 0$ intersects $\psi(S)$ strictly in one point, S_{k+1}. That is,

$$\Gamma_t(t)|_{t=0} = C_t(\Phi(S_k) + t, S_k + t)|_{t=0} = h(\Phi(S_k), S_k) < 0,$$

since $h(\psi(S_k), S_k) = 0$, $\psi(S_k) > \phi(S_k)$ and $h(s, S)$ is increasing in s. Hence Γ is decreasing from $\phi(S_k)$ and since it is strictly convex it has strictly one minimum, say S_{k+1} (the intersection point, cf. Lemma 3.6.5), such that

$$S_{k+1} > S_k. \tag{3.6.3.11}$$

The fact that $\phi_S(S) < 1$ and $\psi_S(S) < 1$ implies for all $S_k < S < S_{k+1}$ that

$$\phi(S) < \phi(S_k) + (S - S_k), \tag{3.6.3.12}$$

$$\psi(S) > \psi(S_{k+1}) + (S - S_{k+1}). \tag{3.6.3.13}$$

Hence, for every $S_k \leq S < S_{k+1}$, relations (3.6.3.12) and (3.6.3.13) imply together that

$$\phi(S) < \phi(S_k) + (S_{k+1} - S_k) + (S - S_{k+1}) = \psi(S_{k+1}) + (S - S_{k+1}) < \psi(S).$$

In conclusion, $\phi(S) < \psi(S)$ for all $S_k \leq S < S_{k+1}$, that is, there are no stationary points for C in (S_k, S_{k+1}). Since S_k is an increasing sequence (cf. relation (3.6.3.11)) and it is bounded by the intersection point of the two functions $\phi(S) = \psi(S)$[2], S_k is convergent, hence the subroutine converges monotonically, too, and we have $C(s_{k+1}, S_{k+1}) < C(s_k, S_k)$ for all $k \geq 0$.

In the case when $\psi(S_k) < \phi(S_k)$ we proceed exactly in the same way, obtaining a decreasing sequence $S_{k+1} < S_k$. In conclusion, the subroutine always converges to a stationary point S_n such that $\psi_S(S_n) < 0$. In view of Proposition 3.6.4 this implies that the subroutine always converges to a (local) minimum. This completes the proof.

[1] By "closest minimum" we understand the following: if the starting point is between two stationary points of ϕ, then the subroutine will find the stationary point among those two which is a maximum point for ϕ, that is, a minimum point for C; it is not necessarily the closest in norm (distance).

[2] It is also easy to check that this upper bound is at the same time the lowest upper bound of the sequence

$C(s, S) = c(s)$ and $h(s, S) = 0$ are nonlinear equations, in fact, the first means to compute $s = \phi(S)$, while the second is to compute $s = \psi(S)$. In Section 3.7.3 a detailed explanation is given about the computation of these two functions. Yet, it is important to point out here how we can slightly modify an optimality condition such that the local optimum $\bar{S}$ is not overshot, hence insuring that the iterations are monotone: When the iterations begin at $S_0 < \bar{S}$ (thus the sequence increases monotonically to the right), we replace Step 5 with $0 \leq C(s, S) - c(s) \leq \varepsilon$ and the stopping condition with $0 \leq t_k \leq \varepsilon$, for any $\varepsilon > 0$ (obviously, $h(s, S) < 0$). Analogously, when $S_0 > \bar{S}$ (thus the sequence decreases monotonically to the left), we replace Step 5 with $-\varepsilon \leq C(s, S) - c(s) \leq 0$ and the stopping condition with $-\varepsilon \leq t_k \leq 0$, for any $\varepsilon > 0$.

The core of the algorithm is the subroutine LM which finds the local minimum, but it is equally important to define a search which finds the global optimum in a fast and efficient way. For this purpose the next lemma will be of great help, it will assure that the algorithm is fast and efficient.

Lemma 3.6.7 *The global minimum of $C(s, S)$, $C^* = C(\bar{s}, \bar{S})$ is reached exactly in the global maximum of the function ϕ, $\phi(\bar{S}) = \bar{s}$.*

Proof The proof is very simple and it is based on Proposition 3.6.4 (any local minimum of C is reached in a local maximum of ϕ) and the assumption that c is non increasing on $(-\infty, s^\star]$. $C^* = C(\bar{s}, \bar{S})$ is the global minimum, which means that $\bar{s}$ is a minimum, that is, $C(\bar{s}, \bar{S}) = c(\bar{s})$ (cf. Lemma 3.3.1). Moreover,

$$c(\phi(\bar{S})) = c(\bar{s}) = C(\bar{s}, \bar{S}) \ < \ C(\hat{s}, \hat{S}) = c(\hat{s}) = c(\phi(\hat{S}))$$

for any other local minimum $(\hat{s}, \hat{S})$. Since c is non increasing, it follows that $\phi(\bar{S}) > \phi(\hat{S})$, for all $\hat{S}$ local maximum for ϕ (cf. Proposition 3.6.4). This completes the proof.

3.7 A fast algorithm for the continuous case

Now we are ready to proceed with the description of the algorithm. Before providing a detailed description we give the general idea behind the algorithm in three major steps. **Step 0**: It is the easiest move to start the search at the lower bound $s^\star$ by running the subroutine $LM(s^\star)$, obtaining $S_0 := LM(s^\star)$, with $(\phi(S_0), S_0)$ representing a local minimum for C. (Since $\psi(s^\star) = s^\star > \phi(s^\star)$, follows that $S_0 > s^\star$.) The difficulty arises at this point. Since the subroutine always finds the closest stationary point, we have to step away "far enough" otherwise the subroutine would find back S_0 again and again. The upper bound S^u (cf. Lemma 3.4.2) yields the solution:

Step 1: restart the search at the upper bound, finding an optimum S_1 such that: a) $\phi(S_1) > \phi(S_0)$ (that is $C(\phi(S_1), S_1) < C(\phi(S_0), S_0)$, cf. Lemma 3.6.7) and b) $\phi(S_1) > \phi(S)$ for all $S \in (S_1, S^u]$. **Step 2**: With the help of the new maximum, $\phi(S_1)$ we can restart the search in S_0, obtaining S_2, such that $\phi(S_2) > \phi(S_1)$. We construct thus iterations from both of the ends of the feasibility interval of the form $[S_{k-1}, S_k]$ (S_{k-1}, S_k local optima's) until the two ends meet, reducing the interval to 0. The last found optima is the optimal policy.

3.7.1 The policy improvement step: relevance levels

There's one more question we still have to answer before giving the algorithm, and that is: how to find the closest maximum for ϕ, which is bigger than the previously found maximum. That is, find the point where ϕ increases to the level of the last maximum (say, S_k^u and S_k^l) and in that point restart the subroutine LM. In Step 1 we proceed in the following way: Let the feasibility interval be (S_{k-1}, S_{k-2}) at this moment, with $\hat{s}_{k-1} := \phi(S_{k-1}) > \phi(S_{k-2})$, the last found maxima for ϕ. The aim is to find the level crossing point:

$$S_k^u := \sup\{S < S_{k-2} : \phi(S) = \hat{s}_{k-1}\}. \tag{3.7.3.1}$$

Due to $\phi_S(S) \leq 1$ (cf. Lemma 3.6.2), the 45 degree secant in the point $(S_{k-2}, \hat{s}_{k-1})$ intersects ϕ in strictly one point, say S^1, with $S^1 < S_{k-2}$ and $\phi(S^1) \leq \hat{s}_{k-1}$. Taking this secant repeatedly in the points $(S^n, \hat{s}_{k-1}), \ldots$ we obtain a decreasing sequence $\ldots < S^n < \ldots < S^1$, which converges to S_k^u. (Later we start the subroutine $LM(S_k^u)$ in this point, obtaining the new maxima $\hat{s}_k > \hat{s}_{k-1}$ with $\hat{s}_k := \phi(S_k)$, and the new feasibility interval $[S_{k-1}, S_k]$.) Let us summarize this in the subroutine:

Subroutine: ISU$(S_{k-2}, \hat{s}_{k-1})$

1. $n := 0$; $S_0 := S_{k-2}$; $t_0 = 1$;

while $t_n > \varepsilon$ do begin

2. $n := n + 1$;
3. t_n is the solution of $0 \leq C(\hat{s}_{k-1} - t, S_{n-1} - t) - c(\hat{s}_{k-1} - t) \leq \varepsilon$;
4. $S_n := S_n - t_n$;

end.

5. $ISU := S_n$;

Step 2 works similarly, with the difference that the level crossing point is given by

$$S_k^l := \inf\{S > S_{k-1} : \phi(S) = \hat{s}_k\},$$

thus we obtain an increasing sequence $\{S^n\}$, (starting from S_{k-1}), such that we take secants first in the point $(S_{k-1}, \phi(S_{k-1}))$, which intersects the line $\hat{s}_k$ in S^1. $\phi_S(S) < 1$ insures that $\phi(S^1) \leq \hat{s}_k$. Taking this secant repeatedly in the points $(S^n, \phi(S^n)), \ldots$ we obtain an increasing sequence $S^1 < \ldots < S^n < \ldots$, which converges to S_k^l. (Start then the subroutine $LM(S_k^l)$ in this point, obtaining the new maxima $\hat{s}_{k+1} > \hat{s}_k$ with $\hat{s}_{k+1} := \phi(S_{k+1})$, and the new feasibility interval $[S_{k+1}, S_k]$). The subroutine is given as follows:

Subroutine: ISL($S_{k-1}, \hat{s}_k$)

1. $n := 0$; $S_0 := S_{k-1}$; $\delta_0 := 1$;

while $\delta_n > \varepsilon$ do begin

2. $n := n + 1$;
3. $\delta_n := \hat{s}_k - s$, where s is the solution of $0 \leq C(s, S_{n-1}) - c(s) \leq \varepsilon$;
4. $S_n := S_{n-1} + \delta_n$;

end.

5. $ISL := S_n$;

Having established these subroutines, the algorithm itself is simple. The search stops when the feasibility interval reduces to zero.

3.7.2 The algorithm

1. (input ε); $S_0 := LM(s^\star)$; $\hat{s}_0 := \phi(S_0)$; $c_0 := c(\hat{s}_0)$;
2. **while** $c_0 \geq c(S_0 + \Delta)$ **do** $\Delta := 2 \cdot \Delta$;

   $S_1 := S_0 + \Delta$;

   if $\phi(S_1) < \hat{s}_0$ then $RP := ISU(S_1)$; $S_1 := LM(RP)$;

   $\hat{s}_1 := \phi(S_1)$;

   $k := 1$;
3. **while** $S_k - S_{k-1} > \varepsilon$ **do begin**

4. $RP := ISL(S_{k-1}, \hat{s}_k)$;
5. $S_{k+1} := LM(RP)$; $\hat{s}_{k+1} := \phi(S_{k+1})$;
6. $RP := ISU(S_k, \hat{s}_{k+1})$;
7. $S_{k+2} := LM(RP)$; $\hat{s}_{k+2} := \phi(S_{k+2})$;
8. $k := k + 2$;

end;

9. $S^* := S_k$; $s^* := \hat{s}_k$;

The algorithm always converges to the global minimum of C, the optimal policy is (s^*, S^*). An example for the iterations made, using the functions ϕ and ψ, is given in figure 3.2. The parameters are, as follows: $L = 1$, $\lambda = 1$, $h = 1$, $p = 10$, $K = 1$. The individual demands are distributed with a Gamma distribution, with parameters $\alpha = \beta = 200$. In this case, the global optimum is found in four iterations, numbered

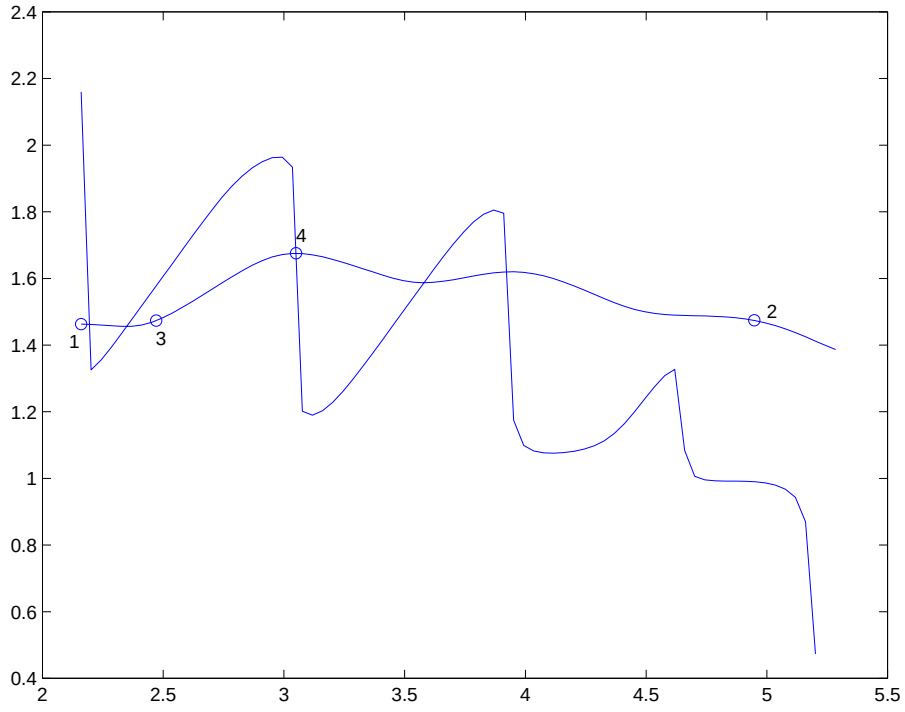

Figure 3.2: Some iterations of the algorithm

on the graph as they follow. After having found the first local maximum of ϕ, S_0, of value (1), the algorithm finds the upper bound for the optimal order-up-to level, S_1, of function value (2). Since (2) is larger than (1), (2) automatically defines the new relevance level. The next step is starting $ISL(S_0, (2))$, obtaining S_3, the level crossing point, (3). $LM(S_3)$ finds (4), which turns out to be the global maximum for ϕ, since $ISU(S_2, (4))$ finds back the same point, reducing the feasibility interval to zero. The optimal policy is $(1.6754, 3.0503)$.

Remark 3.7.1 *Steps 5 and 7 require the computation of $\hat{s}_k := \phi(S_k)$, where $\phi(S_k)$ is calculated for instance with bisection (see Section 3.7.3). To insure that that we do not overshoot the global minimum, just as we did for the sake of monotonicity before, we approximate $\hat{s}_k := \phi(S_k)$ such that $-\varepsilon \leq C(s, S_k) - c(s) \leq 0$, for any $\varepsilon > 0$. This will ensure that $\hat{s}_k \leq \hat{s}^*$, for every $k \geq 1$, where $\hat{s}^*$ represents the global maximum of ϕ. This means in fact, that the algorithm will never overshoot the global minimum of C (see Lemma 3.6.7), that is, for the found optimal policy $C(s^*, S^*) \geq min_{0 \leq s < S} C(s, S)$.*

3.7.3 Calculating $\phi(S)$ and $\psi(S)$

We owe the reader one more explanation, that is, how we solve the nonlinear equations $C(s, S_0) = c(s)$ and $h(s, S_0) = 0$ (which is, in fact, calculating $s = \phi(S_0)$ and $s = \psi(S_0)$) for a fixed S_0. For most of the software packages it is standard to solve equations which have only got a single root (cf. Lemma 3.3.1 this is the case for $\phi(S)$, and cf. Lemma 3.5.3 for $\psi(S)$). However, we propose an approximation, which is easy to evaluate, so much the more for instance in Step 7 the precision of the value of ϕ does not influence the precision of the found local maximum in Step 8, thus it does not influence the precision of the optimal policy. The precision of the optimal policy is only important in the last evaluation of the subroutine LM. Since we can choose any $\varepsilon > 0$, any precision can be attained. We describe now a bisection method because it is more accessible, although there are faster methods, such as the Newton method.

Subroutine PHI(S):

$l := \hat{s}_k$; $s_1 := \hat{s}_k - l/2$; $i := 0$;

while $l > \varepsilon$ **do begin**

$l := l/2$; $i := i + 1$;

If $C(s_i, S) < c(s_i)$

then $s_{i+1} := s_i + l/2$;
else $s_{i+1} := s_i - l/2$;

end {while}

We know from Lemma 3.3.1, relation (3.3.3.7) that $C(s,S) > c(s)$ if and only if $s > \bar{s}$ (where $C(\bar{s},S) = c(\bar{s})$). The rest of the routine speaks for itself. The subroutine PSI(S) is similar to PHI(S), except that in the if case the condition is $h(s_i,S) < 0$.

3.7.4 Speed of convergence

The algorithm basically consists of repeated evaluations of the subroutines LM, ISU, ISL, PHI, PSI, and the functions $C(s,S)$ and $c(s)$. The evaluations of the functions $C(s,S)$ and $c(s)$ are done with a Laplace transform inversion method (Den Iseger (2002)): due to their convolution structure, their Laplace transform is easy to calculate, which is then inverted. The results are accurate (up to machine precision) and they are obtained in fractions of time. The subroutines PHI and PSI use a simple bisection or Newton method for finding the unique solution of a nonlinear equation. The results are accurate up to an ε precision, for any $\varepsilon > 0$. However, it pays off not to choose ε very small, since it does not effect the convergence of the algorithm to the global minimum, while a larger ε can make the algorithm even faster. The three subroutines, LM, ISU, and ISL have the same speed of convergence, since in a neighborhood of the limit point they are very similar. Denoting with l_n the distance from the limit point at the nth iteration, we obtain for the different subroutines the following expressions for l_{n+1}:

- subroutine LM: $l_{n+1} = (1/(1-\psi_S(\hat{S}_k)))l_n$, with $\psi_S(\hat{S}_k) < 0$.
- subroutine ISU: $l_{n+1} = (1/(1-\phi_S(RP)))l_n$, with $\phi_S(RP) < 0$.
- subroutine ISL: $l_{n+1} = (1-\phi_S(RP))l_n$, with $\phi_S(RP) > 0$,

where $\hat{S}_k$ is the intersection point of ϕ and ψ, and RP is the point where ϕ crosses the actual relevance level. This means that the subroutines converge linearly, such that $l_{n+1} = \alpha l_n$, with $0 < \alpha < 1$. It is also important to remark that if N is the number of the local optima (N is always finite), then the algorithm will execute a local search at most N times. **Figure** 3.3 is used for determining the speed of convergence of subroutine LM in a neighborhood of the intersection point S_k, when LM converges from the left to the right.

Furthermore, from the speed of convergence expressions we can deduce the following relations:

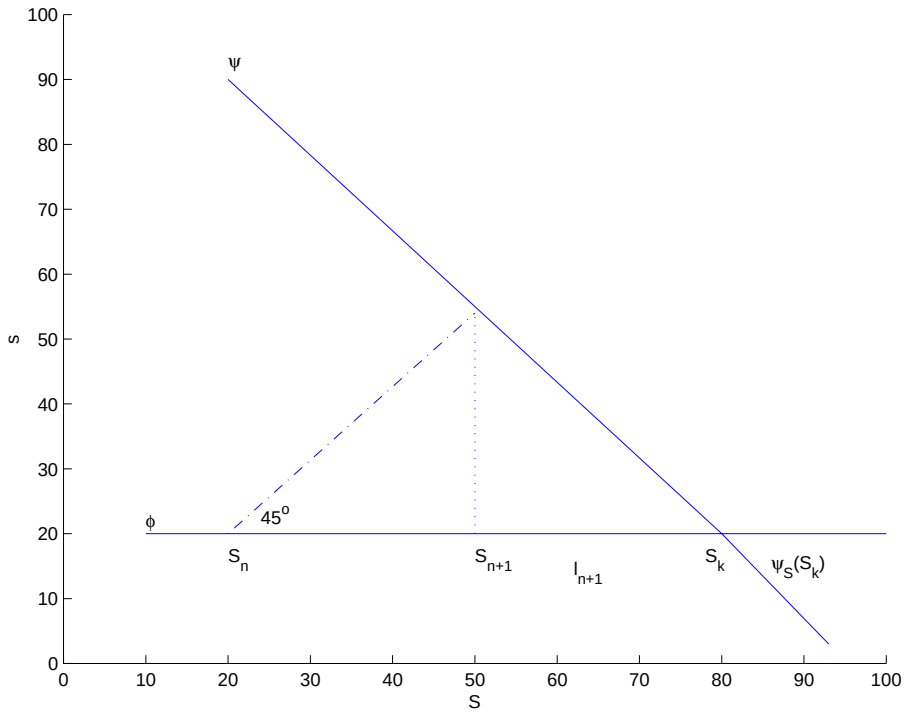

Figure 3.3: Determining the speed of convergence for subroutine LM

LM: If the subroutine goes from to the left to the right we have: $l_{n+1} - l_n = \psi_S(\hat{S}_k)l_{n+1}$ and $S_{n+1} - S_n = l_n - l_{n+1}$ imply, that taking a precision $\varepsilon > 0$ yields $\varepsilon = \psi_S(\hat{S}_k)l_{n+1}$. The Taylor expansion of ϕ in the point $\hat{S}_k$ yields $\phi(S_{n+1}) - \phi(\hat{S}_k) = 1/2\phi_{SS}(\hat{S}_k)l_{n+1}^2$, since $\phi_S(\hat{S}_k) = 0$ and $l_{n+1} = \hat{S}_k - S_{n+1}$. Substituting this into the speed of convergence expression, we obtain for the convergence of the ϕ values to the local maximum, that

$$\phi(S_{n+1}) - \phi(\hat{S}_k) = \frac{\varepsilon^2\phi_{SS}(\hat{S}_k)}{2\psi_S^2(\hat{S}_k)}, \tag{3.7.3.2}$$

where $\phi_{SS}(\hat{S}_k) < 0$. When the subroutine converges from the right to the left we obtain the same result.

ISU: $l_{n+1} - l_n = \phi_S(RP)l_{n+1} = \phi(S_{n+1}) - \phi(RP)$ $(\phi_S(RP) < 0)$, that is, $\phi(RP) - \phi(S_{n+1}) = \varepsilon$, for an $S_n - S_{n+1} = l_n - l_{n+1} = \varepsilon$ step size $(\forall\varepsilon > 0)$.

ISL: $l_n - l_{n+1} = \phi_S(RP) l_n$ $(0 < \phi_S(RP) < 1)$, that is,

$$l_n - l_{n+1} = \frac{\phi_S(RP)}{1 - \phi_S(RP)} l_{n+1}.$$

Having $\phi_S(RP) l_{n+1} = \phi(RP) - \phi(S_{n+1})$ and $S_{n+1} - S_n = l_n - l_{n+1}$, we obtain for any $\varepsilon > 0$ that

$$\phi(RP) - \phi(S_{n+1}) = \varepsilon(1 - \phi_S(RP)),$$

with $0 < \phi_S(RP) < 1$.

With regard to the speed of convergence of the subroutines ISU and ISL, the reader might wonder what happens at the last iteration of the algorithm, when the feasibility interval reduces to zero, yielding $\phi_S(S) = 0$ at the last iterations. In particular, consider the case when $(S_k, \hat{s}_k)$, the global maximum is found, such that $S_k < S_{k-1}$ (obviously, $\hat{s}_{k-1} < \hat{s}_k$). At this instance the feasibility interval is (S_k, S_{k-1}), $|S_k - S_{k-1}| > \varepsilon$. At the next step, $ISU(S_{k-1}, \hat{s}_k)$ should find back $S_{k+1} = S_k$, yielding $|S_{k+1} - S_k| < \varepsilon$. Observe, that in a neighborhood of S_k $\phi_S(S_n) = 0$, yielding $l_{n+1} = l_n$ (check the expression for speed of convergence). However, this also means that $S_{n+1} = S_n$, that is, the subroutine ISU stops. Now, the subroutine LM is run in this point, but its speed of convergence depends on ψ_S, having $\psi_S(S) << 0$ in a neighborhood of S_k. In conclusion, the algorithm terminates without any problem.

3.8 The Zheng–Federgruen algorithm for discrete order quantity (s, S) models

The algorithm is presented for the case of periodic review, that is, a discrete time (s, S) model, with i.i.d. and integer valued one period demand, the objective being to minimize long run average costs over an infinite horizon. At the end of the article they give suggestions for the extension of the algorithm for continuous review models and discounted costs. Since their periodic review system is equivalent to a discrete – time system consideration (see Section 2.8 for a short discussion), we denote with $\mathbf{Y}_n$ the nth – period demand, with common distribution $p_j := I\!P\{\mathbf{Y}_n = j\}$, for $j = 0, 1, \ldots$. Furthermore $G(y)$ denotes the one-period expected cost (holding and backorder cost), when the inventory position is y. It is assumed that $-G$ is unimodal with $\lim_{|y| \uparrow \infty} G(y) > \min_y G(y) + K$. Under these assumptions an (s, S) policy is

optimal when minimizing long run average costs (Veinott (1966), Zheng (1991)), given by the expression

$$C(s,S) = \frac{1}{M(S-s)} \left(K + \sum_{j=0}^{S-s-1} m(j) G(S-j) \right),$$

where m and M are defined recursively as

$$m(0) = (1-p_0)^{-1} \qquad\qquad M(0) = 0$$
$$m(j) = \sum_{l=0}^{j} p_l m(j-l) \qquad \text{and} \qquad M(j) = M(j-1) + m(j-1),$$

$j = 1, 2, \ldots$. This cost expression is just the discrete time version of (3.3.3.2), while G takes the place of c of our approach. The smallest and largest minimizers of G are defined as

$$y_1 := \min\{y : G(y) = \min_x G(x)\} \text{ and } y_2 := \max\{y : G(y) = \min_x G(x)\}, \qquad (3.8.3.1)$$

respectively. The algorithm is completely built on the properties of the cost function C, and the bounds for the optimal $s^\star$ and $S^\star$, which we list here.

1,(Lemma 1 of Z&F) For any given (fixed) order-up-to level S

(i) a reorder level $s^0 < y_1$ (cf.(3.8.3.1)) is optimal, i.e. $C(s^0, S) = \min_{s<S} C(s, S)$ if and only if

$$G(s^0) \geq C(s^0, S) \geq G(s^0 + 1). \qquad (3.8.3.2)$$

Multiple optima of s for fixed S occur when at least one of the inequalities in (3.8.3.2) holds as an equality.

(ii) there exists an optimal reorder level s^0 such that $s^0 < y_1$ and (3.8.3.2) holds.

(iii) denote with $s_l^0 < y_1$ and $s_u^0 < y_1$ the smallest and largest optimal reorder level, respectively. Then

$$G(s_l^0) > C(s_l^0, S) = C(s_u^0, S) > G(s_u^0 + 1). \qquad (3.8.3.3)$$

As we already mentioned previously, Lemma 3.3.1 is just the continuous version of (i). The difference is, that in the discrete case the solution of the 'marginal cost equation' (3.3.3.5) is not unique. Moreover (3.8.3.3) is analogous to relations (3.3.3.7) and (3.3.3.7) of Lemma 3.3.1, which we deduced from the marginal cost relation (see also **Figure** 3.1).

2,(Corollary 1 of Z&F) For any given order-up-to level S the reorder level

$$s^0 = \max\{y < y_1 : C(y, S) \leq G(y)\} \tag{3.8.3.4}$$

is an optimal and (3.8.3.2) holds.

3,(Bounds for $s^\star$) The smallest optimal reorder level $s_l^\star \leq y_1 - 1$. Furthermore, if s_0 satisfies (3.8.3.2) for an arbitrary S then $s^0 \leq s_u^\star$. We omit the alternative lower bounds provided by Zheng and Federgruen, since they are not needed for the optimization algorithm itself.

4,(Bounds for $S^\star$) and policy improvement step.

(i) Lower bound of Veinott and Wagner (1965): There exist an optimal policy $(s^\star, S^\star)$ with $S^\star \geq y_2$.

(ii) Upper bound of Zheng and Federgruen: For every optimal policy $(s^\star, S^\star)$, $S^\star \leq S_u := \max\{y \geq y_2 : G(y) \leq c^*\}$.

(iii) Improvement step for the upper bound: Any upper bound C of the optimal average cost C^* defines an upper bound $S_C := \max\{y \geq y_2 : G(y) \leq C\}$, such that $S_u \leq S_{C_1} \leq S_{C_2}$ if $C^* \leq C_1 \leq C_2$.

The upper bound S^u, given by relation (3.4.3.5) (derived in Lemma 3.4.2) is tighter than the one given in point (ii). However, we ought to remark that, in an algorithm such as the one of Zheng and Federgruen, which uses iterative improvements of the upper bound, it is more reasonable to use the upper bound $S_u := \sup\{S > s^\star : C^* \geq c(S)\}$. The reason for this is that the function $K(1 - F_Y(x))$ is not monotone, which implies that the consequent iterations might not improve the upper bound S^u monotonically either. The difference between the two upper bounds is shown in **Figure** 3.4.

Point (iii) is the direct consequence of the definition of the upper bound and the unimodality of $-G$; or, in the continuous case the unimodality of $-c$.

5,(Lemma 3 of Z&F) Updating the optimal S, and consequently the optimal s: For any given order-up-to level S^0 let s^0 be an optimal reorder level. Then,

(i) $\min_{s<S} C(s, S) < \min_{s<S} C(s, S^0)$ if and only if $C(s^0, S) < C(s^0, S^0)$

(ii) if (3.8.3.2) holds with S^0 and $C(s^0, S') < C(s^0, S^0)$ for some S' then $s' := \min\{y \geq s^0 : C(y, S') > G(y + 1)\} \leq y_1$ is optimal for S' (hence (3.8.3.2) holds with (s', S')).

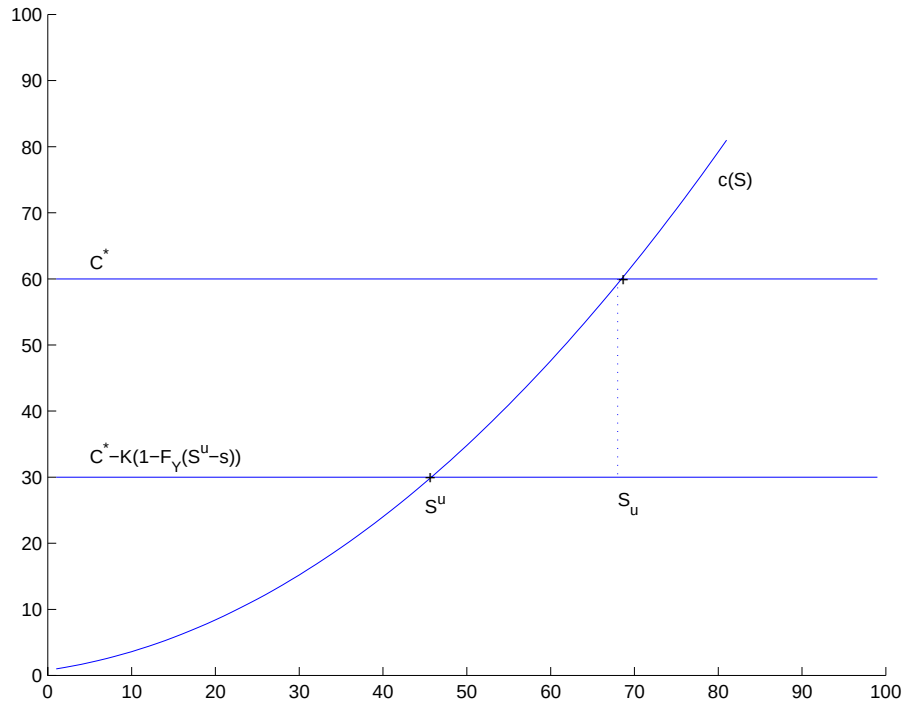

Figure 3.4: Upper bounds for the optimal order-up-to level S

This last step is again the analogue of the marginal cost relation, as it is deduced from relation (3.8.3.2). The continuous version works also the same: when an improvement for the local minimal order-up-to level S is found, one needs to find the corresponding optimal reorder level.

The algorithm is based on the properties listed above. First we give the algorithm then we explain how it works and finally we emphasize the essential differences with the continuous case.

3.8.1 The algorithm

Let y^* be a minimum point of G, then

Step 0. $s := y^*$; $S_0 := y^*$;

Repeat $s := s - 1$ until $C(s, S_0) \leq G(s)$;

$s_0 := s;\ \ C^0 := C(s_0, S_0);\ \ S^0 := S_0;\ \ S := S^0 + 1;$

Step 1. While $G(S) \leq C^0$ do begin

If $C(s, S) < C^0$ then begin

$S^0 := S;$

While $C(s, S^0) \leq G(s+1)$ do $s := s + 1;$

$C^0 := C(s, S^0)$

end;

$S := S + 1$

end;

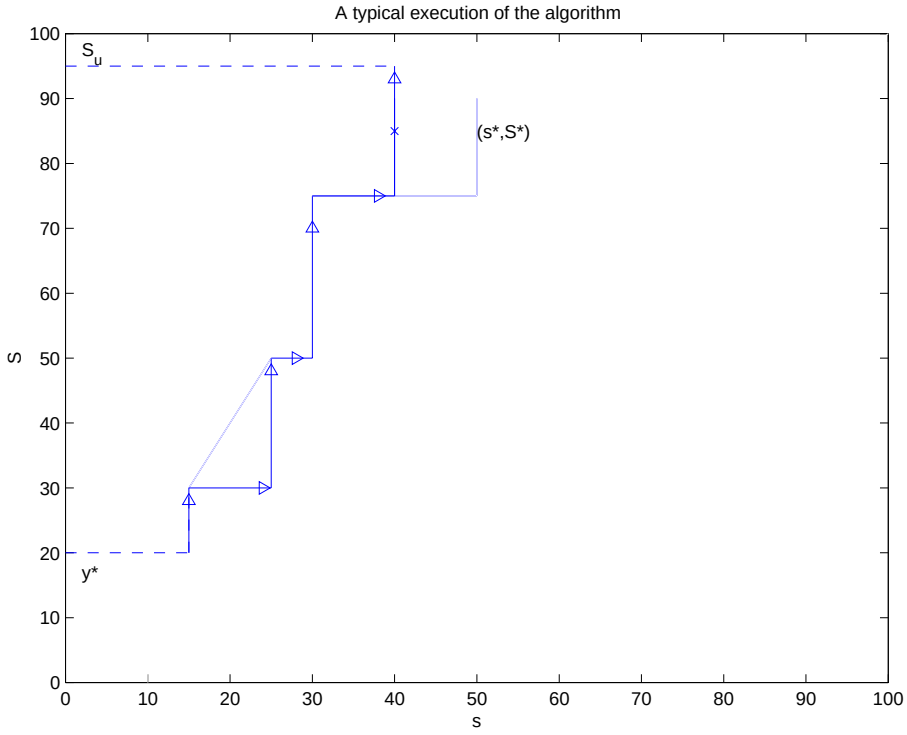

Figure 3.5: The Zheng–Federgruen algorithm

The algorithm starts S^0 in y^*, a lower bound of the optimal order-up-to level (see (4.i)), y^* being an arbitrary minimum of G. Since y^* is also an upper bound

of the (global) optimal reorder level, the optimal reorder level s^0, corresponding to S^0 , is found by decreasing s from y^* with enumeration until $c(s, S^0) \leq G(s)$ (see 2 and relation (3.8.3.4)). Now that the first local optimum (s^0, S^0) is found, in the discrete order size case one can simply proceed with enumeration, since there is no risk of skipping an optimum. **Figure** 3.5 depicts a typical execution of the algorithm. In the continuous case, on the contrary, there is the possibility of skipping the optimum, even with a small stepsize. Now (Step 1) the improvement step (5.i) is employed: increase S from S^0 with step sizes of one (thus: enumeration), while compare $C(s^0, S)$ and $C(s^0, S^0)$. As soon as improvement is achieved the value of S^0 is updated and a new optimal reorder level s^0 is determined: increment the value of s^0 until $C(s^0, S^0) > G(s^0 + 1)$ (see (5.ii)). In view of (4.iii), every time Step 1 restarts C^0 is updated, which represents an upper bound for the (global) minimum C^*. This upper bound defines an upper bound for the (global) optimal order-up-to level, hence the algorithm stops as soon as the upper bound is overshot, that is, $G(S) > C^0$. The stopping condition is also different in our continuous version algorithm: our algorithm defines already in the initialization step a search interval, which is defined by the lower and upper bounds of the optimal order-up-to level. During the execution of the algorithm the iterations are made alternatively from the beginning and the end of this search interval, which eventually reduces to zero. This is our stopping condition.

Thus, the main difference between our and theirs is that the continuous case prohibits enumeration: hence, every time the Zheng-Federgruen algorithm uses enumeration we need to construct a subroutine, which replaces the enumeration step while achieving policy improvement. This issue makes most of time continuous optimization a more difficult problem, and this is the point where our algorithm is completely different from the algorithm of Zheng and Federgruen. However, in the discrete case our algorithm is easily adjustable to use enumeration. In the continuous case, this leads sometimes to divergence of the idea's behind the algorithms, however the two algorithms remain very similar on a abstract level.

For the sake of completeness we also mention the complexity results of the Zheng–Federgruen algorithm, although for obvious reasons we cannot compare it to our complexity results. Assuming that y^* is known, the algorithm requires no more than $B = 2(\Delta_1 + 1)^2 + 2(\Delta_1 + 1) - (\Delta_2 + 1)^2 + 3\Delta_3$ elementary operations and $(\Delta_1 + 2)$ evaluations of the G–function, where $\Delta_1 := S_u - s_0$, $\Delta_2 := y^* - s_0$, $\Delta_3 := s^\star - s_0$. Moreover, assuming also that the values $G(s_0 + 1), \ldots, G(S_u + 1)$ are known, and letting $R(\Delta_1)$ denote the number of elementary operations needed for the algorithm, it follows that $R(\Delta_1) \leq 2.4$ and $\lim_{\Delta_1 \uparrow \infty} R(\Delta_1) = 2$ (cf. Theorem 1 of Zheng and Federgruen (1991)).

3.9 The Federgruen –Zipkin algorithm with continuous demand sizes

This algorithm is a direct extension of the discrete order quantity algorithm of Federgruen and Zipkin (1984a). Similarly to the Zheng–Federgruen algorithm, this one also tackles the periodic review case, where $G(y)$ stands for the one period expected cost, as before, with y denoting the the inventory position after ordering but before demand, while x is the inventory level at the beginning of the period. Let f and F denote the p.d.f. and c.d.f. of the one period demand, respectively. Here it is also assumed that $-G$ is continuous and strictly unimodal. Furthermore, the one-step total expected cost when x is the initial state, is defined as

$$\gamma(x,y) := \begin{cases} G(Y), & \text{if} \quad y = x \\ G(y) + K, & \text{if} \quad y > x. \end{cases}$$

If $R = (s, S)$ is the policy which prescribes the action $y = S$ when $x \leq s$ and $y = x$ when $x > s$, then $y_R(x)$ denotes the value of y prescribed by policy R in state x, and $\gamma_R(x) = \gamma(x, y_R(x))$. With this notation, g_R stands for the average cost incurred with policy R, while g^* is the minimal average cost over all policies. It is shown, through a renewal theoretic argument, that

$$g_R = \frac{[k_s(S) + K]}{U(S-s)},$$

where $U(S-s)$ is just the renewal function related to the sequence of one-period demands (see also Section 2.2.2), and $k_s(S)$, is given by the relation

$$k_s(y) = G(y) + \int_0^{y-s} G(y-u)m(u)du, \quad y > s,$$

with m the derivative of U. According to the policy iteration technique used for the algorithm, a 'relative cost function' v_R is defined as

$$v_R(x) := \begin{cases} [k_s(x) + K] - g_R U(S-s) & \text{if} \quad x > s \\ K, & \text{if} \quad x \leq s. \end{cases}$$

The continuous analogue of the optimality equation (see for instance (3.2.3.2) is proven in Lemma 1, that is,

$$v(x) = \gamma_R(x) - g + \int_0^{\infty} v(y_R(x) - u)f(u)du, \text{ for all } x, \qquad v(S) = 0, \qquad (3.9.3.1)$$

where $v(S) = 0$ serves as a normalization condition. Furthermore, a policy-iteration algorithm also requires so–called test quantities, in order to search for improved

policies. According to the standard theory these quantities (see also relation (3.2.3.1)) are defied as

$$I_R(x, y) = \gamma(x, y) - g_R + \int_0^\infty v_R(y - u) f(u) du, \qquad y \geq x. \tag{3.9.3.2}$$

Since the algorithm uses the test functions $I_R(x, y)$ only for certain values of (x, y), the expression (3.9.3.2) simplifies considerably in these cases. First of all, for any pair of policies R, $I_R(x, y_R(x)) = v_R(x)$. Further, for $y > x$ follows that

$$I_R(x, y) = K + G(y) - g_R + \int_0^\infty v_R(y - u) f(u) du,$$

and if also $y > s$, then, through $\gamma_R(y) = G(y)$ and (3.9.3.1), the previous becomes

$$I_R(x, y) = K + v_R(y), \qquad y > x, \qquad y > s. \tag{3.9.3.3}$$

The definition of v_R, together with (3.9.3.2) yields finally the expression for the case $x \leq s$:

$$I_R(x, y) = K + G(x) - g_R. \tag{3.9.3.4}$$

The algorithm is initialized with the bounds on s and S identified by Veinott (1966): $L^* \leq s \leq M^* \leq S \leq U^*$, where M^* is the unique minimum of $G(y)$, U^* is the unique solution to $G(y) = G(M^*) + K$ for $y > M*$, and L^* is the unique solution to $G(y) = G(M^*) + K$ for $y < M*$. The upper bound for the optimal order-up-to level used with the two previous approaches are tighter. Thus the algorithm is stated as follows:

Step 0 *Initialize:* Compute L^*, M^*, U^*, and choose an initial policy $R = (s, S)$ such that $L^* \leq s \leq M^* \leq S \leq U^*$.

Step 1 *Policy evaluation:* Compute g_R.

Step 2 *Policy improvement:*

(a) Find $M^* \leq S' \leq U^*$ satisfying $v_R(S') = \min_{M^* \leq y \leq U^*} v_R(y)$.

(b)

(i) If $G(s) - g_R > v_R(S')$ then
$s' := \min\{x \in (s, M^*] : v_R(x) - K = v_R(S')\}$ if it exists;
otherwise set $s' = M^*$ (cf. (3.9.3.3))

(ii) If $G(s) - g_R \leq v_R(S')$ then set $s' \in [L^*, s]$ to be the root of $G(x) = g_R$, if no root exists set $s' = L^*$.

Step 3 Return to Step 1.

It seems rather difficult to give a geometrical interpretation to the policy improvement step 2.(b), yet, it is possible to relate this step to the algorithm of Bázsa and Den Iseger (2001b) (see also Section 3.6.2): The cases (i) and (ii) can be interpreted as the two conditions for being under or below the graph of the function $\phi(S)$, exactly what we also exploit in our procedure. Furthermore, the equation $G(x) = g_R$ is in fact $C(s,S) = c(s)$, that is, calculating $\phi(S)$. We proved that a solution of this equation does exist, moreover it is unique, and we also gave a simple procedure for finding this root.

A termination test can also be incorporated into the algorithm to ensure finite convergence to an ε-optimal policy R (that is, $g_R \leq g^* + \varepsilon$) for any precision $\varepsilon > 0$. Defining thus

$$v_R^+ = \sup_{s \leq x \leq M^*} v_R(x),$$

and

$$\delta_R = \min\{v_R(S') + K - v_R^+, G(s) - g_R\},$$

let Step 3 include the following termination test:

If $G(s) - g_R > v_R(S')$: $\delta_R = v_R^+ - v_R(S') - K$.

If $G(s) - g_R \leq v_R(S')$: $\delta_R = g_R - G(S)$.

Stop if $\|\delta\| \leq \varepsilon$ then $R' = (s', S')$ is $\varepsilon -$ optimal. Else, $R = R'$ and go to Step 1.

The policy improvement step does not always select a policy achieving the best one-step improvement, as in standard policy-iteration (Federgruen and Zipkin (1985)). Strictly speaking, Step 2(a) itself requires a search procedure, since v_R need not be convex, except if the renewal function $U(x)$ is concave (Sahin (1982)). However, convergence to the optimum is still guaranteed if Step 2(a) is modified to set S' to be any value $y \in [M^*, U^*]$ such that $v_R(y) < v_R(S) = 0$; if no such value exists, set $S' = S$. The search for an optimal order up to level S is completely different in the Bázsa–Den Iseger algorithm. There the very convenient properties of the functions $\phi(S)$ and $\psi(S)$ are exploited, while always obtaining a policy improvement.

3.10 Discussion

Ongoing research on supply chains has increased the interest in continuous-demand models. Clark and Scarf (1960) were the first to show that the optimal policy for

a two-echelon inventory system, with finite horizon, can be computed by decomposing the problem into two separate single-location problems. For the depot an (s, S) policy solves the problem, and constitutes an optimal order policy for the whole system. However, the lack of an efficient optimization algorithm for such single location problems results in suboptimality of the overall solution. Although recent research has extended the results of Clark and Scarf (see, for instance, Eppen and Schrage (1981), Federgruen and Zipkin (1984), Rosling (1989)), a truly efficient algorithm for the continuous order quantity (s, S) policy was missing until now.

To our knowledge, the only successful attempt to tackle the continuous demand case directly, without previous discretization, is the optimization algorithm of Federgruen and Zipkin(1985). This algorithm is a direct extension of a similar algorithm for discrete order sizes (Federgruen and Zipkin (1984a)), the dominating idea being that of a policy iteration method from Markov decision theory. Although standard policy iteration always results in policy improvement, this algorithm does not always achieve the best one-step improvement. Yet, convergence is always obtained, and the algorithm does itself represent a big improvement upon the known algorithms up to date.

The state of the art for (s, S) models with discrete demand size was much more advanced: Zheng and Federgruen(1991) present an improvement in the complexity of the calculation of an optimal (s, S) policy, developing an efficient and intuitive algorithm. Although their algorithm does not apply to continuous order quantity systems (since the continuous case prohibits enumeration), some of the ideas translate nicely into our algorithm. However, the similarities between these two algorithms do not really mirror the similarities between the two Federgruen–Zipkin algorithms (discrete and continuous case, from 1984 and 1985, respectively), since our algorithm has a radically different approach: Based on the optimality relations we construct two aid functions, $s = \phi(S)$ and $s = \psi(S)$. It turns out that these aid functions have two simple but crucial properties: (1) both of them increase slower than 45 degrees; and (2) ψ always intersects ϕ in a maximum or minimum of ϕ, and these are the only stationary points for ϕ. In fact, the algorithm merely exploits the 'geometrical properties' of these two functions. The algorithm converges monotonically, such that at every iteration a policy improvement is obtained. Since every iteration finds a local minimum of the expected average cost, the number of iterations is at most N, where $N < \infty$ represents the number of local minimums. Besides, the algorithm is also easy to understand, every step can be followed using a simple graphical representation.

Chapter 4

Inventories in interaction with the world

4.1 Introduction

Although very general, the analysis of a single item inventory system, as it was done in Chapter 2, can be extended substantially. The first point to tackle would be incorporating nonstationary policies. The second is to consider more general demand processes, or even more general systems. For example, demand could depend on an exogenous factor (Song and Zipkin (1993), Sigman (1990)) or the dependency could be of endogenous nature (Lovejoy (1992), Glynn and Sigman (1992)). In fact, such a system leads to a solution method for networks, since there the primal difficulty is to deal with the various inter-dependencies of the components of the networks. In the present chapter we only give examples of single-item single-echelon systems, with more complicated dependency structures. The next chapter depicts a two echelon decentralized system, which can be solved with the theory developed in the chapters prior to that one, for more general demand processes than before (Axsäter (1997)).

The classical renewal reward theorem is extensively used in operation research to calculate the expected long run average cost of a system. As its name suggests, this theorem is only applicable to so called renewal reward processes (Ross (1970)), that is, $(\mathbf{T}_n, \mathbf{Y}_n), n = 1, 2, \ldots$ are i.i.d., where $\mathbf{T}_n$ stand for the interarrival times and $\mathbf{Y}_n$ is the reward earned at the nth arrival, and $\mathbf{X}(t) = \sum_{n=1}^{\mathbf{N}(t)} \mathbf{Y}_n$. The renewal reward theorem asserts (Ross (1970)) that if $I\!E|\mathbf{Y}_n|$ and $I\!E\mathbf{T}_n$ are finite then $\mathbf{X}(t)/t$ converges to $I\!E\mathbf{Y}_1/I\!E\mathbf{T}_1$ in L^1 as $t \to \infty$. This is indeed a powerful result, but its limitation is exactly the fact that $(\mathbf{T}_n, \mathbf{Y}_n), n = 1, 2, \ldots$ has to be an i.i.d.

sequence. Under the simultaneous influence of the works of Song and Zipkin (1993) and Glynn and Sigman (1992), we aim to develop a more powerful reward theorem for general stochastic processes. Song and Zipkin (1993) considered inventory models in which the demand process is described by a Markov modulated Poisson process, that is, the arrival process of customers is driven by an exogenous state-of-the-world variable: when the world is in state i, demand follows a Poisson process with rate λ_i. Glynn and Sigman (1992), on the other hand, develop a reward theorem for synchronous processes: a process with identically distributed but dependent cycles. However many operations research problems cannot be modeled with this approach. For example inventory models: it is rather easy and not excessively restrictive to make the necessary assumptions and show that **IP** and **D** are synchronous processes. However, the 'synchronousness' property in general does not preserve under measurable functions of more than one variable, hence it is very difficult to prove that the netstock process **IN**, needed for the actual reward theorem, is also synchronous.

Our approach is to consider general stochastic processes, for which all the relevant information about the past can be described by a Harris chain. This means, that there exists an embedded Harris chain (its points being a subset of the arrival points of the stochastic process) such that by conditioning on the present state of the Harris chain, the future of the stochastic process will be independent of its past. From this point on, the stochastic process will inherit some very advantageous properties from the embedded Harris chain: it can be shown that it has a general regenerative property, which is much less restrictive than the classical regenerative property defined in Chapter 2. Now, this general regeneration does preserve under measurable functions, moreover, the average cost expression we deduced is in terms of simple 'arrival cycles', instead of the regeneration cycles. This is very important, because in general the regeneration cycles, as we have seen in Section 2.2 for classical regenerative processes, do not coincide with the simple arrival cycles, consequently being considerably more difficult to determine. As it turns out, many relevant operations research problems can be modeled with this technique, such as an inventory model where demand is described by a general stochastic process. As a solution method we suggest a Markov Decision technique. With this technique a nonstationary optimal policy can be obtained, for instance when demand is nonstationary.

This chapter is organized in the following way. Sections 4.2 up to 4.5 provide the background and essential properties needed for the limit theorems. Section 4.5.4 brings the main theorem, while Section 4.6 provides examples of relevant models from inventory theory, which can be solved with the new technique.

4.2 Measure theoretic background

Let us start with a review of the measure theoretic terminology, which we will use throughout the following sections. The definitions are taken from the book of Thorisson (2000). A random element in a measurable space $(E, \mathcal{E})$, defined on a probability space $(\Omega, \mathcal{F}, I\!P)$, is a measurable mapping $\mathbf{Y}$ from $(\Omega, \mathcal{F}, I\!P)$ to $(E, \mathcal{E})$, that is,

$$\{\mathbf{Y} \in A\} \in \mathcal{F}, \quad \text{for all} \quad A \in \mathcal{E},$$

where

$$\{\mathbf{Y} \in A\} := \{\omega \in \Omega : \mathbf{Y}(\omega) \in A\} =: \mathbf{Y}^{-1}A.$$

Definition 4.2.1 *It is also said that* $\mathbf{Y}$ *is supported by* $(\Omega, \mathcal{F}, I\!P)$*, and* $\mathbf{Y}$ *is an* $\mathcal{F}/\mathcal{E}$ *measurable mapping from* Ω *to* E*.*

The distribution of a random element $\mathbf{Y}$ (under $I\!P$) is the probability measure on $(E, \mathcal{E})$ induced by $I\!P\mathbf{Y}^{-1}$. Since $I\!P\{\mathbf{Y} \in A\} = I\!P\mathbf{Y}^{-1}A$ for all $A \in \mathcal{E}$, we use the notation $I\!P\{\mathbf{Y} \in \cdot\}$. A random element $\mathbf{Y}$ is *canonical* if $\mathbf{Y}$ is the identity mapping, that is, if

$$(\Omega, \mathcal{F}) = (E, \mathcal{E}) \quad \text{and} \quad \mathbf{Y}(\omega) = \omega, \ \omega \in \Omega. \quad \text{Then} \quad I\!P\{\mathbf{Y} \in \cdot\} = I\!P.$$

A random element $\hat{\mathbf{Y}}$ in $(E, \mathcal{E})$, defined on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{I\!P})$ is a *copy* or *representation* of $\mathbf{Y}$ if

$$\hat{I\!P}\{\hat{\mathbf{Y}} \in \cdot\} = I\!P\{\mathbf{Y} \in \cdot\}, \quad \text{that is,} \quad \hat{\mathbf{Y}} \stackrel{d}{=} \mathbf{Y}.$$

A random element $\mathbf{Y}$ *always has a canonical representation*, the canonical random element on $(E, \mathcal{E}, I\!P\{\mathbf{Y} \in \cdot\})$.
A *random variable* $\mathbf{Y}$ is a random element in $(I\!R, \mathcal{B})$, where $I\!R$ is the set of real numbers and $\mathcal{B}$ denotes its Borel subsets (i.e. $\mathcal{B}$ is the σ-algebra generated by the open sets). The following definition is from the book of Thorisson (2000).

Definition 4.2.2 *The random variable* $\mathbf{Y}_1$ *is spread out if there exists a finite* $n \in I\!N$ *and a function* $g \in \mathcal{B}^+$ *such that* $\int_{I\!R} g(x)dx > 0$ *and, with* $\mathbf{Y}_2, \ldots, \mathbf{Y}_n$ *i.i.d. copies of* $\mathbf{Y}_1$*,*

$$I\!P\{\mathbf{Y}_1 + \ldots + \mathbf{Y}_n \in B\} = F_Y^{n*}(B) \geq \int_B g(x)dx, \qquad B \in \mathcal{B}.$$

This means (Asmussen (1987)), that F_Y is spread out if F_Y^{n*} (for some $n \in I\!N$) has a component G ($0 \neq G \leq F_Y$) which is absolutely continuous, i.e. G has a density g with respect to the Lebesgue measure.

Observation 4.2.3 *Suppose now that F_Y is spread out, and F_Y^{n*} has an absolute continuous component G. Then, since the derivative of the convolution function $G * F_Y$ is given by $g * F_Y$, it follows that for every integer $m \geq n$, F_Y^{m*} has an absolute continuous component, $G * F_Y^{(m-n)*}$.*

4.2.1 Conditional independence

Let $\mathbf{Y}_0, \mathbf{Y}_1$ and $\mathbf{Y}_2$ be random elements in $(E_0, \mathcal{E}_0), (E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2)$ defined on $(\Omega, \mathcal{F}, I\!P)$. Say that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are *conditionally independent given* $\mathbf{Y}_0$ if

$$I\!P\{\mathbf{Y}_1 \in \cdot, \mathbf{Y}_2 \in \cdot | \mathbf{Y}_0\} = I\!P\{\mathbf{Y}_1 \in \cdot | \mathbf{Y}_0\} I\!P\{\mathbf{Y}_2 \in \cdot | \mathbf{Y}_0\}, \tag{4.2.4.1}$$

that is, for all bounded $\mathcal{E}_i/\mathcal{B}$ measurable functions $f_i, i = 0, 1, 2$, it holds that

$$I\!E(f_0(\mathbf{Y}_0) f_1(\mathbf{Y}_1) f_2(\mathbf{Y}_2)) = I\!E\left(f_0(\mathbf{Y}_0) I\!E(f_1(\mathbf{Y}_1)|\mathbf{Y}_0) I\!E(f_2(\mathbf{Y}_2)|\mathbf{Y}_0)\right).$$

This is equivalent to

$$I\!P\{\mathbf{Y}_2 \in \cdot | \mathbf{Y}_0, \mathbf{Y}_1\} = I\!P\{\mathbf{Y}_2 \in \cdot | \mathbf{Y}_0\}.$$

Due to the latter expression it is also said that $\mathbf{Y}_2$ *depends on* $\mathbf{Y}_1$ *only through* $\mathbf{Y}_0$.

4.2.2 Modes of convergence

Let $\{\mathbf{Y}_n : n \in I\!N\}$ be a sequence of random variables, and $\mathbf{Y}$ a random variable, all defined on $(\Omega, \mathcal{F}, I\!P)$. Let us summarize the types of convergence used in the following sections (Williams (1991)):

- **almost surely:** $\mathbf{Y}_n \to \mathbf{Y}$ a.s. if $I\!P\{\mathbf{Y}_n \to \mathbf{Y}\} = 1$ as $n \to \infty$;
- **in probability:** $\mathbf{Y}_n \xrightarrow{d} \mathbf{Y}$ if for every $\varepsilon > 0$ $I\!P\{|\mathbf{Y}_n - \mathbf{Y}| > \varepsilon\} \to 0$ as $n \to \infty$;
- **$\mathcal{L}^p$ convergence:** $\mathbf{Y}_n \xrightarrow{p} \mathbf{Y}$ if $\mathbf{Y}_n, \mathbf{Y} \in \mathcal{L}^p, n \in I\!N$ and $I\!E(|\mathbf{Y}_n - \mathbf{Y}|^p) \to 0$ as $n \to \infty$;
- **total variation:** $\mathbf{Y}_n \xrightarrow{\text{tv}} \mathbf{Y}$ if $\sup_{A \in \mathcal{E}} |I\!P\{\mathbf{Y}_n \in A\} - I\!P\{\mathbf{Y} \in A\}| \to 0$ as $n \to \infty$;

Almost sure convergence and $\mathcal{L}^p$–convergence (for $p \geq 1$) imply convergence in probability.

4.2.3 Uniform integrability

As we will show later, a necessary and sufficient condition for our limit theorem is uniform integrability (U.I.). This follows directly from the fact that uniform integrability is a necessary and sufficient condition for L^1–convergence (Williams (1991)).

Definition 4.2.4 *A family of random variables* $\mathbf{Y}_n$, $n \in I\!N \cup \{0\}$ *(or,* $\mathbf{Y}_s$, $s \in [0, \infty)$*) is uniform integrable if*

$$\sup_{n \geq 0} I\!E\left(|\mathbf{Y}_n| 1_{\{\mathbf{Y}_n > x\}}\right) \to 0, \qquad x \to \infty.$$

The following result can be found in the book of Williams (1991).

Theorem 4.2.5 *If* $(\mathbf{Y}_n)$ *is a sequence in* L^1 *and* $\mathbf{Y} \in \mathrm{L}^1$*, then* $I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \longrightarrow 0$ *if and only if the following two conditions are satisfied:*

$$(i) \;\; \mathbf{Y}_n \longrightarrow \mathbf{Y} \text{ in probability,}$$

$$(ii) \text{ the sequence } \{\mathbf{Y}_n\} \text{ is U.I.}$$

Scheffé's Lemma (Williams (1991)) proves to be very useful, since the result which is more often needed than $I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \longrightarrow 0$ is $I\!E(\mathbf{Y}_n) \longrightarrow I\!E(\mathbf{Y})$.

Lemma 4.2.6 *Suppose that* $\mathbf{Y}_n, \mathbf{Y} \in \mathrm{L}^1$ *such that* $\mathbf{Y}_n \to \mathbf{Y}$ *a.e. Then*

$$I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \to 0 \text{ if and only if } I\!E(|\mathbf{Y}_n|) \to I\!E(|\mathbf{Y}|).$$

Since almost surely convergence implies convergence in probability (see the previous subsection), and having $\mathbf{Y}_n, \mathbf{Y} \in \mathrm{L}^1$ with $\mathbf{Y}_n \to \mathbf{Y}$ a.s., it follows that $I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \to 0$ if and only if $I\!E(|\mathbf{Y}_n|) \to I\!E(|\mathbf{Y}|)$ if and only if the sequence $(\mathbf{Y}_n)$ is U.I. Moreover, uniform integrability and a.s. convergence imply $I\!E(\mathbf{Y}_n) \to I\!E(\mathbf{Y})$. If $\mathbf{Y}_n$ are positive then no distinction is needed.

4.2.4 σ–finite measure

For the sake of completeness we introduce the notion of a σ–finite measure, which, although will not be used directly, will be present in some of the statements. A measure μ on a field $\mathcal{F}$ in Ω is σ–finite (Billingsley (1986)) if $\Omega = A_1 \bigcup A_2 \bigcup \ldots$ for some finite or countable sequence of $\mathcal{F}$–sets, satisfying $\mu(A_k) < \infty$. A σ–finite measure can be finite or infinite; a finite measure is by definition σ–finite. An important result is that if μ is a σ–finite measure on the field $\mathcal{F}$, then $\mathcal{F}$ cannot contain an uncountable, disjoint collection of sets of positive μ–measure.

4.3 General stochastic processes

A stochastic process with index set $\mathbb{I}$ and state space $(E, \mathcal{E})$ is a family $\mathbf{X} := (\mathbf{X}_s)_{s \in \mathbb{I}}$, where the $\mathbf{X}_s$ are random elements defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and all taking values in $(E, \mathcal{E})$. Now, rather than regarding $\mathbf{X}$ as a family of random elements in $(E, \mathcal{E})$, we can equivalently regard $\mathbf{X}$ as a random mapping (Thorisson (2000)), that is, a *single random element* in $(E^{\mathbb{I}}, \mathcal{E}^{\mathbb{I}})$, defined by

$$\mathbf{X}(\omega) = \{\mathbf{X}_s(\omega) : s \in \mathbb{I}\}, \ \ \omega \in \Omega.$$

In the rest of the thesis we restrict ourselves to $\mathbb{I} = [0, \infty)$ (one – sided continuous time). The *paths* of $\mathbf{X}$ are the realizations $\mathbf{X}(\omega)$, $\omega \in \Omega$ of the random mapping $\mathbf{X}$. Most of the time there are restrictions put on the path, for our case, that they are right continuous with left limits. More generally, one can say that they lie in a subset H of $E^{[0,\infty)}$ (Thorisson (2000)). In this case it is more natural to say that $\mathbf{X}$ is a random element in $(H, \mathcal{H})$, instead of $(E, \mathcal{E})$, where $\mathcal{H}$ is the σ-algebra on H, generated by the projection mapping taking $x \in H$ to $x_t \in E$, for all $t \in [0, \infty)$. $\mathcal{H}$ is also called the trace of H on $\mathcal{E}^{[0,\infty)}$ (Thorisson (2000)), because

$$\mathcal{H} = \mathcal{E}^{[0,\infty)} \bigcap H := \{A \bigcap H : A \in \mathcal{E}^{[0,\infty)}\}.$$

In conclusion, for every $t > 0$ $\mathbf{X}_t$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ if and only if $\mathbf{X}$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(H, \mathcal{H})$, and $(H, \mathcal{H})$ is called the *path space* of $\mathbf{X}$. It is important to realize that H need not be an element of $\mathcal{E}^{[0,\infty)}$. In particular, H is not an element of $\mathcal{E}^{[0,\infty)}$ in the standard settings in continuous time when $(E, \mathcal{E})$ is Polish[1]. Then H is one of the sets

$C_E([0, \infty))$ continuous maps from $[0, \infty)$ to E,

$D_E([0, \infty))$ right–continuous maps from $[0, \infty)$ to E with left–hand limits,

$R_E([0, \infty))$ right–continuous maps from $[0, \infty)$ to E.

Since later we will consider stochastic processes with path set $D_E([0, \infty))$, we mention here that $D_E([0, \infty))$ can be metrized such that the Borel σ–algebra $\mathcal{D}_E([0, \infty))$ is the trace of $D_E([0, \infty))$ on $\mathcal{E}^{[0,\infty)}$.

Furthermore, the distribution of a stochastic process $\mathbf{X}$ with path space $(H, \mathcal{H})$ is the distribution of $\mathbf{X}$ as a random element is $(H, \mathcal{H})$.

[1] A measurable space $(E, \mathcal{E})$ is Polish if E is a complete separable metric space and $\mathcal{E}$ is generated by the open sets.

4.3.1 Shift measurability

Observing a continuous time stochastic process at a random time means the following. Let $\mathbf{T}$ be a random time in $[0, +\infty)$; by $\mathbf{X}_\mathbf{T}$ we mean the E valued mapping defined on Ω as $\mathbf{X}_\mathbf{T}(\omega) := \mathbf{X}_{\mathbf{T}(\omega)}(\omega)$, for all $\omega \in \Omega$. This mapping need not be $\mathcal{F}/\mathcal{E}$ measurable! (Thorisson (2000), see also Definition 4.2.1). To take care of this measurability problem, one needs to impose a *canonical joint measurability* condition (Thorisson (2000)):

Definition 4.3.1 *The process* $\mathbf{X}$ *is canonically jointly measurable if the mapping taking* $(x,t) \in H \times [0,\infty)$ *to* $x_t \in E$ *is* $\mathcal{H} \otimes \mathcal{B}[0,\infty)/\mathcal{E}$ *measurable.*

This condition suffices for drawing the conclusion that if $\mathbf{X}'$ is also a stochastic process such that $(\mathbf{X}, \mathbf{T}) \stackrel{d}{=} (\mathbf{X}', \mathbf{T})$, then $\mathbf{X}_\mathbf{T} \stackrel{d}{=} \mathbf{X}'_\mathbf{T}$. However, rather than observing a stochastic process *at* a random time, we need most of the time to observe the whole process *from* that time onwards. Canonical joint measurability is insufficient for this purpose, hence the definition is extended in the following manner (Thorisson (2000)):

Definition 4.3.2 *The path set* H *of a continuous time stochastic process* $\mathbf{X}$ *is internally shift–invariant if*

$$\{\phi_t x : x \in H\} = H, \text{ for all } t \in [0,\infty),$$

where $\phi_t x = (x_{t+s})_{s\in[0,\infty)}$, *for all* $x \in H$. *The process* $\mathbf{X}$ *is said to be* **shift-measurable** *if its path set* H *is internally shift-invariant and if the mapping taking* $(x,t) \in H \times [0,\infty)$ *to* $\phi_t x \in H$ *is* $\mathcal{H} \otimes \mathcal{B}[0,\infty)/\mathcal{H}$ *measurable.*

Finally, a stochastic process with internally shift-invariant path space is shift measurable if and only if it is canonically jointly measurable. The standard cases, where the paths are right continuous with left limits and the state space $(E, \mathcal{E})$ is Polish, are all covered by shift measurability, in fact, completeness of E is not even necessary (Thorisson (2000)).

4.3.2 Back to convergence

For the shift measurable stochastic processes $\mathbf{X}$ and $\mathbf{X}'$, plain total variation convergence is defined similarly to that for random elements:

$$\|I\!P\{\phi_t\mathbf{X} \in \cdot\} - I\!P\{\phi_t\mathbf{X}' \in \cdot\}\| \to 0, \text{ as } t \to \infty.$$

If $\mathbf{U}$ is a uniform random variable on $[0,1]$, then the Cesaro (or time-average) total variation convergence is defined as

$$\|I\!P\{\phi_{\mathbf{U}t}\mathbf{X} \in \cdot\} - I\!P\{\phi_{\mathbf{U}t}\mathbf{X}' \in \cdot\}\| \to 0, \text{ as } t \to \infty.$$

4.3.3 Invariant σ–field

Ergodicity for stochastic processes is defined through the so called *invariant field* or invariant σ-algebra. The invariant field (Thorisson (2000)) consists of path sets in $\mathcal{H}$ that do not depend on where the origin is placed; it is formally defined as:

$$\mathcal{I} = \{A \in \mathcal{H} \,:\, \phi_t^{-1} A = A,\ 0 \le t < \infty\}. \tag{4.3.4.1}$$

This is a σ-algebra because if A is the union of the sets A_k, satisfying $\phi_t^{-1} A_k = A_k$, then $\phi_t^{-1} A = A$; $\mathcal{I}$ is also closed under complementation and it contains H.

4.3.4 Regeneration

Let us introduce some properties for general stochastic processes, which generalize the classical regenerative property introduced in Chapter 2, Section 2.2. The first concept we introduce is called wide-sense regeneration, first defined by Smith (1955), using the term 'equilibrium process'. At that time the term and the property remained unnoticed, and later it was rediscovered independently by Asmussen and Thorisson. Lag–l regeneration is somewhat more restrictive than wide-sense regeneration (Asmussen and Thorisson (1987)), one-dependent regeneration is noted in the dissertation of Glynn in 1982, and can be found in the article of Sigman (1990). To our best knowledge wide-sense k-dependence is only noted for the first time by Bázsa and Den Iseger (2001).

A shift measurable stochastic process $\mathbf{X}$ is *wide-sense regenerative* with regeneration times $\mathbf{S} = (\mathbf{S}_n)_0^\infty$ (Thorisson (2000), Chapter 10, Section 4) if

$$\phi_{\mathbf{S}_n}(\mathbf{X}, \mathbf{S}) \stackrel{d}{=} (\mathbf{X}^0, \mathbf{S}^0), \quad n \in I\!N \cup \{0\}, \tag{4.3.4.2}$$

where $(\mathbf{X}^0, \mathbf{S}^0)$ is the zero-delayed version of $(\mathbf{X}, \mathbf{S})$, and

$$\phi_{\mathbf{S}_n}(\mathbf{X}, \mathbf{S}) \text{ is independent of } (\mathbf{S}_0, \ldots, \mathbf{S}_n), \quad n \in I\!N \cup \{0\}. \tag{4.3.4.3}$$

The shift $\phi_{\mathbf{S}_n}$ has to be interpreted here as a two-dimensional shift which places the origin to $\mathbf{S}_n$. Furthermore, the process $\mathbf{X}$ is called *k-dependent* (Glynn and Sigman (1992)) with $k \in I\!N$ and $\mathbf{S}$, if for each $n \in I\!N$ $(\mathbf{D}, \mathbf{C}_1, \ldots, \mathbf{C}_n)$ and $(\mathbf{C}_{n+k+1}, \ldots)$ are independent, where $\mathbf{D} := (\mathbf{X}_s)_{s \in [0, \mathbf{S}_0)}$ is the delay, and $\mathbf{C}_n := (\mathbf{X}_{\mathbf{S}_{n-1}+s})_{s \in [0, \mathbf{L}_n)}$, with $\mathbf{L}_n = \mathbf{S}_n - \mathbf{S}_{n-1}$, $n \in I\!N$. The interpretation of the stochastic processes $\mathbf{D}$ and $\mathbf{C}_n$ is to think of them as random elements entering an absorbing state Δ when vanishing, where Δ is external to the state space. The cycle lengths $\mathbf{L}_1, \mathbf{L}_2, \ldots$ and the delay length $\mathbf{S}_0$ are all obtained by the same measurable mapping from their respective cycles $\mathbf{C}_1, \mathbf{C}_2, \ldots$ and delay $\mathbf{D}$ (Thorisson (2000)).

As a combination of the two notions, wide-sense regenerative and k-dependent processes, we define a third regeneration related notion, which generalizes the wide-sense regeneration, as follows:

Definition 4.3.3 *The process* $\mathbf{X}$ *is said to be wide-sense* k*-dependent with* $k \in \mathbb{N}$ *and the sequence* $\mathbf{S}$ *if* $(\mathbf{X}, \mathbf{S})$ *satisfies (4.3.4.2) and*

$$\phi_{\mathbf{S}_n}(\mathbf{X}, \mathbf{S}) \text{ is independent of } (\mathbf{S}_0, \ldots, \mathbf{S}_{n-k}), \quad n \in \mathbb{N}. \tag{4.3.4.4}$$

Observation 4.3.4 *A very important difference between the wide-sense regenerative and classical regenerative properties is the following (Thorisson (2000)): If* $(\mathbf{X}, \mathbf{S})$ *is classical regenerative the path process* $(\phi_s \mathbf{X})_{s \in [0,\infty)}$ *with state space* $(H, \mathcal{H})$ *is in general not classical regenerative (unless* $\mathbf{X}$ *is a non-random constant), but it is wide-sense regenerative with regeneration times* $\mathbf{S}$*. The wide-sense regenerative, as well the wide-sense* k*-dependent regenerative properties are also preserved under measurable functions* f*, defined on* $(H, \mathcal{H})$ *into some measurable space.*

Thorisson (2000) also notes here that although the Markovian property is not preserved under measurable functions, *for any general stochastic process* $\mathbf{X}$ *the path process* $(\phi_s \mathbf{X})_{s \in [0,\infty)}$ *is always a Markov process!*

4.3.5 Cost or reward cycles

Let us now associate with the shift measurable stochastic process $\mathbf{X}$ a random sequence $0 \leq \mathbf{t}_0 < \mathbf{t}_1 < \ldots$, with $\mathbf{t}_n \to \infty$ as $n \to \infty$. Moreover, let f be a measurable (cost) function, and denote the cost of a cycle, related to the process $\mathbf{X}$, as

$$\mathbf{J}_n = \mathbf{J}_n(f) := \int_{\mathbf{t}_{n-1}}^{\mathbf{t}_n} f(\phi_t \circ \mathbf{X}) dt. \tag{4.3.4.5}$$

There have been numerous attempts in the literature to establish limit theorems in order to determine the average cost expressions

$$\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t f(\phi_s \circ \mathbf{X}) ds.$$

Establishing limit theorems, which hold under weak assumptions, that is, applicable for a wide class of mathematical models, is the aim of this chapter. Therefore, we list the known results which are relevant as forerunners of our extensions. But first, we need to introduce the concept of synchronous processes and Harris processes.

4.4 Two specific stochastic processes

In the remaining part of this chapter we assume that every stochastic process is shift measurable.

4.4.1 Synchronous processes

Consider now a stochastic process $\mathbf{X}$, defined on $(\Omega, \mathcal{F}, I\!P)$, with Polish state space $(E, \mathcal{E})$, and a Polish path space $(H, \mathcal{H})$ of right continuous maps from $[0, +\infty)$ to E with left limits. Let us define now a so called cemetery state Δ external to E, and $\mathcal{E}$ is now endowed with the one-point compactification topology (Glynn and Sigman (1992)).

Definition 4.4.1 [2] *The stochastic process* $\mathbf{X}$ *is said to be a synchronous process with respect to the random sequence* $0 \leq \mathbf{t}_0 < \mathbf{t}_1 < \ldots$, *if* $\{\mathbf{X}_n : n \geq 1\}$ *forms a stationary sequence in* $(H, \mathcal{H})$, *where*

$$\mathbf{X}_n(t) := \begin{cases} \mathbf{X}(\mathbf{t}_{n-1} + t), & \text{if} \quad 0 \leq t < \mathbf{T}_n, \\ \Delta & \text{if} \quad t \geq \mathbf{T}_n. \end{cases} \tag{4.4.4.1}$$

Let $I\!P^0$ denote the probability measure under which $\mathbf{X}$ is non-delayed, that is, $I\!P^0\{\mathbf{X} \in B\} = I\!P\{\phi_{\mathbf{t}_0} \circ \mathbf{X} \in B\}$ (or $\mathbf{t}_0 = 0$). We refer to $\mathbf{t}_n$ as the *synch-times* for $\mathbf{X}$.

Definition 4.4.2 *The process* $\mathbf{X}$ *is called positive recurrent if* $I\!E(\mathbf{T}_1) < \infty$, *null recurrent otherwise. Moreover,* $\mathbf{X}$ *is called ergodic if it is positive recurrent and the invariant* σ*-field,* $\mathcal{I}$, *defined by relation (4.3.4.1) of* $\{\mathbf{T}_n, \mathbf{X}_n\}$ *is trivial. Furthermore,* $\lambda := 1/I\!E(\mathbf{T}_1)$ *is called the rate of the synch-times, while* $\hat{\lambda} := 1/I\!E(\mathbf{T}_1|\mathcal{I})$ *is called the conditional rate.*

Although synchronous processes start over probabilistically at the synch-times, the future is not necessarily independent of the past, in contrast with regenerative processes. Therefore, the synch-times do not form a renewal process. Due to this dependency, one needs to establish extra conditions in order to be able to construct limit theorems, similar to those known for classical regenerative processes (e.g. Ross (1970)).

[2] The definitions and basic properties of (related to) synchronous processes were taken from the article of Glynn and Sigman (1992)

4.4.2 Harris chains and Harris processes

A discrete or continuous time stochastic process $\mathbf{X}$, with state space $(E, \mathcal{E})$ and general path space $(H, \mathcal{H})$ is a Markov process (Thorisson (2000)) if the future depends on the past only through the present. The Markov process is time–homogeneous if the conditional distribution of $\phi_t \mathbf{X}$ given the value of $\mathbf{X}_t$ does not depend on t. For time–homogeneous Markov processes, the transition kernel P^t, $t \in [0, \infty)$ is defined by

$$P^t(x, A) := I\!P\{\mathbf{X}_{s+t} \in A | \mathbf{X}_s = x\}, \quad x \in E, A \in \mathcal{E}, s \in [0, \infty). \tag{4.4.4.2}$$

Now we define Harris chains (discrete time) and Harris processes (continuous time) consecutively (Thorisson (2000)).

Definition 4.4.3 (Regeneration sets) *A discrete- time Markov process $\mathbf{X} = (\mathbf{X}_k)_0^\infty$, with state space $(E, \mathcal{E})$ and one-step transition probabilities P, is a Harris chain if it has a regeneration set, that is if there is a set $A \in \mathcal{E}$ such that the hitting time of the set A,*

$$\tau_A := \inf\{n \in I\!N \cup \{0\} : \mathbf{X}_n \in A\}$$

is finite with probability one for all initial distributions, and there is an $l > 0$, a $p \in (0, 1]$, and a probability measure μ on $(E, \mathcal{E})$ with

$$I\!P\{\mathbf{X}_l \in \cdot | \mathbf{X}_0 = x\} = P^l(x, \cdot) \geq p\, \mu(\cdot), \quad x \in A. \tag{4.4.4.3}$$

Remark 4.4.4 *If E is finite or countable, and $\mathbf{X}$ is irreducible and recurrent then $\mathbf{X}$ is a Harris chain.*

Now, in order to extend regeneration to continuous time we need the strong Markov property: a shift measurable Markov process $\mathbf{X} = (\mathbf{X}_s)_{s \in [0, \infty)}$ with semigroup of transition probabilities $P^s, 0 \leq s < \infty$ is a strong Markov process (Thorisson (2000)) if the Markov property holds at all stopping times τ, that is,

$$\begin{aligned} &\phi_\tau \mathbf{X} \text{ depends on } (\mathbf{X}_s)_{s \in [0, \tau]} \text{ only through } \mathbf{X}_\tau \text{ and} \\ &I\!P\{\phi_\tau \mathbf{X} \in \cdot | \mathbf{X}_\tau = x\} = I\!P\{\mathbf{X} \in \cdot | \mathbf{X}_0 = x\}, \quad x \in E. \end{aligned}$$

Now the definition of a continuous time Harris process stays the same as for Harris chains, in the sense that it is defined to be a strong Markov process which has a regeneration set with property (4.4.4.3). However the definition of a regeneration set for the continuous time case is different (Thorisson (2000)):

Definition 4.4.5 *A set $A \in \mathcal{E}$ is a regeneration set for the strong Markov process $\mathbf{X}$, if the hitting time τ_A is measurable and finite with probability one for all initial distributions, such that $\mathbf{X}_{\tau_A} \in A$, and if there is an $l > 0$, a $p \in (0,1]$, and a probability measure μ on $(E,\mathcal{E})$ such that (4.4.4.3) holds.*

Intuitively, (4.4.4.3) means, that whenever $\mathbf{X}$ enters A it lag–l regenerates l time units later with probability p. Thorisson (2000) proves that with a so called conditional splitting one can construct an increasing sequence $\mathbf{S} = (\mathbf{S}_n)_0^\infty$ such that $(\mathbf{X}, \mathbf{S})$ is lag–l regenerative, and the distribution of its zero-delayed version (under $I\!P^0$) $(\mathbf{X}^0, \mathbf{S}^0)$ does not depend on the initial distribution of $\mathbf{X}$.

There is also an other, alternative definition for Harris processes, used often in the literature (see for example Sigman(1992)):

Definition 4.4.6 (φ-recurrence) *A Markov process $\mathbf{X}$, with Polish state space $(E,\mathcal{E})$, satisfying the strong Markov property, is called Harris recurrent if there exists a non-trivial σ-finite measure φ on $(E,\mathcal{E})$, such that for any $B \in \mathcal{E}$, with $\varphi(B) > 0$, the total time spent by $\mathbf{X}$ in the set B is infinite with probability one for all initial distributions, that is,*

$$I\!P_z\left(\int_0^\infty 1_B \circ \mathbf{X}(t)dt = \infty\right) = 1 \text{ for all } z \text{ starting states.}$$

In the discrete case it can be shown (Orey (1971)) that φ–recurrence for some φ, and the existence of regeneration sets are equivalent properties. However, in continuous time this relation has not been proven yet. We conclude this section with some of the properties of Harris processes which are important for our limit theorems.

(i) A Harris chain is aperiodic if the inter-regeneration times are aperiodic; and this holds independently of the choice of the regeneration set and of l and p at (4.4.4.3).

(ii) Glynn (1994) showed that if $\mathbf{X}$ has a stationary distribution , then $\mathbf{X}$ is a Harris process if and only if for all initial distributions and all $A \in \mathcal{H}$,

$$I\!P\{\phi_{\mathbf{U}t}\mathbf{X} \in A\} \to I\!P\{\mathbf{X}^* \in A\}, \text{ as } t \to \infty,$$

where $\mathbf{X}^*$ is a stationary version of $\mathbf{X}$ and $\mathbf{U}$ is uniform on $[0,1]$.

(iii) Sigman (1990) proves that Harris processes are one-dependent regenerative, possessing a unique invariant σ–finite measure μ. Conversely, if a Markov process is positive recurrent one-dependent regenerative, then it is a positive recurrent Harris process.

Now we are ready to proceed with the limit results.

4.5 Limit theorems

4.5.1 Birkhoff's ergodic Theorem

In order to state the already known limit results we make use of Birkhoff's ergodic theorem, which can be found in the book of Billingsley (1995). The following concepts are strongly related to the one of invariant σ–fields, dealt with in subsection 4.3.3. A mapping $T : \Omega \to \Omega$, with $(\Omega, \mathcal{F}, P)$ the underlying probability space, is called a measure-preserving transformation if it is measurable $\mathcal{F}/\mathcal{F}$ and $P(T^{-1}A) = P(A)$ for all A in $\mathcal{F}$ (in the definition of the invariant σ–field in Section 4.3.3, the measure-preserving transformation considered is the shift–map $\phi_1 = T$ in discrete time). The set $A \in \mathcal{F}$ is invariant under T if $T^{-1}A = A$, and it is non–trivial invariant if $0 < P(A) < 1$. Likewise for stochastic processes, the transformation T is called *ergodic* if there are no non–trivial invariant sets in $\mathcal{F}$. A measurable function is *invariant* if $f(Tw) = f(w)$ for all $w \in \Omega$; A is invariant if and only if 1_A is.

Theorem 4.5.1 *(Birkhoff's ergodic theorem) Let T be a measure-preserving transformation on the triplet $(\Omega, \mathcal{F}, I\!P)$, and g a measurable and integrable function. Then*

$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=1}^{n} g(T^{k-1}w) = I\!E(g|\mathcal{I}) \quad a.s., \tag{4.5.4.1}$$

where $\mathcal{I}$ is the invariant σ-field. If T is ergodic then $I\!E(g|\mathcal{I}) = I\!E(g)$ a.s.

This theorem leads us to the first limit result. Consider a synchronous process $\mathbf{X}$ with synch-times $\mathbf{t}_n, n = 0, 1, \ldots$, a measurable cost function f, and the cost functions $\mathbf{J}_n$, defined by relation (4.3.4.5). Recall (see Section 4.3.1) that both the random times $\mathbf{t}_n, n = 0, 1, \ldots$ (as well interarrival times $\mathbf{T}_n$) and the process $\mathbf{X}$ are defined on a probability space $(\Omega, \mathcal{F}, I\!P)$ (in particular, $\mathbf{X}$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$). Define now the function

$$g := \int_{\mathbf{t}_0}^{\mathbf{t}_1} f(\phi_s \circ \mathbf{X})ds = \mathbf{J}_1 \qquad \text{(that is, } g(\omega) = \mathbf{J}_1(\omega)\text{)}, \tag{4.5.4.2}$$

and the transformation T on $(\Omega, \mathcal{F}, I\!P)$ as $g(T\omega) \stackrel{d}{=} g(\omega)$. Since $\mathbf{X}$ is synchronous with synch-times $\mathbf{t}_n, n = 0, 1, \ldots$, it follows that T is a measure preserving transformation. In order to perform the transformations $\mathbf{J}_k = \mathbf{J}_1(T^{k-1}w)$, we consider the canonical versions of $\mathbf{J}$ (likewise for the cycle lengths $\mathbf{T}_n$).

Corollary 4.5.2 *If $I\!E(\mathbf{T}_n|\mathcal{I}) > 0$, and if $\mathbf{X}$ is positive recurrent, $I\!E(\mathbf{J}_1(|f|)) < \infty$ and $\int_0^{\mathbf{t}_0} f(\phi_s \circ \mathbf{X}ds < \infty$ a.s. then*

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t f(\phi_s \circ \mathbf{X})ds = \frac{I\!E(\mathbf{J}_1|\mathcal{I}_J)}{I\!E(\mathbf{T}_1|\mathcal{I}_T)} \quad a.s., \tag{4.5.4.3}$$

where $\mathcal{I}_J$ is the invariant σ–field obtained by applying the transformation T to $\mathbf{J}$, and likewise, $\mathcal{I}_T$ is obtained by applying the transformation T to the cycle lengths process. If in addition $\{\mathbf{T}_n\}$ is ergodic, then

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t f(\phi_s \circ \mathbf{X})ds = \frac{I\!E(\mathbf{J}_1|\mathcal{I}_J)}{I\!E(\mathbf{T}_1)}. \tag{4.5.4.4}$$

The above statement follows directly by the following relation

$$\frac{1}{\mathbf{N}(t)}\int_0^{\mathbf{N}(t)} f(\phi_s \circ \mathbf{X})ds \le \frac{1}{t}\int_0^t f(\phi_s \circ \mathbf{X})ds \le \frac{1}{\mathbf{N}(t)+1}\int_0^{\mathbf{N}(t)+1} f(\phi_s \circ \mathbf{X})ds.$$

See also Theorem A1 of Glynn and Sigman (1992). Using the notation

$$\pi(f) := I\!E\left(\frac{I\!E(\mathbf{J}_1|\mathcal{I}_J)}{I\!E(\mathbf{T}_1|\mathcal{I})}\right), \tag{4.5.4.5}$$

π defines a measure on $(H, \mathcal{H})$, which is called the *stationary probability measure for* $\mathbf{X}$: under π, the shift $\phi = (\phi_s)$ is measure preserving on $(H, \mathcal{H})$. In particular, if $\mathbf{X}$ has distribution π, then $\mathbf{X}$ is time stationary (Glynn and Sigman (1992)). Observe that if $\{\mathbf{T}_n\}$ is ergodic then, by Corollary 4.5.2, $\pi(f) = \lambda I\!E(\mathbf{J}_1)$. The question is now: under what kind of conditions can one obtain convergence in L^1 for the limit (4.5.4.3)?

4.5.2 The results of Glynn and Sigman for synchronous processes

Define now the functional

$$\mu_t(f) := \frac{1}{t}\int_0^t I\!E f(\phi_s \circ \mathbf{X})ds. \tag{4.5.4.6}$$

A necessary and sufficient condition for L^1 convergence, $\mu_t(f) \to \pi(f)$, has been given in subsection 4.2.3. The results established there yield that

$$\mu_t(f) \to \pi(f) \iff \{\frac{1}{t}\int_0^t f(\phi_s \circ \mathbf{X})ds : t \ge 0\} \text{ U.I.} \tag{4.5.4.7}$$

In particular, the Cesaro averaged distributions converge weakly. Glynn and Sigman (1992) established additional conditions under which $\mu_t(f)$ converges uniformly (over a class of functions) to $\pi(f)$. Their main result reads as follows:

Proposition 4.5.3 *If $\mathbf{X}$ is a positive recurrent synchronous process, $g \in \mathrm{L}^1_+(\pi)$ is such that $(1/t)I\!E\int_0^{t\wedge\mathbf{t}_0} g(\phi_s \circ \mathbf{X})ds \to 0$ (where $\mathbf{t}_0$ stands for the delay) and*

$$\left\{\frac{1}{t}\int_0^t g(\phi_s \circ \mathbf{X})ds \,:\, t \ge 0\right\} \text{ is uniform integrable} \tag{4.5.4.8}$$

under the non-delay distribution, then

$$\sup_{|f|\leq g} |\mu_t(f) - \pi(f)| \to 0. \tag{4.5.4.9}$$

In particular, if either there exists an $\varepsilon > 0$ *such that* $I\!P^0\{\mathbf{T}_n > \varepsilon\} = 1$*, or the cycles* $\mathbf{X}_n$ *form a k-dependent process, then (4.5.4.8) holds for all* $g \in L_1^+(\pi)$ *which satisfy* $I\!E \int_0^{\mathbf{t}_0} g(\phi_s \circ \mathbf{X})ds < \infty$.

In accordance with relation (4.5.4.7), uniform integrability, that is, condition (4.5.4.8) is necessary and sufficient; in particular k-dependence implies uniform integrability (cf. Proposition 3.1, Glynn and Sigman (1992))).

4.5.3 General stochastic processes with an embedded Harris chain

The following section depicts a limit theorem for shift measurable stochastic processes $\mathbf{X}$, with state space $(D, \mathcal{D})$, derived by Bázsa and Den Iseger (2001). The stochastic process is general in the sense that any kind of dependency structure of the underlying point process $(\mathbf{t}_n)_0^\infty$ (arrivals) is allowed, as long as this dependency structure can be modeled through a Harris chain. This means, that we define an embedded Harris chain at the points $\mathbf{t}_n$, such that $\mathbf{A}_n := \mathbf{A}(\mathbf{t}_n)$ (with state space $(E, \mathcal{E})$ and path space $(H, \mathcal{H})$), such that it 'takes care' of the dependency structure of $\mathbf{X}$, that is, $\mathbf{X}(\mathbf{t}_n)$ is conditionally independent of the past given $\mathbf{A}_n$. Thus, we consider mathematical models, where such an embedded Harris chain can be constructed. As it is illustrated later, a large class of models in operations research can be covered by this construction. Moreover, the limit theorem derived for these types of models has the advantage of using exclusively the simple cycles determined by two consecutive points of the process $\mathbf{X}$, $\mathbf{T}_n = \mathbf{t}_n - \mathbf{t}_{n-1}$, which we call 'simple arrival cycles'.

Consider for instance a marketing problem: if the total demand for a planning horizon is known, then the magnitude of the demand up to the present gives a lot of information about demand in the remaining of the planning horizon. Define therefore the embedded Harris chain $\mathbf{A}$ as the total demand up to the present. Conditioning on $\mathbf{A}_n$ at point $\mathbf{t}_n$ yields the desired independence. Certainly, we need to have the conditions implying that $\mathbf{A}$ is indeed a Harris chain.

Sigman (1990) introduces the notion of marked point processes governed by a Harris chain, however his motivation and consequently his model is different. The examples he considers essentially consist of marked point processes with an underlying renewal sequence, while his limit theorem is given in terms of regeneration cycles, which in general are different from the simple arrival cycles (hence more difficult to determine).

Independence realized with A

Let us now give the formal conditions which yield the desired properties of the Harris chain $\mathbf{A}$, which realize the independence in our model. Throughout this chapter we assume the following:

Assumption 4.5.4 *The Harris chain $\mathbf{A}_n = \mathbf{A}(\mathbf{t}_n)$ satisfies the following conditions:*

$$\begin{aligned} &\mathbf{T}_k \text{ depends on } \{\mathbf{T}_j, j<k\} \text{ only through } \mathbf{A}_{k-1}, \\ \text{and} \quad &\mathbf{A}_k \text{ depends on } \{\mathbf{T}_i, i \le j < k\} \text{ only through } \mathbf{A}_j, \end{aligned} \tag{4.5.4.10}$$

$$\mathbf{J}_k \text{ depends on } \{\mathbf{T}_j, j<k\} \text{ only through } \mathbf{A}_{k-1}. \tag{4.5.4.11}$$

and, for all $n \in \mathbb{N}$,

$$I\!P(\mathbf{T}_n | \mathbf{A}_{n-1} = s) = I\!P(\mathbf{T}_1 | \mathbf{A}_0 = s), \text{ for a.e. } s \in E \tag{4.5.4.12}$$

$$I\!P(\phi_{\mathbf{t}_n}(\mathbf{X}, \mathbf{S}) | \mathbf{A}_{n-1} = s) = I\!P(\phi_{\mathbf{t}_0}(\mathbf{X}, \mathbf{S}) | \mathbf{A}_0 = s) \text{ for a.e. } s \in E. \tag{4.5.4.13}$$

Conditions (4.5.4.10) and (4.5.4.11) provide precisely the desired independence: $\mathbf{A}$ *contains all the relevant information about the past.* Hence, conditioning on $\mathbf{A}$ yields independence of the cycles. Furthermore, a sort of 'time- homogeneousness' property is assumed for the conditional distributions of the cycles, given the realizations of $\mathbf{A}$.

4.5.4 A limit result with an embedding technique

The aim of this section is to prove that the limit theorem 4.5.3 also holds for the general stochastic process $\mathbf{X}$ with the embedded Harris chain $\mathbf{A}$, defined by relations (4.5.4.10) – (4.5.4.13). Knowing that every Harris chain forms a one-dependent process, we will show that this implies together with relations (4.5.4.10) and (4.5.4.11) that the process $\mathbf{X}$ (hence also $\mathbf{J}$, cf. Observation 4.3.4) forms a wide-sense one-dependent process, which suffices to satisfy all the conditions of Proposition 4.5.3, thus we obtain the desired limit results.

Remark 4.5.5 *In 1992 Glynn and Sigman (1992) established limit theorems for Harris chains, however, although the limit theorems in the present paper are related to Harris chains they are more general: they are established for general stochastic processes, of which dependency structure can be modeled through Harris chains satisfying Assumption 4.5.4.*

In Section 4.4.2, property (iii) established that for every Harris chain there exists a set of points $\{\mathbf{N}\}$, such that $(\mathbf{A}, \mathbf{N})$ is one-dependent regenerative. Define now

$$\mathbf{S}_k := \mathbf{t}_{\mathbf{N}_k}, \tag{4.5.4.14}$$

the arrival moments (in continuous time!) of the process $\mathbf{X}$ corresponding to the regeneration times of $\mathbf{A}$ in discrete time. Furthermore, by the definition of the process $\mathbf{A}$ (cf. relations (4.5.4.10) and (4.5.4.11)) we have that if $\mathbf{A}_{k-1}$ is independent of $\mathbf{A}_j$, then $\mathbf{A}_{k-1}$ is independent of $\mathbf{T}_j$ $j = 0, \ldots, k-2$, which in turn implies that $(\mathbf{T}_k, \mathbf{J}_k)$ (or equivalently, $(\mathbf{T}_k, \phi_{\mathbf{t}_{k-1}}\mathbf{X})$) is independent of $\mathbf{T}_j$, $j = 0, \ldots, k-2$. Letting $\mathbf{S}$ denote the sequence $\{\mathbf{S}_k : k \in I\!N \cup \{0\}\}$, with $\mathbf{S}_k$ defined by relation (4.5.4.14), we can state the following theorem:

Theorem 4.5.6 $(\mathbf{X}, \mathbf{S})$, *or equivalently* $(\mathbf{J}, \mathbf{S})$, *is a wide-sense one-dependent process, and* $\{\mathbf{S}\}$ *is ergodic. Moreover the long run expected average cost of the system is given by*

$$I\!E\bar{\mathbf{J}}_1 \,/\, I\!E\mathbf{S}_1, \tag{4.5.4.15}$$

where

$$\bar{\mathbf{J}}_1 := \sum_{k=1}^{\mathbf{N}_1} \mathbf{J}_k.$$

Proof: The discussion before the statement proves that $(\mathbf{X}, \mathbf{S})$ is indeed wide–sense one–dependent. The ergodicity of $\mathbf{S}$ follows from the one–dependence property:

$$\lim_{t\uparrow\infty} \frac{\mathbf{N}(t)}{t} = \lim_{n\uparrow\infty} \frac{n}{\mathbf{S}_n} = \lim_{n\uparrow\infty} \frac{n}{\sum_{k=1}^{n}(\mathbf{S}_k - \mathbf{S}_{k-1})}.$$

Define $\mathbf{L}_k := \mathbf{S}_k - \mathbf{S}_{k-1}$, and split the summation in the above expression as follows

$$\sum_{k=1}^{2n} \mathbf{L}_{2k} + \sum_{k=1}^{2n} \mathbf{L}_{2k-1}.$$

The one–dependence property implies that $\{\mathbf{L}_{2k}\}$ and $\{\mathbf{L}_{2k+1}\}$ are i.i.d. sequences, hence they are both ergodic processes with rates λ_1 and λ_2, respectively. This yields after some elementary calculations that

$$\lim_{t\uparrow\infty} \frac{\mathbf{N}(t)}{t} = \frac{2\lambda_1\lambda_2}{\lambda_1 + \lambda_2} =: \lambda,$$

hence $\{\mathbf{S}\}$ is ergodic. Applying now Proposition 3.1 of Glynn and Sigman (1992) yields that, if the expected cost of the delay cycle is finite then in the long run expected average cost expression we can ignore this term, considering only the process which started in the first regeneration point, $\mathbf{S}_0$ (see Appendix 4.7 for the proof). Moreover, if $I\!E\mathbf{S}_1 > 0$ and the expected cost of this first cycle is also finite, then the necessary and sufficient uniform integrability condition (4.5.4.8) is satisfied (see Proposition 4.5.3 and the remark afterwards), hence the long run average cost of the system is given by relation (4.5.4.15). This completes the proof.

Remark 4.5.7 *Although the statement of Proposition 3.1 of Glynn and Sigman (1992) requires one-dependency, its proof only uses the conditions of wide-sense one-dependency, which makes it possible for us to apply it for our case.*

Consider the identity

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_1}\mathbf{J}_k\right) = I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right) + I\!E\left(\sum_{k=1}^{\mathbf{N}_1}(\mathbf{J}_k - I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))\right).$$

By the construction, $\mathbf{N}_1$ is a stopping time with respect to $\mathbf{A}_k$, that is, $\{\mathbf{N}_1 \leq k\} \subseteq \sigma(\mathbf{A}_j : j \leq k)$; on the other hand, $\mathbf{J}_k - I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})$ is independent of $\sigma(\mathbf{A}_j : j \leq k-1)$ (Williams (1991)), hence $\mathbf{N}_1$ and $\{\mathbf{J}_k - I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\}$ are independent for every k. This yields that

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_1}(\mathbf{J}_k - I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))\right) = I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k - I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))\right) = 0.$$

This means that we only need to deal with

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right). \tag{4.5.4.16}$$

Define now the functions Y_n and Z_n, defined on E as

$$Y_n(s) := I\!E(\mathbf{J}_n|\mathbf{A}_{n-1} = s) \quad \text{and} \quad Z_n(s) := I\!E(\mathbf{T}_n|\mathbf{A}_{n-1} = s) \tag{4.5.4.17}$$

(functions of the realizations of $\mathbf{A}_{n-1}$) for all $n \in I\!N$. Assume that

$$\int_E Y_1(u)\pi_\infty(du) < \infty, \quad \text{and} \quad \int_E Z_1(u)\pi_\infty(du) > 0, \tag{4.5.4.18}$$

where π_∞ is the limiting distribution in the Cesaro sense (cf. Section 2.4) of the Harris chain $\mathbf{A}$, given by

$$\pi_\infty := \lim_{n\uparrow\infty}\frac{1}{n}\sum_{k=1}^{n}\pi_k. \tag{4.5.4.19}$$

We are now ready to state the main result of this section.

Theorem 4.5.8 *Under the conditions of Assumption 4.5.4 and (4.5.4.18) the expected first cycle is finite, that is, $I\!E\bar{\mathbf{J}}_1 < \infty$ and $I\!E\mathbf{S}_1 > 0$. Moreover, the expected long run average cost expression (4.5.4.15) can be expressed in terms of the arrival cycles:*

$$\frac{I\!E\bar{\mathbf{J}}_1}{I\!E\mathbf{S}_1} = \frac{\int_E I\!E(\mathbf{J}_1|\mathbf{A}_0 = s)\pi_\infty(ds)}{\int_E I\!E(\mathbf{T}_1|\mathbf{A}_0 = s)\pi_\infty(ds)}, \tag{4.5.4.20}$$

for almost all initial states of $\mathbf{A}$, or equivalently, of the process $\mathbf{X}$, w.r.t. π_∞.

Proof: As mentioned before, the proof of the finiteness of the expected first cycle is given in Appendix 4.7. In order to prove that the long run average cost is indeed given by expression (4.5.4.20), observe that having obtained expression (4.5.4.16), we have

$$\frac{1}{I\!E\mathbf{N}_1} I\!E\bar{\mathbf{J}}_1 = \frac{1}{I\!E\mathbf{N}_1} I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right).$$

Conditioning on $\mathbf{A}_{k-1}$ with distribution π_{k-1}, the previous expression is equal to

$$\int_E \frac{1}{I\!E\mathbf{N}_1} I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}=u)\right)\pi_{k-1}(du).$$

Using again condition (4.5.4.13) this can be evaluated as

$$\int_E Y_1(s)\left(\frac{1}{I\!E\mathbf{N}_1}\sum_{k=1}^{I\!E\mathbf{N}_1}\pi_{k-1}(ds)\right).$$

Since $\mathbf{N}_1$ constitutes a regeneration point for the Harris chain $\mathbf{A}$ it yields that

$$\frac{1}{I\!E\mathbf{N}_1}\sum_{k=1}^{I\!E\mathbf{N}_1}\pi_{k-1} = \pi_\infty,$$

where π_∞ was defined by relation (4.5.4.19). In conclusion,

$$I\!E\bar{\mathbf{J}}_1 = I\!E\mathbf{N}_1\int_E Y_1(s)\pi_\infty(ds) < \infty \tag{4.5.4.21}$$

by condition (4.5.4.18). Note that $I\!E\mathbf{N}_1 < \infty$ since $\mathbf{A}$ is positive recurrent. Similarly as for $\mathbf{J}_k$ we can use the same argumentation for $\mathbf{T}_k$, obtaining

$$I\!E\mathbf{S}_1 = I\!E\mathbf{N}_1\int_E Z_1(s)\pi_\infty(ds) > 0, \tag{4.5.4.22}$$

by condition (4.5.4.18). This immediately yields expression (4.5.4.20) for the expected long run average cost, completing the proof.

Remark 4.5.9 *1. If $\mathbf{A}_0$ is distributed with the invariant distribution, Theorem 4.5.8 remains true (see Proposition 4.3 of).*

2. The Harris chain $\mathbf{A}$ from Section 4.5.3 can consist of two chains: $\mathbf{A}^1$ satisfying conditions (4.5.4.10) and (4.5.4.12) and $(\mathbf{A}^1, \mathbf{A}^2)$ satisfying conditions (4.5.4.11), (4.5.4.13). Certainly, $(\mathbf{A}^1, \mathbf{A}^2)$ needs to be a Harris chain as well.

Although at first sight it might look difficult to prove that $\mathbf{A} := (\mathbf{A}^1, \mathbf{A}^2)$ (or more generally the collection $\mathbf{A} := (\mathbf{A}^1, \ldots, \mathbf{A}_n)$) is a Harris chain, as it is stated in the previous remark, the argument is rather simple: it the consequence of result (ii) of Section 4.4.2. Hence, if we can prove that $\mathbf{A}$ is convergent in Cesaro total variation (that is, the time stationary version of $\mathbf{A}$ exists) it implies that it is a Harris chain. In case of inventory models, as discussed in Chapter 2, Section 2.4, it is often the case that the joint time- average distribution of $(\mathbf{IP}, \mathbf{N}_\Delta)$ exists. This is the result we will use later to show for several models that the embedded Harris chain exists.

4.6 Models 'under control': Examples

4.6.1 Solving the models: an MDP formulation

This section shows how to formulate a decision process when the underlying dynamics follow one of the general processes discussed above, just as a standard Markov decision process rests on a (discrete-state) Markov chain.The key idea is the construction of the Harris chain $\mathbf{A}$ in Section 4.5.3. This section is aimed as a guideline for solving models of the type of the previous sections with generalized Markov decision theory, that is, Markov decision processes with continuous state space. Since the theory of Markov decision processes is vast, this section is only meant to formulate the problem, and not to give precise conditions under which an optimal solution exists, and there is also no exact solution procedure provided. Besides, the problem formulated below is very general, one needs to solve the optimization algorithms for each specified problem.

Knowing that every Harris chain possesses a unique invariant σ–finite measure π_∞^R (cf. property (iii) of Section 4.4.2) defined by relation (4.5.4.19), the problem is formulated with the normalization

$$\int_E \pi_\infty^R(ds) = 1. \tag{4.6.4.1}$$

We proved that the average cost of the system is given by

$$g(R) = \frac{\int_E I\!E(\mathbf{J}_1|\mathbf{A}_0 = s)\pi_\infty^R(ds)}{\int_E I\!E(\mathbf{T}_1|\mathbf{A}_0 = s)\pi_\infty^R(ds)} + \int_E \int_{\mathcal{E}} K^R(s,y)P^R(s,dy)\pi_\infty^R(ds), \tag{4.6.4.2}$$

where P is the transition kernel associated with the Harris chain A (defined by relation (4.4.4.2)), with $P^R\,\pi_\infty^R = \pi_\infty^R$. There is a nonnegative cost $K \geq 0$ associated with the control of the system, which also depends on the transitions, hence on the policy R. Take for instance an inventory system with a positive order policy (that is, no disposal allowed). Then $K^R(x,y) = K1_{\{x<y\}}$. As usual, a policy R^* is optimal if

$g(R^*) \leq g(R)$, for all stationary policies R. We ought to remark here that we indeed are looking for nonstationary control policies, however this nonstationarity will be realized through the transitions of the embedded Harris chain $\mathbf{A}$: the control policy depends on the state of $\mathbf{A}$. The optimization problem is hence of the form:

$$\begin{cases} \min_R \left\{ \frac{\int_E I\!E(\mathbf{J}_1|\mathbf{A}_0=s)\pi_\infty^R(ds)}{\int_E I\!E(\mathbf{T}_1|\mathbf{A}_0=s)\pi_\infty^R(ds)} + \int_E \int_{\mathcal{E}} K^R(s,y)P^R(s,dy)\pi_\infty^R(ds) \right\}, \\ P^R\,\pi_\infty^R = \pi_\infty^R, \\ \int_E \pi_\infty^R(ds) = 1. \end{cases} \tag{4.6.4.3}$$

In the case when E is finite or countable, the problem reduces to a classical semi Markov decision problem. Methods for solving semi Markov decision problems can be found in any standard textbook (see for example Tijms (1994), Ross (1983)); one can use policy iteration or value iteration (in the latter case, with a data transformation technique the problem is reduced to a simple Markov decision problem). In the continuous state space case, in the inventory applications the state space can almost always be assumed compact, but certainly Borel. For standard solution techniques the reader is referred to Dynkin and Yushkevich (1979) or De Leve, Federgruen, and Tijms (1977).

Note that Federgruen and Zipkin (1985) solve a similar continuous state space Markov decision problem for the optimization of a stationary (s, S) policy (only depending on the inventory position process).

4.6.2 Demand forecasting with time series

One of the forecasting techniques which accounts for seasonal and trend factors in the demand process is the autoregressive moving average modeling (ARMA). It is believed that these discrete- time models are the most likely to be found in the real world (Granger (1989)). The ARMA process, unlike the first- order autoregressive model or exponential smoothing model, is not a Markov process, since $\mathbf{D}_{k+1}$ depends not just on $\mathbf{D}_k$ but also on values at certain earlier times, say $\mathbf{D}_{k-1}$ and $\mathbf{D}_{k-2}$ (Zipkin (2000)). It also depends on scalar noise factors over earlier time periods, say ε_k and ε_{k-1} (ε_k are independent over all k, with 0 mean for all k). Thus the dynamics of the system is given by

$$\mathbf{D}_k = \sum_{j=1}^{m} \alpha_j \mathbf{D}_{k-j} + \sum_{j=0}^{m} \beta_j \varepsilon_{k-j}.$$

For the sake of simplicity, in this example we will only consider the following dynamics, characteristic for seasonal demand processes:

$$\mathbf{D}_k = \alpha_1 \mathbf{D}_{k-1} + \alpha_2 \mathbf{D}_{k-2} + \varepsilon_k + \beta \varepsilon_{k-1},$$

The stability condition is standard (as in Zipkin (2000)). Observe that the above dynamics is time-homogeneous since α_i, $i = 1, 2$ and β are constants. Defining now $\mathbf{A}_{k-1} := (\mathbf{D}_{k-1}, \mathbf{D}_{k-2}, \varepsilon_{k-1})$ (with $\mathbf{D}_{-1} := 0$) it is a Harris chain: the Markov property is satisfied by the construction of the demand process, while the stability condition leads to stationarity.

Now the joint process $(\mathbf{A}, \mathbf{IP})$ realizes the desired independence (4.5.4.11) and the homogeneousness like property (4.5.4.13), that is for all n

$$I\!E\left(f(\mathbf{IP}_n - \sum_{k=n+1}^{n+L-1} \mathbf{D}_k)|(\mathbf{A}_n, \mathbf{IP}_n) = (s, u)\right)$$

are the same for almost every (s, u) (with $s = (s_1, s_2)$), follows from stationarity. One alternative in order to show that $(\mathbf{A}, \mathbf{IP})$ is also a Harris chain, is to follow again statement (ii) of Section 4.4.2 and the theory of Section 2.4 to prove that its distribution converges to its time stationary version in Cesaro total variation. However, following this method, we only know for sure that the time average distribution exists if we assume that the control policy is stationary, that is, $\mathbf{IP}$ has a limiting distribution. Since the time- average distribution of $\mathbf{A}$ exists, the time average distribution of $(\mathbf{A}, \mathbf{IP})$ also exists (cf. Section 2.4). Hence, the long run expected average cost is given by the expression

$$I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}\left(I\!E\left(f(\mathbf{IP}_0 - \sum_{k=1}^{L-1} \mathbf{D}_k|(\mathbf{A}, \mathbf{IP})_0)\right)\right).$$

However, the second alternative, that is, using Definition 4.4.3 yields the possibility to construct nonstationary policies. Although a proof, following Definition 4.4.3 can be somewhat more tedious, the result is worth the trouble. To illustrate this, in the next section we consider a nonhomogeneous compound Poisson process together with a nonstationary $(s(\lambda), S(\lambda))$ policy.

4.6.3 Inventory systems with Harris–modulated demand

Let us assume that demand is a Harris–modulated time–nonhomogeneous stochastic process, that is, the arrivals of customers is governed by a Harris chain $\mathbf{A} := \{\mathbf{A}(\mathbf{t}_n) : n \in I\!N \cup \{0\}\}$. (For instance, the rate of the process, $\lambda(t)$ is described by a Harris chain $\mathbf{A}_n := \lambda(\mathbf{t}_n)$ $n \in I\!N \cup \{0\}$). Furthermore we assume that the individual demand $\mathbf{Y}_n$ is i.i.d., independent of $\mathbf{A}$ and the associated cumulative distribution function F_Y is spread-out, such that the density function g in Definition 4.2.2 has positive support on all sets of positive measure. Assume further that $\mathbf{A}$ realizes conditions (4.5.4.10)

and (4.5.4.12), meaning that conditioning on $\mathbf{A}$ realizes the independence among the cycles, while the cycles become identically distributed for the same realizations of $\mathbf{A}$. The applied decision rule is a so called Harris–modulated (s, S) policy, defined in every decision moment $\mathbf{t}_n, n \in I\!N \cup \{0\}$ as

$$\Gamma(\mathbf{A}_n) = \begin{cases} S(\mathbf{A}_n) & \text{if} \quad \widetilde{\mathbf{IP}}_n \leq s(\mathbf{A}_n), \\ \mathbf{IP}(\mathbf{t}_n) & \text{if} \quad \widetilde{\mathbf{IP}}_n > s(\mathbf{A}_n), \end{cases} \tag{4.6.4.4}$$

where $\widetilde{\mathbf{IP}}_n := \mathbf{IP}_{n-1} - \mathbf{Y}_n$. The action space is assumed compact, such that $\alpha := \sup_a\{s(\mathbf{A}) : \mathbf{A} = a\} < \beta := \inf_a\{S(\mathbf{A}) : \mathbf{A} = a\}$. In this setting $\mathbf{IP}$ depends on $\mathbf{A}$ only through the decision parameters $s(\mathbf{A})$ and $S(\mathbf{A})$. We aim to show now that $(\mathbf{A}, \mathbf{IP})$ forms a Harris chain, which realizes the desired independence conditions (4.5.4.11) and (4.5.4.13). That is, we want to show that there exist a regeneration set $B \times C$, associated with the joint process $(\mathbf{A}, \mathbf{IP})$, such that $\varphi(B, C) > 0$ implies φ-recurrence (see Definition 4.4.6 and the observation thereafter):

$$I\!P\left\{\sum_{n=1}^{\infty} (1_{B\times C} \circ (\mathbf{A}_n, \mathbf{IP}_n)) = \infty\right\} = 1. \tag{4.6.4.5}$$

The reason for using this method in order to prove that $(\mathbf{A}, \mathbf{IP})$ is a Harris chain, instead of using (ii) of Section 4.4.2, is that in this case neither $\mathbf{A}$ nor $\mathbf{IP}$ has a limiting distribution, thus we would need additional information for proving that the joint limiting distribution in the Cesaro sense of $\mathbf{A}$ and $\mathbf{IP}$ exists.

Since $\mathbf{A}$ is a Harris chain, there exists a regeneration set B, together with $l_A > 0$, $\mathbf{A} > 0$ and a probability measure μ_A satisfying (4.4.4.3), such that the hitting time τ_A of B is finite with probability one. Consider the sequence $\{\tau_A^k : k \in I\!N\}$ generated by the hitting times of the set B, such that

$$\tau_A^k := \inf\{m \in I\!N : \mathbf{A}_m \in B \text{ and } m > \tau_A^{k-1}\}, \quad k \geq 2, \quad \tau_A^1 := \tau_A.$$

Since the sequence $\{\tau_A^k : k \in I\!N\}$ is a subset of the arrival times $\{\mathbf{t}_n : n \in I\!N\}$, define $\widehat{\mathbf{IP}}_k := \mathbf{IP}_{\tau_A^k}$. Since $\mathbf{Y}$ has a spread-out distribution, it means (see Definition 4.2.2) that there exists an $n_0 \in I\!N$ (finite) such that $F_Y^{n_0 *}$ has an absolutely continuous component with a density g. Consequently, supposing that an average order cycle $\mathbf{T}$ is completed in k arrivals, we have that $I\!P\{\mathbf{T} = k\} = (F_Y^{k*} - F^{(k+1)*})(S(\mathbf{A}_0) - s(\mathbf{A}_k)) > 0$ if $k \geq n_0$. Considering $C := (\alpha, \beta)$ yields that there exist $m, k \geq n_0$ such that $I\!P\{\widehat{\mathbf{IP}}_k \in C\}$ is greater or equal than

$$\inf_a I\!P\{S(a) - \mathbf{D}_m \in C\} I\!P\{\text{replenishment order placed at time } \mathbf{t}_{k-m}\} > 0,$$

since $\mathbf{D}_m = F_Y^{m*}$ has an absolutely continuous component for $m > n_0$, yielding $F_Y^{m*}(x) > 0$ for all $x > 0$. We can conclude thus that there exist a $p > 0$ such

that $I\!P\{\widehat{\mathbf{IP}}_k \in C\} > p$. Combining this with relation (4.6.4.5) yields (by a slightly modified Borel-Cantelli lemma, Theorem 4.4. of Billingsley (1986)) that

$$I\!P\left\{\sum_{n=1}^{\infty}(1_{B\times C}\circ(\mathbf{A}_n,\mathbf{IP}_n))=\infty\right\}=I\!P\left\{\sum_{k=n_0}^{\infty}\left(1_C\circ\widehat{\mathbf{IP}}_k\right)=\infty\right\}=1.$$

Finally we can conclude that $(\mathbf{A},\mathbf{IP})$ is a Harris chain satisfying conditions (4.5.4.10) –(4.5.4.13), yielding that the long run average cost of the system is given by

$$g_R=\frac{1}{I\!E_{\mathbf{A}^c_\infty}(I\!E(\mathbf{T}_1|\mathbf{A}_0))}I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}\left(I\!E\left(\int_0^{\mathbf{t}_1}f(\mathbf{IP}_0-\mathbf{D}_L(t)|(\mathbf{A},\mathbf{IP})_0)\right)\right),$$

where $I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}$ denotes the expectation w.r.t. the limiting distribution in the Cesaro sense of the distribution of $(\mathbf{A}_n,\mathbf{IP}_n)$. Solving the optimization problem (4.6.4.3) for this g_R yields a dynamic (nonstationary) optimal policy $(s(\lambda),S(\lambda))$.

Markov modulated arrivals

Markov modulated demand processes have been considered in several articles in the literature, however their approach is quite different from ours. The closest to our approach is that of Sigman (1990), however his motivation and consequently his results are different. As mentioned earlier, he proves that every Harris process is a one-dependent process, but the limit result he deduces is in terms of the one-dependent regeneration cycles.

Song and Zipkin (1993) consider an exogenous 'world-driven' Poisson demand: when the world is in state i, demand follows a Poisson distribution with rate λ_i. Their model is a discrete-state dynamic program with two state variables, the world and the inventory position. Their results can be obtained directly from our model by assuming Poisson arrivals, discrete state space, and setting $\mathbf{A}_n := \lambda(\mathbf{t}_n)$. Then the average cost expression is obtained in terms of the conditional expectations given the stationary distribution of $(\lambda_n,\mathbf{IP}_n)$. Note that assuming that $\lambda(t)$ is periodic yields immediately that the stationary distribution of $(\lambda_n,\mathbf{IP}_n)$ exists, given that $\mathbf{IP}_n$ is positive recurrent (see condition (d) of Section 2.3).

Lovejoy (1992) considers demand processes as functions of a Markovian information process. This information process may depend on the past of the demand process as well as on an exogenous variable. Further it concentrates on the efficacy of myopic policies.

4.6.4 Inventory with returns

Consider a general single item inventory model which allows returns, with the following characteristics. Demand up to time t, $\mathbf{D}(t)$, is a compound renewal process with

arrival process $\{\mathbf{t}_n : n \in I\!N \cup \{0\}\}$ and individual demands $\{\mathbf{Y}_n, n \in I\!N \cup \{0\}\}$. We assume that control actions, denoted by Γ, are only permitted at purchase arrivals, that is, at times $\{\mathbf{t}_n : n \in I\!N\}$; furthermore, Γ is a stationary policy, depending on the inventory position process and the Harris chain $\mathbf{A}$, as defined later. There is also a so called returns process $\mathbf{R}$, a nonhomogeneous compound Poisson process with fixed batch return sizes μ. Denote the total amount of items on the market at time t as $\mathbf{A}(t)$ (that is, demand minus returns) and let us assume that the returns rate is a, given $\mathbf{A} = a$. Thus, fixing a sample path of $\mathbf{A}$, $\mathbf{R}$ is just a compound Poisson process with a variable rate. Assuming that every item is returned is not very restrictive, since we can use the following correction: introduce the binary variable $(\mathbf{y}/\mathbf{n})$ (yes or no), for deciding whether the returned item is suitable for remanufacturing or not. Hence, the returns process has the form $\sum_{k=1}^{\mathbf{N}_r(t)} (\mathbf{y}/\mathbf{n})\mu$ (we assume here that non-suitable items can be instantaneously disposed of). Since we only need to concentrate on the times $\mathbf{t}_n$ of purchase arrivals, it is useful to introduce the notations $\mathbf{IP}_n := \mathbf{IP}(\mathbf{t}_n^+)$ and $\mathbf{A}_n := \mathbf{A}(\mathbf{t}_n^+)$. By the definition of the model it follows now that

$$\begin{aligned} \mathbf{IP}_{n+1} &= \Gamma(\mathbf{IP}_n - \mathbf{Y}_{n+1} + \mathbf{R}(\mathbf{t}_n, \mathbf{t}_{n+1}]), \\ \mathbf{A}_{n+1} &= \mathbf{A}_n + \mathbf{Y}_{n+1} - \mathbf{R}(\mathbf{t}_n, \mathbf{t}_{n+1}]. \end{aligned} \tag{4.6.4.6}$$

This means that the $\{\mathbf{IP}_n\}$ and $\{\mathbf{A}_n\}$ are Markov processes. A common assumption for inventory models (see for instance condition (c) in Section 2.3) to assume that the embedded Markov chain $\mathbf{IP}_n$ is positive recurrent – since our model assumes continuous state space we assume φ–recurrence (see Section 4.4.2), hence $\mathbf{IP}_n$ is a Harris chain. Having assumed that the control policy is stationary, the limiting distribution of $\mathbf{IP}_n$ exists. This implies that $(\mathbf{A}, \mathbf{IP})$ converges to its stationary version in Cesaro total variation, hence it is a Harris chain (statement (ii), Section 4.4.2). Furthermore, the equivalent of the classical flow-conservation law remains valid:

$$\mathbf{IN}(t+L) = \mathbf{IP}(t) - (\mathbf{D}(t, t+L] - \mathbf{R}(t, t+L]). \tag{4.6.4.7}$$

Since $(\mathbf{A}, \mathbf{IP})$ is a Harris chain, we aim to show that it satisfies the conditions of Assumption 4.5.4. Since $\{\mathbf{t}_n\}$ is a renewal process, conditions (4.5.4.10) and (4.5.4.12) follow. Since $\mathbf{R}$ is a compound Poisson process an $\mathbf{D}$ is a compound renewal process, conditions (4.5.4.11) and (4.5.4.13) are satisfied too. Hence, applying Theorem 4.5.8, it follows that the long run average cost of the system is given by the expression

$$\frac{1}{I\!E\mathbf{T}_1} I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty} \left(\int_0^{\mathbf{t}_1} f(\mathbf{IP}_0 - \mathbf{D}_L(t) + \mathbf{R}_L(t))dt \middle| (\mathbf{A}, \mathbf{IP})_0) \right),$$

where $I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}$ stands for the expectation with respect to the joint time average distribution of the Harris chain $(\mathbf{A}, \mathbf{IP})$.

4.6.5 Generalized Semi – Markov Processes (GSMP's)

A Generalized Semi – Markov process has less restrictive assumptions than a continuous time Markov Chain. The following intuitive summarizing definition is taken from the lecture notes of Haas (2000). That is, a GSMP $\{\mathbf{X}(t) : t \geq 0\}$ makes stochastic state transitions when one or more events associated with the occupied state occur: events associated with a state compete to trigger the next state transition; each event has its own distribution for determining the next state; new events can be scheduled at each state transition; for each new event a *clock* is set with a reading that indicates the time until the next transition (when the clock runs down to 0 the event occurs); old events don't trigger state transitions but are associated with the next state, their clocks continue to run down; cancelled events don't trigger a state transition and are not associated with the next state, their clock readings are discarded; clocks can run down at state dependent speed. The mathematical definition of a GSMP has the following elements: the set of states S is countable, there is a finite set of events $E := \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$, and $E(s)$ is the set of events scheduled to occur in state $s \in S$. Furthermore, $p(s'; s, E^*)$ is the probability that the new state is s' given that the events in E^* simultaneously occur in s. If $E^* = \{\mathbf{e}^*\}$ for some $\mathbf{e}^* \in E(s)$, then write $p(s'; s, \mathbf{e}^*)$. $r(s, \mathbf{e})$ denotes the nonnegative finite speed at which clock for $\mathbf{e}$ runs down in state s, and $F(\cdot; s', \mathbf{e}', s, E^*)$ the distribution function used to set the clock for the new event $\mathbf{e}$, when the simultaneous occurrence of the events in E^* triggers a state transition from s to s'. μ is the initial distribution function for the clock and state readings. It is assumed however, that μ is such that the initial state s is chosen according to a distribution ν, and for each event $\mathbf{e} \in E(s)$ the clock is set independently according to $F_0(\cdot\,; \mathbf{e}, s)$. Hence, formally, $\mathbf{X} = \{(S_n, \mathbf{C}_n) : n \geq 0\}$, where S_n is the state after the nth transition and $\mathbf{C}_n = (\mathbf{C}_{n,1}, \ldots, \mathbf{C}_{n,M})$ is a clock-reading vector after the nth transition. The solution method for such a GSMP model is generally simulation.

A simple example for such a GSMP is the GI/G/1 queue. Assuming that the interarrival distribution F_a and the service time distribution F_s are continuous, such that an arrival and a service completion never occur simultaneously, we can model the queue as a GSMP in the following manner. Let $\mathbf{X}(t)$ the number of jobs in service or waiting in queue at time t. Then $S = \{0, 1, 2, \ldots\}$, $E = \{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1$ is the event arrival, and $\mathbf{e}_2$ is the event completion of service. $E(s) = \{\mathbf{e}_1\}$ if $s = 0$ and $E(s) = \{\mathbf{e}_1, \mathbf{e}_2\}$ if $s > 0$. Furthermore $p(s+1; s, \mathbf{e}_1) = 1$ and $p(s-1; s, \mathbf{e}_2) = 1$. $F(x; s', \mathbf{e}', s, \mathbf{e}) = F_a(x)$ if $\mathbf{e}' = \mathbf{e}_1$ and $F_s(x)$ if $\mathbf{e}' = \mathbf{e}_2$, while $r(s, \mathbf{e}) = 1$ for all s and $\mathbf{e}$. Finally, $\nu(1) = 1$, $F_0(\cdot\,; \mathbf{e}_1, s) = F_a(\cdot)$ and $F_0(\cdot; \mathbf{e}_2, s) = F_s(\cdot)$.

With the modeling technique of general stochastic process with an embedded

Harris chain one can also model GSMP's in the following way. Extend the state space with the states S_n, and the Harris chain with the clock-reading vectors $\mathbf{A} \times \{\mathbf{C}_1, \mathbf{C}_2, \dots\}$. We need to assume here that the clocks are set independently, which yields that this is then indeed a Harris chain. Suppose that $\mathbf{e}' = \mathbf{e}_n$. Then $\mathbf{T}_k = \mathbf{C}_n(k)$, the nth clock at time $\mathbf{t}_k$. Then $\mathbf{C}_j(k+1) = \mathbf{C}_j(k) - \mathbf{T}_k$, $j \neq n$, and $\mathbf{C}_n(k+1)$ is distributed with $F(\cdot\,; s', \mathbf{e}', s, \mathbf{e}_n)$. The transition probabilities for the Harris chain are given by $I\!P\{\mathbf{A}_{k+1}, \mathbf{A}_k; \mathbf{e}' = \mathbf{e}_n\}$. The event $\mathbf{e}' = \mathbf{e}_n$ is known at time $\mathbf{t}_k$, $\mathbf{e}' = \mathbf{e}_n$ if and only if $\mathbf{C}_n = \min\{\mathbf{C}_1, \mathbf{C}_2, \dots\}$. In fact, given the clocks at time $\mathbf{t}_k$, $\mathbf{T}_k$ is deterministic, i.e. $\mathbf{T}_k = \min\{\mathbf{C}_1(k), \mathbf{C}_2(k), \dots\}$.

There are many more models which can be included under the framework of general stochastic processes with an embedded Harris chain, among others multi–echelon models, which, most of the time, prove to be even more difficult to solve exactly than the single echelon models considered so far. Such a model is a two- level decentralized distribution system, consisting of one depot and several retailers. Since this example nicely combines the theory of all the previous chapters, we work it out in the next chapter.

4.7 Discussion

The essential difference between semi-Markov processes and the 'Harris-modulated stochastic processes' described in Section 4.5.3 is, that while in the case of semi-Markov processes the jump can depend on the state of the process when the jump occurs, for Harris–modulated stochastic processes the allowed dependency can be more complex. For instance, the expected interarrival time can depend on the next state (where the process jumps to) instead of depending solely on the state where the process has been before the jump. This means that the class of models which can be analyzed is considerably extended. Moreover, the established limit theorem (cf. Section 4.5.4, that is, the average cost expression (4.5.4.20), is given in terms of the simple arrival cycles. Within this framework, the analysis of many important models becomes straightforward (only identifying the embedded Harris chain), which otherwise would cost a laborious work of many pages; for other models, the analysis has only been performed under simplifying assumptions.

It is very interesting to observe how things fall into their places and how do all the models considered so far are related to each other. In the first chapter performance measures, among others an average cost expression was deduced under the assumption that the limiting distribution in the Cesaro sense of the joint process $(\mathbf{IP}, \mathbf{N}_\Delta)$ exists. The condition of our limit theorem 4.5.8 and the average cost expression

(4.5.4.20) was the existence of a Harris chain satisfying the initial assumptions 4.5.4. Now, do these results really relate to each other? Statement (*ii*) of Section 4.4.2 yields the answer: in order to prove that the Markov process constructed according to the conditions of Assumption 4.5.4 is a Harris chain, one can use the necessary and sufficient condition that 'the process is converging in Cesaro total variation to its stationary version'. This means exactly that if and only if the limiting distribution in the Cesaro sense exists. Although it might not be necessary, we like to stress that this is a necessary and sufficient condition, that is, Theorem 4.5.8 holds if and only if the Cesaro sense limiting distribution exists. Furthermore, for the case of inventory models, the Harris chain realizing the independence condition is always the joint process of **IP** and an embedded chain **A** of the arrival process of customers. Since the chain **A** is driving the arrival process, it can be intuitively seen as a better candidate for **N**, inheriting many of its properties. Besides, when **N** has independent increments, **N** itself can play the role of the embedded Harris chain, leading thus to the condition of Chapter 2 of the existence of the Cesaro limiting distribution of $(\mathbf{IP}, \mathbf{N}_\Delta)$.

A sufficient condition for the existence of a Cesaro limiting distribution of the joint process $(\mathbf{IP}, \mathbf{N}_\Delta)$ is that either of them possesses a limiting distribution while the other a Cesaro limiting distribution. However, this is NOT a necessary condition, therefore we also make use of alternative definitions and properties of Harris chains. For instance, when considering nonhomogeneous demand, we want to consider a nonstationary control policy. This implies that neither **A** not **IP** will possess a limiting distribution, however it can be shown that for the joint Markov process $(\mathbf{A}, \mathbf{IP})$ a regeneration set exists, implying that $(\mathbf{A}, \mathbf{IP})$ is a Harris chain. Although this implies that the joint process $(\mathbf{A}, \mathbf{IP})$ possesses a Cesaro limiting distribution, it would be difficult to prove its existence without further information (specification of the problem).

Appendix

The cost of the delay cycle

Our proof for the average cost expression, in Section 4.5.4 uses Proposition 3.1 of Glynn and Sigman (1992) . In order to make the use of this proposition 'legal', we need to prove that the expected cost of the delay cycle is finite, that is, $I\!E \int_0^{\mathbf{S}_0} g(\phi_s \circ \mathbf{X})ds < \infty$. By the findings of Section 4.5.4 (relation (4.5.4.16)) this is equivalent with proving that

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_0} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right) < \infty. \tag{4.7.4.1}$$

The idea of the proof is exactly as that of Proposition 4.2 of Glynn and Sigman (1992). Let

$$r(z) := I\!E_z \sum_{k=1}^{\mathbf{N}_0} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}),$$

and $\varepsilon := \{z : r(z) < \infty\}$, and we need to show (cf.Proposition 4.2, Glynn and Sigman (1992)) that $\pi_\infty(\varepsilon) = 1$. Denoting with $I\!E^0$ the expectation associated with the probability measure under which $\mathbf{X}$ is non-delayed (see Section 4.4.1),

$$\begin{aligned} \pi_\infty(\varepsilon) &= \frac{1}{I\!E\mathbf{N}_1} I\!E^0\left(\sum_{k=1}^{\mathbf{N}_1} 1_{\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))<\infty\}}\right) \qquad (4.7.4.2)\\ &= \frac{1}{I\!E\mathbf{N}_1}\sum_{k=1}^{\infty} I\!P^0\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})) < \infty, \mathbf{N}_1 > k\}. \end{aligned}$$

On the other hand, using the notation $z_k := I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})$, we have

$$\begin{aligned} I\!E^0(r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})), \mathbf{N}_1 > k) &= I\!E^0\left(I\!E^0_{z_k}\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right), \mathbf{N}_1 > k\right) \\ &= I\!E^0\left(I\!E^0_{z_k}\left(\sum_{k=1}^{\infty} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}), \mathbf{N}_1 > k\right)\right) \\ &= I\!E^0\left(I\!E^0\left(\sum_{n=k}^{\infty} I\!E(\mathbf{J}_n|\mathbf{A}_{n-1}), \mathbf{N}_1 > k\right)\right) \\ &= I\!E^0\left(\sum_{n=k}^{\infty} I\!E(\mathbf{J}_n|\mathbf{A}_{n-1})\right) \\ &\leq I\!E^0\left(\sum_{n=1}^{\infty} I\!E(\mathbf{J}_n|\mathbf{A}_{n-1})\right) < \infty, \end{aligned}$$

which means that $I\!P^0\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})) < \infty, \mathbf{N}_1 > k\} = 1$. In the very first equality of the above evaluation we use the fact that under the non-delay expectation $I\!E^0$, we can consider $r(z)$ as the expected cost of the first regeneration cycle, started in an arbitrary state, z_k. Having thus

$$\sum_{k=1}^{\infty} I\!P^0\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})) < \infty, \mathbf{N}_1 > k\} = \sum_{k=1}^{\infty} I\!P^0\{\mathbf{N}_1 > k\} = I\!E\mathbf{N}_1,$$

it follows immediately by relation (4.7.4.2) that $\pi_\infty(\varepsilon) = 1$. This proves thus that the expected cost of the delay-cycle is finite.

Chapter 5

A two level decentralized distribution system

5.1 Introduction

Within supply chain management the most commonly encountered problem is the modeling and optimization of the so called multi echelon production/distribution systems. These systems are also called networks. The simplest structure is a *series system*, where the output of each production or stocking point represents the input of the successive stage, hence each stage supplies the next one (Zipkin (2000)). There is only one finished product at the end of the system, where demand occurs. Next in complexity is the *assembly system*: here there is also one finished product, although there may be several raw materials, all supplied exogenously. These are assembled into components, then in further stages assembled further finally into the end product. A *distribution system* if represented on a diagram looks like a reversed assembly system (Zipkin (2000)). Further there are *tree systems*, which allow features of both assembly and distribution systems and *general systems* which allow more complex features. In this chapter we consider a distribution system, restricting the analysis to only two stages, which is very commonly encountered in the literature.

Information and control can be *centralized* or *decentralized* (localized). In the latter case each retailer sees only its own demands and the warehouse sees only the incoming order streams. The warehouse applies a first-come-first-served rule, and all the locations apply local policies. On the other hand, with a centralized control, the warehouse already accounts for demands when they occur at the retailers, thus it requires fully centralized information. Hence, the essential difference between

centralized and decentralized control systems is, that the latter relies on the history rather then the current status of the system to make crucial decisions (Zipkin (2000)). The system analyzed in this chapter is a decentralized two-level distribution system.

The classic model of a multi-level distribution system is METRIC, developed by Sherbrooke (1968). He also assumes only two stages, demand at the retailers Poisson, all locations using a base stock -$(S-1,S)$- policy. Variations on the METRIC models are also developed. Rosenbaum (1981) developed a heuristic model for a two-level system, assuming normally distributed demand. Deuermeyer and Schwarz (1981) also consider a two-level decentralized distribution system: one warehouse and multiple identical retailers, where external demand is Poisson (unit), and retailers apply (s,Q) policies. They perform a direct analysis of the steady-state stochastic behaviour of the system. They also incorporate into the model an 'expected delay from warehouse to retailer due to inadequate warehouse stock'. The setup of their model resembles our two-level distribution model the most. However, the techniques used and their results are completely different, since the aim of this model is to provide approximations, while we give exact average cost expressions.

The most exact evaluations of policies or performance measures for distribution systems are concerned with the case of centralized information. Cachon (1998) gives a very good reference of the recently developed research. Cachon himself uses a game theory approach, assuming that each location selects its reorder point such as to minimize its own costs given that each other player will do the same. Axsäter (1997) evaluates exact echelon stock (R,Q) policies (thus centralized control), assuming compound Poisson demand.

The model considered in this chapter is a truly decentralized two level distribution system: the retailers do not compete, and the dependency between the single retailers is only through the additional waiting time incurred by the event when the warehouse is out of stock. We also derive an exact expression for the distribution of the remaining waiting time due to warehouse out-of-stock, which also represents an important service measure. The technique we use combines the theory of the previous chapters.

5.2 The model

Consider a two level distribution system, consisting of one warehouse and N retailers. The warehouse and retailers follow (s,nQ) control policies, that is, the warehouse applies a policy with parameters (s_0,Q_0), while at retailer i the policy parameters are (s_i,Q_i), $i=1,\ldots,N$. All the processes and characteristics describing or related to the retailers will be indexed with i, $i=1,\ldots,N$, while for the warehouse we

use index 0. Demand, denoted by $\mathbf{D}_i$, at each retailer is described by a compound renewal process with i.i.d. individual demands $\{\mathbf{Y}_i^n : n \in I\!N\}$, $i = 1, \ldots, N$. Let further G_i denote the cdf. of the interarrival times and F_i the order size distribution for $i = 1, \ldots, N$. L_i, $i = 0, \ldots, N$ stands for the leadtime of a replenishment order. As it was discussed in Section 2.2, identifying the replenishment moments $\tau_i = \{\tau_i^n : n \in I\!N\}$ at the individual retailers, $(\mathbf{IP}_i, \tau_i)$ form a regenerative process for all $i = 1, \ldots, N$. The distribution of regeneration cycle lengths, that is, the time elapsed between two consecutive replenishment orders at retailer i, $\Phi_i(t)$, is given by

$$\sum_{k=0}^{\infty} G_i^{(k+1)*} I\!P\{\mathbf{Y}_1 + \ldots + \mathbf{Y}_k < s + Q_i - \mathbf{IP}_{\tau_{i-1}}^-, \mathbf{Y}_1 + \ldots + \mathbf{Y}_{k+1} \geq s + Q_i - \mathbf{IP}_{\tau_i}^-\} \tag{5.2.5.1}$$

If $\mathbf{Y}$ stands for the undershoot, than $\mathbf{Y} \stackrel{d}{=} F_Y^e$, the invariant distribution related to the renewal sequence of the individual demands, and then, $s + Q_i - \mathbf{IP}_{\tau_i}^- \stackrel{d}{=} \mathbf{Y}$ mod Q_i. Let us denote the stochastic counting process associated with the (renewal) sequence τ_i by $\mathbf{N}_i(t)$, $i = 1, \ldots, N$. The size of a replenishment order at retailer i is

$$\mathbf{R}_i^n := \mathbf{c}_i^n Q_i, \text{ with } \mathbf{c}_i^n := \inf\{k \geq 0 : kQ_i > s_i - \mathbf{IP}_i^n + \mathbf{Y}_i^{n+1}\},$$

the distribution of $\mathbf{c}_i^n$ being independent of n.

The superposition of N marked point processes is a marked point process whose points are the union of those N processes. The new mark space is the union of the N mark spaces. With this definition, demand at the warehouse is the superposition of the replenishment processes $(\tau_i, \mathbf{R}_i)$ of the retailers, that is,

$$\mathbf{D}_0(0,t] = \sum_{i=1}^{N} \sum_{n=1}^{\mathbf{N}_i(t)} \mathbf{R}_i^n = \sum_{j=1}^{\mathbf{N}_0(t)} \mathbf{R}^j, \tag{5.2.5.2}$$

where $\mathbf{N}_0$ is the superposition of the independent renewal processes $\mathbf{N}_i$, and $\mathbf{R}^j$ are the marks belonging to the original points, that is, the replenishment orders placed on time $\mathbf{t}_j = \tau_{\mathbf{N}_0(t)}$. Let $\widetilde{\mathbf{R}}_i(t)$ denote the cumulative process of replenishment orders of retailer i, up to time $t > 0$, that is,

$$\widetilde{\mathbf{R}}_i(t) := \sum_{n=1}^{\mathbf{N}_i(t)} \mathbf{R}_i^n,$$

$i = 1, \ldots, N$. Although the parameters differ at every retailer, the behaviour of the single systems are identical, therefore we need to perform the analysis for any one of the N retailers, and for the warehouse. Let us start with the analysis of the warehouse.

5.3 The warehouse

In order to compute the long run average cost of the warehouse, related to the net-stock process, we make use of the flow conservation law: $\mathbf{IN}_0(t+L_0) = \mathbf{IP}_0(t) - \mathbf{D}_0(t, t+L_0]$. In the spirit of Section 2.4, Chapter 2, to be able to use an efficient procedure, we need to find out whether the inventory position process and the leadtime demand are asymptoticly independent. The size of the individual demand at the warehouse, $\mathbf{R}^j$ (see relation 5.2.5.2) equals nQ_i with $i \in \{1, \ldots, N$ and $n \in I\!N$, thus an integer multiple of any the batch sizes of the retailers. Defining q as the largest common factor of $Q_0, Q_1, \ldots, Q_N$ (Axsäter (1997)), the state space of the inventory position process at the warehouse is $\{s_0 + q, \ldots, s_0 + kq, \ldots, Q_0\}$. Furthermore, as it was deduced in Section 2.4.2 of Chapter 2, the inventory position process associated with an (s, nQ) policy is always uniformly distributed, for any demand process, given the system was started in the invariant distribution. However, $\mathbf{IP}_0$ is unfortunately not a Markov process! Hence it is also not a Harris chain. To see why it is not a Markov process, let us examine $I\!P\{\mathbf{IP}_0^n \in \cdot \,|\mathbf{IP}_0^{n-1} = x_{n-1}, \ldots, \mathbf{IP}_0^0 = x_0\}$. $\mathbf{IP}_0^n = \mathbf{IP}_0^{n-1} - \mathbf{R}^n$ if $\mathbf{IP}_0^{n-1} - \mathbf{R}^n > s_0$, otherwise an order is placed. Let us assume that $\mathbf{R}^n$ is the replenishment order of retailer i_0, that is $\mathbf{R}^n = \mathbf{R}_{i_0}^m$, $m = \mathbf{N}_{i_0}(\mathbf{t}_n)$. Since $\tau_{i_0}^{\mathbf{N}_{i_0}(\mathbf{t}_n)}$ is a regeneration point, there are no dependencies. But what about the other retailers? At point $\mathbf{t}_n$ the residual lives $\mathbf{A}_i(\mathbf{t}_n) \geq 0$, $i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, N$ give information about the inventory position processes $\mathbf{IP}_i$. In this way, the inventory position of the warehouse $\mathbf{IP}_0^n$ depends on the whole sample paths of the past, hence $I\!P\{\mathbf{IP}_0^n \in \cdot \,|\mathbf{IP}_0^{n-1} = x_{n-1}, \ldots, \mathbf{IP}_0^0 = x_0\}$ does not equal $I\!P\{\mathbf{IP}_0^n \in \cdot \,|\mathbf{IP}_0^{n-1} = x_{n-1}\}$, hence it is not Markov. On the other hand, conditioning on $\mathbf{A} := (\mathbf{A}_1, \ldots, \mathbf{A}_N)$ yields the desired independence $I\!P\{\mathbf{IP}_0^n \in \cdot \,|\mathbf{IP}_0^{n-1} = x_{n-1}, \ldots, \mathbf{IP}_0^0 = x_0\} = I\!P\{\mathbf{IP}_0^n \in \cdot \,|\mathbf{IP}_0^{n-1} = x_{n-1}\}$.

As in Section 2.7.3, th stochastic counting process related to the leadtime demand at the warehouse, $\mathbf{N}_0(t, t + L_0]$ depends on the joint residual life $\mathbf{A}(t)$. On the other hand, since the individual demands $\mathbf{R}^j$ are not identically distributed, with a similar reasoning as before, the leadtime demand also depends on $\mathbf{IP}_0$ through $\mathbf{A}$. Conditioning on $\mathbf{A}$ yields again independence. The question which arises now: is $\mathbf{A}$ a Harris chain, even more, is $(\mathbf{IP}_0, \mathbf{A})$ a Harris chain? Since $\mathbf{IP}_0$ has a limiting distribution (the system started in the invariant distribution), if $\mathbf{A}$ is a Harris chain, it will follow through statement (ii) of Section 4.4.2 that the joint process $(\mathbf{IP}_0, \mathbf{A})$ is a Harris chain too. Hence it only remains to verify if $\mathbf{A}$ is a Harris chain. If it is, we can use the results of Chapter 4 and apply the limit theorem for stochastic processes with an embedded Harris chain to derive the long run average cost for the warehouse.

The limiting distribution of leadtime demand at the warehouse is given thus by

$$\mathbf{D}_0^{\infty}(0, L_0] \stackrel{d}{=} \sum_{i=1}^{N} \sum_{n=1}^{\mathbf{N}_i^{\infty}(0,L_0]} \mathbf{R}_i^n,$$

where $\mathbf{N}_i^{\infty}(0, L_0]$ is characterized by relation (2.7.2.10) (see Section 2.7.3) as

$$\lim_{t\uparrow\infty} \phi_t \mathbf{N}_i(0, L_0] = \lim_{t\uparrow\infty} (\mathbf{N}_i(t + L_0) - \mathbf{N}_i(t)) \stackrel{d}{=} \mathbf{N}_i^0(L_0 - \mathbf{A}_i^{\infty}),$$

where $\mathbf{N}_i^0$ is the stochastic counting process associated with the sequence $\{\tau_i\}$, with a renewal in zero. Hence, the demand process $\mathbf{D}_0$ of the warehouse depends on the inventory position process through the residual life processes $\mathbf{A}_i(t) := \mathbf{t}_{\mathbf{N}_i(t)+1} - t$. Proposition 5.1 of Sigman (1990) helps us to come around this difficulty: assuming that all the Φ_i's are spread out (cf. Definition 4.2.2), the process $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_N)$ is a positive recurrent Harris chain which converges weakly to its stationary distribution (denote with $\mathbf{A}_{\infty}$ the random variable distributed with this stationary distribution). Now the joint process $(\mathbf{IP}_0(t), \mathbf{A}(t))$ is a Markov process, moreover, due to the spread out cycles assumption, and property (ii) of Section 4.4.2, it is a Harris chain, having a stationary distribution, given by the distribution of $(\mathbf{IP}_0, \mathbf{A})_{\infty}^c$. The limit theorem of Chapter 4 yields now for the average cost of the warehouse

$$g_R = \frac{1}{I\!E_{\mathbf{A}_{\infty}^c}(I\!E(\mathbf{T}_1|\mathbf{A}))} I\!E_{(\mathbf{A},\mathbf{IP}_0)_{\infty}^c} \left(I\!E \left(\int_0^{\mathbf{t}_1} f(Q_0\mathbf{U} - \mathbf{D}_0(t, t + L_0]|(\mathbf{A}, \mathbf{IP}_0)) \right) \right).$$

Observation 5.3.1 *The reason for considering identical retailers most of the time in the literature is, that the non-identical retails assumption generates a dependency structure, which is difficult to deal with by a 'traditional' limit theorem. Although Axsäter (1997) considers non-identical retailers, he assumes that demand is compound Poisson, which, by the memoryless property maintains the desired independence properties under which the inventory position process* $\mathbf{IP}_0$ *remains Markovian. The real difficulty only arises when demand is compound renewal.*

It remains now to analyze the behaviour of the relevant stochastic processes at the retailers.

5.4 The retailers

The flow conservation law for the retailers is given by

$$\mathbf{IN}_i(t + L_i + \mathbf{W}(t)) = \mathbf{IP}_i(t) + \mathbf{D}_i(t, t + L_i + \mathbf{W}(t)], \tag{5.4.5.1}$$

where $\mathbf{W}(t)$ is the additional remaining waiting time at time t incurred by the event when the warehouse is out of stock. By this definition of the waiting time it is clear that the flow conservation law remains valid. In general the difficulty is to determine the distribution of the remaining waiting time $\mathbf{W}(t)$. If the distribution of $\mathbf{W}(t)$ is known, and we can prove that the embedded processes comprising all the relevant information about the past form a Harris chain, then we can perform the analysis on the same line as before for the warehouse.

Let us first attempt to determine the distribution of the remaining waiting time at t, $\mathbf{W}(t)$, since, intuitively, that will also reveal the precise dependency structure. The state space of $\mathbf{W}(t)$ is $[0, L^0]$, and its distribution can be characterized by the following relation:

$$\begin{aligned} I\!P\{\mathbf{W}(t) \leq w\} &= I\!P\{\mathbf{IN}_0(t+w) + \mathbf{D}_0(t, t+w] > 0\} && (5.4.5.2) \\ &= I\!P\{\mathbf{IP}_0(t+w-L_0) - \mathbf{D}_0(t+w-L_0, t] > 0\}. && (5.4.5.3) \end{aligned}$$

Intuitively, relation (5.4.5.2) expresses the fact that if the remaining waiting time is w, then the incoming order at the warehouse at time $t+w$ raises the net inventory of the warehouse $\mathbf{IN}_0$ to a positive level, if no demand is coming in at the warehouse during the time $[t, t+w]$. Hence, the distribution of the remaining waiting time only depends on the retailer in question indirectly, through the demand $\mathbf{D}_0(t, t+w]$. In relation (5.4.5.3) the flow conservation law is used for $\mathbf{IN}_0(t+w)$.

The demand, as defined earlier, $\mathbf{D}_0(t+w-L_0, t] = \sum_{k=1}^{N} \widetilde{\mathbf{R}}_k(t+w-L_0, t]$. All the $\widetilde{\mathbf{R}}_k, k = 1, \ldots, i-1, i+1, \ldots, N$ are independent, and we can determine the superposition $\mathbf{D}_0$ as before. Yet, $\widetilde{\mathbf{R}}_i(t+w-L_0, t]$ (the replenishment orders of the retailer in question) equals $\widetilde{\mathbf{R}}_i(t+w-L_0, \tau_{\mathbf{N}_i(t)}]$, having $\tau_i := \tau_{\mathbf{N}_i(t)}$ dependent on $\mathbf{IP}_i$, exactly what we want to avoid in relation (5.4.5.1). Fixing now a w and knowing that the retailer applies an (s_i, nQ_i) policy we have

$$\mathbf{IP}_i(t) = (\mathbf{IP}_i(t-(L_0-w)) - \mathbf{D}_i(t-(L_0-w), t]) \bmod Q_i.$$

Define now

$$\widehat{\mathbf{IP}}_i(t-(L_0-w), t] = \mathbf{IP}_i(t-(L_0-w)) - \mathbf{D}_i(t-(L_0-w), t],$$

and it follows that

$$\begin{aligned} \widetilde{\mathbf{R}}_i(t-(L_0-w), t] &= \mathbf{IP}_i(t) - \widehat{\mathbf{IP}}_i(t-(L_0-w), t] \\ &= Q_i \left\{ (\mathbf{IP}_i(t) + \mathbf{D}_i(t-(L_0-w), t]) \text{ div } Q_i \right\}. && (5.4.5.4) \end{aligned}$$

This expression yields that $\mathbf{W}$ depends on $\mathbf{IP}_i$ through $\mathbf{D}_0$. $\mathbf{IP}_i$ depends on $\mathbf{D}_0$ through $\mathbf{A}$ and $\widetilde{\mathbf{R}}_i(t+w-L_0, t]$. And finally, $\widetilde{\mathbf{R}}_i(t+w-L_0, t]$ is given by relation

(5.4.5.4). Hence, the joint process $(\mathbf{IP}_i, \mathbf{A})$ comprises all the relevant information about the past, for a fixed w. Thus we need to condition on the remaining waiting time $\mathbf{W}(t)$. As we have seen in the previous section, $(\mathbf{IP}_i, \mathbf{A})$ is a Harris chain. Hence, the average cost of retailer i is given by the expression

$$\frac{I\!E_{(\mathbf{A},\mathbf{IP}_i)^c_\infty} \int_0^{L_0} \left(I\!E \left(\int_0^{\mathbf{t}_1} f(\mathbf{IP}_i - \mathbf{D}_i(t, t+L_0+w] | (\mathbf{A}, \mathbf{IP}_i)) \right) \right) F_W(dw)}{I\!E_{\mathbf{A}^c_\infty}(I\!E(\mathbf{T}_1|\mathbf{A}))}.$$

Observation 5.4.1 *The waiting time* $\mathbf{W}$ *with distribution given by (5.4.5.3), is not only important for the calculation of long run average costs, but it yields service measure, frequently used in practice: the probability that a customer has to wait more than certain amount of time* $T > 0$.

Remark 5.4.2 *The analysis of the two level distribution system described above remains valid also in the case when the system starts in an arbitrary state, not in the invariant distribution. In this case we need the additional assumption of a finite expected delay-cycle cost.*

5.5 Discussion

There are several reasons why this approach works well compared, for example, to the model of Deuermeyer and Schwarz (1981). Let us start with a basic modeling issue: the additional waiting time (besides the fixed transportation time from the warehouse to the retailer) is approximated by Deuermeyer and Schwarz by its expectation: D_j is the expected delay from the warehouse to retailer j due to inadequate warehouse stock. Then, they define the expected effective leadtime to retailer j by adding D_j to the fixed transportation time. Our approach is to define $\mathbf{W}(t)$ as the additional *remaining* waiting time at time t of a retailer due to inadequate warehouse stock. At first sight, the approximation of Deuermeyer and Schwarz might look simpler, thus easier to determine, however, our modeling has a great advantage: the classical flow conservation law remains valid with $L_i + \mathbf{W}(t)$, while this is not true in the other case. Hence, having determined the distribution of $\mathbf{W}(t)$ we only need to solve a classical single location single item model. The remaining single item single location problem is still a very complex one because of the induced dependency structure. Applying the embedded Harris chain method developed in Chapter 4, the problem can be solved straightforwardly. This is strongly related to an other idea exploited in the analysis, concerning the superposition of several independent renewal processes: the joint process of the residual lives of these renewal processes forms a Harris chain, thus it fits nicely into the theory of Chapter 4. Because of the dependency structure

induced by general renewal processes, most of the time models in the literature are limited to the identical retailers case or Poisson arrivals. In the identical retailers case the history of orders is irrelevant because all the replenishment orders are equal. In the case of Poisson arrivals the memoryless property significantly simplifies the analysis. Otherwise, assuming general compound renewal demand at the non-identical retailers triggers the following problem: the inventory position process related to the warehouse will not possess the Markovian property, it depends on the whole history of the sample paths through the residual life processes. However, since the inventory position process and the residual life jointly form a Harris chain, as mentioned above, the theory of Chapter 4 yields the solution.

Chapter 6

Concluding remarks

The main concern of this thesis has been to extend already known results for single item single location problems. These single location models can serve as components of a complex supply chain, and as such they need sophisticated solution techniques. Often, the approximate solution for a supply chain is rough, because of the lack of a technique, which would be able to deal with induced dependency structures on the components, and provide exact performance measure expressions. Although the specialist literature on single item single location inventory models is very extensive, nonetheless, the listed models cannot handle certain demand processes or policies (or a combination of a demand process with a certain policy). Moreover, single item single location models with exogenous or endogenous dependency structures are scarce in the specialists literature, especially on this area there is a need for more advanced models.

The first chapter revisits simple single item single location models, with exogenous demand processes. The mainstream research on such models can be divided into a renewal theoretic and a Markovian approach. The main idea behind the 'renewal theoretic' approach is to identify the regeneration points of the inventory position process at the time points of order placement. The related asymptotic analysis is a tedious evaluation of algebraic expressions. The accent is rather on developing efficient approximations, which is motivated by the objectives and computational capacities of computers at that time. Zipkin (1986), ahead of his time, develops a new perspective of stochastic single item inventory models. He is the first to observe that all the basic policies have the same properties and one can perform the analysis on a single general policy which comprises all the basic policies, while his results remain valid under constant as well stochastic leadtimes. With this approach long run average costs and service measures can be deduced straightforwardly. However, Zipkin's

results can be extended further by relaxing his assumption on the existence of the limiting distribution of the stochastic counting process $\mathbf{N}$ (Bázsa and Den Iseger (2001)). The advantage of the latter approach lies on one hand in the approach of treating constant and stochastic leadtimes jointly (but under the no order crossing condition), on the other hand, extending the class of demand processes to nonstationary point processes. In fact, this approach also extends the combination of policy/demand processes: for instance it can handle an (s, S) policy with unit demand, such that the stochastic counting process $\mathbf{N}$ converges in the limit to a stationary one. Zipkin's model can only handle the combination of periodic chain/regenerative demand process. Furthermore Bázsa and Den Iseger (2001) deduce a surprising equivalence between the stochastic leadtimes model of Zipkin and a model with constant leadtimes but nonstationary demand, concluding: if in certain cases it is reasonable to use stationary policies in combination with stochastic leadtimes, then it is just as reasonable to use stationary policies in combination with nonstationary demand.

Ongoing research on supply chains has increased the interest in continuous-demand models. Clark and Scarf (1960) were the first to show that the optimal policy for a two-echelon inventory system, with finite horizon, can be computed by decomposing the problem into two separate single-location problems. For the depot an (s, S) policy solves the problem, and constitutes an optimal order policy for the whole system. However, the lack of an efficient optimization algorithm for such single location problems results in suboptimality of the overall solution. To our knowledge, the only successful attempt to tackle the continuous demand case directly, without previous discretization, is the optimization algorithm of Federgruen and Zipkin (1985), the dominating idea being that of a policy iteration method from Markov decision theory. Although standard policy iteration always results in policy improvement, this algorithm does not always achieve the best one-step improvement. Yet, convergence is always obtained, and the algorithm does itself represent a big improvement upon the known algorithms up to date. The algorithm developed in Chapter 3 is, however, more efficient and much simpler, in the sense that each step is ensued by elementary geometrical considerations. Based on the optimality relations we construct two aid functions, $s = \phi(S)$ and $s = \psi(S)$. It turns out that these aid functions have two simple but crucial properties: (1) both of them increase slower than 45 degrees; and (2) ψ always intersects ϕ in a maximum or minimum of ϕ, and these are the only stationary points for ϕ. The algorithm converges monotonically, such that at every iteration a policy improvement is obtained. Since every iteration finds a local minimum of the expected average cost, the number of iterations is at most N, where $N < \infty$ represents the number of local minimums. Besides, the algorithm is also easy to understand, every step can be followed using a simple graphical

representation.

The essential difference between semi-Markov processes and the 'Harris-modulated stochastic processes' described in Chapter 4 is, that while in the case of semi-Markov processes the jump can depend on the state of the process when the jump occurs, for Harris–modulated stochastic processes the allowed dependency can be more complex. For instance, the expected interarrival time can depend on the next state (where the process jumps to) instead of depending solely on the state where the process has been before the jump. This means that the class of models which can be analyzed is considerably extended. Moreover, the the average cost expression derived for such models is given in terms of the simple arrival cycles. Within this framework, the analysis of many important models becomes straightforward (only identifying the embedded Harris chain), which otherwise would cost a laborious work of many pages; for other models, the analysis has only been performed under simplifying assumptions. The specialist literature is quite modest on this line of research. One of the cases studied, for instance by Song and Zipkin Song (1993), is the so called Markov -modulated demand process, that is: the rate of the Poisson arrivals changes with the state of an exogenous, independent Markov chain, called the *world*. However, in case of endogenous dependencies the problem becomes considerably more difficult to solve. We tackle this problem by identifying an embedded Harris chain, which carries all the relevant information about the past. Conditioning on this Harris chain, yields independence, hence from this point on, the problem reduces to the 'simple' model of Chapter 2.

It is very interesting to observe how things fall into their places and how do all the models considered so far are related to each other. In the first chapter performance measures, among others an average cost expression was deduced under the assumption that the limiting distribution in the Cesaro sense of the joint process $(\mathbf{IP}, \mathbf{N}_\Delta)$ exists. The condition of the limit theorem of Chapter 4 was the existence of a Harris chain satisfying certain assumptions. Now, statement (ii) of Section 4.4.2 yields the following: in order to prove that the Markov process constructed according to certain assumptions is a Harris chain, one can use the necessary and sufficient condition that 'the process is converging in Cesaro total variation to its stationary version'. This means exactly that if and only if the limiting distribution in the Cesaro sense exists. Although it might not be necessary, we like to stress that this is a necessary and sufficient condition, that is, the limit theorem holds if and only if the Cesaro sense limiting distribution exists. Furthermore, for the case of inventory models, the Harris chain realizing the independence condition is always the joint process of $\mathbf{IP}$ and an embedded chain $\mathbf{A}$ of the arrival process of customers. Since the chain $\mathbf{A}$ is driving the arrival process, it can be intuitively seen as a better candidate for $\mathbf{N}$, inheriting

many of its properties. Besides, when $\mathbf{N}$ has independent increments, $\mathbf{N}$ itself can play the role of the embedded Harris chain, leading thus to the condition of Chapter 2 of the existence of the Cesaro limiting distribution of $(\mathbf{IP}, \mathbf{N}_\Delta)$.

A sufficient condition for the existence of a Cesaro limiting distribution of the joint process $(\mathbf{IP}, \mathbf{N}_\Delta)$ is that either of them possesses a limiting distribution while the other a Cesaro limiting distribution. However, this is NOT a necessary condition, therefore we also make use of alternative definitions and properties of Harris chains. For instance, when considering nonhomogeneous demand, we want to find an optimal nonstationary control policy. This implies that neither $\mathbf{A}$ not $\mathbf{IP}$ will possess a limiting distribution, however it can be shown that for the joint Markov process $(\mathbf{A}, \mathbf{IP})$ a regeneration set exists, implying that $(\mathbf{A}, \mathbf{IP})$ is a Harris chain. Although this implies that the joint process $(\mathbf{A}, \mathbf{IP})$ possesses a Cesaro limiting distribution, it would be difficult to prove its existence without further specification of the problem.

The last chapter demonstrates how the newly developed techniques for single item single location models can be employed in order to solve a two–level decentralized distribution system. Because the hidden interrelation among the components induces a dependency structure on each component, so far only approximate performance measure expressions were derived in the specialists literature. Yet, applying the theory of Chapter 4 helps us come around the difficulties: Chapter 5 derives exact average cost and service measure (waiting time) expressions. There are several reasons why this approach works well compared, for example, to the model of Deuermeyer and Schwarz (1981). First of all, there is a basic modeling issue: the additional waiting time (besides the fixed transportation time from the warehouse to the retailer) is approximated by Deuermeyer and Schwarz by its expectation: D_j is the expected delay from the warehouse to retailer j due to inadequate warehouse stock. Then, they define the expected effective leadtime to retailer j by adding D_j to the fixed transportation cost. Our approach is to define $\mathbf{W}(t)$ as the additional *remaining* waiting time at time t of a retailer due to inadequate warehouse stock. At first sight, the approximation of Deuermeyer and Schwarz might look simpler, thus easier to determine, however, our modeling has a great advantage: the classical flow conservation law remains valid with $L_i + \mathbf{W}(t)$, while this is not true in the other case. Hence, having determined the distribution of $\mathbf{W}(t)$ we only need to solve a classical single location single item model. The remaining single item single location problem is still a very complex one because of the induced dependency structure. Applying the embedded Harris chain method developed in Chapter 4, the problem can be solved straightforwardly.

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Nederlandse samenvatting

Steeds meer winkels zien de laatste jaren af van het aanhouden van voorraden, vooral in de drukke binnensteden waar het ruimtetekort het meest prangend is. Dit beeld sluit gaan aan bij de meest recente trend op het gebied van het beheer van de zogenaamde logistieke ketens: de just-in-time (JIT) aanpak. Dit betekent dat voorraden pas op de fabrieksvloer of in de winkel aankomen alleen als en wanneer het nodig is. De reductie in levertijden, de kosten en tijd voor het opstarten van het productieproces hebben een enorme invloed op het functioneren van de logistieke keten. Vooral in de telecommunicatie en informatietechnologie sector is gebleken dat het aanhouden van teveel voorraad een kostbare misrekening is. Zo heeft bijvoorbeeld het Amerikaanse IT bedrijf Cisco Systems een bedrag van US$ 2.25 miljard moeten afschrijven op haar voorraad. "Voorraad is een kwaad", zo luidt een bekende JIT slogan.

Na de gebeurtenissen van 11 september 2001, die aanleiding waren voor verscherpte veiligheidsmaatregelen bij grensovergangen, havens en vliegvelden, deden goederen er veel langer over dan verwacht om op hun uiteindelijke bestemming te komen. In een artikel in The Economist van 22 september 2001 wordt vastgesteld dat "iedere perceptie dat er sprake is van verhoogd risico de balans doet doorslaan in de richting van een JIC (just-in-case) strategie, in plaats van een JIT, om voorraden aan te houden in geval er sprake is van een verstoring van de logistieke keten. De balans is nog verder doorgeslagen door de sterke daling van de rente, die de voorraadkosten meer dan gehalveerd hebben, en zodoende de behoefte aan een JIT systeem verder afgezwakt hebben". De waarheid is inderdaad dat geen algemeen geldend model is voor voorraadmodel dat in alle gevallen toepasbaar is. Uiteraard kan JIT zonder meer worden toegepast indien de levertijd en opstartkosten verwaarloosbaar klein zijn: dit weegt zwaarder dan de bovengenoemde risicofactor. Maar hoewel JIT in de telecommunicatie en informatietechnologie sector zijn waarde bewezen heeft, kan in het geval van een ander type product of omgeving een geheel andere aanpak noodzakelijk blijken. De "oude" modellen zijn derhalve nog wel degelijk waardevol.

In voorraadbeheer is het in principe niet voldoende om te weten waar in het algemeen mogelijkheden tot verbetering te vinden zijn, omdat sommige alternatieven in een specifieke situatie erg kostbaar kunnen zijn. Men moet daarom een inschatting kunnen maken van de opbrengsten in een dergelijke specifieke situatie en weten welke methoden tot verandering ter beschikking staan. Een logistieke keten kan veelal geoptimaliseerd worden door het te reduceren tot een reeds eenvoudigere één-produkt-één-locatie problemen (zie hoofdstuk 5). Er kunnen echter zichtbare of verborgen relaties bestaan tussen de componenten van de logistieke keten; een dergelijke afhankelijkheidsstructuur is een reden om ook nauwkeurigere oplossingsmethoden in ogenschouw te nemen. Hoe minder precies men namelijk de deelcomponenten benadert, hoe slechter de algehele oplossing voor de gehele logistieke keten zal zijn. Andersom geldt iets soortgelijks: een exacte oplossing van de één-produkt-één-locatie deelproblemen kan een garantie zijn voor een goede globale oplossing van de logistieke keten. Op basis van deze gedachte is het doel van dit proefschrift gebaseerd: we geven een hernieuwde kijk op de basale componenten van de logistieke keten, te beginnen met de meest eenvoudige modellen, zoals het model met exogene vraag zonder de complicerende afhankelijkheidsstructuren. Vervolgens stappen we over op modellen met verschillende vormen van exogene en endogene afhankelijkheidsstructuren. Tenslotte laten we zien hoe voor een specifieke logistieke keten deze nieuwe technieken toegepast kunnen worden om een globaal exacte oplossing te kunnen krijgen. De behoefte aan exacte oplossingen is niet alleen ingegeven door de enorme toename aan mogelijkheden van de huidige computers, maar ook door de ontwikkeling door Den Iseger van een techniek om zogenaamde Laplace transformaties te inverteren, waarmee willekeurig exacte oplossing verkregen kunnen worden. Dit algoritme stelt ons in staat om met dezelfde precisie gesloten vorm uitdrukkingen te ontwikkelen voor de prestatiemaatstaven van de logistieke keten.

In hoofdstuk 2 van dit proefschrift geven we allereerst een overzicht van de meest eenvoudige één-produkt-één-locatie modellen, met een stochastisch, exogene vraag. Er bestaat een schat aan literatuur over dit onderwerp, omdat onderzoekers hun analyses veelal beperkten tot een enkele klasse van modellen. Hoewel er misschien al een overdaad aan onderzoek op dit gebied bestaat, is er nog voldoende ruimte voor verbetering. De modellen kunnen worden onderverdeeld in verschillende klassen, op basis van de belangrijkste kenmerken: de gebruikte voorraadpolitiek (bijvoorbeeld (s, S), (s, Q) regels; *continuous* of *periodic review*), de aard van de levertijd (constant of stochastisch), het type vraag (eenheid of willekeurig, discreet of continu individuele vraag; Poisson aankomsten of meer algemeen). In 1986 voltooide Zipkin als eerste een analyse van een model dat al deze karakteristieken tezamen beschouwde, gebruikmakend van slechts zwakke regulariteitsvoorwaarden. Zijn bijdrage is vooral in het oog

springend door de asymptotische analyse onder de aanname van stochastische levertijden. De nieuwe resultaten in dit hoofdstuk staan een analyse toe van algemenere vraagprocessen dan die Zipkin beschouwde. Een duidelijk voorbeeld is het tijds-niet-homogene samengestelde Poisson process, dat niet aan de regulariteitsvoorwaarden van Zipkin voldoet, maar wel in ons model past. Hoewel dit voorbeeld op zich al belangwekkend is, geeft het ook een nieuwe kijk op stochastische levertijden; tegelijkertijd geeft het ook aan dat er een interessante equivalentie bestaat tussen Zipkin's order-aankomst mechanisme met stochastische levertijden, en een model met constante levertijd, maar niet-stationaire vraag. De verbeteringen die we realiseren voor de "eenvoudige" modellen zijn een voorproef op het model in hoofdstuk 4, dat een uitbreiding op de "eenvoudige" modellen vormt met complicerende afhankelijkheidsstructuren.

Het volgende hoofdstuk gaat over het optimaliseren van beslissingsregels voor een één-produkt-één-locatie voorraadmodel. Zo hebben met name de zogenaamde (s, S) regels al jarenlang van veel onderzoekers de nodige aandacht gekregen. Zheng en Federgruen hebben in 1991 een optimalisatie-algoritme gepubliceerd voor (s, S) regels met discrete ordergrootte, wat toendertijd op dit gebied een doorbraak betekende. Hun algoritme werkt echter niet voor een continue ordergrootte, zodat er in de literatuur nog steeds een efficient algoritme voor het continue geval ontbrak. Hoofdstuk 3 voorziet een oplossing voor dit tekort, terwijl ook een overzicht en vergelijking van de reeds bekende algoritmen gegeven wordt. Het optimaliseren van (s, S) regels met een continue toestandsruimte wordt gezien als een moeilijk probleem, omdat de techniek op basis van Markov beslissingsprocessen om fijne aanpassingen vraagt. Het model uit dit hoofdstuk is daarentegen erg eenvoudig, daar iedere stap een zuiver geometrische motivatie heeft. Bovendien convergeert het algoritme op monotone wijze en levert iedere iteratie een verbetering op. De resulterende voorradpolitiek is ε-optimaal voor een gegeven $\varepsilon > 0$.

In hoofdstuk 4 vervolgen we de analyse van één-produkt-één-locatie met endogene en/of exogene afhankelijkheden. Een voorbeeld van een dergelijk model met een exogene afhankelijkheidsstructuur is een model waarbij de vraag gebaseerd is op *Markov-modulated* aankomsten. Song en Zipkin noemen een dergelijk vraagproces een *world-dependent* vraagproces: de rate van de Poisson-aankomsten varieert op basis van een oxogeen, onafhankelijk (discrete toestand) stochastisch process, de wereld genoemd. De analyse wordt een stuk gecompliceerder met endogene afhankelijkheden. Toch zijn de resultaten in dit hoofdstuk verrassend simpel, het resultaat van een modelleringsidee: we nemen aan dat er een zogenaamde Harrisketen (intuitief gezien een positieve recurrente Markovketen met continue toestandsruimte) bestaat die alle relevante informatie over het verleden bevat. In de praktijk is een dergelijk proces re-

delijk simpel te vinden. Beschouw bijvoorbeeld het geval waarbij de levertijd van het verleden afhangt via het residual life proces. Een dergelijk probleem treedt op wanneer we periodic review beslissingsregels combineren met een samensgestelde renewal vraag (dus niet Poisson), of het voorbeeld in hoofdstuk 5. Het is bekend dat het residual life proces een Harrisketen is, dus sluit het uitstekend aan bij onze behoefte: we hebben immers een ingebedde Harrisketen gevonden die alle relevante informatie over het verleden bevat. De uiteindelijke gemiddelde kosten formules zijn gedefinieerd in termen van aankomstcycli, dat wil zeggen in een zo simpel mogelijke vorm, terwijl de optimalisaties worden uitgevoerd op de overgangen in de voorraadpositie en de ingebedde Harrisketen. Het laatste betekent dat niet-stationaire optimale regels kunnen worden afgeleid.

Tenslotte worden de nieuwe methoden en technieken die zijn ontwikkeld voor één-produkt-één-locatie problemen toegepast om een gedecentraliseerd distributiesysteem met twee niveau's op te lossen. Het systeem bestaat uit een opslagplaats en verscheidene niet-identieke verkooppunten. De externe vraag bij de verkooppunten is een samengesteld renewal proces. Het gebruik van een gedecentraliseerd beleid betekent dat men het probleem oplost voor iedere component afzonderlijk: elk verkooppunt weet alleen zijn eigen vraag en de opslagplaats ziet alleen maar de stroom van binnenkomende bestellingen. De afzonderlijke componenten behoren echter niet tot de klasse van eenvoudige één-produkt-één-locatie modellen, omdat de verborgen relaties tussen de componenten voor alle componenten afhankelijksheidsstructuren impliceren. Er waren daarom tot nu toe voor dergelijke gedecentraliseerde systemen alleen nog maar benaderende prestatiemaatstaven afgeleid. Wanneer we echter de theorie uit hoofdstuk 4 toepassen, kunnen we de moeilijkheden omzeilen: in hoofdstuk 5 leiden we exacte uitdrukkingen af voor de gemiddelde kosten en serviceniveau (wachttijd).

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