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THE RATE OF CONVERGENCE TO OPTIMALITY OF THE LPT RULE

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The LPT rule is a heuristic method to distribute jobs among identical machines so as to minimize the makespan of the resulting schedule. If the processing times of the jobs are assumed to be independent identically distributed random variables, then (under a mild condition on the distribution) the absolute error of this heuristic is known to converge to 0 almost surely. In this note we analyse the asymptotic behaviour of the absolute error and its first and higher moments to show that under quite general assumptions the speed of convergence is proportional to appropriate powers of $(\log\log n)/n$ and $1/n$. Thus, we simplify, strengthen and extend earlier results obtained for the uniform and exponential distribution.

1. Introduction

Suppose that *n jobs* with *processing times* $p_1, \ldots, p_n$ have to be distributed among *m uniform machines*. Let s_i be the *speed* of machine i $(i=1,\ldots,m)$. If the sum of the processing times assigned to machine i is denoted by $Z_n(i)$ $(i=1,\ldots,m)$, then a common objective is to minimize the *makespan* $Z_n^{(m)} = \max_i \{Z_n(i)/s_i\}$. For this NP-hard problem many heuristics have been proposed and analyzed; we refer to [Graham et al. 1979; Rinnooy Kan 1984] for a survey. Among them, the *LPT rule* in which jobs are assigned to the first available machine in order of decreasing p_j is a particularly simple and attractive one. The value $Z_n^{(m)}(\text{LPT})$ produced by this rule is related to the optimal solution value $Z_n^{(m)}(\text{OPT})$ for the case that $s_i = 1$ for all i by [Graham 1969]

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$$\frac{Z_n^{(m)}(\mathrm{LPT})}{Z_n^{(m)}(\mathrm{OPT})} \le \frac{4}{3} - \frac{1}{3m} . \tag{1}$$

Computational evidence, however, suggests that this *worst case* analysis is unnecessarily pessimistic in that problem instances for which (1) is satisfied as an equality appear to occur only rarely.

To achieve a better understanding of this phenomenon, let us assume the processing times p_j ($j=1,\dots,n$) to be independent, identically distributed *random variables.* The relation between the random variables $\underline{Z}_n^{(m)}(\mathrm{OPT})$ and $\underline{Z}_n^{(m)}(\mathrm{LPT})$ can then be subjected to a *probabilistic analysis.* In [Frenk & Rinnooy Kan 1984] it was shown that (under mild conditions on the distribution of the p_j) the *absolute error*

$$\underline{Z}_n^{(m)}(\mathrm{LPT}) - \underline{Z}_n^{(m)}(\mathrm{OPT}) \tag{2}$$

converges to 0 *almost surely* as well as in *expectation*. Thus, the heuristic is *asymptotically optimal* in a strong (*absolute* rather than *relative*) sense, which provides an explanation for its excellent computational behaviour.

In [Frenk & Rinnooy Kan 1984], the speed at which the absolute error converges to 0 was analyzed for the special cases of the uniform and exponential distribution respectively. Here we extend and generalize the results for almost sure convergence and convergence in expectation by showing that for a large class of distributions (essentially those with $F(x)=x^a$ ($0\le x\le 1$, $0<a<\infty$)), this speed is proportional to appropriate powers of $(\log\log n)/n$ and $1/n$ respectively. This implies that, although the optimality of the LPT rule could only be established asymptotically, the convergence of the absolute error to 0 at least occurs reasonably fast. In some sense, to be explained later, these results are the best possible ones obtainable for this heuristic.

The main result for the case of almost sure convergence, is described and proved in Section 2. The case of convergence in expectation is dealt with in Section 3, where we bound first and higher moments of the expected absolute error. The proof in this Section is of a particularly attractive simplicity. Some extensions and conjectures are briefly examined in Section 4.

2. Almost sure convergence

In [Frenk & Rinnooy Kan 1984], it is shown that the absolute error of the LPT rule (2) is bounded (up to a multiplicative constant) by

$$\underline{D}_n(\alpha) = \max_{1\le k\le n} \left\{ \underline{p}_{k:n} - \frac{1}{\alpha} \sum_{j=1}^{k} \underline{p}_{j:n} \right\} \tag{3}$$

where $\underline{p}_{1:n} \le \underline{p}_{2:n} \le \cdots \le \underline{p}_{n:n}$ are the *order statistics* of the processing times and $\alpha = 1 + (m-1)s_1/s_m$. Let us assume that the distribution function of the processing times is given by

$$F(x)=x^a \quad (0\le x\le 1,\ 0<a<\infty). \tag{4}$$

In that case, $\underline{p}_{k:n} \stackrel{\mathrm{d}}{=} \underline{U}^b_{k:n}$, where $\underline{U}_{k:n}$ $(k=1,\ldots,n)$ are the order statistics of n independent random variables uniformly distributed on $[0,1]$, and $b=1/a$.

To study the asymptotic behaviour of $\underline{D}_n(\alpha)$, let us define the random variable $\underline{T}_n$ to be the index $p\in\{1,\ldots,n\}$ for which the maximum in (3) is actually achieved. Hence, $\underline{T}_n=p$ implies that

$$\underline{U}^b_{p:n}-\frac{1}{\alpha}\sum_{j=1}^{p}\underline{U}^b_{j:n}\ge \underline{U}^b_{k:n}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{U}^b_{j:n} \quad (k=1,\ldots,p-1) \tag{5}$$

i.e., that

$$\alpha\underline{U}^b_{k:n}+\sum_{j=k+1}^{p}\underline{U}^b_{j:n}-\alpha\underline{U}^b_{p:n}\le 0 \quad (k=1,\ldots,p-1) \tag{6}$$

so that (by addition of these inequalities)

$$\sum_{k=1}^{p-1}(\alpha+k-1)\underline{U}^b_{k:n}\le(\alpha-1)(p-1)\underline{U}^b_{p:n}. \tag{7}$$

Thus, $\Pr\{\underline{T}_n=p\}$ is bounded from above by the probability of (7).

Now, it is easily verified that

$$\begin{aligned}&\Pr\{\underline{U}^b_{k:n}\le x_k\ (k=1,\ldots,p-1)\mid \underline{U}^b_{p:n}=y\}\\ &\quad=\Pr\{(\underline{U}_{k:p-1}(y^a))^b\le x_k\ (k=1,\ldots,p-1)\}\end{aligned} \tag{8}$$

where, for any $z\in[0,1]$, $\underline{U}_{k:n}(z)$ $(k=1,\ldots,p-1)$ are the order statistics of $p-1$ independent random variables, uniformly distributed on $[0,z]$ [Karlin & Taylor 1981, p. 103].

Let $F_p(y)=\Pr\{\underline{U}^b_{p:n}\le y\}$. Then (7) and (8) imply that

$$\begin{aligned}&\Pr\{\underline{T}_n=p\}\\ &\quad\le\int_0^1\Pr\left\{\sum_{k=1}^{p-1}(\alpha+k-1)(\underline{U}_{k:p-1}(y^a))^b\le(\alpha-1)(p-1)y\right\}F_p(\mathrm{d}y)\end{aligned} \tag{9}$$

Since $(\underline{U}_{k:p-1}(y^a))^b/y \stackrel{\mathrm{d}}{=} \underline{U}^b_{k:p-1}$, (9) is bounded by

$$\Pr\left\{\sum_{k=1}^{p-1}(\alpha+k-1)\underline{U}^b_{k:p-1}\le(\alpha-1)(p-1)\right\}. \tag{10}$$

Lemma 1 in Appendix 1 implies that for certain constants $C=C(\alpha)$, $c=c(\alpha)$

$$\Pr\{\underline{T}_n=p\}\le C\,\mathrm{e}^{-cp}. \tag{11}$$

To derive our main result from the Borel-Cantelli lemma, we now use (11) to bound

$$\Pr\left\{\underline{D}_n(\alpha)\ge\left[\frac{D\log_2 n}{n}\right]^b\right\} \tag{12}$$

(where D is a constant to be chosen later and $\log_2 n=\log\log n$) by

$$\Pr\{\underline{T}_n \geq \log n\}$$
$$+\Pr\left\{\max_{1\leq k\leq \log n}\left\{\underline{U}^b_{k:n}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{U}^b_{j:n}\right\}\geq\left(\frac{D\log_2 n}{n}\right)^b\right\} \tag{13}$$

The first term in (13) is $O(n^{-c})$ from (11). We again condition on the value $\underline{U}^b_{\log n:n}$ being greater or smaller than $(2(\log n)/n)^b$, to bound the second term by

$$\Pr\left\{\underline{U}^b_{\log n:n}\geq\left(\frac{2\log n}{n}\right)^b\right\}$$
$$+\int_0^{(2\log n/n)^b}\Pr\left\{\max_{1\leq k\leq \log n}\left\{\underline{U}^b_{k:n}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{U}^b_{j:n}\right\}\right.$$
$$\left.\geq\left(\frac{D\log_2 n}{n}\right)^b\,\middle|\,\underline{U}^b_{\log n:n}=y\right\}F_{\log n}(\mathrm{d}y). \tag{14}$$

The first term in (14) is $O(n^{-1/4})$ (cf. [de Haan & Taconis 1979]). To bound the second term, we observe that the term within the integral is bounded for every $y\in(0,1)$ by (cf. (8))

$$\Pr\left\{\max_{1\leq k\leq \log n-1}\left\{\underline{U}^b_{k:\log n-1}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{U}^b_{j:\log n-1}\right\}\geq\frac{1}{y}\left(\frac{D\log_2 n}{n}\right)^b\right\}$$
$$+\Pr\left\{\left(1-\frac{1}{\alpha}\right)-\frac{1}{\alpha}\sum_{j=1}^{\log n-1}\underline{U}^b_{j:\log n-1}\geq\frac{1}{y}\left(\frac{D\log_2 n}{n}\right)^b\right\}, \tag{15}$$

so that the integral itself is bounded by

$$\Pr\left\{\max_{1\leq k\leq \log n-1}\left\{\underline{U}^b_{k:\log n-1}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{U}^b_{j:\log n-1}\right\}\geq\left(\frac{D\log_2 n}{2\log n}\right)^b\right\}$$
$$+\Pr\left\{\frac{1}{\alpha}\sum_{j=1}^{\log n-1}\underline{U}^b_{j:\log n-1}\leq 1-\frac{1}{\alpha}\right\}. \tag{16}$$

The second probability in (16) converges exponentially to 0 (in $\log n$). We again bound the first probability by conditioning on the index $\underline{T}(\log n-1)$ (where the maximum is attained) being greater or smaller than $d\log_2 n$, for a constant d still to be chosen. From (11), the probability of the former event is $O((\log n)^{-cd})$. The remaining conditional probability is bounded by

$$\Pr\left\{\underline{U}^b_{d\log_2 n:\log n-1}\geq\left(\frac{D\log_2 n}{2\log n}\right)^b\right\}. \tag{17}$$

For $d=D/4$, this term is $O((\log n)^{-D/16})$ (cf. [de Haan & Taconis 1979]).

Collecting all our upper bounds on (12), we conclude that, if $D=2\max\{16,4/c\}$, then

$$\Pr\left\{\underline{D}_n(\alpha)\geq\left(\frac{D\log_2 n}{n}\right)^b\right\}=O(1/(\log n)^2). \tag{18}$$

Define $k_n = \mathrm{e}^n$. The Borel–Cantelli lemma implies immediately that

$$\limsup_{n\to\infty}\left(\frac{k_n}{\log_2 k_n}\right)^b \underline{D}_{k_n}(\alpha) < \infty \qquad \text{(a.s.).} \tag{19}$$

We show in the Appendix (Lemma 2) that $\underline{D}_n(\alpha)$ is almost surely nonincreasing in n. It follows that

$$\limsup_{n\to\infty}\left(\frac{n}{\log_2 n}\right)^b \underline{D}_n(\alpha) < \infty \qquad \text{(a.s.)} \tag{20}$$

and we have proved the main result of this section.

Theorem 1. *If the distribution function of the processing times equals $F(x) = x^a$ $(0 \le x \le 1,\ 0 < a < \infty)$, then*

$$\limsup_{n\to\infty}\left(\frac{n}{\log_2 n}\right)^{1/a} (\underline{Z}_n^{(m)}(\mathrm{LPT}) - \underline{Z}_n^{(m)}(\mathrm{OPT})) < \infty \qquad (a.s.).$$

This speed of convergence result is the best possible one that can be derived from the upper bound (3), as can be seen from the fact that

$$\Pr\left\{\underline{U}_{1:n} \ge \frac{\log_2 n}{n} \text{ i.o.}\right\} = 1. \tag{21}$$

It can be shown [Karp 1983] that the speed of convergence to optimality for the LPT rule is at least $1/n$ for the case that $a = 1$. In the next section, we shall see that this lower bound is also an upper bound when we consider convergence in expectation.

3. Convergence in expectation

Again, we assume that $F(x) = x^a$ $(0 \le x \le 1,\ 0 < a < \infty)$. With $\underline{T}_n$ as defined before,

$$E(\underline{D}_n(\alpha)^q) \le \Pr\{\underline{T}_n = n\} + E\left(\left(\max_{1\le k\le n-1}\left\{\underline{U}_{k:n}^b - \frac{1}{\alpha}\sum_{j=1}^k \underline{U}_{j:n}^b\right\}\right)^q\right). \tag{22}$$

As before we condition on the value of the largest order statistic to bound the second term by

$$\begin{aligned}
&E\left(E\left(\left(\max_{1\le k\le n-1}\left\{\underline{U}_{k:n}^b - \frac{1}{\alpha}\sum_{j=1}^k \underline{U}_{j:n}^b\right\}\right)^q \middle| \underline{U}_{n:n}\right)\right)\\
&= E\left(\underline{U}_{n:n}^{qb} E\left(\left(\max_{1\le k\le n-1}\left\{\left(\frac{\underline{U}_{k:n}}{\underline{U}_{n:n}}\right)^b - \frac{1}{\alpha}\sum_{j=1}^k \left(\frac{\underline{U}_{j:n}}{\underline{U}_{n:n}}\right)^b\right\}\right)^q \middle| \underline{U}_{n:n}\right)\right)\\
&= E(\underline{U}_{n:n}^{qb}) E\left(\left(\max_{1\le k\le n-1}\left\{\underline{U}_{k:n-1}^b - \frac{1}{\alpha}\sum_{j=1}^k \underline{U}_{j:n-1}^b\right\}\right)^q\right).
\end{aligned} \tag{23}$$

Hence, for n sufficiently large, (11) and (23) together imply that

$$E(\underline{D}_n(\alpha)^q) \le \mathrm{e}^{-cn} + \frac{n}{n+qb} E(\underline{D}_{n-1}(\alpha)^q). \tag{24}$$

Let $h_n = (n+1)^{qb} E(\underline{D}_n(\alpha)^q)$. Then (24) implies that

$$h_n \le (n+1)^{qb} \mathrm{e}^{-cn} + \mathrm{e}^{(qb/n)^2} h_{n-1}. \tag{25}$$

This implies that h_n is bounded by a constant and we have proved the main result of this section.

Theorem 2. *If the distribution function of the processing times equals* $F(x) = x^a$ $(0 \le x \le 1,\ 0 < a < \infty)$, *then*

$$\limsup_{n\to\infty} n^{q/a} E((\underline{Z}_n^{(m)}(\mathrm{LPT}) - \underline{Z}_n^{(m)}(\mathrm{OPT}))^q) < \infty.$$

Identical results for the case that $s_i = 1$ for all i are derived in a different fashion in [Boxma 1984], see also [Coffman et al. 1984]. Our proofs, especially the above one, are different and much simpler. Again, they are the sharpest possible ones in the sense of the previous section. It is worth noting that this is the first time that bounds on higher moments have been derived for a heuristic of this nature.

4. Extensions and concluding remarks

Theorems 1 and 2 can both be extended to the case that $F(x) = \Theta(x^a)$, i.e., there exist positive constants ε, L and U such that

$$Lx^a \le F(x) \le Ux^a \quad \text{for } x \in [0, \varepsilon). \tag{26}$$

In the case of almost sure convergence, this is done by showing that one may restrict oneself in the maximization (3) to $k \in \{1, \dots, [\varepsilon n]\}$ (as in [Frenk & Rinnooy Kan 1984]). This maximization involves only the smaller order statistics and for those we are essentially in the situation analyzed in Section 2.

In the case of convergence in expectation, our technique requires that

$$E\underline{p}^{q(1+b)+1} < \infty \tag{27}$$

for the extension of Theorem 3 to hold. We strongly suspect, however, that this condition is not essential. Details of the proofs for these results can be found in Appendix 2.

These unusually strong results, as well as other recent ones in this area confirm the remarkable amenability of the LPT rule to a probabilistic analysis. Extensions to other priority rules involving order statistics of processing times seem feasible and interesting.

Appendix 1

Lemma 1. *For every $\beta \geq 1$ there are positive constants $C = C(\beta)$ and $c = c(\beta)$ such that*

$$\Pr\left\{\sum_{k=1}^{m} (\beta + k)\underline{U}_{k:m}^{1/a} \leq \beta m\right\} \leq C\,\mathrm{e}^{-cm}.$$

Proof. If $a \geq 1$, then $\underline{U}_{k:m}^{1/a} \geq \underline{U}_{k:m}$ a.s.

Also, if $a < 1$, we obtain from Hölder's inequality [Goffman & Pedrick 1965, p. 2] that (take $p = 1/a$, $q = 1/(1-a)$, $y_k = (\beta + k)^a \underline{U}_{k:m}$, $x_k = 1$)

$$\sum_{k=1}^{m} (\beta + k)^a \underline{U}_{k:m} \leq m^{1-a}\left(\sum_{k=1}^{m} (\beta + k)\underline{U}_{k:m}^{1/a}\right)^a \quad \text{a.s.} \tag{A.1}$$

This implies that

$$\begin{aligned}
&\Pr\left\{\sum_{k=1}^{m} (\beta + k)\underline{U}_{k:m}^{1/a} \leq \beta m\right\} \\
&= \Pr\left\{\left(\sum_{k=1}^{m} (\beta + k)\underline{U}_{k:m}^{1/a}\right)^a \leq \beta^a m^a\right\} \\
&\leq \Pr\left\{\sum_{k=1}^{m} (\beta + k)^a \underline{U}_{k:m} \leq \beta^a m\right\}.
\end{aligned} \tag{A.2}$$

Hence we consider the distribution of

$$\sum_{k=1}^{m} (\beta + k)^a \underline{U}_{k:m} \quad (a \leq 1).$$

Since $(\underline{U}_{1:m}, \ldots, \underline{U}_{m,m}) \stackrel{d}{=} (\underline{S}_1/\underline{S}_{m+1}, \ldots, \underline{S}_m/\underline{S}_{m+1})$ with $\underline{S}_i = \sum_{j=1}^{i} \underline{V}_j$ and $\underline{V}_j$ independent exponentially distributed random variables with parameter $\lambda = 1$ $(j = 1, \ldots, m)$ ([Karlin & Taylor 1981, p. 103]), we can rewrite the right hand side of (A.2) as

$$\Pr\left\{\sum_{k=1}^{m+1} (\beta + k)^a (\underline{S}_{m+1} - \underline{S}_k) \geq \underline{S}_{m+1}\left(\sum_{k=1}^{m} (\beta + k)^a - \beta^a m\right)\right\}.$$

Now for every $\varepsilon > 0$, there exists some $m_0 = m_0(\varepsilon)$ such that, for every $m \geq m_0(\varepsilon)$, the above probability is bounded from above by

$$\begin{aligned}
&\Pr\left\{\sum_{l=1}^{m+1} (1 + l + \beta)^{a+1} \underline{V}_l \geq ((1-\varepsilon)m)^{a+1} \underline{S}_{m+1}\right\} \\
&= \Pr\left\{\sum_{l=1}^{m+1} c_{l,m+1} \underline{V}_l \geq 0\right\}
\end{aligned}$$

with

$$c_{l,m+1} = ((1 + l + \beta)/(1-\varepsilon)m)^{a+1} - 1. \tag{A.4}$$

Clearly

$$-1 < c_{l,m+1} \le ((m+2+\beta)/(1-\varepsilon)m)^{a+1} - 1 \quad (l = 1, \dots, m+1)$$

and this implies for $\lambda \in [0, \frac{1}{2}(2(1-\varepsilon)/3)^{a+1}]$ and $m \ge \max(m_0(\varepsilon), 2(\beta+2))$ that

$$\Pr\left\{\sum_{l=1}^{m+1} c_{l,m+1} \underline{V}_l \ge 0\right\} \le E\left(\exp\left(\lambda \sum_{l=1}^{m+1} c_{l,m+1} \underline{V}_l\right)\right)$$

$$= \prod_{l=1}^{m+1} 1/(1-\lambda c_{l,m+1}). \tag{A.5}$$

From the Taylor expansion of $\log(1+x)$ around $x=0$, we then show that the above term is bounded by

$$\exp\left(\lambda \sum_{l=1}^{m+1} c_{l,m+1} + \lambda^2 \sum_{l=1}^{m+1} c_{l,m+1}^2\right).$$

Since

$$\lim_{m\to\infty} \left(\sum_{l=1}^{m+1} c_{l,m+1}\right) \Big/ m = 1/((a+1)(1-\varepsilon)^{a+1}) \tag{A.6}$$

and

$$\lim_{m\to\infty} \left(\sum_{l=1}^{m+1} c_{l,m+1}^2\right) \Big/ m = 1/((2a+3)(1-\varepsilon)^{2a+2}) - 2/((a+2)(1-\varepsilon)^{a+1}) + 1, \tag{A.7}$$

the desired result follows from the appropriate choice of positive values for λ and ε.

Lemma 2.

$$\sum_{n=1}^{\infty} \Pr\{\underline{D}_{n+1}(\alpha) > \underline{D}_n(\alpha)\} < \infty.$$

Proof. It is easy to verify that $D_{n+1}(\alpha) \le \underline{D}_n(\alpha)$ unless (perhaps) if the new processing time is larger than all the previous ones.

Hence,

$$\Pr\{\underline{D}_{n+1}(\alpha) > \underline{D}_n(\alpha)\} \le \Pr\left\{\underline{p}_{n+1} - \frac{1}{\alpha}\sum_{j=1}^{n+1} \underline{p}_j > 0\right\}$$

$$= \int_0^{\infty} F^{n*}((\alpha-1)y) F(\mathrm{d}y) \tag{A.8}$$

so that

$$\sum_{n=1}^{\infty} \Pr\{\underline{D}_{n+1}(\alpha) > \underline{D}_n(\alpha)\} \le \int_0^{\infty} U((\alpha-1)y) F(\mathrm{d}y) \tag{A.9}$$

with $U(x) = \sum_{n=1}^{\infty} F^{n*}(x)$ the *renewal function* ([Feller 1971; Van Dulst & Frenk

1984]). The result now follows from

$$\lim_{x\to\infty} \frac{U(x)}{x} = \int_0^\infty x\,F(\mathrm{d}x) < \infty \tag{A.10}$$

and the local boundedness of $U(x)$ on $(0, \infty)$.

Appendix 2

The purpose of this appendix is to describe brief proofs of two results listed in Section 4.

We shall prove that Theorems 1 and 2 can be extended to the case that, for $x \in [0, \varepsilon)$ $(\varepsilon > 0)$,

$$Lx^a \le F(x) \le Ux^a \quad \text{with } 0 < L \le U < \infty. \tag{A.11}$$

Theorem 1a.

$$\limsup_{n\to\infty} \left(\frac{n}{\log_2 n}\right)^{1/a} (\underline{Z}_n^{(m)}(\mathrm{LPT}) - \underline{Z}_n^{(m)}(\mathrm{OPT}) < \infty \quad \text{(a.s.).} \tag{A.12}$$

Proof. As before, we consider

$$\underline{D}_n(\alpha) = \max_{1\le k\le n} \left\{\underline{p}_{k:n} - \frac{1}{\alpha}\sum_{j=1}^{k} \underline{p}_{j:n}\right\} \tag{A.13}$$

and distinguish between the case that $k \in \{1, \ldots, [\varepsilon n]\}$ and $k \in \{[\varepsilon n]+1, \ldots, n\}$. With respect to the latter range, we showed in [Frenk & Rinnooy Kan 1984] that, for every sequence $d(n)\uparrow\infty$,

$$\lim_{n\to\infty} d(n) \max_{[\varepsilon n]<k\le n} \left\{p_{k:n} - \frac{1}{\alpha}\sum_{j=1}^{k} p_{j:n}, 0\right\} = 0 \qquad \text{(a.s.).} \tag{A.14}$$

With respect to the former range, we have that for every $D > 0$, $\varepsilon \in (0, 1)$,

$$\begin{aligned}
&\Pr\left\{\max_{1\le k\le[\varepsilon n]} \left\{\underline{p}_{k:n} - \frac{1}{\alpha}\sum_{j=1}^{k} \underline{p}_{j:n}\right\} \ge \left(\frac{D\log_2 n}{n}\right)^{1/a}\right\}\\
&= Pr\left\{\max_{1\le k\le[\varepsilon n]} \left\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\sum_{j=1}^{k} F^{-1}(\underline{U}_{j:n})\right\} \ge \left(\frac{D\log_2 \mathrm{n}}{\mathrm{n}}\right)^{1/a}\right\}\\
&\le \Pr\left\{\underline{U}_{[\varepsilon n]:n} \le 2\varepsilon,\ \max_{1\le k\le[\varepsilon n]} \left\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\sum_{j=1}^{k} F^{-1}(\underline{U}_{j:n})\right\} \ge \left(\frac{D\log_2 n}{n}\right)^{1/a}\right\}\\
&\quad + \Pr\{\underline{U}_{[\varepsilon n]:n} > 2\varepsilon\}\\
&\le \Pr\left\{\underline{U}_{[\varepsilon n]:n} \le 2\varepsilon,\ \max_{1\le k\le[\varepsilon n]} \left\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\sum_{j=1}^{k} F^{-1}(\underline{U}_{j:n})\right\} \ge \left(\frac{D\log_2 n}{n}\right)^{1/a}\right\}\\
&\quad + \mathrm{e}^{-(\varepsilon/4)n}.
\end{aligned} \tag{A.15}$$

Now (A.11) implies that the first term on the right-hand side is bounded by

$$\Pr\left\{\max_{1\le k\le[\varepsilon n]}\left\{(\underline{U}_{k:n})^{1/a}-\frac{1}{\alpha^*}\sum_{j=1}^{k}(\underline{U}_{j:n})^{1/a}\right\}\ge\left(\frac{D^*\log_2 n}{n}\right)^{1/a}\right\}$$
$$\le\Pr\left\{\max_{1\le k\le n}\left\{(\underline{U}_{k:n})^{1/a}-\frac{1}{\alpha^*}\sum_{j=1}^{k}(\underline{U}_{j:n})^{1/a}\right\}\ge\left(\frac{D^*\log_2 n}{n}\right)^{1/a}\right\} \quad (A.16)$$

where $\alpha^*=\alpha\bar{U}/\bar{L}$ and $D^*=D/\bar{U}^{1/a}$, with

$$0<\bar{L}x^{1/a}\le F^{-1}(x)\le\bar{U}x^{1/a}<\infty \quad (A.17)$$

for x sufficiently small. But with (A.16) we are essentially back in the situation analysed in Section 2, and we can copy the arguments there and use (A.14) to prove Theorem 1a.

Theorem 2a. *If*

$$E\underline{p}^{q(1+b)+1}<\infty, \quad (A.18)$$

then

$$\limsup_{n\to\infty}(n+1)^{qb}E((\underline{Z}_n^{(m)}(\mathrm{LPT})-\underline{Z}_n^{(m)}(\mathrm{OPT}))^q)<\infty. \quad (A.19)$$

Proof. For every $q>0$ and $\varepsilon\in(0,1)$,

$$E\left(\left(\max_{1\le k\le n}\left\{\underline{p}_{k:n}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{p}_{j:n}\right\}\right)^q\right)$$
$$\le E\left(\left(\max\left(\max_{1\le k\le[\varepsilon n]}\left\{\underline{p}_{k:n}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{p}_{j:n}\right\},\underline{p}_{n:n}-\frac{1}{\alpha}\sum_{j=1}^{[\varepsilon n]}\underline{p}_{j:n}\right)\right)^q\right)$$
$$\le E\left(\left(\max_{1\le k\le[\varepsilon n]}\left\{\underline{p}_{k:n}-\frac{1}{\alpha}\sum_{j=1}^{k}\underline{p}_{j:n}\right\}\right)^q\right)$$
$$+E\left(\left(\max\left\{\underline{p}_{n:n}-\frac{1}{\alpha}\sum_{j=1}^{[\varepsilon n]}\underline{p}_{j:n},0\right\}\right)^q\right). \quad (A.20)$$

The first term on the right hand side of (A.20) can be bounded by

$$E\left(\left(\max_{1\le k\le[\varepsilon n]}\left\{F^{-1}(\underline{U}_{k:n})-\frac{1}{\alpha}\sum_{j=1}^{k}F^{-1}(\underline{U}_{j:n})\right\}\right)^q I_{\{\underline{U}_{[\varepsilon n]:n}\ge 2\varepsilon\}}\right)$$
$$+E\left(\left(\max_{1\le k\le[\varepsilon n]}\left\{F^{-1}(\underline{U}_{k:n})-\frac{1}{\alpha}\sum_{j=1}^{k}F^{-1}(\underline{U}_{j:n})\right\}\right)^q I_{\{\underline{U}_{[\varepsilon n]:n}<2\varepsilon\}}\right)$$
$$\le E((F^{-1}(\underline{U}_{[\varepsilon n]:n}))^q I_{\{\underline{U}_{[\varepsilon n]:n}\ge 2\varepsilon\}})$$
$$+E\left(\left(\max_{1\le k\le[\varepsilon n]}\left\{F^{-1}(\underline{U}_{k:n})-\frac{1}{\alpha}\sum_{j=1}^{k}F^{-1}(\underline{U}_{j:n})\right\}^q I_{\{\underline{U}_{[\varepsilon n]:n}<2\varepsilon\}}\right)\right). \quad (A.21)$$

As in the previous proof, (A.11) implies that the second term is $O(n^{-qb})$. The first term can easily be seen to be $O(n^{-qb})$.

The second term on the right hand side of (A.20) can be bounded by conditioning on $\underline{p}_{n:n}$ being smaller or greater than $\beta n > 0$. In the former case, the conditional expectation can be seen to be bounded by

$$\Pr\left\{\frac{1}{\alpha}\sum_{j=1}^{[\varepsilon n]} \underline{p}_{j:n} \leq \beta n\right\} \cdot O(n^q). \tag{A.22}$$

In the latter case, it is bounded by

$$E(\underline{p}_{n:n}^q I_{\underline{p}_{n:n} \geq \beta n}) \leq n \int_{\beta n}^{\infty} y^q F(\mathrm{d}y). \tag{A.23}$$

Now (A.11) implies that, for an appropriate choice of β, the probability in (A.22) decreases to 0 exponentially fast. The remaining term (A.23) then implies the need for (A.18) to hold for the theorem to be satisfied.

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